Supplementary Material for "Bartlett Correction of Frequency Domain Empirical Likelihood for Time Series with Unknown Innovation Variance"

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In this supplementary material, we give proofs of to the Lemmas 1–2 and Theorems 1–2. First, we begin by deriving the stochastic expansion of the profiled frequency domain empirical likelihood (FDEL).

Stochastic Expansion

To prove Lemma 2 and Theorems 1–2, we need the stochastic expansion of the profiled FDEL firstly. Based on the Lagrange multiplier argument, the profiled FDEL is

$$\tilde{l}_n(\beta_0) = 4\sum_{j=1}^{n} \log(1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0))\,, \tag{1}$$

where $\tilde{t}$ is the solution of the nonlinear equation

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\tilde{m}_j(\beta_0)}{1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0)} = 0\,. \tag{2}$$

To establish the stochastic expansion of the profiled FDEL $\tilde{l}(\beta_0)$, we first prove that the magnitude $\tilde{t}(\beta_0) = O_p(n^{-1/2})$. As $w_j = n^{-1}(1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0))^{-1}$, we have $1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0) \geq 0$ and

$$|1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0)|^{-1} \geq (1+|\tilde{t}(\beta_0)|\max_j|\tilde{m}_j(\beta_0)|)^{-1}\,. \tag{3}$$

Substituting $1/(1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0)) = 1-\tilde{t}(\beta_0)\tilde{m}_j(\beta_0)/(1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0))$ into the (2) gives

$$\begin{aligned} 0 &= \left|\frac{1}{n}\sum_{j=1}^{n}\frac{\tilde{t}(\beta_0)\tilde{m}_j^2(\beta_0)}{1+\tilde{t}(\beta_0)\tilde{m}_j(\beta_0)} - \frac{1}{n}\sum_{j=1}^{n}\tilde{m}_j(\beta_0)\right| \\ &\geq |\tilde{t}(\beta_0)|\frac{1}{n}\sum_{j=1}^{n}\frac{\tilde{m}_j^2(\beta_0)}{1+|\tilde{t}(\beta_0)|\max_{1\leq j\leq n}|\tilde{m}_j(\beta_0)|} - \left|\frac{1}{n}\sum_{j=1}^{n}\tilde{m}_j(\beta_0)\right|\,, \end{aligned}$$

using the (3). Therefore, we have

$$|\tilde{t}(\beta_0)|\left(\frac{1}{n}\sum_{j=1}^{n}\tilde{m}_j^2(\beta_0) - \max_{1\leq j\leq n}|\tilde{m}_j(\beta_0)||\bar{\tilde{m}}|\right) \leq |\bar{\tilde{m}}|\,.$$

Under **RC**, the standard argument in Owen (2001) gives $\max_{1\leq j\leq n}|\tilde{m}_j(\beta_0)| = o_p(n^{1/2})$. Applying the results of Lemma 1, we have $\bar{\tilde{m}} = O_p(n^{-1/2})$ and $\frac{1}{n}\sum_j \tilde{m}_j^2(\beta_0) = O_p(1)$. Hence,

$$\tilde{t}(\beta_0) = O_p(n^{-1/2})\,.$$

Applying Taylor's expansion to the (2), we get the stochastic expansion of $\tilde{t}(\beta_0)$,

$$\tilde{t}(\beta_0) = \frac{\bar{\tilde{m}}}{\lambda_2} - \frac{\bar{\tilde{m}}\Delta_2}{\lambda_2^2} + \frac{\lambda_3\bar{\tilde{m}}^2}{\lambda_2^3} + \frac{\bar{\tilde{m}}\Delta_2^2}{\lambda_2^3} - 3\frac{\lambda_3\bar{\tilde{m}}^2\Delta_2}{\lambda_2^4} + 2\frac{\lambda_3^2\bar{\tilde{m}}^3}{\lambda_2^5} + \frac{\bar{\tilde{m}}^2\Delta_3}{\lambda_2^3} - \frac{\lambda_4\bar{\tilde{m}}^3}{\lambda_2^4} + O_p(n^{-2})\,. \quad (4)$$

Substituting the (4) into the (1) leads to the stochastic expansion of $\tilde{l}(\beta_0)$,

$$\begin{aligned} \frac{1}{n}\tilde{l}_n(\beta_0) &= 2\frac{\bar{\tilde{m}}^2}{\lambda_2} - 2\frac{\bar{\tilde{m}}^2\Delta_2}{\lambda_2^2} + \frac{4}{3}\frac{\lambda_3\bar{\tilde{m}}^3}{\lambda_2^3} + 2\frac{\bar{\tilde{m}}^2\Delta_2^2}{\lambda_2^3} + \frac{4}{3}\frac{\bar{\tilde{m}}^3\Delta_3}{\lambda_2^3} - 4\frac{\lambda_3\bar{\tilde{m}}^3\Delta_2}{\lambda_2^4} \\ &\quad +2\frac{\lambda_3^2\bar{\tilde{m}}^4}{\lambda_2^5} - \frac{\lambda_4\bar{\tilde{m}}^4}{\lambda_2^4} + O_p(n^{-5/2})\,. \end{aligned} \quad (5)$$

By equating the terms in the (5) to the terms of $\tilde{SR}^2$ of the same orders, we obtain the (12).

Proofs of Lemmas 1–2 and Theorems 1–2

In this section, we give the proofs of our lemmas and theorems.

Proof of Lemma 1: From the definition,

$$\begin{aligned} \lambda_4 &= \frac{1}{n}\sum_{j=1}^{n}\mathrm{cum}(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) \\ &\quad +\frac{4}{n}\sum_{j=1}^{n}\mathrm{cum}(\tilde{m}_j(\beta_0))\mathrm{cum}(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) \\ &\quad +\frac{3}{n}\sum_{j=1}^{n}\mathrm{cum}^2(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) + \frac{6}{n}\sum_{j=1}^{n}\mathrm{cum}^2(\tilde{m}_j(\beta_0))\mathrm{cum}(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) \\ &\quad +\frac{1}{n}\sum_{j=1}^{n}\mathrm{cum}^4(\tilde{m}_j(\beta_0)) + O(n^{-1})\,. \end{aligned} \quad (6)$$

Note that this expansion for λ_4 is not the same as the one with known variance σ_ϵ^2 given in Chen *et al.* (2016) because the extra terms with $\mathrm{cum}(\tilde{m}_j(\beta_0))$ cannot be ignored. Based on a calculation similar to that given in Lemma 2 of Chen *et al.* (2016), we have

$$\mathrm{cum}(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) = 6\sigma_\epsilon^8 p_j(\beta_0)^4 + O(n^{-1})\,, \quad (7)$$

$$\mathrm{cum}(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) = 2\sigma_\epsilon^6 p_j(\beta_0)^3 + O(n^{-1})\,, \quad (8)$$

and

$$\mathrm{cum}(\tilde{m}_j(\beta_0),\tilde{m}_j(\beta_0)) = \sigma_\epsilon^4 p_j(\beta_0)^2 + O(n^{-1})\,. \quad (9)$$

Substituting the equations (9) and (7)–(9) into the (6), we have

$$\lambda_4 = \frac{24}{n}\sigma_\epsilon^8\sum_{j=1}^{n}p_j(\beta_0)^4 + O(n^{-1})\,.$$

Applying similar arguments to those used to derive λ_4, it can be shown that

$$\lambda_2 = \frac{2}{n}\sigma_\epsilon^4\sum_{j=1}^{n}p_j(\beta_0)^2 + O(n^{-1})\,,$$

$$\lambda_3 = \frac{6}{n}\sigma_\epsilon^6 \sum_{j=1}^n p_j(\beta_0)^3 + O(n^{-1})\,.$$

Next, we consider the orders of $\mathrm{Var}(\Delta_k)$. By Chebyshev's inequality, for any $\delta > 0$, $P(|\Delta_k| \geq \delta) \leq \mathrm{E}(\Delta_k^2)/\delta^2$, and as $\mathrm{E}(\Delta_k) = 0$, the orders of Δ_k can be directly obtained by computing the orders of $\mathrm{Var}(\Delta_k)$. The case $k = 1$ is omitted because it is simpler. For $k = 2$,

$$\begin{aligned}\mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^n \tilde{m}_j(\beta_0)^2\right) = & \frac{1}{n^2}\sum_{j=1}^n\sum_{k=1}^n [\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0), \tilde{m}_k(\beta_0)) \\ & +2\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0))\mathrm{cum}(\tilde{m}_k(\beta_0)) \\ & +2\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0), \tilde{m}_k(\beta_0))\mathrm{cum}(\tilde{m}_j(\beta_0)) \\ & +4\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0))\mathrm{cum}(\tilde{m}_j(\beta_0))\mathrm{cum}(\tilde{m}_k(\beta_0)) \\ & +2\mathrm{cum}^2(\tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0))] + O\left(n^{-1}\right)\,. \end{aligned} \tag{10}$$

For the double summation in the (10), the terms corresponding to $j = k$ can be collected as

$$\frac{1}{n^2}\sum_{j=1}^n \mathrm{Var}(\tilde{m}_j(\beta_0)) = \frac{20}{n^2}\sigma_\epsilon^8 \sum_{j=1}^n p_j(\beta_0)^4 + O(n^{-2})\,, \tag{11}$$

using the equations (9) and (7)–(9).

By applying an argument similar to that used in the second equality of Lemma 2 in Chen *et al.* (2016), for $j \neq k$, we have

$$\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0)) = \frac{1}{n}\kappa_{\epsilon,4} p_j(\beta_0) p_k(\beta_0) + O(n^{-2})\,, \tag{12}$$

$$\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0)) = \frac{2}{n}\kappa_{\epsilon,4}\sigma_\epsilon^2 p_j(\beta_0)^2 p_k(\beta_0) + O(n^{-2})\,, \tag{13}$$

$$\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0), \tilde{m}_k(\beta_0)) = \frac{2}{n}\kappa_{\epsilon,4}\sigma_\epsilon^2 p_j(\beta_0) p_k(\beta_0)^2 + O(n^{-2})\,, \tag{14}$$

and

$$\mathrm{cum}(\tilde{m}_j(\beta_0), \tilde{m}_j(\beta_0), \tilde{m}_k(\beta_0), \tilde{m}_k(\beta_0)) = \frac{4}{n}\kappa_{\epsilon,4}\sigma_\epsilon^4 p_j(\beta_0)^2 p_k(\beta_0)^2 + O(n^{-2})\,, \tag{15}$$

where the $O(n^{-2})$ terms are uniform in j, k.

Substituting the equations (9) and (11)–(15) into the (10) yields

$$\mathrm{Var}(\Delta_2) = \frac{20}{n^2}\sigma_\epsilon^8 \sum_{j=1}^n p_j(\beta_0)^4 + \frac{16}{n^3}\kappa_{\epsilon,4}\sigma_\epsilon^4 \sum_{j\neq k} p_j(\beta_0)^2 p_k(\beta_0)^2 + O(n^{-2})\,.$$

By applying similar but more tedious calculations to Δ_3 and Δ_4, we have

$$\mathrm{Var}(\Delta_3) = \frac{K_3}{n^2}\sigma_\epsilon^{12} \sum_{j=1}^n p_j(\beta_0)^6 + \frac{K_3^{'}}{n^3}\kappa_{\epsilon,4}\sigma_\epsilon^8 \sum_{j\neq k} p_j(\beta_0)^3 p_k(\beta_0)^3 + O(n^{-2})\,,$$

$$\mathrm{Var}(\Delta_4) = \frac{K_4}{n^2}\sigma_\epsilon^{16} \sum_{j=1}^n p_j(\beta_0)^8 + \frac{K_4^{'}}{n^3}\kappa_{\epsilon,4}\sigma_\epsilon^{12} \sum_{j\neq k} p_j(\beta_0)^4 p_k(\beta_0)^4 + O(n^{-2})\,,$$

where K_3, K_4, $K_3^{'}$ and $K_4^{'}$ are some constants. This completes the proof of Lemma 1.

Proof of Lemma 2: To derive $\kappa_1(\sqrt{n}\tilde{SR})$, first note that

$$\mathrm{cum}(\bar{\tilde{m}}) = \frac{1}{n^2}\sum_{j=1}^{n}\frac{p_j(\beta_0)}{g_j(\beta_0)}b_I(\omega_j) + O(n^{-2})\,, \tag{16}$$

$$\mathrm{cum}(\bar{\tilde{m}}^2) = \mathrm{cum}(\bar{\tilde{m}},\bar{\tilde{m}}) + (\mathrm{cum}(\bar{\tilde{m}}))^2 = \frac{1}{n}\frac{1}{2}\lambda_2 + O(n^{-2})\,, \tag{17}$$

and

$$\begin{aligned}\mathrm{cum}(\bar{\tilde{m}}\Delta_2) &= \mathrm{cum}(\bar{\tilde{m}},\Delta_2)\\ &= \frac{1}{n^2}\sum_{j=1}^{n}\frac{p_j(\beta_0)^3}{g_j(\beta_0)^3}[\mathrm{cum}(I_n(\omega_j),I_n(\omega_j),I_n(\omega_j))\\ &\quad +2\mathrm{cum}(I_n(\omega_j),I_n(\omega_j))\mathrm{cum}(I_n(\omega_j))] + O(n^{-2})\\ &= \frac{4\sigma_\epsilon^6}{n^2}\sum_{j=1}^{n}p_j(\beta_0)^3 + O(n^{-2}) = \frac{2}{3}\frac{\lambda_3}{n} + O(n^{-2})\,. \end{aligned} \tag{18}$$

By using the equations (16)–(18), we have

$$\begin{aligned}\kappa_1(\sqrt{n}\tilde{SR}) &= \mathrm{cum}(\sqrt{n}(\tilde{R}_1+\tilde{R}_2+\tilde{R}_3+\tilde{R}_n))\\ &= \sqrt{n}\mathrm{cum}\left\{\sqrt{2}\frac{\bar{\tilde{m}}}{\lambda_2^{1/2}} + \frac{\sqrt{2}}{3}\frac{\lambda_3\bar{\tilde{m}}^2}{\lambda_2^{5/2}} - \frac{\sqrt{2}}{2}\frac{\bar{\tilde{m}}\Delta_2}{\lambda_2^{3/2}}\right\} + O(n^{-2})\\ &= \frac{1}{n^{3/2}}\frac{\sqrt{2}}{\lambda_2^{1/2}}\sum_{j=1}^{n}\frac{b_I(\omega_j)}{g_j(\beta_0)}p_j(\beta_0) - \frac{1}{n^{1/2}}\frac{\sqrt{2}}{6}\frac{\lambda_3}{\lambda_2^{3/2}} + O(n^{-3/2})\,.\end{aligned}$$

Hence, the (13) is established. For $\kappa_2(\sqrt{n}\tilde{SR})$, consider

$$\begin{aligned}\kappa_2(\sqrt{n}\tilde{SR}) =&\mathrm{cum}(\sqrt{n}\tilde{SR},\sqrt{n}\tilde{SR})\\ =&n\left\{\mathrm{cum}(\tilde{R}_1,\tilde{R}_1)+2\mathrm{cum}(\tilde{R}_1,\tilde{R}_2)+2\mathrm{cum}(\tilde{R}_1,\tilde{R}_3)+\mathrm{cum}(\tilde{R}_2,\tilde{R}_2)\right\}\\ &+O(n^{-2})\,.\end{aligned} \tag{19}$$

Using the equations (7)–(9) and (12)–(15), we have

$$\begin{aligned}\mathrm{cum}(\tilde{R}_1,\tilde{R}_1) &= \mathrm{cum}\left(\sqrt{2}\frac{\bar{\tilde{m}}}{\lambda_2^{1/2}},\sqrt{2}\frac{\bar{\tilde{m}}}{\lambda_2^{1/2}}\right) = \frac{1}{n} + \frac{4\sigma_\epsilon^2}{\lambda_2}\frac{1}{n^3}\sum_{j=1}^{n}\frac{p_j(\beta_0)^2}{g_j(\beta_0)}b_I(\omega_j)\\ &\quad+\frac{2}{\lambda_2}\frac{1}{n^2}\sum_{j\neq k}\frac{p_j(\beta_0)p_k(\beta_0)}{g_j(\beta_0)g_k(\beta_0)}\mathrm{cum}(I_n(\omega_j),I_n(\omega_k)) + O(n^{-3})\,,\end{aligned} \tag{20}$$

$$\begin{aligned}\mathrm{cum}(\tilde{R}_1,\tilde{R}_2) &= \frac{2}{3}\frac{\lambda_3}{\lambda_2^3}\mathrm{cum}(\bar{\tilde{m}},\bar{\tilde{m}}^2) - \frac{1}{\lambda_2^2}\mathrm{cum}(\bar{\tilde{m}},\bar{\tilde{m}}\Delta_2)\\ &= \frac{1}{n^2}\left\{\frac{2}{9}\frac{\lambda_3^2}{\lambda_2^3} - \frac{1}{2}\frac{\lambda_4}{\lambda_2^2} - 2\frac{\kappa_{\epsilon,4}}{\sigma_\epsilon^4}\right\} + O(n^{-3})\,,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathrm{cum}(\tilde{R}_1, \tilde{R}_3) &= \frac{3}{4}\frac{1}{\lambda_2^3}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}\Delta_2^2) + \frac{2}{3}\frac{1}{\lambda_2^3}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}^2\Delta_3) - \frac{5}{3}\frac{\lambda_3}{\lambda_2^4}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}^2\Delta_2) \\ &+ \frac{8}{9}\frac{\lambda_3^2}{\lambda_2^5}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}^3) - \frac{1}{2}\frac{\lambda_4}{\lambda_2^4}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}^3) \\ &= \frac{1}{n^2}\left\{\frac{11}{16}\frac{\lambda_4}{\lambda_2^2} - \frac{1}{3}\frac{\lambda_3^2}{\lambda_2^3} + \frac{3}{2}\frac{\kappa_{\epsilon,4}}{\sigma_\epsilon^4}\right\} + O(n^{-3})\,, \end{aligned} \tag{22}$$

and

$$\begin{aligned}\mathrm{cum}(\tilde{R}_2, \tilde{R}_2) &= \frac{2}{9}\frac{\lambda_3^2}{\lambda_2^5}\mathrm{cum}(\bar{\tilde{m}}^2, \bar{\tilde{m}}^2) - \frac{2}{3}\frac{\lambda_3}{\lambda_2^4}\mathrm{cum}(\bar{\tilde{m}}\Delta_2, \bar{\tilde{m}}^2) + \frac{1}{2}\frac{1}{\lambda_2^3}\mathrm{cum}(\bar{\tilde{m}}\Delta_2, \bar{\tilde{m}}\Delta_2) \\ &= \frac{1}{n^2}\left\{-\frac{1}{9}\frac{\lambda_3^2}{\lambda_2^3} + \frac{5}{24}\frac{\lambda_4}{\lambda_2^2} + \frac{\kappa_{\epsilon,4}}{\sigma_\epsilon^4}\right\} + O(n^{-3})\,. \end{aligned} \tag{23}$$

Substituting the equations (20)–(23) into the (19) yields

$$\begin{aligned}\kappa_2(\sqrt{n}\tilde{SR}) &= 1 + \frac{1}{n}\left\{-\frac{1}{3}\frac{\lambda_3^2}{\lambda_2^3} + \frac{7}{12}\frac{\lambda_4}{\lambda_2^2} + \frac{4\sigma_\epsilon^2}{\lambda_2}\frac{1}{n}\sum_{j=1}^{n}\frac{p_j(\beta_0)^2}{g_j(\beta_0)}b_I(\omega_j)\right. \\ &\left. + \frac{2}{\lambda_2}\sum_{j\neq k}\frac{p_j(\beta_0)p_k(\beta_0)}{g_j(\beta_0)g_k(\beta_0)}\mathrm{cum}(I_n(\omega_j), I_n(\omega_k))\right\} + O(n^{-2})\,.\end{aligned}$$

For $\kappa_3(\sqrt{n}\tilde{SR})$, note that

$$\begin{aligned}\kappa_3(\sqrt{n}\tilde{SR}) &= \mathrm{cum}(\sqrt{n}\tilde{SR}, \sqrt{n}\tilde{SR}, \sqrt{n}\tilde{SR}) \\ &= n^{3/2}\{\mathrm{cum}(\tilde{R}_1, \tilde{R}_1, \tilde{R}_1) + 3\mathrm{cum}(\tilde{R}_1, \tilde{R}_1, \tilde{R}_2)\} + O(n^{-3/2})\,.\end{aligned} \tag{24}$$

Using the equations (7)–(9) and (12)–(15), we have

$$\mathrm{cum}(\tilde{R}_1, \tilde{R}_1, \tilde{R}_1) = \frac{1}{n^2}\sqrt{2}\frac{\lambda_3}{\lambda_2^{3/2}} + O(n^{-3})\,. \tag{25}$$

For the second term on the right-hand side of the (24), it can be shown that

$$3\mathrm{cum}(\tilde{R}_1, \tilde{R}_1, \tilde{R}_2) = 2\sqrt{2}\frac{\lambda_3}{\lambda_2^{7/2}}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}, \bar{\tilde{m}}^2) - 3\sqrt{2}\frac{1}{\lambda_2^{5/2}}\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}, \bar{\tilde{m}}\Delta_2)\,.$$

Again using the equations (7)–(9) and (12)–(15), we have

$$\mathrm{cum}(\bar{\tilde{m}}, \bar{\tilde{m}}, \bar{\tilde{m}}^2) = \frac{\lambda_2^2}{2n^2} + O(n^{-3})\,,$$

$$\mathrm{cum}(\overline{\tilde{m}}, \bar{\tilde{m}}, \bar{\tilde{m}}\Delta_2) = \frac{2}{3n^2}\lambda_2\lambda_3 + O(n^{-3})\,.$$

Thus, we have

$$3\mathrm{cum}(\tilde{R}_1, \tilde{R}_1, \tilde{R}_2) = -\frac{\sqrt{2}}{n^2}\frac{\lambda_3}{\lambda_2^{3/2}} + O(n^{-3})\,. \tag{26}$$

Substituting the equations (25) and (26) into the (24), we have $\kappa_3(\sqrt{n}\tilde{SR}) = O(n^{-3/2})$.

For $\kappa_4(\sqrt{n}\tilde{SR})$, we have

$$\begin{aligned}
\kappa_4(\sqrt{n}\tilde{SR}) &= \mathrm{cum}(\sqrt{n}\tilde{SR}, \sqrt{n}\tilde{SR}, \sqrt{n}\tilde{SR}, \sqrt{n}\tilde{SR}) \\
&= \mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1) + 4\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_2) \\
&\quad + 4\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_3) + 6\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_2, \sqrt{n}\tilde{R}_2) \\
&\quad + O(n^{-2})\,. \qquad (27)
\end{aligned}$$

Using the equations (7)–(9) and (12)–(15), it can be shown that

$$\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1) = \frac{1}{n}\left\{\frac{\lambda_4}{\lambda_2^2} + 12\frac{\kappa_{\epsilon,4}}{\sigma_\epsilon^4}\right\} + O(n^{-2})\,, \qquad (28)$$

$$\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_2) = \frac{1}{n}\left\{-\frac{3}{2}\frac{\lambda_4}{\lambda_2^2} - 6\frac{\kappa_{\epsilon,4}}{\sigma_\epsilon^4}\right\} + O(n^{-2})\,, \qquad (29)$$

$$\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_3) = \frac{1}{n}\frac{3}{4}\frac{\lambda_4}{\lambda_2^2} + O(n^{-2})\,, \qquad (30)$$

and

$$\mathrm{cum}(\sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_1, \sqrt{n}\tilde{R}_2, \sqrt{n}\tilde{R}_2) = \frac{1}{n}\left\{\frac{1}{3}\frac{\lambda_4}{\lambda_2^2} + 2\frac{\kappa_{\epsilon,4}}{\sigma_\epsilon^4}\right\} + O(n^{-2})\,. \qquad (31)$$

Substituting the equations (28)–(31) into the (27), we have $\kappa_4(\sqrt{n}\tilde{SR}) = O(n^{-2})$. This completes the proof of Lemma 2.

Proof of Theorem 1: The proof is standard and can be found in Chan *et al.* (2014). □

Proof of Theorem 2: Based on the cumulant expansions in Lemma 2,

$$\begin{aligned}
\mathrm{E}(\tilde{l}(\beta_0)) = \mathrm{E}(n\tilde{SR}^2) &= \kappa_2(\sqrt{n}\tilde{SR}) + \kappa_1(\sqrt{n}\tilde{SR})^2 = 1 + \frac{\tilde{k}_{1,1}^2 + \tilde{k}_{2,2}}{n} + O(n^{-2}) \\
&= 1 + \frac{\tilde{b}}{n} + O(n^{-2})\,.
\end{aligned}$$

Next, we derive the coverage error of the Bartlett-corrected confidence interval based on the Edgeworth expansion of $\sqrt{n}\tilde{SR}^* := \sqrt{n}\tilde{SR}(1 - \tilde{b}/2n)$. Specifically,

$$\begin{aligned}
& P(\tilde{l}(\beta_0) \le (1 + \tilde{b}/n)\chi^2_{1,1-\alpha}) = P((\sqrt{n}\tilde{SR}^*)^2 + O_p(n^{-2}) \le \chi^2_{1,1-\alpha}) \\
&= \int_{-\sqrt{\chi^2_{1,1-\alpha}}}^{\sqrt{\chi^2_{1,1-\alpha}}} \phi(x)\,dx + \int_{-\sqrt{\chi^2_{1,1-\alpha}}}^{\sqrt{\chi^2_{1,1-\alpha}}} \left\{\frac{\tilde{r}_1^*(x)}{\sqrt{n}} + \frac{\tilde{r}_2^*(x)}{n} + \frac{\tilde{r}_3^*(x)}{n^{3/2}}\right\}\phi(x)\,dx + O(n^{-2}) \\
&= 1 - \alpha + O(n^{-2})\,, \qquad (32)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{r}_1^*(x) &= \sqrt{n}\left\{\kappa_1(\sqrt{n}\tilde{SR}^*)x + \frac{1}{6}\kappa_3(\sqrt{n}\tilde{SR}^*)(x^3 - 3x)\right\}, \\
\tilde{r}_2^*(x) &= \frac{n}{2}\left\{\kappa_2(\sqrt{n}\tilde{SR}^*) - 1 + \kappa_1(\sqrt{n}\tilde{SR}^*)\right\}(x^2 - 1)\,.
\end{aligned}$$

The last equality in the (32) holds because the integral of $\tilde{r}_2^*(x)$ equals $O(n^{-1})$ based on the equations (13) and (14). Moreover, the integrals of terms with order $n^{-1/2}$ and $n^{-3/2}$ are equal to zero because of the oddness of polynomials $\tilde{r}_1^*(x)$ and $\tilde{r}_3^*(x)$. This completes the proof of Theorem 2.

Proof of Lemma 3: It is sufficient to show that $\hat{\tilde{k}}_{1,1} = \tilde{k}_{1,1} + O_p(n^{-1/2})$ and $\hat{\tilde{k}}_{2,2} = \tilde{k}_{2,2} + O_p(n^{-1/2})$. Note that the convergences hold point-wise on the interior of the compact parameter space Θ. Under **RC**, the $\sqrt{n}$-consistency of $\hat{\beta}$ implies $\hat{\lambda}_k = \lambda_k + O_p(n^{-1/2})$, $k = 2, 3, 4$. Thus, $\hat{\tilde{k}}_{2,2} = \tilde{k}_{2,2} + O_p(n^{-1/2})$. For $\hat{\tilde{k}}_{1,1}$, we argue that $\hat{b}_I(\omega) = b_I(\omega) + O_p(n^{-1/2})$, where the error is uniform in $\omega \in \Pi$. We consider the decomposition $|b_I(\omega) - \hat{b}_I(\omega)| \leq T_1(\omega) + T_2(\omega)$, where

$$T_1(\omega) = \left| b_I(\omega) + \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} |u| \gamma_{\hat{\theta}}(u) \exp(\imath u \omega) \right| ,$$

$$T_2(\omega) = \left| -\frac{1}{2\pi} \sum_{u=-\infty}^{\infty} |u| \gamma_{\hat{\theta}}(u) \exp(\imath u \omega) + \frac{1}{2\pi} \sum_{u=-[n/2]+1}^{[n/2]+1} |u| \gamma_{\hat{\theta}}(u) \exp(\imath u \omega) \right| .$$

Next, we prove that $\sup_{\omega \in \Pi} T_1(\omega) = O_p(n^{-1/2})$. By applying the Taylor expansion, we have

$$\begin{aligned} \sup_{\omega \in \Pi} T_1(\omega) &\leq \frac{1}{2\pi} \sup_{\omega \in \Pi} \sum_{u=-\infty}^{\infty} |u| |\gamma_{\hat{\theta}}(u) - \gamma_{\theta_0}(u)| |\exp(\imath u \omega)| \\ &\leq \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} |u| |\gamma_{\hat{\theta}}(u) - \gamma_{\theta_0}(u)| \\ &= \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} |u| |(\hat{\theta} - \theta_0)' \frac{\partial}{\partial \theta} \gamma_{\theta^*}(u)| \\ &\leq \frac{1}{2\pi} \|\hat{\theta} - \theta_0\| \sum_{u=-\infty}^{\infty} |u| \|\frac{\partial}{\partial \theta} \gamma_{\theta^*}(u)\| \\ &= O_p(n^{-1/2}) , \end{aligned}$$

where $\|\theta^* - \theta_0\| \leq \|\hat{\theta} - \theta_0\|$ and the last equality follows from Regularity Condition (a) and $\|\hat{\theta} - \theta_0\| = O_p(n^{-1/2})$. As $\sup_{\omega \in \Pi} T_2(\omega) \leq (2\pi)^{-1} \sum_{u=[n/2]+1}^{\infty} |u| |\gamma_{\hat{\theta}}(u)| = o_p(n^{-1/2})$, it follows that $\hat{b}_I(\omega) = b_I(\omega) + O_p(n^{-1/2})$, where the error term is uniform in ω. Hence, $\hat{\tilde{k}}_{1,1} = \tilde{k}_{1,1} + O_p(n^{-1/2})$. Under Condition (b), by using the similar arguments as above, we can show that $\widehat{\text{cum}}(I_n(\omega_j), I_n(\omega_k)) = \text{cum}(I_n(\omega_j), I_n(\omega_k)) + O_p(n^{-1/2})$ and thus $\hat{\tilde{k}}_{2,2} = \tilde{k}_{2,2} + O_p(n^{-1/2})$.

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