Supplementary Material to "Model selection for the robust efficient signal processing observed with small Lévy noise". *

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This supplementary document contains Appendix in which we prove all technical results.

1 Appendix

A.1 Proof of Proposition 3

We use here the same method as in Konev and Pergamenshchikov (2009). First, note that from the definitions (19) and (35) we obtain

$$\widehat{\tau}_{j,\varepsilon} = \tau_j + \varepsilon\,\eta_j\,, \tag{A.1}$$

where

$$\tau_j = \int_0^1 S(t)\,\mathrm{Tr}_j(t)\mathrm{d}t \quad \text{and} \quad \eta_j = \int_0^1 \mathrm{Tr}_j(t)\,\mathrm{d}\check{\xi}_t\,.$$

So, we have

$$\widehat{\varkappa}_\varepsilon = \sum_{j=[1/\varepsilon]+1}^{n} \tau_j^2 + 2\varepsilon\,\check{M}_\varepsilon + \varepsilon^2 \sum_{j=[1/\varepsilon]+1}^{n} \eta_j^2\,, \tag{A.2}$$

where $\check{M}_\varepsilon = \sum_{j=[1/\varepsilon]+1}^{n} \tau_j\,\eta_j$. Note that for the continuously differentiable functions (see, for example, Lemma A.6 in Konev and Pergamenshchikov (2009a))

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the Fourier coefficients (τ_j) for any $n \ge 1$ satisfy the following inequality

$$\sum_{j=[1/\varepsilon]+1}^{\infty} \tau_j^2 \le 4\varepsilon \left(\int_0^1 |\dot{S}(t)| \mathrm{d}t \right)^2 \le 4\varepsilon \|\dot{S}\|^2 \,. \tag{A.3}$$

We recall, that $\dot{S}$ is the derivative of S. To estimate the term $\check{M}_\varepsilon$ note, that it can be represented as

$$\check{M}_\varepsilon = \check{I}_1 \left(\sum_{j=[1/\varepsilon]+1}^{n} \tau_j \mathrm{Tr}_j \right) \,.$$

Therefore, using the last equality in (19) and orthonormality of the functions $(\mathrm{Tr}_j)_{j \ge 1}$, we obtain that

$$\mathbf{E}_Q \, \check{M}_\varepsilon^2 = \check{\varkappa}_Q \sum_{j=[1/\varepsilon]+1}^{n} \tau_j^2 \le 4\varepsilon \, \varkappa_Q \|\dot{S}\|^2 \,.$$

Moreover, taking into account that for $j \ge 2$ the expectation $\mathbf{E}\, \eta_j^2 = \check{\varkappa}_Q$ we can represent the last term in (A.2) as

$$\varepsilon^2 \sum_{j=[1/\varepsilon]+1}^{n} \eta_j^2 = \check{\varkappa}_Q (\varepsilon^2 n - \varepsilon^2 [1/\varepsilon]) + \varepsilon \, B_{2,\varepsilon}(x') \,,$$

where $x' = (x'_j)_{1 \le j \le n}$ with $x'_j = \varepsilon \mathbf{1}_{\{1/\varepsilon < j \le n\}}$ and the function $B_{2,\varepsilon}(x')$ is defined in (20). We remind that $n = [1/\varepsilon^2]$ and that for the trigonometric basis (14) the upper bound $\phi^* = \sqrt{2}$. Therefore, in view of Proposition 2 and the definition of $\overline{a}$ in (27) we obtain that

$$\mathbf{E}_Q \left| \varepsilon^2 \sum_{j=[\sqrt{1/\varepsilon}]+1}^{n} \eta_j^2 - \check{\varkappa}_Q \right| \le \varepsilon \left(2 \, \varkappa_Q + \sqrt{U_Q} + 4 \sqrt{\varkappa_Q} \right) \,.$$

So, we obtain the bound (36). Hence Proposition 3. □

A.2 Property of Lemma 1

In vue of the definition of $\mathrm{Err}_\varepsilon(\lambda)$ and the equation (19) one has

$$\mathrm{Err}_\varepsilon(\lambda) = \sum_{j=1}^{n} \left((\lambda(j) - 1)\theta_j + \varepsilon \lambda(j) \overline{\xi}_j \right)^2 + \sum_{j=n+1}^{\infty} \theta_j^2$$

$$\ge 2\varepsilon \sum_{j=1}^{n} (\lambda(j) - 1)\theta_j \lambda(j) \overline{\xi}_j + \varepsilon^2 \sum_{j=1}^{n} \lambda^2(j) \overline{\xi}_j^2 \,.$$

Moreover, using here the definition (20) we obtain

$$\mathbf{E}_Q \, \mathrm{Err}_\varepsilon(\lambda) \geq \varepsilon^2 \sum_{j=1}^{n} \lambda^2(j) \mathbf{E}_Q \, \overline{\xi}_j^2 = P_\varepsilon(\lambda) - \varepsilon^2 B_{1,\varepsilon}(\lambda^2) \,,$$

where $\lambda^2 = (\lambda^2(j))_{1\leq j\leq n}$. Now, Proposition 1 implies Lemma 1. $\square$

A.3 Proof of Proposition 4

First of all note that, the density (59) on the process ξ is bounded with respect to $\theta_j \in \mathbb{R}$ and for any $1 \leq j \leq d$

$$\limsup_{|\theta_j|\to\infty} f(\xi,\theta) \,=\, 0 \,. \qquad \text{a.s.}$$

Now, we set

$$\widetilde{\Phi}_j = \widetilde{\Phi}_j(x,\theta) = \frac{\partial \left(f(x,\theta)\Phi(\theta)\right)/\partial\theta_j}{f(x,\theta)\Phi(\theta)} \,.$$

Taking into account the condition (60) and integrating by parts yield

$$\widetilde{\mathbf{E}} \left(((\widehat{g} - g(\theta))\widetilde{\Phi}_j \right) = \int_{\mathcal{X}\times\mathbb{R}^d} ((\widehat{g}(x) - g(\theta)) \frac{\partial}{\partial\theta_j} \left(f(x,\theta)\Phi(\theta)\right) \mathrm{d}\theta \, \mathbf{P}_\xi(\mathrm{d}x)$$

$$= \int_{\mathcal{X}\times\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} g_j'(\theta) \, f(x,\theta)\Phi(\theta)\mathrm{d}\theta_j \right) \left(\prod_{i\neq j} \mathrm{d}\theta_i \right) \, \mathbf{P}_\xi(\mathrm{d}x) = \lambda_j \,.$$

Now by the Bouniakovskii-Cauchy-Schwarz inequality we obtain the following lower bound for the quadratic risk

$$\widetilde{\mathbf{E}}((\widehat{g} - g(\theta))^2 \geq \frac{\Lambda_j^2}{\widetilde{\mathbf{E}}\widetilde{\Phi}_j^2} \,.$$

To study the denominator in the left hand of this inequality note that in view of the representation (59)

$$\frac{1}{f(y,\theta)} \frac{\partial \, f(y,\theta)}{\partial\theta_j} = \frac{1}{\varrho_1} \int_0^1 \, \phi_j(t) \, \mathrm{d}w_t \,.$$

Therefore, for each $\theta \in \mathbb{R}^d$,

$$\mathbf{E}_\theta \, \frac{1}{f(y,\theta)} \frac{\partial \, f(y,\theta)}{\partial\theta_j} \, = 0$$

and

$$\mathbf{E}_\theta \, \left(\frac{1}{f(y,\theta)} \frac{\partial \, f(y,\theta)}{\partial\theta_j} \right)^2 = \frac{1}{\varrho_1^2} \int_0^1 \, \phi_j^2(t)\mathrm{d}t = \frac{1}{\varrho_1^2} \|\phi_j\|^2 \,.$$

Taking into account that

$$\widetilde{\Phi}_j = \frac{1}{f(x,\theta)} \frac{\partial\, f(x,\theta)}{\partial \theta_j} + \frac{1}{\Phi(\theta)} \frac{\partial\, \Phi(\theta))}{\partial \theta_j}\,,$$

we get

$$\widetilde{\mathbf{E}} \widetilde{\Phi}_j^2 = \frac{1}{\varrho_1^2}\, \|\phi_j\|^2 + I_j\,.$$

Hence Proposition 4. $\square$

A.4 The absolute continuity of distributions for the Lévy processes.

In this section we study the absolute continuity for the the Lévy processes defined as

$$\mathrm{d}y_t = S(t)\mathrm{d}t + \mathrm{d}\xi_t\,, \quad 0 \le t \le T\,, \tag{A.4}$$

where $S(\cdot)$ is any arbitrary nonrandom square integrated function, i.e. from $\mathbf{L}_2[0,T]$ and $(\xi_t)_{0\le t\le T}$ is the Lévy process of the form (7) with nonzero constants ϱ_1 and ϱ_2. We denote by $\mathbf{P}_y$ and $\mathbf{P}_\xi$ the distributions of the processes $(y_t)_{0\le t\le 1}$ and $(\xi_t)_{0\le t\le 1}$ on the Skorokhod space $\mathbf{D}[0,T]$. Now for any $0 \le t \le T$ and $(x_t)_{0\le t\le T}$ from $\mathbf{D}[0,T]$ we set

$$\Upsilon_t(x) = \exp\left\{ \int_0^t \frac{S(u)}{\varrho_1^2}\, \mathrm{d}x_u^c - \int_0^t \frac{S^2(u)}{2\varrho_1^2}\, \mathrm{d}u \right\}\,, \tag{A.5}$$

where x^c is the continuous part of the process x defined in (59). Now we study the measures $\mathbf{P}_y$ and $\mathbf{P}_\xi$ in $\mathbf{D}[0,T]$.

Proposition A.1. *For any $T > 0$ the measure $\mathbf{P}_y \ll \mathbf{P}_\xi$ in $\mathbf{D}[0,T]$ and the Radon-Nikodym derivative is*

$$\frac{\mathrm{d}\mathbf{P}_y}{\mathrm{d}\mathbf{P}_\xi}(\xi) = \Upsilon_T(\xi)\,.$$

Proof. Note that to show this proposition it suffices to check that for any $0 = t_0 < \ldots < t_n = T$ any $b_j \in \mathbb{R}$ for $1 \le j \le n$

$$\mathbf{E}\, \exp\left\{ i \sum_{l=1}^{n} b_j (y_{t_j} - y_{t_{j-1}}) \right\} = \mathbf{E}\, \exp\left\{ i \sum_{l=1}^{n} b_j (\xi_{t_j} - \xi_{t_{j-1}}) \right\} \Upsilon_T(\xi)\,.$$

taking into account that the processes $(y_t)_{0\le t\le T}$ and $(\xi_t)_{0\le t\le T}$ have the independent homogeneous increments, to this end one needs to check only that for any $b \in \mathbb{R}$ and $0 \le s < t \le T$

$$\mathbf{E}\, \exp\left\{ i\, b(y_t - y_s) \right\} = \mathbf{E}\, \exp\left\{ i\, b(\xi_t - \xi_s) \right\} \frac{\Upsilon_t(\xi)}{\Upsilon_s(\xi)}\,. \tag{A.6}$$

To check this equality note that the process

$$\Upsilon_t(\xi) = \exp\left\{ \int_0^t \frac{S(u)}{\varrho_1}\,\mathrm{d}w_u - \int_0^t \frac{S^2(u)}{2\varrho_1^2}\,\mathrm{d}u \right\}$$

is the gaussian martingale. From here we directly obtain the squation (A.6). Hence Proposition A.1. $\square$

A.5 Proof of the limit equality (94)

First, setting $\zeta_\varepsilon = \sum_{j=1}^d \kappa_j^2\, a_j$, we obtain that

$$\left\{ S_\kappa \notin W_{k,\mathbf{r}} \right\} = \{ \zeta_\varepsilon > \mathbf{r} \}\ .$$

Moreover, note that one can check directly that

$$\lim_{\varepsilon\to 0} \mathbf{E}\,\zeta_\varepsilon = \lim_{\varepsilon\to 0} \frac{1}{v_\varepsilon} \sum_{j=1}^d s_j^*\, a_j = \check{\mathbf{r}} = (1-\check{\gamma})\mathbf{r}\,.$$

So, for sufficiently small ε we obtain that

$$\left\{ S_\kappa \notin W_{k,r} \right\} \subset \left\{ \widetilde{\zeta}_\varepsilon > \mathbf{r}_1 \right\}\,,$$

where $\mathbf{r}_1 = \mathbf{r}\check{\gamma}/2$, $\widetilde{\zeta}_\varepsilon = \zeta_\varepsilon - \mathbf{E}\,\zeta_\varepsilon = v_\varepsilon^{-1} \sum_{j=1}^d s_j^* a_j \widetilde{\eta}_j$ and $\widetilde{\eta}_j = \eta_j^2 - 1$ Through the correlation inequality (see, Proposition A.1 in Galthouk and Pergamenshchikov (2013)) we can get that for any $p \ge 2$

$$\mathbf{E}\,\widetilde{\zeta}_\varepsilon^p \le (2p)^{p/2} \mathbf{E}|\widetilde{\eta}_1|^p\, v_\varepsilon^{-p} \left(\sum_{j=1}^d (s_j^*)^2 a_j^2 \right)^{p/2} = \mathrm{O}(\, v_\varepsilon^{-\frac{p}{4k+2}}\,)\,,$$

as $\varepsilon \to 0$. Therefore, for any $\iota > 0$ using the Chebychev inequality for $p > (4k+2)\iota$ we obtain that

$$v_\varepsilon^\iota \mathbf{P}(\widetilde{\zeta}_\varepsilon > \mathbf{r}_1) \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

Hence the equality (94). $\square$

A.6 Proof of Proposition 5

Substituting (19) and taking into account the definition (95) one gets

$$\|\check{S} - S\|^2 = \sum_{j=1}^\infty (1-\check{\lambda}(j))^2\,\theta_j^2 - 2\check{M}_\varepsilon + \varepsilon^2 \sum_{j=1}^\infty \check{\lambda}^2(j)\,\check{\xi}_j^2\,,$$

where $\check{M}_\varepsilon = \varepsilon \sum_{j=1}^\infty (1 - \check{\lambda}(j))\, \check{\lambda}(j)\, \theta_j\, \overline{\xi}_j$. Note now that for any $Q \in \mathcal{Q}^*_\varepsilon$ the expectation $\mathbf{E}_{Q,S}\, \check{M}_\varepsilon = 0$ and, in view of the upper bound (9),

$$\sup_{Q\in\mathcal{Q}^*_\varepsilon} \mathbf{E}_{Q,S} \sum_{j=1}^\infty \check{\lambda}^2(j)\, \overline{\xi}^2_j \le \varsigma^* \sum_{j=1}^\infty \check{\lambda}^2(j)\,.$$

Therefore,

$$\mathcal{R}^*_\varepsilon(\check{S}, S) \,\le\, \sum_{j=\check{j}_*}^\infty (1 - \check{\lambda}(j))^2\, \theta^2_j + \frac{1}{\upsilon_\varepsilon} \sum_{j=1}^\infty \check{\lambda}^2(j)\,, \tag{A.7}$$

where $\check{j}_* = j_*(\check{\alpha})$. Setting

$$\mathbf{u}_\varepsilon = \upsilon_\varepsilon^{2k/(2k+1)} \sup_{j\ge\check{j}_*} (1 - \check{\lambda}(j))^2 / a_j\,,$$

we obtain that for each $S \in W^k_r$

$$\Upsilon_{1,\varepsilon}(S) = \upsilon_\varepsilon^{2k/(2k+1)} \sum_{j=\check{\iota}}^\infty (1 - \check{\lambda}(j))^2\, \theta^2_j \,\le\, \mathbf{u}_\varepsilon \sum_{j=\check{\iota}}^\infty a_j\, \theta^2_j \,\le\, \mathbf{u}_\varepsilon\, r\,.$$

Taking into account that $\check{r} \to r$, we obtain that

$$\limsup_{\varepsilon\to 0} \sup_{S\in W^k_r} \Upsilon_{1,\varepsilon}(S) \,\le\, \frac{r^{1/(2k+1)}}{\pi^{2k}(\mathrm{d}_k)^{2k/(2k+1)}} := \Upsilon^*_1\,.$$

To estimate the last term in the right hand of (A.7), we set

$$\Upsilon_{2,\varepsilon} = \frac{1}{\upsilon_\varepsilon^{1/(2k+1)}} \sum_{j=1}^n \check{\lambda}^2(j)\,.$$

It is easy to check that

$$\limsup_{\varepsilon\to 0} \Upsilon_{2,\varepsilon} \le \frac{2(r\mathrm{d}_k)^{1/(2k+1)}\, k^2}{(k+1)(2k+1)} := \Upsilon^*_2\,.$$

Therefore, taking into account that by the definition of the Pinsker constant in (53) the sum $\Upsilon^*_1 + \Upsilon^*_2 = l_*(\mathbf{r})$, we arrive at the inequality

$$\lim_{\varepsilon\to 0} \upsilon_\varepsilon^{2k/(2k+1)} \sup_{S\in W^k_r} \mathcal{R}^*_\varepsilon(\check{S}, S) \le l_*(\mathbf{r})\,.$$

Hence Proposition 5. $\square$

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