Supplementary material to "Semi-parametric transformation boundary regression models"*

Natalie Neumeyer, Leonie Selk and Charles Tillier

Department of Mathematics, University of Hamburg

July 31, 2019

C Proofs of asymptotic results in the fixed design case

C.1 Proof of Lemma 5

To prove Lemma 5, we first need the following technical lemma.

Lemma 1. *Assume model (2) holds under assumptions (**A1'**), (**A2'**) and (**A4'**). Then we have*

$$\sup_{x\in[0,1]} \min_{\substack{i\in\{1,\dots,n\}\\ |x_{i,n}-x|\le b_n}} |\varepsilon_{i,n}| = o_P(1).$$

Proof. The proof is similar to the proof of Lemma A.2 in Drees et al. (2018) but some adaptations are needed to deal with non-equidistant fixed design points. Let $Z_1, Z_2, \dots$ be iid with the same distribution as $-\varepsilon_{i,n}$ with cumulative distribution function U. To prove the result, we shall show that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{x\in[0,1]} \min_{\substack{i\in\{1,\dots,n\}\\ |x_{i,n}-x|\le b_n}} Z_i > \epsilon\right) = 0, \quad \epsilon > 0.$$

For $n \geq 1$, let $0 < k \leq n$, $x \in [0,1]$ and set $I_n = [x - b_n, x + b_n]$. Assume that exactly k points lie in I_n, say

$$x_{m+1,n} < \dots < x_{m+k,n} \in I_n$$

for some $m < n + 1 - k$. We shall distinguish two cases.

*Financial support by the DFG (Research Unit FOR 1735 *Structural Inference in Statistics: Adaptation and Efficiency*) is gratefully acknowledged.

(1) If $(x_{m,n}, x_{m+k+1,n}) \in [0,1]^2$, it means that

$$2b_n = |I_n| < x_{m+k+1,n} - x_{m,n} = \sum_{j=m}^{m+k} (x_{j+1,n} - x_{j,n}) \le (k+1)\bar{\Delta}_n,$$

since $\bar{\Delta}_n \ge x_{j,n} - x_{j-1,n}$ for any $1 \le j \le n+1$.

(2) If $x_{m,n}$ or $x_{m+k+1,n}$ do not exist, which means that either $x_{m+1,n} = x_{0,n} = 0$ or $x_{m+k+1,n} = x_{n+1,n} = 1$. Consider the first case $x_{m+1,n} = x_{0,n}$ (the extremal case is $x = 0$). Then we have

$$b_n = \frac{|I_n|}{2} < x_{k,n} - x_{0,n} = \sum_{j=0}^{k-1} (x_{j+1,n} - x_{j,n}) \le k\bar{\Delta}_n.$$

A similar inequality holds for $x_{m+k+1,n} = x_{n+1,n} = 1$ (with the extremal case $x = 1$).

In both cases, (**1**) and (**2**) yield to

$$b_n < k\bar{\Delta}_n \Rightarrow k > \frac{b_n}{\bar{\Delta}_n}, \quad n \ge 1.$$

Then, for all $y > 0$, we have with $d_n := \lceil \frac{b_n}{\bar{\Delta}_n} \rceil$

$$\begin{aligned}
\mathbb{P}\left(\sup_{x\in[0,1]} \min_{\substack{i\in\{1,\dots,n\} \\ |x_{i,n}-x|\le b_n}} Z_i > y \right) &\le \mathbb{P}\left(\left\{ \max_{j\in\{1,\dots,n-d_n\}} \min_{i\in\{j,\dots,j+d_n\}} Z_i > y \right\} \right) \\
&\le \sum_{j=1}^{n-d_n} \mathbb{P}\left(\min_{i\in\{j,\dots,j+d_n\}} Z_i > y \right) \\
&= (n-d_n)\mathbb{P}\left(\min_{i\in\{1,\dots,d_n+1\}} Z_i > y \right) \\
&= (n-d_n)\overline{U}(y)^{d_n+1}.
\end{aligned}$$

Thus it remains to show that for all $\epsilon > 0$

$$(n-d_n)\overline{U}(\epsilon)^{d_n+1} \xrightarrow[n\to\infty]{} 0$$

which is true since $d_n \underset{n\to\infty}{\sim} \frac{b_n}{\bar{\Delta}_n}$ and

$$\begin{aligned}
\frac{b_n}{\bar{\Delta}_n}\log(\overline{U}(\epsilon)) + \log(n-d_n) &\le \frac{b_n}{\bar{\Delta}_n}\log(\overline{U}(\epsilon)) + \log(n) \\
&= \log(n)\left(\frac{b_n}{\bar{\Delta}_n \log(n)} \log(\overline{U}(\epsilon)) + 1 \right) \\
&\xrightarrow[n\to\infty]{} -\infty
\end{aligned}$$

since $\overline{U}(\epsilon) < 1$ under (**A1'**) and $\frac{b_n}{\bar{\Delta}_n \log(n)} \xrightarrow[n\to\infty]{} \infty$ under (**A4'**). This concludes the proof.

□

The **proof of Lemma 5** is analogous to the proof of Lemma 3.

C.2 Proof of Theorem 6 in the fixed design case

The first part of the proof is similar to the random design case. Here, we use

$$\begin{aligned}\sup_{\vartheta\in\Theta}|M_n(\vartheta)-M(\vartheta)| \;&\le\; \sup_{\vartheta\in\Theta}\|G_n(\vartheta,\hat h_\vartheta)-\bar G_n(\vartheta,\hat h_\vartheta)\|+\sup_{\vartheta\in\Theta}\|\bar G_n(\vartheta,\hat h_\vartheta)-\tilde G_n(\vartheta,\hat h_\vartheta)\|\\ &\quad+\sup_{\vartheta\in\Theta}\|\tilde G_n(\vartheta,\hat h_\vartheta)-G(\vartheta,\hat h_\vartheta)\|+\sup_{\vartheta\in\Theta}\|G(\vartheta,\hat h_\vartheta)-G(\vartheta,h_\vartheta)\|,\end{aligned}$$

where the definition for M and G is as in the random case, and

$$\bar G_n(\vartheta,h)(y,s) \;=\; \frac{1}{n}\sum_{i=1}^n I\{\Lambda_\vartheta(Y_{i,n})-h(x_{i,n})\le y\}\big(I\{x_{i,n}\le s\}-F_X(s)\big).$$

Further,

$$\begin{aligned}\tilde G_n(\vartheta,h)(y,s) \;&=\; \frac{1}{n}\sum_{i=1}^n F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+h(x_{i,n})))-h_0(x_{i,n})\right)I\{x_{i,n}\le s\} \qquad \text{(C.1)}\\ &\quad-F_X(s)\sum_{i=1}^n F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+h(x_{i,n})))-h_0(x_{i,n})\right)\end{aligned}$$

is a Riemann-sum approximation of $G(\vartheta,h)(y,s)$. Note that for any deterministic function h we have $\tilde G_n(\vartheta,h)=\mathbb{E}[\bar G_n(\vartheta,h)]$. The assertion of the theorem follows from

$$\sup_{\vartheta\in\Theta}\|G_n(\vartheta,\hat h_\vartheta)-\bar G_n(\vartheta,\hat h_\vartheta)\|\le\sup_{s\in[0,1]}|\hat F_{X,n}(s)-F_X(s)|=o(1) \qquad \text{(C.2)}$$

and from Lemmas 2–4 by an application of the arg-max theorem. For (C.2) note that with assumption (**A2”**)

$$\begin{aligned}\sup_{s\in[0,1]}|\hat F_{X,n}(s)-F_X(s)| \;&=\; \sup_{s\in[0,1]}\left|\frac{1}{n}\sum_{i=1}^n I\{x_{i,n}\le s\}-\int_0^s f_X(x)\,dx\right| \qquad \text{(C.3)}\\ &\le\; \sup_{s\in[0,1]}\left|\sum_{i=1}^n\int_{x_{i-1,n}}^{x_{i,n}} f_X(x)\,dx I\{x_{i,n}\le s\}-\int_0^s f_X(x)\,dx\right|+o(1)\\ &=\; \sup_{s\in[0,1]}\left|\int_{\max\{x_{i,n}|x_{i,n}\le s\}}^s f_X(x)\,dx\right|+o(1)\\ &=\; \bar\Delta_n\sup_{x\in[0,1]}f_X(x)+o(1)=o(1). \qquad \text{(C.4)}\end{aligned}$$

□

Lemma 2. *Under the assumptions of Theorem 6 (ii),*

$$\sup_{\vartheta\in\Theta}\|\bar G_n(\vartheta,\hat h_\vartheta)-\tilde G_n(\vartheta,\hat h_\vartheta)\|=o_P(1).$$

Proof. As in the proof of Lemma 10 we assume in what follows that (25) holds. We only consider the difference between the first sum in the definitions of $G_n(\vartheta, h)$ and the first sum in $\tilde{G}_n(\vartheta, h)$ (see (8) and (C.1), respectively). The difference of the second sums can be treated similarly. Applying (25) the first sum in $G_n(\vartheta, \hat{h}_\vartheta)(y, s)$ can be nested as

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} I\{\Lambda_\vartheta(Y_{i,n}) - h_\vartheta(x_{i,n}) \le y - a_n\} I\{x_{i,n} \le s\} \\ \le\ &\frac{1}{n}\sum_{i=1}^{n} I\{\Lambda_\vartheta(Y_{i,n}) - \hat{h}_\vartheta(x_{i,n}) \le y\} I\{x_{i,n} \le s\} \\ \le\ &\frac{1}{n}\sum_{i=1}^{n} I\{\Lambda_\vartheta(Y_{i,n}) - h_\vartheta(x_{i,n}) \le y + a_n\} I\{x_{i,n} \le s\} \end{aligned}$$

while the first sum in $\tilde{G}_n(\vartheta, \hat{h}_\vartheta)(y, s)$ can be nested as

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y - a_n + h_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) I\{x_{i,n} \le s\} \\ \le\ &\frac{1}{n}\sum_{i=1}^{n} F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y + \hat{h}_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) I\{x_{i,n} \le s\} \\ \le\ &\frac{1}{n}\sum_{i=1}^{n} F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y + a_n + h_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) I\{x_{i,n} \le s\}. \end{aligned}$$

Thus we have to consider

$$\begin{aligned} H_{n,\vartheta}^{(1)}(y,s) &= \frac{1}{n}\sum_{i=1}^{n} \Big(I\{\Lambda_\vartheta(Y_{i,n}) - h_\vartheta(x_{i,n}) \le y + a_n\} \\ &\qquad - F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y + a_n + h_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) \Big) I\{x_{i,n} \le s\} \\ H_{n,\vartheta}^{(2)}(y,s) &= \frac{1}{n}\sum_{i=1}^{n} \Big(F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y + a_n + h_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) \\ &\qquad - F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y + h_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) \Big) I\{x_{i,n} \le s\} \end{aligned}$$

and the same terms with $y + a_n$ replaced by $y - a_n$, which can be treated completely analogously. We have to show that $\sup_{\vartheta\in\Theta} \|H_{n,\vartheta}^{(1)}\| = o_P(1)$ and $\sup_{\vartheta\in\Theta} \|H_{n,\vartheta}^{(2)}\| = o(1)$.

Recall condition (**N1**) and note that $\sup_{\vartheta\in\Theta} \sup_{s\in[0,1], y\in C} |H_{n,\vartheta}^{(2)}(y,s)| = o(1)$ follows from uniform continuity of F_0 and of $\Lambda_0 \circ \Lambda_\vartheta^{-1}$ uniformly in ϑ (see (**B5**) and (**B4**)), from the representation $h_\vartheta = \Lambda_\vartheta \circ \Lambda_0^{-1} \circ h_0$ and uniform continuity of Λ_ϑ uniformly in ϑ (see (**B3**)), and $a_n \to 0$.

Now to prove $\sup_{\vartheta\in\Theta} \|H_{n,\vartheta}^{(1)}\| = o_P(1)$, let $\epsilon > 0$ and for the moment fix $s \in [0,1]$, $\vartheta \in \Theta$ and $y \in C$. Choose $\delta > 0$ corresponding to ϵ as in assumption (**B5**). Let n be large enough such that $|a_n| \le \tau$ for τ both from (**B5**) and (**B4**).

Partition $[0,1]$ into finitely many intervals $[s_j, s_{j+1}]$ such that $F_X(s_{j+1}) - F_X(s_j) < \epsilon$ for all j. For the fixed s, denote the interval containing s by $[s_j, s_{j+1}] = [s^\ell, s^u]$.

Now choose a finite sup-norm bracketing of length γ for the class $\mathcal{L}_S = \{\Lambda_\vartheta|_S : \vartheta \in \Theta\}$ according to (10) with γ as in assumption (**B4**) corresponding to the above chosen δ. For the fixed ϑ this gives a bracket $h^\ell \le h_\vartheta \le h^u$ of sup-norm length γ.

Choose a finite sup-norm bracketing of length δ for the class $\mathcal{L}^1_{\tilde{S}} = \{\Lambda_0 \circ \Lambda_\vartheta^{-1}|_{\tilde{S}} : \vartheta \in \Theta\}$ according to (10). For the fixed ϑ this gives a bracket $V^\ell \le \Lambda_0 \circ \Lambda_\vartheta^{-1} \le V^u$.

Then consider the bounded and increasing function

$$D_n(y) = \frac{1}{n}\sum_{i=1}^n F_0(V^\ell(y + a_n + h^\ell(x_{i,n})) - h_0(x_{i,n}))$$

and choose a finite partition of the compact C in intervals $[y_k, y_{k+1}]$ such that $D_n(y_{k+1}) - D_n(y_k) < \epsilon$. For the fixed y, denote the interval containing y by $[y_k, y_{k+1}] = [y^\ell, y^u]$. Note that the brackets depend on n. This is suppressed in the notation because it is not relevant for the remainder of the proof because the number of brackets is $O(\epsilon^{-1})$, uniformly in n.

Now we can nest as follows

$$\begin{aligned}
&I\{\Lambda_0(Y_{i,n}) \le V^\ell(y^\ell + a_n + h^\ell(x_{i,n}))\} I\{x_{i,n} \le s^\ell\} \\
\le\ & I\{\Lambda_\vartheta(Y_{i,n}) - h_\vartheta(x_{i,n}) \le y + a_n\} I\{x_{i,n} \le s\} \\
=\ & I\{Y_{i,n} \le \Lambda_\vartheta^{-1}(y + a_n + h_\vartheta(x_{i,n}))\} I\{x_{i,n} \le s\} \\
\le\ & I\{\Lambda_0(Y_{i,n}) \le V^u(y^u + a_n + h^u(x_{i,n}))\} I\{x_{i,n} \le s^u\},
\end{aligned}$$

and have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n \Big(\mathbb{E}[I\{\Lambda_0(Y_{i,n}) \le V^u(y^u + a_n + h^u(x_{i,n}))\} I\{x_{i,n} \le s^u\}] \\
&\qquad - \mathbb{E}[I\{\Lambda_0(Y_{i,n}) \le V^\ell(y^\ell + a_n + h^\ell(x_{i,n}))\} I\{x_{i,n} \le s^\ell\}] \Big) \\
\le\ & \hat{F}_{X,n}(s^u) - \hat{F}_{X,n}(s^\ell) \\
&+ \frac{1}{n}\sum_{i=1}^n \Big| F_0\left(V^u(y^u + a_n + h^u(x_{i,n})) - h_0(x_{i,n})\right) - F_0\left(V^\ell(y^\ell + a_n + h^\ell(x_{i,n})) - h_0(x_{i,n})\right) \Big| \\
\le\ & 2\epsilon + o(1) \\
&+ \frac{1}{n}\sum_{i=1}^n \Big| F_0\left(V^u(y^u + a_n + h^u(x_{i,n})) - h_0(x_{i,n})\right) - F_0\left(V^\ell(y^u + a_n + h^\ell(x_{i,n})) - h_0(x_{i,n})\right) \Big|
\end{aligned}$$

by (C.3) and the definitions of $[s^\ell, s^u]$ and $[y^\ell, y^u]$. Further, we can bound the last sum by

$$\frac{1}{n}\sum_{i=1}^n \Big| F_0\left(V^u(y^u + a_n + h^u(x_{i,n})) - h_0(x_{i,n})\right)$$

$$
\begin{aligned}
&\quad - F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y^u + a_n + h^u(x_{i,n}))) - h_0(x_{i,n})\right)\Big| \\
&+ \frac{1}{n}\sum_{i=1}^n \Big| F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y^u + a_n + h^\ell(x_{i,n}))) - h_0(x_{i,n})\right) \\
&\quad - F_0\left(V^\ell(y^u + a_n + h^\ell(x_{i,n})) - h_0(x_{i,n})\right)\Big| \\
&+ \frac{1}{n}\sum_{i=1}^n \Big| F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y^u + a_n + h^u(x_{i,n}))) - h_0(x_{i,n})\right) \\
&\quad - F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y^u + a_n + h^\ell(x_{i,n}))) - h_0(x_{i,n})\right)\Big| \\
\leq\ & 3\epsilon
\end{aligned}
$$

using the construction of brackets above (note that $\|V^u - \Lambda_0 \circ \Lambda_\vartheta^{-1}\|_\infty \leq \delta$, $\|\Lambda_0 \circ \Lambda_\vartheta^{-1} - V^\ell\|_\infty \leq \delta$, $\|h^u - h^\ell\|_\infty \leq \gamma$ and recall assumptions (**B5**) and (**B4**)).

Thus $\sup_{\vartheta\in\Theta} \sup_{s\in[0,1], y\in C} |H^{(1)}_{n,\vartheta}(y,s)|$ can be bounded by $O(\epsilon) + o(1)$ plus a finite maximum over the absolute value of terms

$$
\frac{1}{n}\sum_{i=1}^n \Big(I\{\Lambda_0(Y_{i,n}) \leq V^u(y^u + a_n + h^u(x_{i,n}))\} - \mathbb{E}[I\{\Lambda_0(Y_{i,n}) \leq V^u(y^u + a_n + h^u(x_{i,n}))\}]\Big)
$$

and

$$
\frac{1}{n}\sum_{i=1}^n \Big(I\{\Lambda_0(Y_{i,n}) \leq V^\ell(y^\ell + a_n + h^\ell(x_{i,n}))\} - \mathbb{E}[I\{\Lambda_0(Y_{i,n}) \leq V^\ell(y^\ell + a_n + h^\ell(x_{i,n}))\}]\Big).
$$

However, those converge to zero in probability by a simple application of Chebychev's inequality.

This completes the proof of $\sup_{\vartheta\in\Theta} \|H^{(1)}_{n,\vartheta}\| = o_P(1)$ and thus of the lemma. □

Lemma 3. *Under the assumptions of Theorem 6 (ii),*

$$
\sup_{\vartheta\in\Theta} \|G(\vartheta, h_\vartheta) - G(\vartheta, \hat{h}_\vartheta)\| = o_P(1).
$$

Proof. The proof is analogous to the proof of Lemma 11. □

Lemma 4. *Under the assumptions of Theorem 6 (ii),*

$$
\sup_{\vartheta\in\Theta} \|\tilde{G}_n(\vartheta, \hat{h}_\vartheta) - G(\vartheta, \hat{h}_\vartheta)\| = o_P(1).
$$

Proof. According to assumption (**N1**) it suffices to show

$$
\sup_{\vartheta\in\Theta} \sup_{\substack{s\in[0,1]\\ y\in C}} |\tilde{G}_n(\vartheta, \hat{h}_\vartheta)(y,s) - G(\vartheta, \hat{h}_\vartheta)(y,s)| = o_P(1).
$$

Recalling the definitions of $\tilde{G}_n$ in (C.1) and G in (7) we only consider the first sum and first integral, respectively. It holds by the mean value theorem for integration

$$\begin{aligned}
&\Bigg|\frac{1}{n}\sum_{i=1}^{n} F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) I\{x_{i,n}\le s\} \\
&\quad - \int F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(x))) - h_0(x)\right) I\{x\le s\} f_X(x)\,dx\Bigg| \\
=&\Bigg|\sum_{i=1}^{n}\Big(\frac{1}{n}F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) \\
&\quad - \int_{x_{i-1,n}}^{x_{i,n}} F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(x))) - h_0(x)\right) f_X(x)dx\Big) I\{x_{i,n}\le s\} \\
&\quad - \int_{\max\{x_{i,n}|x_{i,n}\le s\}}^{s} F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(x))) - h_0(x)\right) f_X(x)\,dx\Bigg| \\
\le& \sum_{i=1}^{n}\Bigg|\frac{1}{n}F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(x_{i,n}))) - h_0(x_{i,n})\right) \\
&\quad - F_0\left(\Lambda_0(\Lambda_\vartheta^{-1}(y+\hat{h}_\vartheta(\xi_{i,n}))) - h_0(\xi_{i,n})\right) f_X(\xi_{i,n})(x_{i,n}-x_{i-1,n})\Bigg| \\
&\quad + O(\bar{\Delta}_n)
\end{aligned}$$

for some $\xi_{i,n}\in[x_{i-1,n},x_{i,n}]$. Now the assertion follows from assumption (**A2”**), uniform continuity of F_0 and of $\Lambda_0\circ\Lambda_\vartheta^{-1}$ (uniformly in ϑ) and from

$$|\hat{h}_\vartheta(x_{i,n}) - \hat{h}_\vartheta(\xi_{i,n})| \le \|\hat{h}_\vartheta - h_\vartheta\|_\infty + |\Lambda_\vartheta(\Lambda_0^{-1}(h_0(x_{i,n})) - \Lambda_\vartheta(\Lambda_0^{-1}(h_0(\xi_{i,n})))|$$

in connection with Lemma 7 and assumptions (**A3**), (**B3**). □

D Identifiability of the model in the fixed design case

To prove identifiability in the case of deterministic covariates as in Remark 3 one starts similarly to the proof in section B of the appendix (main paper) with the cdf of $\varepsilon_{i,n}(\vartheta_1) = \Lambda_{\vartheta_1}(Y_{i,n}) - h_{\vartheta_1}(x_{i,n})$ in y to obtain that $H^{-1}(y+H(h_{\vartheta_0}(x_{i,n}))) - h_{\vartheta_0}(x_{i,n})$ does not depend on $x_{i,n}$ for $y\in(-\infty,0]$. Due to continuity of the functions and $\bar{\Delta}_n\to 0$ one obtains that $H^{-1}(y+H(h_{\vartheta_0}(x))) - h_{\vartheta_0}(x)$ does not depend on $x\in[0,1]$ for $y\in(-\infty,0]$. The remainder of the proof is as in section B.

References

Drees, H., Neumeyer, N. and Selk, L. (2018). Estimation and hypotheses tests in boundary regression models. *Bernoulli*, to appear.

http://www.bernoulli-society.org/index.php/publications/bernoulli-journal

E Figures and Tables

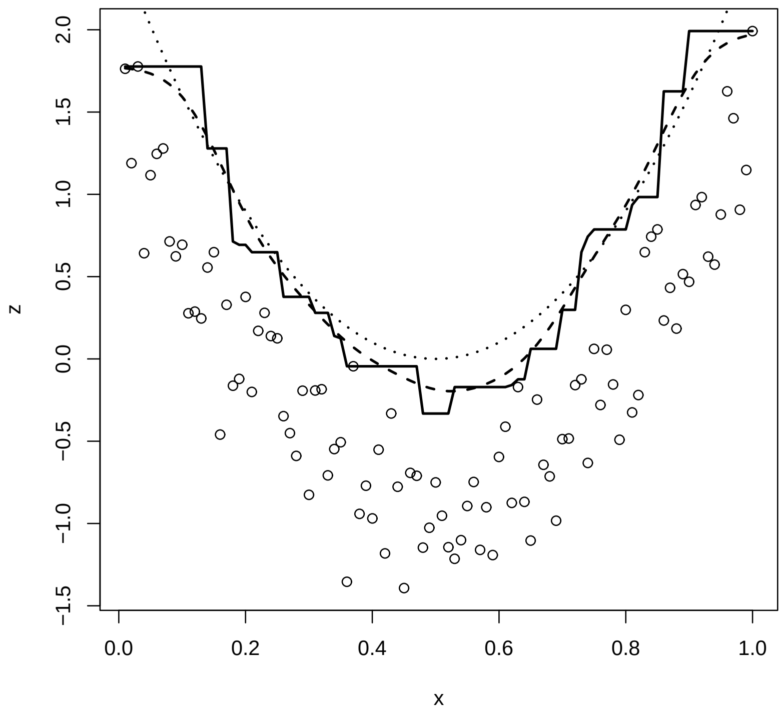

Figure 5: Data corresponding to the model in Figure 1. The true curve is dotted, while the local constant estimator is given by the solid line and the smoothed estimator (with bandwidths $b_n = n^{-1/3}$ and $a_n = b_n/2$) by the dashed line.

$n = 50$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	0.192 0.197 (0.102)	**0.001** 0.009 (0.037)	-0.038 -0.139 (0.118)	0.225 0.208 (0.118)
$\vartheta_0 = 0.5$	0.778 0.691 (0.191)	**0.378** 0.402 (0.092)	0.239 0.302 (0.410)	0.858 0.798 (0.274)
$\vartheta_0 = 1$	1.290 1.340 (0.233)	**0.728** 0.741 (0.232)	0.388 0.264 (1.000)	1.370 1.350 (0.308)
$\vartheta_0 = 1.5$	**1.750** 1.790 (0.195)	1.160 1.290 (0.368)	0.507 0.292 (1.810)	1.790 1.790 (0.222)
$\vartheta_0 = 2$	1.940 2.060 (0.201)	1.590 1.750 (0.478)	0.585 0.424 (2.880)	**1.970** 2.060 (0.141)

$n = 100$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	**0.017** 0.018 (0.037)	0.080 0.079 (0.014)	0.061 0.073 (0.022)	-0.020 -0.004 (0.020)
$\vartheta_0 = 0.5$	**0.496** 0.517 (0.028)	0.338 0.346 (0.042)	0.516 0.578 (0.080)	0.521 0.548 (0.032)
$\vartheta_0 = 1$	0.973 0.979 (0.044)	0.745 0.745 (0.092)	0.906 1.050 (0.225)	**1.030** 1.020 (0.054)
$\vartheta_0 = 1.5$	1.480 1.460 (0.059)	1.210 1.230 (0.123)	1.310 1.510 (0.412)	**1.510** 1.490 (0.060)
$\vartheta_0 = 2$	**1.960** 2.000 (0.059)	1.690 1.740 (0.144)	1.550 1.860 (0.822)	1.920 1.940 (0.058)

Table 6: *Mean, median and MISE for Model (12) for* $n = 50$ *and* $n = 100$ *with* $a_n = b_n/2$.

$n = 50$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	0.156 0.167 (0.076)	-0.050 -0.062 (0.031)	**0.022** 0.103 (0.095)	0.191 0.198 (0.0,86)
$\vartheta_0 = 0.5$	0.713 0.646 (0.130)	**0.324** 0.336 (0.088)	0.268 0.407 (0.348)	0.781 0.695 (0.197)
$\vartheta_0 = 1$	1.260 1.310 (0.191)	0.655 0.646 (0.242)	0.447 0.511 (0.919)	**1.330** 1.350 (0.258)
$\vartheta_0 = 1.5$	**1.720** 1.780 (0.188)	1.100 1.180 (0.365)	0.619 0.559 (1.660)	**1.720** 1.780 (0.177)
$\vartheta_0 = 2$	**1.970** 2.060 (0.141)	1.550 1.660 (0.442)	0.726 0.619 (2.630)	1.960 2.060 (0.111)

$n = 100$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	**0.001** 0.050 (0.044)	0.129 0.128 (0.023)	0.028 0.037 (0.016)	-0.014 -0.042 (0.015)
$\vartheta_0 = 0.5$	0.467 0.474 (0.033)	0.282 0.287 (0.063)	**0.497** 0.533 (0.057)	0.481 0.486 (0.034)
$\vartheta_0 = 1$	0.934 0.942 (0.043)	0.674 0.649 (0.130)	0.878 0.999 (0.190)	**0.965** 0.960 (0.049)
$\vartheta_0 = 1.5$	1.420 1.390 (0.056)	1.120 1.130 (0.185)	1.320 1.500 (0.336)	**1.440** 1.400 (0.053)
$\vartheta_0 = 2$	**1.910** 1.920 (0.071)	1.590 1.610 (0.228)	1.560 1.850 (0.790)	1.850 1.790 (0.079)

Table 7: *Mean, median and MISE for Model (12) for* $n = 50$ *and* $n = 100$ *with* $a_n = b_n/20$.

$n = 50$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	0.141 0.010 (0.309)	**0.020** 0.012 (0.008)	-0.056 -0.053 (0.016)	0.137 0.028 (0.260)
$\vartheta_0 = 0.5$	0.519 0.549 (0.039)	**0.518** 0.506 (0.023)	0.521 0.532 (0.040)	0.546 0.574 (0.038)
$\vartheta_0 = 1$	1.010 1.010 (0.085)	**1.000** 0.996 (0.040)	0.996 0.998 (0.071)	1.040 1.030 (0.077)
$\vartheta_0 = 1.5$	1.530 1.530 (0.125)	**1.500** 1.490 (0.066)	**1.500** 1.510 (0.113)	1.550 1.570 (0.110)
$\vartheta_0 = 2$	1.960 2.060 (0.118)	**2.010** 2.040 (0.069)	1.950 2.000 (0.156)	1.970 2.050 (0.093)

$n = 100$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	0.019 0.009 (0.022)	**0.006** 0.000 (0.004)	0.043 0.038 (0.007)	-0.014 -0.007 (0.013)
$\vartheta_0 = 0.5$	0.522 0.524 (0.023)	**0.505** 0.498 (0.013)	0.562 0.555 (0.020)	0.528 0.524 (0.022)
$\vartheta_0 = 1$	1.030 1.030 (0.042)	**1.010** 1.000 (0.021)	1.080 1.080 (0.038)	1.030 1.020 (0.042)
$\vartheta_0 = 1.5$	1.550 1.550 (0.061)	**1.510** 1.510 (0.030)	1.600 1.590 (0.055)	1.550 1.550 (0.061)
$\vartheta_0 = 2$	2.040 2.060 (0.066)	**2.000** 2.000 (0.037)	2.070 2.070 (0.061)	2.020 2.050 (0.058)

Table 8: *Mean, median and MISE for Model (13) for* $n = 50$ *and* $n = 100$ *with* $a_n = b_n/2$.

$n = 50$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	0.097 0.031 (0.223)	**0.005** 0.016 (0.008)	-0.037 -0.030 (0.013)	-0.087 -0.001 (0.174)
$\vartheta_0 = 0.5$	0.487 0.506 (0.039)	0.479 0.462 (0.021)	**0.506** 0.508 (0.036)	0.514 0.522 (0.035)
$\vartheta_0 = 1$	0.976 0.978 (0.092)	0.965 0.962 (0.044)	**0.984** 0.997 (0.074)	1.020 1.020 (0.078)
$\vartheta_0 = 1.5$	**1.499** 1.469 (0.120)	1.440 1.430 (0.063)	1.450 1.470 (0.119)	1.530 1.500 (0105)
$\vartheta_0 = 2$	1.920 1.990 (0.105)	**1.960** 1.960 (0.069)	1.940 1.940 (0.127)	1.930 1.970 (0.086)

$n = 100$	TKS	TCM	TKSCM	TCMKS
$\vartheta_0 = 0$	0.017 0.004 (0.016)	**0.004** 0.000 (0.004)	0.042 0.039 (0.007)	-0.010 -0.007 (0.007)
$\vartheta_0 = 0.5$	0.530 0.537 (0.021)	**0.507** 0.497 (0.011)	0.563 0.554 (0.019)	0.534 0.539 (0.021)
$\vartheta_0 = 1$	1.020 1.020 (0.042)	**1.000** 1.000 (0.020)	1.080 1.070 (0.035)	1.020 1.010 (0.039)
$\vartheta_0 = 1.5$	1.550 1.560 (0.064)	**1.510** 1.510 (0.031)	1.600 1.600 (0.054)	1.560 1.550 (0.064)
$\vartheta_0 = 2$	2.050 2.060 (0.069)	**2.020** 2.040 (0.041)	2.090 2.100 (0.064)	2.030 2.060 (0.059)

Table 9: *Mean, median and MISE for Model (13) for* $n = 50$ *and* $n = 100$ *with* $a_n = b_n/20$.

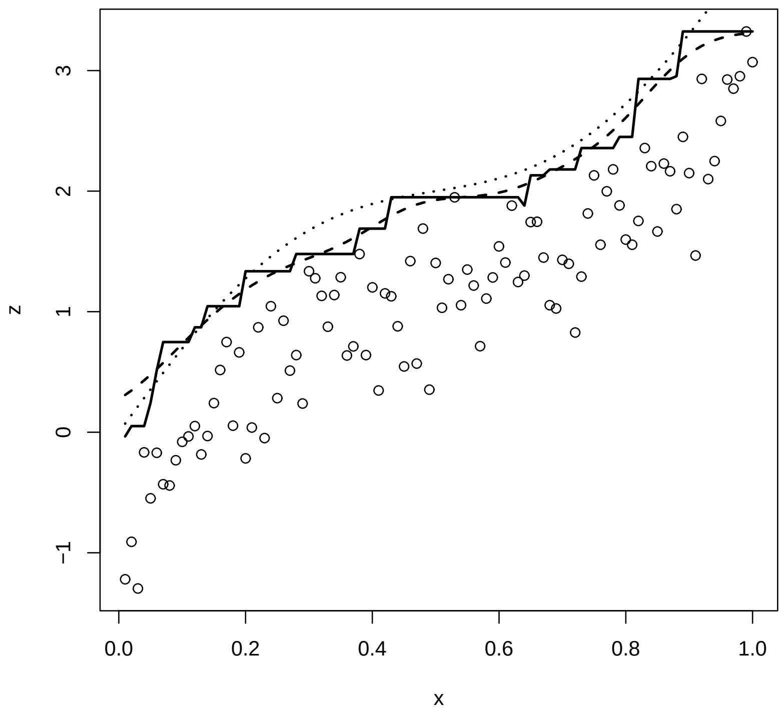

Figure 6: The setting is similar to Figure 5 with model from Figure 2 in the main paper.

Method	Pearson	Kendall	Spearman
Original data	-0.634	-0.456	-0.612
True parameter ϑ_0	0.001	0.001	0.001
TKS	0.001	0.001	0.001
TCM	0.009	0.006	0.008
TKSCM	0.005	0.002	0.004
TCMKS	0.002	0.001	0.001

Table 10: *Pearson's, Kendall's and Spearman's correlation coefficients (the average over* 1000 *iterations) between the covariates and the errors for the model* (13) *when* $n = 100$*. The first line corresponds to the correlations for the original data while the second line is for the true transformation parameter (*$\vartheta_0 = 0.5$*). The last four lines correspond to the correlations for each estimator.*