Supplementary material on "Comparing the marginal densities of two strictly stationary linear processes"

Paul Doukhan . Ieva Grublytė .
Denys Pommeret . Laurence Reboul

1 Representation of the value of $(\widetilde{Q}_1^*)_\infty^{(1)}(0)$ with respect to the quantity σ_ϵ^2

The following figure shows how the derivative $(\widetilde{Q}_1^*)_\infty^{(1)}(0)$ is very sensible to the value of σ_ϵ^2.

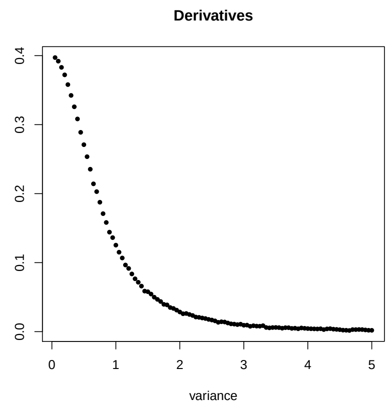


Fig. 10 Representation of $(\widetilde{Q}_1^*)_\infty^{(1)}(0)$ with respect to σ_ϵ^2 for $n = 500$.

2 Simulation study under transformations of ARFIMA model

In this section, we complete the previous numerical study by considering two transformations of ARFIMA processes. First, an exponential transformation with $K(x) = L(x) = \exp(x)$, yielding log-normal margins. Second, a quadratic

P. Doukhan University Cergy-Pontoise, AGM-UMR 8080, 2 Bd. Adolphe Chauvin 95000 Cergy-Pontoise - France
I. Grublytė Institute of Mathematics and Informatics of Vilnius University, 4 Akademijos, LT-08663 Vilnius - Lithuania
D. Pommeret (corresponding author) Univ. Lyon 1, ISFA, LSAF, and Aix Marseille Univ, CNRS, Centrale Marseille, I2M, 163 Avenue de Luminy, 13009 Marseille - France denys.pommeret@univ-amu.fr
L. Reboul Aix Marseille Univ, CNRS, Centrale Marseille, I2M, 163 Avenue de Luminy, 13009 Marseille - France

transformation with $K(x) = L(x) = x^2$, yielding scaled chi-square distributed margins. From Sang and Sang (2016), Theorem 2.2, if a transformation of an ARFIMA stationary process has a power rank equal to 1, then under mild condition the memory parameter of the transformed process is this of the ARFIMA process. We will use this property to construct our test statistic .

2.1 Models

We place in the same condition than Section 6.1; that is, we simulate $n \in \{100, 200, 500, 1000, 2000\}$ independent ARFIMA processes X and Y with margin distributions $\mathcal{M}(0, \delta_X, \mu_X, 1)$ and $\mathcal{M}(0, \delta_Y, \mu_Y, 1)$, respectively. We deduce the following two transformed processes:

$$X' = \exp(X), Y' = \exp(Y), \text{ and } X'' = X^2, Y'' = Y^2.$$

Based on the observations of X', Y' or X'', Y'', we investigate several null and alternative distributions.

- Null hypothesis: both process X', Y' (resp. X'', Y'') have same margins.
- Mean deviation alternative hypotheses: the original simulated ARFIMA have different means $\mu_X \neq \mu_Y$.
- Long-memory deviation alternative hypotheses: the original simulated ARFIMA have different memory parameters $\delta_Y \neq \delta_X$.

2.2 Test statistics

According to the support of both transformed processes considered here, we used the exponential distribution with parameter 1 as reference measure ν, with associated Laguerre polynomials. Then we have

$$\tilde{Q}_k(x) = L_k(x) \exp(x),$$

where L_k is the kth order Laguerre polynomial with $L_0 = 1$ and $L(x) = 1 - x$. Writing $h'(x) = \tilde{Q}_1^*(\exp(x))$ and $h''(x) = \tilde{Q}_1^*(x^2)$ we obtain

$$\begin{aligned} (h'^{(1)}_\infty(0) &= \mathbb{E}(\exp(-X')X'(X' - 2)) \\ (h''^{(1)}_\infty(0) &= \mathbb{E}(2(X''(X''^2 - 2)\exp(-X''^2)). \end{aligned}$$

If we assume that both previous quantities are non null, that is both transformations have power rank equal to 1, then from Theorem 2.2. of Sang and Sang (2016) it follows that X', Y' and X'', Y'' have the same memory parameters than X and Y. In that case we can estimate δ_X and δ_Y directly from the observed transformed processes X', Y' or X'', Y''.

The limit variance is still estimated from (25) or (26).

2.3 Empirical levels

We consider as null model the case where the processes X and Y are the same and are governed by the model $\mathcal{M}(0, \delta_X, 0, 1)$. In that case, X' and Y' have the same log-normal distribution, and X'' and Y'' have the same scaled chi-square distribution. Different values of the long memory parameter are considered, namely $\delta_X \in \{0.01, 0.1, 0.2, 0.3, 0.4, 0.45, 0.49\}$.

Empirical levels for log-normal processes X', Y', and for chi-square processes X'', Y'', are represented on Figure 11. For large values of $\delta_X = \delta_Y$, that is values close to 0.5, the convergence to the nominal 5% is unstable, and very slow, especially for the exponential transformation. This is due to the previous loss of stationarity phenomena.

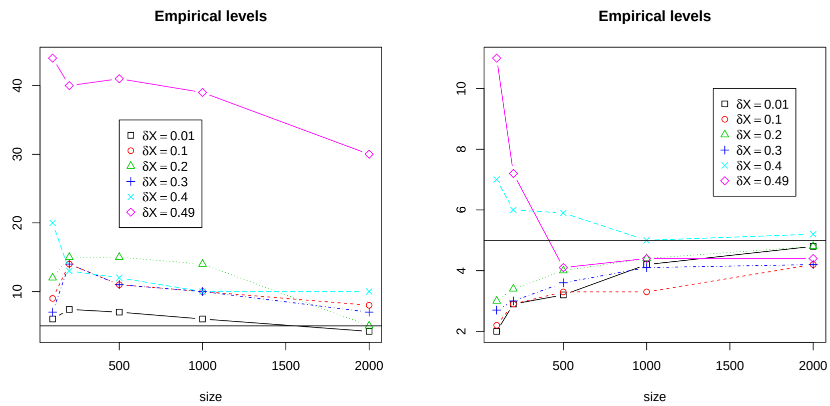


Fig. 11 Empirical levels for $\delta_X = 0.01, 0.1, 0.2, 0.3, 0.40.49$. X is $\mathcal{M}(0, \delta_X, 0, 1)$, Y is $\mathcal{M}(0, \delta_Y, 0, 1)$, and $\delta_X = \delta_Y$. The observed data are $K(X)$ and $L(Y)$. The test is $H_0 : f_{K(X)} = f_{L(Y)}$. On left, $K(x) = L(x) = \exp(x)$. On right, $K(x) = L(x) = x^2$.

2.4 Empirical powers

Long memory deviations First alternatives considered are memory deviations where the process X is governed by the model $\mathcal{M}(0, \delta_X, 0, 1)$ and the process Y is governed by $\mathcal{M}(0, \delta_Y, 0, 1)$, with $\delta_X \neq \delta_Y$. Figure 12 shows the empirical powers obtained with small or large difference between δ_X and δ_Y. We can observe how these deviations are detected through the transformed processes X', Y' and X'', Y''.

We observed similar results for exponential and quadratic transformations. On the one hand, the more memory parameters are different and the higher the power seems. Thus the strongest power was obtained when $\delta_Y = 0.01$ and $\delta_X = 0.49$, which contrasts here a process close to the short memory to a process closer to non stationarity.

For both transformations, we can also observe a low power when both memory parameters are close to 0, and the lower power is obtained when $\delta_Y = 0.01$ and $\delta_X = 0.1$, which is the closest case to the short memory case.

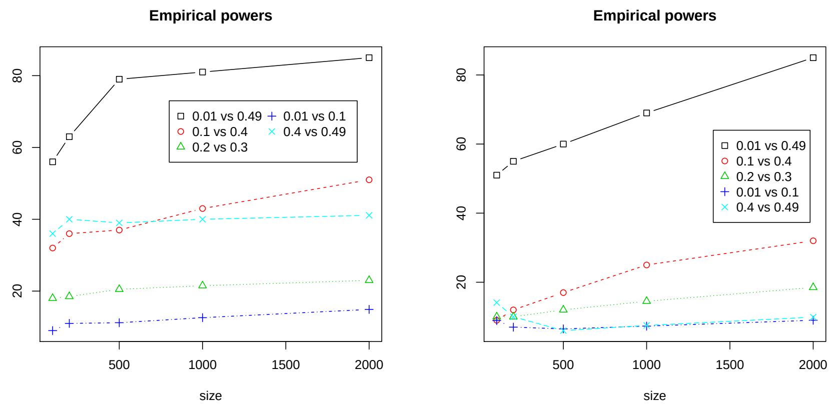
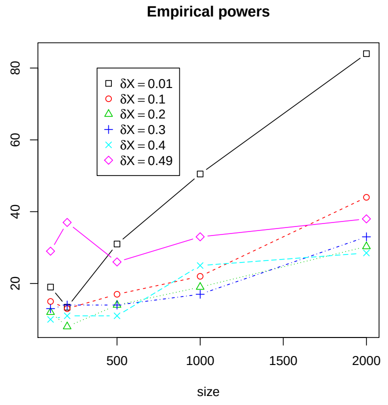
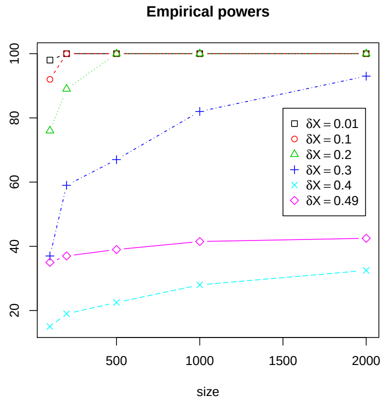


Fig. 12 Empirical powers. X is $\mathcal{M}(0, \delta_X, 0, 1)$, Y is $\mathcal{M}(0, \delta_Y, 0, 1)$, and $(\delta_Y, \delta_X) \in \{(0.01, 0.49), (0.1, 0.4), (0.2, 0.3), (0.01, 0.1), (0.4, 0.49)\}$. The observed data are $K(X)$ and $L(Y)$. The test is $H_0 : f_{K(X)} = f_{L(Y)}$. On left, $K(x) = L(x) = \exp(x)$. On right, $K(x) = L(x) = x^2$.

Mean deviation Second alternatives considered are mean deviations where the process X is governed by the model $\mathcal{M}(0, \delta_X, 0, 1)$ and the process Y is governed by $\mathcal{M}(0, \delta_X, \mu_Y, 1)$, with $\mu_Y \neq 0$.

Figure 13 (resp. 14) shows the empirical powers obtained with the exponential (resp. quadratic) transformation. As expected, the power is greater for larger values of μ_Y. The alternative with $\mu_Y = 0.1$ is not well detected, excepted for $\delta_X = 0.01$, but this case correspond to an instability of the test statistic. For large values of δ_X the power decreases, as shown in both Figures 13-14, when $\delta_X = 0.4$ or $\delta_X = 0.49$, which can be explained by the non stationarity proximity.





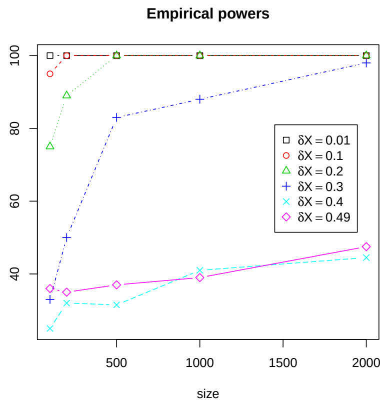


Fig. 13 Empirical powers. X is $\mathcal{M}(0, \delta_X, 0, 1)$ and Y is $\mathcal{M}(0, \delta_Y, \mu_Y, 1)$, with $\mu_Y = 0.1, 1, 10$. The observed data are $X' = \exp(X), Y' = \exp(Y)$. First figure, $\mu_Y = 0.1$. Second figure: $\mu_Y = 1$. Third figure: $\mu_Y = 10$.

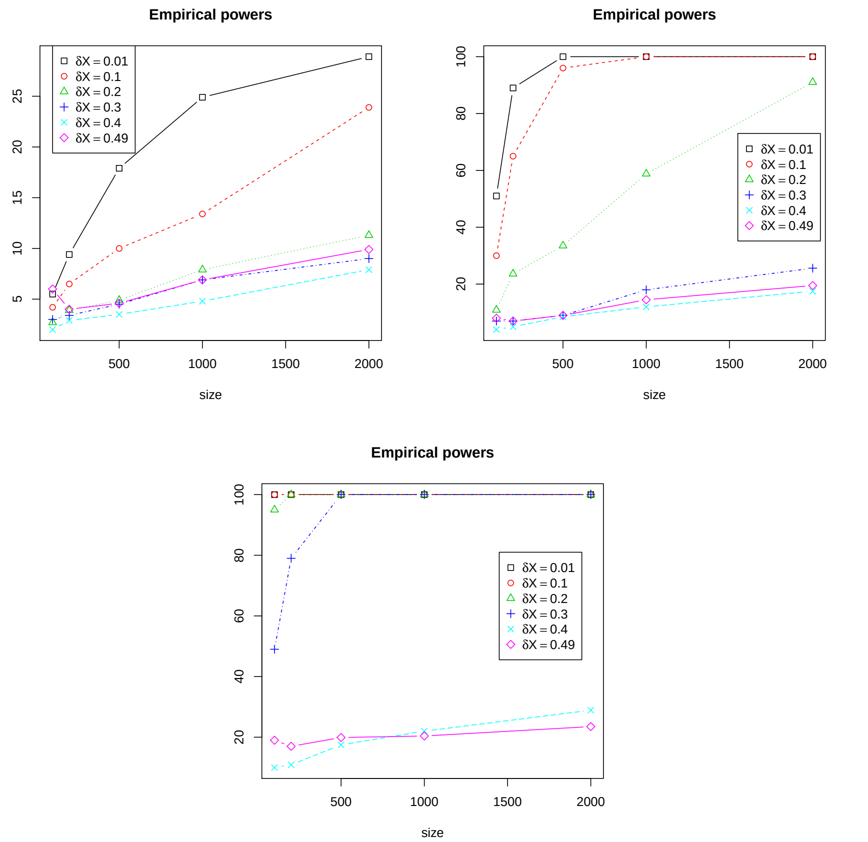


Fig. 14 Empirical powers. X is $\mathcal{M}(0, \delta_X, 0, 1)$ and Y is $\mathcal{M}(0, \delta_Y, \mu_Y, 1)$, with $\mu_Y = 0.1, 1, 10$. The observed data are $X'' = X^2, Y'' = Y^2$. First figure, $\mu_Y = 0.1$. Second figure: $\mu_Y = 1$. Third figure: $\mu_Y = 10$.

3 Proof of Proposition 4

Following the proof of Doukhan et al. (2015)'s Theorem 2, one has

$$\mathbb{P}_1(K_n \geq 2) \leq \sum_{k=2}^{d(n)} \frac{1}{\log n} \mathbb{E}_1 \left(|U_n^{(k)}|^2 \right).$$

Let $Q_k^a(.) = Q_k(.) - a_k$ and $Q_k^b(.) = Q_k(.) - b_k$. We have

$$U_n^{(k)} = \frac{1}{u_n} \sum_{s=1}^{n} \widetilde{Q}_k^a(X_s) - \frac{1}{u_n} \sum_{s=1}^{n} \widetilde{Q}_k^b(Y_s) + n\gamma_k / u_n. \tag{34}$$

So,

$$\mathbb{E}_1 \left(U_n^{(k)} \right)^2 = A_n^*(k) + B_n^*(k) + n^2 \gamma_k^2 / u_n^2,$$

where $A_n^*(k)$ and $B_n^*(k)$ are defined as in (29) with $\mathbb{E}_1$ in place of $\mathbb{E}_0$. As in the proof of (A) in Propositions 1 to 3 one has under (i) and (ii)

$$\frac{1}{\log n}\sum_{k=2}^{d(n)}(A_n^*(k)+B_n^*(k))\to 0.$$

Moreover under H_1^*

$$\frac{1}{\log n}\sum_{k=2}^{d(n)} n^2\gamma_k^2/u_n^2\to 0$$

so that $\mathbb{P}_1(K_n\geq 2)\to 0$. Therefore, using the proofs of Propositions 1 to 3 according to δ_X and δ_Y, we get the limiting law of the statistic under H_1^*.

4 Proof of Proposition 5

Under H_1^{**}, one has by similar arguments $\mathbb{P}_1(K_n>K)$ tends to zero. On the other hand, following the proof of Doukhan et al. (2015)'s Theorem 2

$$\mathbb{P}_1(K_n<K)\leq K\mathbb{P}_1\left(\frac{|U_n^{(K)}|}{\sqrt{\log(n)}}\leq\sqrt{K}\right),$$

with

$$U_n^{(K)}=\frac{1}{u_n}\sum_{s=1}^{n}\widetilde{Q}_K^*(X_s)-\frac{1}{u_n}\sum_{s=1}^{n}\widetilde{Q}_K^*(Y_s)+n\gamma_K/u_n=\sqrt{\log n}D_n(K)+n\gamma_K/u_n.$$

One has

$$\begin{aligned}
K\mathbb{P}_1\left(\frac{|U_n^{(K)}|}{\sqrt{\log(n)}}\leq\sqrt{K}\right) &\leq K\mathbb{P}_1\left(\frac{U_n^{(K)}}{\sqrt{\log(n)}}\leq\sqrt{K}\right)\\
&\leq K\mathbb{P}_1\left(D_n(K)+\frac{n\gamma_K}{u_n\sqrt{\log(n)}}\leq\sqrt{K}\right)\\
&\leq K\mathbb{P}_1\left(D_n(K)\leq\sqrt{K}-\frac{n\gamma_K}{u_n\sqrt{\log(n)}}\cap D_n(K)>0\right)\\
&+K\mathbb{P}_1\left(D_n(K)\leq\sqrt{K}-\frac{n\gamma_K}{u_n\sqrt{\log(n)}}\cap D_n(K)\leq 0\right).
\end{aligned}$$

Under H_1^{**}, there exists some n_0 such that for all $n>n_0$, $\sqrt{K}-n\gamma_K/(u_n\sqrt{\log n})<0$, so for $n>n_0$

$$\mathbb{P}_1(K_n<K)=K\mathbb{P}_1\left(-D_n(K)\geq\frac{n\gamma_K}{u_n\sqrt{\log(n)}}-\sqrt{K}\cap D_n(K)\leq 0\right)$$

$$\le K\mathbb{P}_1\left(-D_n(K) \ge \frac{n\gamma_K}{u_n\sqrt{\log(n)}} - \sqrt{K}\right)$$
$$\le K\frac{\mathbb{E}_1\left(D_n(K)^2\right)}{\left(\frac{n\gamma_K}{u_n\sqrt{\log(n)}} - \sqrt{K}\right)^2}$$

which goes to zero as n goes ti ∞. It follows that $K_n \to K$. Moreover under H_1^{**}

$$U_n^{(K)} = D_n(K) + n\gamma_K/u_n \xrightarrow{\mathcal{P}} \infty,$$

so $\widetilde{N}_n$ goes to infinity.

5 Proof of Lemma 1

It is easily seen that for all $k = 0, 1, \ldots$, the first third derivatives of $\tilde{Q}_k$ exist and are bounded. The same holds for $\tilde{Q}_k^*$ and conditions (iv) and (vii) in Propositions 1-3 are satisfied.

Secondly, using the classical recursion properties of Hermite polynomials we show that the first derivative can be expressed as

$$\begin{aligned}
\tilde{Q}_k'(x) &= \exp(-x^2/2)\left(-xH_k(x) + H_k'(x)\right) \\
&= \exp(-x^2/2)\left(-(H_{k+1}(x) + kH_{k-1}(x)) + kH_{k-1}(x)\right) \\
&= -\exp(-x^2/2)H_{k+1}(x) = -\tilde{Q}_{k+1}(x).
\end{aligned}$$

Since $|\tilde{Q}_{k+1}(x)| < C(k+1)^{-1/12}$ we obtain (v). Moreover we have

$$\begin{aligned}
|\tilde{Q}_k^*(x) - \tilde{Q}_k^*(y)| &= |\tilde{Q}_k(x) - \tilde{Q}_k(y)| \\
&< C(k+1)^{-1/12}|x-y|,
\end{aligned}$$

and it follows that conditions (i) and (ii) are satisfied, that is $\tilde{Q}_k^*$ is q_k-Lipschitz with $q_k = C(k+1)^{-1/12}$ and

$$\frac{1}{i}\sum_{k=1}^{i} q_k^2 < C^2.$$

Finally, let us show that $m_1 = 1$ for the Hermite functions. Writing $h = \tilde{Q}_1^*$ for simplicity we have

$$\begin{aligned}
h_\infty^{(1)}(0) &= \frac{\partial}{\partial m}\mathbb{E}(h(m + X_0))|_{m=0} \\
&= \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial m}\mathbb{E}(\exp(-(m+X_0)^2/2))(m+X_0)|_{m=0} \\
&= \frac{1}{\sqrt{2\pi}}\mathbb{E}(\exp(-(m+X_0)^2/2))(1-(m+X_0)^2)|_{m=0} \\
&= \frac{1}{\sqrt{2\pi}}\mathbb{E}(\exp(-X_0^2/2)(1-X_0^2)),
\end{aligned}$$

which is strictly positive except if X_0 has a Dirac distribution concentrated on 1 which is excluded.

References

Doukhan, P., Pommeret, D., Reboul, L. (2015). Data driven smooth test of comparison for dependent sequences. *Journal of Multivariate Analysis*, 139, 147–165.

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