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Poisson source localization on the plane: cusp case

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Abstract
This work is devoted to the problem of estimation of the localization of Poisson source. The observations are inhomogeneous Poisson processes registered by more than three detectors on the plane. We study the behavior of the Bayes estimators in the asymptotic of large intensities. It is supposed that the intensity functions of the signals arriving in the detectors have cusp-type singularity. We show the consistency, limit distributions, the convergence of moments and asymptotic efficiency of these estimators.

Keywords Inhomogeneous Poisson process · Poisson source · Sensors · Bayes estimators · Cusp-type singularity

1 Introduction

Suppose that we have $k \geq 3$ detectors at the points D_j, $j = 1, \ldots, k$ with the coordinates $\vartheta_j = \left(x_j, y_j\right)$, $j = 1, \ldots, k$ on the plane and a source of emission of Poisson signals at the point D_0 with coordinates $\vartheta_0 = (x_0, y_0)$. We consider the problem of estimation of the position $\vartheta_0 = (x_0, y_0)$ by the observations $X = (X_1, \ldots, X_k)$ of independent Poisson signals $X_j = \left(X_j(t), 0 \leq t \leq T\right)$ received by detectors (see Knoll 2010).

An example of such model is given in Fig. 1.

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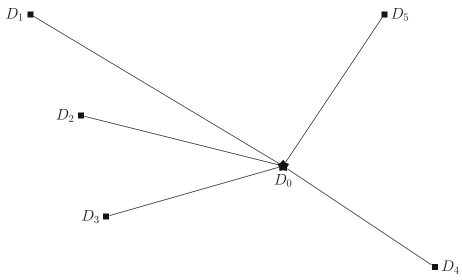

Fig. 1 Model of observations

This is our second work devoted to this problem of identification of localization of the source (see the Introduction of Farinetto et al. (2018), where we give the review of the engineering literature on this subject).

The intensity function $\lambda_{j,n}(\vartheta_0, t)$ of the Poisson process received by the j-th detector taken in this work, in Chernoyarov and Kutoyants (2019) and in Farinetto et al. (2018), is of the form

$$\lambda_{j,n}(\vartheta_0, t) = n\mu_j\left(t - \tau_j\right) \mathrm{I\!I}_{\{t \geq \tau_j\}} + n\lambda_0, \quad 0 \leq t \leq T. \tag{1}$$

Here, $\tau_j = \tau_j(\vartheta_0)$ is the instant of arrival of the Poisson signal at the j-th detector, which is calculated by the formula $\tau_j(\vartheta_0) = \nu^{-1}\left\|\vartheta_j - \vartheta_0\right\|$, where $\nu > 0$ is the known rate of propagation of the signal and $\|\cdot\|$ is Euclidean norm on the plane. The Poisson signals are received in the presence of Poisson noise of the intensity $n\lambda_0 > 0$. We suppose that λ_0 is known because, as usual, the level of noise can be estimated with much higher precision before the experiment and the contribution of the error of its estimation can be negligible with respect to the other errors of estimation. We discuss in the last section the possibility of the joint estimation of ϑ and λ_0. Therefore, we have observations of k independent Poisson processes of intensity functions (1) and have to estimate the position ϑ_0 of the source. The exact calculation of the error of estimation $\mathbf{E}_{\vartheta_0}\|\bar{\vartheta} - \vartheta_0\|^2$ ($\bar{\vartheta}$ is some estimator) in this essentially nonlinear statistical problem is a very difficult problem. Moreover, the most interesting are the situations where the errors of estimation are small. To obtain small errors and have possibility to calculate it, we have to consider one or another type of asymptotics. That is why we introduce the *large parameter* n in the intensity function (1) and study the errors of estimation in the asymptotics $n \to \infty$. This means that the signal and noise are sufficiently large and the estimators $\bar{\vartheta} = \bar{\vartheta}_n$ take values not too far from the true value: $\mathbf{E}_{\vartheta_0}\|\bar{\vartheta} - \vartheta_0\|^2 = o(1)$. Recall that the similar mathematical model can be used in the problem of GPS localization on the plane. In this case, we have k emitters of the Poisson signals and an object which receives these signals. The positions of the emitters are known and the problem is in the estimation of the position of the object by the observations of the signals. The intensity functions of the received Poisson signals depend on the distance between the emitters and the object and the receiver has to define its position by these observations (see, e.g., Luo 2013).

The goal of Chernoyarov and Kutoyants (2019), Farinetto et al. (2018) and of this work is to evaluate the error $\mathbf{E}_{\vartheta_0}\|\tilde{\vartheta} - \vartheta_0\|^2$, where $\tilde{\vartheta}_n$ is the Bayes estimator (BE) with the quadratic loss function. The difference between these three works is in the conditions of regularity of the functions $\lambda_j(\cdot)$ and as a consequence of it, the rates of convergence of the errors are different.

Let us remind this class of models and errors of estimation with the help of the Poisson process with intensity function

$$\lambda_n(\vartheta, t) = 2n \left|\frac{t-\vartheta}{\delta}\right|^{\kappa} 1\!\!1_{\{0\le t-\vartheta\le\delta\}} + 2n\,1\!\!1_{\{t>\vartheta+\delta\}} + n, \quad 0 \le t \le T.$$

Here, the unknown parameter ϑ is one-dimensional, $\vartheta \in (\alpha, \beta) \subset [0, T]$. Choosing the different values of κ, we obtain statistical problems of different regularity. The examples of such intensities are given in Fig. 2, where we put $n = 1$.

The cases (**a**) and (**b**) correspond to the *regular* (smooth, LAN) case. In the case (**c**), we have *cusp*-type singularity. The case (**d**) corresponds to *change-point* model of observations and the case (**e**) is *explosion*-type singularity. The rates of convergence of errors in these cases are

$$(\mathbf{a})\ \mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|^2 \approx \frac{C}{n}, \qquad (\mathbf{b})\ \mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|^2 \approx \frac{C}{n\ln n},$$
$$(\mathbf{c})\ \mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|^2 \approx \frac{C}{n^{\frac{2}{2\kappa+1}}}, \qquad (\mathbf{d})\ \mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|^2 \approx \frac{C}{n^2},$$
$$(\mathbf{e})\ \mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|^2 \approx \frac{C}{n^{\frac{2}{2\kappa+1}}}.$$

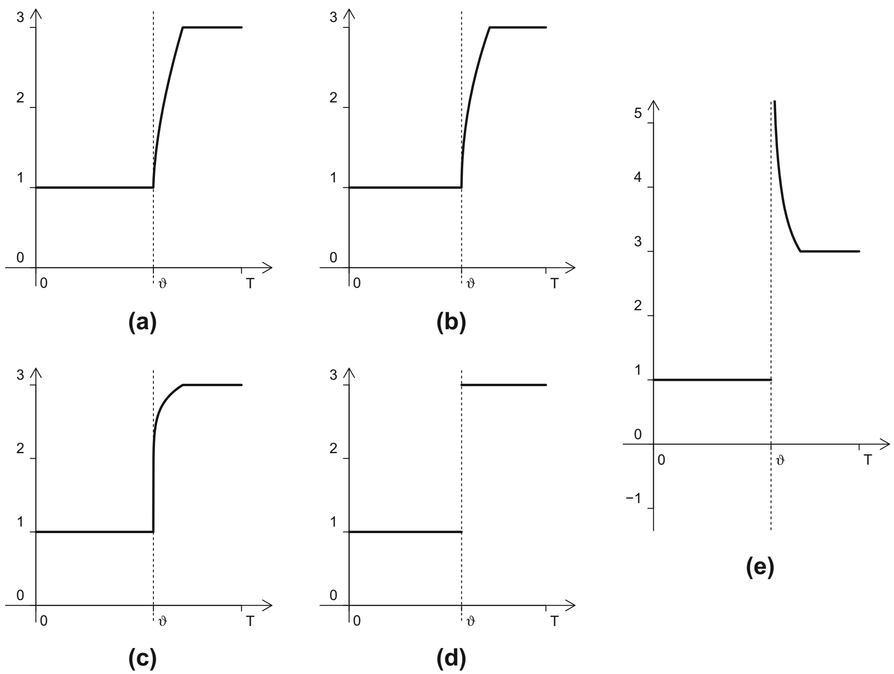

Fig. 2 Intensity functions of different regularity: **a** $\kappa = \frac{5}{8}$, **b** $\kappa = \frac{1}{2}$, **c** $\kappa = \frac{1}{8}$, **d** $\kappa = 0$, **e** $\kappa = -\frac{3}{8}$

For the case (**a**), see Kutoyants (1979); the case (**b**) was considered in Chernoyarov and Kutoyants (2019); for the case (**c**), see Dachian (2003); for the case (**d**), see Kutoyants (1984), while a situation similar to the case (**e**) was studied in Dachian (2011).

Our work is devoted to the problem of source localization in the situation close to the case (**c**) with $\kappa \in (0, 1/2)$. As we explain below, this type of singularity can be an alternative to the well-known change-point-type models with discontinuous intensity functions.

We have to note that the study of MLE and BE in all these cases was done with the help of some general results developed in Ibragimov and Khasminskii (1981) concerning behavior of estimators. Their method is based on the study of the normalized likelihood ratio random fields, which we remind below in the next section.

We have k independent observations of inhomogeneous Poisson processes $X^n = (X_1, \ldots, X_k)$ with intensities (1) depending on $\tau_j(\vartheta_0)$. We suppose that the position of the source $\vartheta_0 \in \Theta$ is unknown and we have to estimate ϑ_0 by the observations X^n. Here, $\Theta \subset \mathcal{R}^2$ is a convex bounded and open set.

It seems that the mathematical study of this class of models was not yet sufficiently developed. The statistical models of inhomogeneous Poisson process with intensity functions having discontinuities along some curves depending on unknown parameters were considered in Sections 5.2 and 5.3 of Kutoyants (1998). Statistical inference for point processes can be found, for example, in Karr (1991), in Snyder and Miller (1991) and in Streit (2010). Our interest to this class of models has the following motivation. There are two large classes of models: regular, where the characteristics of the models are sufficiently smooth with respect to unknown parameter and change-point-type models, where the functions describing the models are discontinuous. The properties of estimators and especially the rate of convergence in these two classes are essentially different. If we consider the problem of detection by a physical device (detector) of the time of positive signals arriving, then the data have strongly increasing at the moment of signal arriving characteristic function. The electrical current of the detector according to physical law cannot have jumps and the pure jump model does not fit to the real data. One way to describe such data can be cusp-type models. Therefore, the statistical study of such models corresponds well to certain real data. What happens if the continuous or change-point-type models used in the situations, where the true signal is of cusp-type were described in Kutoyants (2017) and in Chernoyarov et al. (2018). That is why by our mind, the cusp-type models have to be studied as much as the change-point models. For diffusion processes, such models were studied in Dachian and Kutoyants (2003) and in Kutoyants (2019).

Let us recall the definitions of the MLE and BE. The functions $\mu_j(\cdot)$ are bounded and the constant $\lambda_0 > 0$; therefore, the measures induced by the processes X_j in the space of their realizations are equivalent (see Liptser and Shiryayev 2001). The likelihood ratio function $L(\vartheta, X^n)$ is

$$\ln L\left(\vartheta, X^n\right) = \sum_{j=1}^{k} \int_{\tau_j}^{T} \ln\left(1 + \frac{\mu_j\left(t-\tau_j\right)}{\lambda_0}\right) \mathrm{d}X_j(t) - n \sum_{j=1}^{k} \int_{\tau_j}^{T} \mu_j\left(t-\tau_j\right) \mathrm{d}t.$$

Of course, $\tau_j = \tau_j(\vartheta)$ and the observations $X^n = (X_1, \dots, X_k)$, where $X_j^n = \left(X_j(t), 0 \le t \le T\right), j = 1, \dots, k$ are counting processes from k detectors. The intensity functions $\mu_j(\cdot)$ is given below in (5).

The maximum likelihood estimator (MLE) $\hat{\vartheta}_n$ and Bayesian estimator (BE) $\tilde{\vartheta}_n$ for the quadratic loss function are defined by the "usual" relations

$$L\left(\hat{\vartheta}_n, X^n\right) = \sup_{\vartheta\in\Theta} L\left(\vartheta, X^n\right) \tag{2}$$

and

$$\tilde{\vartheta}_n = \frac{\int_\Theta \vartheta\, p(\vartheta)\, L(\vartheta, X^n)\, \mathrm{d}\vartheta}{\int_\Theta p(\vartheta)\, L(\vartheta, X^n)\, \mathrm{d}\vartheta}. \tag{3}$$

Here, $p(\vartheta), \vartheta \in \Theta$ is the prior density. We suppose that it is positive, continuous function on Θ. In this work, we study the BE only. We do not describe here the behavior of the maximum likelihood estimator (MLE) $\hat{\vartheta}$ for two reasons. First, this estimator is not asymptotically efficient and second, its study requires additional technical lemma which we suppose to prove later. We give some details in the section Discussion below.

2 Main result

Suppose that there exists a source of Poisson signals at some point $\vartheta_0 = (x_0, y_0) \in \Theta \subset \mathcal{R}^2$ and $k \ge 3$ sensors (detectors) on the same plane located at the points $\vartheta_j = \left(x_j, y_j\right), j = 1, \dots, k$. The source was activated at the (known) instant $t = 0$ and the signals from the source (independent inhomogeneous Poisson processes) are registered by all k detectors. The signal arrives at the j-th detector at the instant τ_j. Of course, $\tau_j = \tau_j(\vartheta_0)$ is the time necessary for the signal to arrive in the j-th detector defined by the relation

$$\tau_j(\vartheta_0) = \nu^{-1}\left\|\vartheta_j - \vartheta_0\right\|,$$

where $\nu > 0$ is the known speed of propagation of the signal and $\|\cdot\|$ is the Euclidean norm (distance) in $\mathcal{R}^2$.

The intensity function of the Poisson process $X_j^n = \left(X_j(t), 0 \le t \le T\right)$ registered by the j-th detector is

$$\lambda_{j,n}(\vartheta_0, t) = nS_j\left(t - \tau_j(\vartheta_0)\right) + n\lambda_0, \qquad 0 \le t \le T, \tag{4}$$

where $nS_j\left(t - \tau_j(\vartheta_0)\right)$ is the intensity function of the signal and $n\lambda_0 > 0$ is intensity of the noise. We suppose that the function $S_j(\cdot)$ of the signal can be presented as follows:

$$S_j\left(t - \tau_j\right) = \lambda_j\left(t - \tau_j\right)\left|\frac{t - \tau_j}{\delta}\right|^{\kappa} \mathbb{I}_{\{0 \le t - \tau_j \le \delta\}} + \lambda_j\left(t - \tau_j\right) \mathbb{I}_{\{t - \tau_j > \delta\}}. \tag{5}$$

Here, $\delta > 0$ is some known (small) parameter. This work is devoted to the problem of parameter estimation in the model with cusp-type singularity. This singularity is provided by the choice of the parameter $\kappa \in (0, 1/2)$. Recall that we consider cusp-type models as smoothed version of the change-point models.

The log-likelihood ratio formula in our case is

$$\begin{aligned}
\ln L\left(\vartheta, X^n\right) =& \sum_{j=1}^{n} \int_{\tau_j}^{\tau_j+\delta} \ln\left(1 + \frac{\lambda_j\left(t-\tau_j\right)}{\lambda_0}\left|\frac{t-\tau_j}{\delta}\right|^{\kappa}\right) \mathrm{d}X_j\left(t\right) \\
&+ \sum_{j=1}^{n} \int_{\tau_j+\delta}^{T} \ln\left(1 + \frac{\lambda_j\left(t-\tau_j\right)}{\lambda_0}\right) \mathrm{d}X_j\left(t\right) \\
&- n \int_{\tau_j}^{\tau_j+\delta} \lambda_j\left(t-\tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^{\kappa} \mathrm{d}t - n \int_{\tau_j+\delta}^{T} \lambda_j\left(t-\tau_j\right) \mathrm{d}t,
\end{aligned}$$

where for simplicity of expression, we omitted in $\tau_j\left(\vartheta\right)$ the dependence on ϑ.

Introduce the notations: $\varphi_n = n^{-\frac{1}{2\kappa+1}}$ and for $j = 1, \ldots, k$

$$\begin{aligned}
\tau_j\left(\vartheta_0 + \nu\varphi_n u\right) &= \tau_j\left(\vartheta_0\right) - \varphi_n \langle m_j, u\rangle + \|u\|^2 O\left(\varphi_n^2\right), \\
m_j &= \left(\frac{x_j - x_0}{\rho_j}, \frac{y_j - y_0}{\rho_j}\right), \qquad \rho_j = \left\|\vartheta_j - \vartheta_0\right\|, \quad \left\|m_j\right\| = 1, \\
\alpha_j = \inf_{\vartheta\in\Theta} \tau_j\left(\vartheta\right), \quad & \beta_j = \sup_{\vartheta\in\Theta} \tau_j\left(\vartheta\right), \qquad \mathcal{T}_j = \left[\alpha_j, \beta_j\right].
\end{aligned} \tag{6}$$

Conditions $\mathscr{C}$.

$\mathscr{C}_1$. *The set* Θ *is open, convex, bounded and such that* $0 < \alpha_j < \beta_j < T$ *for all* $j = 1, \ldots, k$.
$\mathscr{C}_2$. *There exists* $\varepsilon > 0$ *such that* $\left\|\vartheta_j - \vartheta_0\right\| \geq \varepsilon$ *for all* $\vartheta_0 \in \Theta$ *and for all* $j = 1, \ldots, k$.
$\mathscr{C}_3$. *The parameters* $\kappa \in \left(0, \frac{1}{2}\right)$ *and* $\delta \in (0, T)$.
$\mathscr{C}_4$. *The functions* $\lambda_j\left(t\right) > 0$ *have continuous derivatives* $\lambda_j'\left(\cdot\right)$ *for all* $j = 1, \ldots, k$.
$\mathscr{C}_5$. *There are at least three detectors which are not on the same line.*

By the condition $\mathscr{C}_2$, we have $\min_j \rho_j > 0$. This condition is quite restrictive because if we take as Θ the region including ϑ_0 and all ϑ_j, we have to suppose that there exists $\varepsilon > 0$ such that the disks $\mathbb{C}_j = \left\{\vartheta_0 : \left\|\vartheta_j - \vartheta_0\right\| \leq \varepsilon\right\}$ are excluded from Θ, but in this case, the set Θ is no more convex. Note that it is possible to modify the proof in such a way that the consistency and convergence to the limit distribution are uniform on compacts $\mathbb{K} \subset \Theta$ which do not include the positions of the detectors ϑ_j. Another point, when we do the re-normalization $\vartheta = \vartheta_0 + \nu\varphi_n u$ with $u \in \mathbb{U}_n = \{u : \vartheta_0 + \nu\varphi_n u \in \Theta\}$, we have to exclude the values u which correspond to $\vartheta \in \mathbb{C}_j$. To avoid such problems, we extend the normalized likelihood ratio random field to include these values u, but the true value ϑ_0 is always separated from ϑ_j.

Introduce the notations: $\lambda_j = \lambda_j(0)$,

$$\mathbb{B}_j = \left\{u:\ \langle m_j, u\rangle < 0\right\}, \qquad \mathbb{B}_j^c = \left\{u:\ \langle m_j, u\rangle \geq 0\right\}, \quad \gamma_j = \frac{\lambda_j}{\delta^\kappa \sqrt{\lambda_0}},$$

$$J_j(u) = J_{j,-}(u)\, 1\!\!\mathrm{I}_{\{u\in\mathbb{B}_j\}} + J_{j,+}(u)\, 1\!\!\mathrm{I}_{\left\{u\in\mathbb{B}_j^c\right\}}, \quad u \in \mathcal{R}^2,$$

$$J_{j,-}(u) = \gamma_j \int_0^\infty \left[\left|s + \langle m_j, u\rangle\right|^\kappa 1\!\!\mathrm{I}_{\{s>-\langle m_j,u\rangle\}} - |s|^\kappa\right] \mathrm{d}W_j(s),$$

$$J_{j,+}(u) = \gamma_j \int_{-\langle m_j,u\rangle}^\infty \left[\left|s + \langle m_j, u\rangle\right|^\kappa - |s|^\kappa 1\!\!\mathrm{I}_{\{s>0\}}\right] \mathrm{d}W_j(s),$$

$$R_j(u) = R_{j,-} 1\!\!\mathrm{I}_{\{u\in\mathbb{B}_j\}} + R_{j,+} 1\!\!\mathrm{I}_{\left\{u\in\mathbb{B}_j^c\right\}}, \quad u \in \mathcal{R}^2,$$

$$R_{j,-} = \gamma_j^2 \int_0^\infty \left[|s-1|^\kappa 1\!\!\mathrm{I}_{\{s>1\}} - |s|^\kappa\right]^2 \mathrm{d}s,$$

$$R_{j,+} = \gamma_j^2 \int_{-1}^\infty \left[|s+1|^\kappa - |s|^\kappa 1\!\!\mathrm{I}_{\{s>0\}}\right]^2 \mathrm{d}s.$$

Here, $W_j(\cdot)$, $j = 1, \ldots, k$ are independent Wiener processes. The limit likelihood ratio field is

$$Z(u) = \exp\left\{\sum_{j=1}^k \left[J_j(u) - \frac{\left|\langle m_j, u\rangle\right|^{2\kappa+1}}{2} R_j(u)\right]\right\}, \qquad u \in \mathcal{R}^2.$$

Note that this is a product of k *independent* random fields

$$Z(u) = \prod_{j=1}^k Z_j(u), \qquad Z_j(u) = \exp\left\{J_j(u) - \frac{\left|\langle m_j, u\rangle\right|^{2\kappa+1}}{2} R_j(u)\right\}.$$

Introduce as well the random vector $\tilde{\zeta}$, which has the same distribution as the limit of the normalized BE

$$\tilde{\zeta} = \nu \frac{\int_{\mathcal{R}^2} u Z(u)\, \mathrm{d}u}{\int_{\mathcal{R}^2} Z(u)\, \mathrm{d}u}.$$

Remark that if all detectors are on the same line, then the consistent identification is impossible because the same signals come from the symmetric with respect to this line possible locations of the source.

We have the following minimax lower bound on the mean square errors of all estimators $\bar{\vartheta}_n$: Let the conditions $\mathcal{C}$ be fulfilled; then, for any $\vartheta_0 \in \Theta$

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \sup_{\|\vartheta-\vartheta_0\|\leq\delta} n^{\frac{2}{2\kappa+1}} \mathbf{E}_\vartheta \left\|\bar{\vartheta}_n - \vartheta\right\|^2 \geq \mathbf{E}_{\vartheta_0} \left\|\tilde{\zeta}\right\|^2.$$

For proof see, for example, Theorem 2.12.1 of Ibragimov and Khasminskii (1981). The simple explication of the construction of such bounds can be given as follows. Let us introduce a prior density $p_\delta(\vartheta)$ on the set $\Theta_\delta = \{\vartheta : \|\vartheta - \vartheta_0\| < \delta\}$ and denote $\tilde\vartheta_{\delta,n}$, the corresponding Bayes estimators. Suppose as well that we have the uniform convergence of the second moments of this estimator. Then for any estimator $\bar\vartheta_n$, we can write

$$\sup_{\|\vartheta-\vartheta_0\|\leq\delta} n^{\frac{2}{2\kappa+1}} \mathbf{E}_\vartheta \left\|\bar\vartheta_n - \vartheta\right\|^2 \geq n^{\frac{2}{2\kappa+1}} \int_{\Theta_\delta} \mathbf{E}_\vartheta \left\|\bar\vartheta_n - \vartheta\right\|^2 p_\delta(\vartheta)\,\mathrm{d}\vartheta$$

$$\geq n^{\frac{2}{2\kappa+1}} \int_{\Theta_\delta} \mathbf{E}_\vartheta \|\tilde\vartheta_{\delta,n} - \vartheta\|^2 p_\delta(\vartheta)\,\mathrm{d}\vartheta \longrightarrow \int_{\Theta_\delta} \mathbf{E}_\vartheta \|\tilde\zeta\|^2 p_\delta(\vartheta)\,\mathrm{d}\vartheta \longrightarrow \mathbf{E}_{\vartheta_0}\|\tilde\zeta\|^2$$

as $n \to \infty$ and then $\delta \to 0$.

The numerical calculations of the limit variances of MLE and BE for the cusp-type models can be found in Kordzakhia et al. (2018) and in Dachian et al. (2018).

We call the estimator $\bar\vartheta_n$ asymptotically efficient, if for all $\vartheta_0 \in \Theta$, we have the equality

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{\|\vartheta-\vartheta_0\|\leq\delta} n^{\frac{2}{2\kappa+1}} \mathbf{E}_\vartheta \left\|\bar\vartheta_n - \vartheta\right\|^2 = \mathbf{E}_{\vartheta_0}\|\tilde\zeta\|^2.$$

Theorem 1 *Let the conditions* $\mathcal{C}$ *be fulfilled; then, the BE* $\tilde\vartheta_n$ *is uniformly consistent, converges in distribution*

$$n^{\frac{1}{2\kappa+1}}\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \tilde\zeta,$$

for any $p > 0$

$$\lim_{n\to\infty} n^{\frac{p}{2\kappa+1}} \mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|^p = \mathbf{E}_{\vartheta_0}\left\|\tilde\zeta\right\|^p,$$

and BE is asymptotically efficient.

Proof The properties of estimators mentioned in this theorem will be verified with the help of the approach developed in Ibragimov and Khasminskii (1981). Let us note that we already used a similar method in Chernoyarov and Kutoyants (2019) and in Farinetto et al. (2018). For the convenience of understanding, we remind it here once more. Introduce the normalized likelihood ratio random field

$$Z_n(u) = \frac{L(\vartheta_0 + \nu\varphi_n u, X^n)}{L(\vartheta_0, X^n)}, \qquad u \in \mathbb{U}_n = \{u:\ \vartheta_0 + \nu\varphi_n u \in \Theta\}$$

where the normalizing function $\varphi_n = n^{-\frac{1}{2\kappa+1}}$.

Suppose that we already proved the convergence

$$Z_n(\cdot) \Longrightarrow Z(\cdot).$$

Then, the limit distribution of the BE can be obtained as follows (see Ibragimov and Khasminskii 1981). Below, we change the variables $\vartheta = \vartheta_u = \vartheta_0 + \nu\varphi_n u$.

$$\tilde{\vartheta}_n = \frac{\int_{\Theta} \theta p(\theta) L(\theta, X^n) \,\mathrm{d}\theta}{\int_{\Theta} p(\theta) L(\theta, X^n) \,\mathrm{d}\theta} = \vartheta_0 + \nu\varphi_n \frac{\int_{\mathbb{U}_n} up(\theta_u) L(\theta_u, X^n) \,\mathrm{d}u}{\int_{\mathbb{U}_n} p(\theta_u) L(\theta_u, X^n) \,\mathrm{d}u}$$
$$= \vartheta_0 + \nu\varphi_n \frac{\int_{\mathbb{U}_n} up(\theta_u) Z_n(u) \,\mathrm{d}u}{\int_{\mathbb{U}_n} p(\theta_u) Z_n(u) \,\mathrm{d}u}.$$

Hence,

$$\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_0\right) = \nu \frac{\int_{\mathbb{U}_n} up(\theta_u) Z_n(u) \,\mathrm{d}u}{\int_{\mathbb{U}_n} p(\theta_u) Z_n(u) \,\mathrm{d}u} \Longrightarrow \nu \frac{\int_{\mathcal{R}^2} uZ(u) \,\mathrm{d}u}{\int_{\mathcal{R}^2} Z(u) \,\mathrm{d}u} = \tilde{\zeta}.$$

Recall that $p(\theta_u) \to p(\vartheta_0) > 0$.

The properties of $Z_n(u)$ required in Theorem 1.10.2 of Ibragimov and Khasminskii (1981) are checked in three lemmas below. Remind that this approach to the study of the properties of these estimators was applied in Kutoyants (1979, 1998). Here, we use some there obtained inequalities. □

Lemma 1 *Let the conditions $\mathscr{C}$ be fulfilled; then, the finite-dimensional distributions of the random field $Z_n(u)$, $u \in \mathbb{U}_n$ converge to the finite-dimensional distributions of the limit random field $Z(u)$, $u \in \mathcal{R}^2$ and this convergence is uniform on compacts $\mathbb{K} \in \Theta$.*

Proof Let us denote $\mathrm{d}\pi_{j,n}(t) = \mathrm{d}X_j(t) - n\left[S_j\left(t - \tau_j(\vartheta_0)\right) + \lambda_0\right]\mathrm{d}t$ and put $\vartheta_u = \vartheta_0 + \nu\varphi_n u$, $\tau_j = \tau_j(\vartheta_0)$. Then, we can write

$$\ln Z_n(u) = \sum_{j=1}^{k} \int_0^T \ln\left(\frac{S_j\left(t - \tau_j(\vartheta_u)\right) + \lambda_0}{S_j\left(t - \tau_j\right) + \lambda_0}\right) \mathrm{d}\pi_{j,n}(t)$$
$$- n\sum_{j=1}^{k} \int_0^T \left[\frac{S_j\left(t - \tau_j(\vartheta_u)\right) + \lambda_0}{S_j\left(t - \tau_j\right) + \lambda_0} - 1\right.$$
$$\left. - \ln\left(\frac{S_j\left(t - \tau_j(\vartheta_u)\right) + \lambda_0}{S_j\left(t - \tau_j\right) + \lambda_0}\right)\right]\left[S_j\left(t - \tau_j\right) + \lambda_0\right]\mathrm{d}t$$
$$= \sum_{j=1}^{k} \int_0^T F_j(t, \vartheta_u) \,\mathrm{d}\pi_{j,n}(t) - n\sum_{j=1}^{k} \int_0^T G_j(t, \vartheta_u) \,\mathrm{d}t$$

with obvious notation.

Let $u \in \mathbb{B}_j$. Then, $\tau_j(\vartheta_u) > \tau_j$. Following the same arguments as those given in Dachian (2003), we obtain the asymptotic ($n \to \infty$) relations:

$$J_{j,n}(u) = \int_0^T F_j(t, \vartheta_u)\,\mathrm{d}\pi_{j,n}(t) = \int_{\tau_j}^{\tau_j+\delta} F_j(t, \vartheta_u)\,\mathrm{d}\pi_{j,n}(t)\,(1 + o(1))$$

$$I_{j,n}(u) = n\int_0^T G_j(t, \vartheta_u)\,\mathrm{d}t = n\int_{\tau_j}^{\tau_j+\delta} G_j(t, \vartheta_u)\,\mathrm{d}t\,(1 + o(1))\,.$$

For $t \in \left[\tau_j, \tau_j - \varphi_n\langle m_j, u\rangle\right]$ as $\varphi_n \to 0$, we obtain the expansions

$$\lambda_j(t - \tau_j(\vartheta_u)) = \lambda_j(0) + \left(t - \tau_j(\vartheta_u)\right)\lambda_j'(0)\,(1 + o(1)) = \lambda_j + o(1)$$

$$\lambda_j(t - \tau_j(\vartheta_u)) = \lambda_j(t - \tau_j) + \varphi_n\langle m_j, u\rangle\lambda_j'(t - \tau_j) + O\left(\varphi_n^2\right)\|u\|^2\,,$$

$$\left|\frac{t - \tau_j(\vartheta_u)}{\delta}\right|^\kappa = \delta^{-\kappa}\left|t - \tau_j + \varphi_n\langle m_j, u\rangle + O\left(\varphi_n^2\right)\right|^\kappa$$

$$= \delta^{-\kappa}\left|t - \tau_j + \varphi_n\langle m_j, u\rangle\right|^\kappa + O\left(\varphi_n^2\right).$$

Here, we used the inequality $|a + b|^\kappa \le |a|^\kappa + |b|^\kappa$.

Further, for $\tau_j \le t \le \tau_j - \varphi_n\langle m_j, u\rangle$ and $\|u\| < L$ for any $L >$, we can write

$$\ln\left(\frac{S_j(t - \tau_j(\vartheta_u)) + \lambda_0}{S_j\left(t - \tau_j\right) + \lambda_0}\right) = \ln\left(\frac{\lambda_0}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}\right)$$

$$= -\ln\left(1 + \frac{\lambda_j}{\lambda_0}\left|\frac{t - \tau_j}{\delta}\right|^\kappa\right)(1 + O(\varphi_n))$$

$$= -\frac{\lambda_j}{\lambda_0}\left|\frac{t - \tau_j}{\delta}\right|^\kappa\left(1 + O\left(\varphi_n^{2\kappa}\right)\right).$$

For $t \in \left[\tau_j - \varphi_n\langle m_j, u\rangle, \delta\right]$, the similar relations are

$$\ln\left(\frac{S_j(t - \tau_j(\vartheta_u)) + \lambda_0}{S_j\left(t - \tau_j\right) + \lambda_0}\right) = \ln\left(\frac{\lambda_j\left(t - \tau_j(\vartheta_u)\right)\left|\frac{t-\tau_j(\vartheta_u)}{\delta}\right|^\kappa + \lambda_0}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}\right)$$

$$= \ln\left(1 + \frac{\lambda_j\left(t - \tau_j(\vartheta_u)\right)\left|\frac{t-\tau_j(\vartheta_u)}{\delta}\right|^\kappa - \lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}\right)$$

$$= \frac{\lambda_j\left(t - \tau_j\right)}{S\left(t - \tau_j\right) + \lambda_0}\left[\left|\frac{t - \tau_j + \varphi_n\langle m_j, u\rangle}{\delta}\right|^\kappa - \left|\frac{t - \tau_j}{\delta}\right|^\kappa\right]\left(1 + O\left(\varphi_n^{2\kappa}\right)\right).$$

Therefore, (below $\tau_{j,u} = \tau_j - \varphi_n\langle m_j, u\rangle$)

$$\begin{aligned}
\mathbf{E}_{\vartheta_0}\left(J_{j,n}(u)\right)^2 &= \int_0^T F_j(t,\vartheta_u)^2\,\lambda_{j,n}(\vartheta_0,t)\,\mathrm{d}t\\
&= \frac{\lambda_j^2 n}{\lambda_0}\int_{\tau_j}^{\tau_{j,u}}\left|\frac{t-\tau_j}{\delta}\right|^{2\kappa}\mathrm{d}t + o(1)\\
&\quad + \int_{\tau_{j,u}}^{\delta}\frac{n\lambda_j\left(t-\tau_j\right)^2}{\left[S\left(t-\tau_j\right)+\lambda_0\right]^2}\left[\left|\frac{t-\tau_j+\varphi_n\langle m_j,u\rangle}{\delta}\right|^{\kappa}-\left|\frac{t-\tau_j}{\delta}\right|^{\kappa}\right]^2\mathrm{d}t\\
&= \frac{\lambda_j^2 n}{\lambda_0\delta^{2\kappa}}\varphi_n^{2\kappa+1}\int_0^{-\langle m_j,u\rangle}|s|^{2\kappa}\,\mathrm{d}s + o(1)\\
&\quad + \frac{n}{\delta^{2\kappa}}\varphi_n^{2\kappa+1}\int_{-\langle m_j,u\rangle}^{\frac{\delta-\tau_j}{\varphi_n}}\frac{\lambda_j(s\varphi_n)^2}{S_j(s\varphi_n)+\lambda_0}\left[\left|s+\langle m_j,u\rangle\right|^{\kappa}-|s|^{\kappa}\right]^2\mathrm{d}s\\
&= \gamma_j^2\left|\langle m_j,u\rangle\right|^{2\kappa+1}\int_0^1|v|^{2\kappa}\,\mathrm{d}v\\
&\quad + \gamma_j^2\left|\langle m_j,u\rangle\right|^{2\kappa+1}\int_1^{-\frac{\delta-\tau_j}{\langle m_j,u\rangle\varphi_n}}\left[|v-1|^{\kappa}-|v|^{\kappa}\right]^2\mathrm{d}v + o(1)\\
&= \gamma_j^2\left|\langle m_j,u\rangle\right|^{2\kappa+1}\int_0^{-\frac{\delta-\tau_j}{\langle m_j,u\rangle\varphi_n}}\left[|v-1|^{\kappa}\,1\!\!1_{\{v\geq 1\}}-|v|^{\kappa}\right]^2\mathrm{d}v + o(1)\\
&= \gamma_j^2\left|\langle m_j,u\rangle\right|^{2\kappa+1}R_n + o(1),
\end{aligned}$$

where we changed the variables $t = \tau_j + s\varphi_n$ and $s = -v\langle m_j,u\rangle$. Recall that $n\varphi_n^{2\kappa+1} = 1$ and $\gamma_j^2 = \lambda_j^2\lambda_0^{-1}\delta^{-2\kappa}$. Note that for any $L > 0$, we have uniform convergence

$$\sup_{-\langle m_j,u\rangle\leq s\leq L}\left|\frac{\lambda_j(s\varphi_n)^2}{S_j(s\varphi_n)+\lambda_0}-\frac{\lambda_j^2}{\lambda_0}\right|\longrightarrow 0$$

and the corresponding integral above is converging.

Hence for $u \in \mathbb{B}_-$, we obtain the following limit

$$\begin{aligned}
R_n &= \int_0^{-\frac{\delta-\tau_j}{\langle m_j,u\rangle\varphi_n}}\left[|v-1|^{\kappa}\,1\!\!1_{\{v\geq 1\}}-|v|^{\kappa}\right]^2\mathrm{d}v\\
&\longrightarrow \int_0^{\infty}\left[|v-1|^{\kappa}\,1\!\!1_{\{v\geq 1\}}-|v|^{\kappa}\right]^2\mathrm{d}v = R_{j,-}.
\end{aligned}$$

These arguments allow us to write the representation

$$J_{j,n}(u) = \gamma_j\int_0^{\frac{\delta-\tau_j}{\varphi_n}}\left[\left|s+\langle m_j,u\rangle\right|^{\kappa}\,1\!\!1_{\{s\geq-\langle m_j,u\rangle\}}-|s|^{\kappa}\right]\mathrm{d}W_{j,n}(s) + o(1).$$

Here,

$$W_{j,n}(s) = \frac{1}{\sqrt{\lambda_0 n\varphi_n}}\left[X_j\left(\tau_j + s\varphi_n\right) - X_j\left(\tau_j\right) - \int_{\tau_j}^{\tau_j+s\varphi_n} \lambda_{j,n}(\vartheta_0, v)\,\mathrm{d}v\right],$$

$$\mathbf{E}_{\vartheta_0} W_{j,n}(s)^2 = \frac{n}{\lambda_0 n\varphi_n}\int_{\tau_j}^{\tau_j+s\varphi_n} \lambda_{j,n}(\vartheta_0, v)\,\mathrm{d}v = s + o(1),$$

$$\mathbf{E}_{\vartheta_0} W_{j,n}(s) = 0, \qquad \mathbf{E}_{\vartheta_0} W_{j,n}(s_1)\, W_{j,n}(s_2) = s_1 \wedge s_2 + o(1).$$

The standard central limit theorem provides us the corresponding convergence of stochastic integrals. For any $u_1, \ldots, u_M \in \mathbb{B}_j$, we have the joint asymptotic normality of the stochastic integrals

$$Y_{j,n} \equiv \Big(J_{j,n}(u_1), \ldots, J_{j,n}(u_M)\Big) \Longrightarrow Y_j \equiv \Big(J_j(u_1), \ldots, J_j(u_M)\Big),$$

where

$$J_j(u) = \gamma_j \int_0^\infty \left[\left|s + \langle m_j, u\rangle\right|^\kappa \mathrm{1\!I}_{\{s \geq -\langle m_j, u\rangle\}} - |s|^\kappa\right] \mathrm{d}W_j(s).$$

Moreover, the similar arguments give us the convergence

$$\mathbf{Y}_n \equiv \Big(Y_{1,n}, \ldots, Y_{k,n}\Big) \Longrightarrow \mathbf{Y} \equiv \Big(Y_1, \ldots, Y_k\Big). \tag{7}$$

Consider now the values $u \in \mathbb{B}_j^c$. Then, $\tau_j(\vartheta_u) \leq \tau_j(\vartheta_0)$ or asymptotically $\tau_j(\vartheta_0) - \varphi_n\langle m_j, u\rangle + O\left(\varphi_n^2\right) \leq \tau_j(\vartheta_0)$. The similar arguments allow us to verify the convergence (7) with the limit process

$$J_j(u) = \gamma_j \int_{-\langle m_j, u\rangle}^\infty \left[\left|s + \langle m_j, u\rangle\right|^\kappa \mathrm{1\!I}_{\{s \leq 0\}} + \left[\left|s + \langle m_j, u\rangle\right|^\kappa - |s|^\kappa\right]\right] \mathrm{d}W(s).$$

Therefore, we have the convergence of finite-dimensional distributions of the stochastic integrals.

For the ordinary integral $I_{j,n}(u)$, we have the similar representation ($u \in \mathbb{B}$, $G_{j,t} = G_j(t, u)$)

$$\begin{aligned} I_{j,n}(u) &= n\int_0^{\tau_j} G_{j,t}\,\mathrm{d}t + n\int_{\tau_j}^{\tau_j(\vartheta_u)} G_{j,t}\,\mathrm{d}t + n\int_{\tau_j(\vartheta_u)}^{\tau_j+\delta} G_{j,t}\,\mathrm{d}t \\ &\quad + n\int_{\tau_j+\delta}^{\tau_j(\vartheta_u)+\delta} G_{j,t}\,\mathrm{d}t + n\int_{\tau_j(\vartheta_u)+\delta}^{T} G_{j,t}\,\mathrm{d}t \\ &= n\int_{\tau_j}^{\tau_j(\vartheta_u)} G_{j,t}\,\mathrm{d}t + n\int_{\tau_j(\vartheta_u)}^{\tau_j+\delta} G_{j,t}\,\mathrm{d}t + n\int_{\tau_j+\delta}^{\tau_j(\vartheta_u)+\delta} G_{j,t}\,\mathrm{d}t + o(1). \end{aligned}$$

For $t \in [0, \tau_j]$, we have $G_j(t, u) = 0$ and for $t \in [\tau_j + \delta, T]$, the function $G_j(t, u)$ has continuous bounded derivative and we can write

$$n \int_{\tau_j(\vartheta_u)+\delta}^{T} G_{j,t} \, \mathrm{d}t \leq C n \varphi_n^2 \|u\|^2 = o(1).$$

Consider the case $t \in [\tau_j, \tau_j(\vartheta_u)]$. Using expansion $\ln(1+x) = x - \frac{x^2}{2} + O(x^3)$, we can write

$$\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} - 1 - \ln\left(\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)}\right) = \frac{\lambda_0}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}$$
$$- 1 - \ln\left(\frac{\lambda_0}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}\right) = \frac{\lambda_j^2}{2\lambda_0^2\delta^{2\kappa}}\left|t - \tau_j\right|^{2\kappa}(1 + o(1)).$$

For $t \in [\tau_j(\vartheta_u), \tau_j + \delta]$, we have

$$\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} - 1 - \ln\left(\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)}\right) = \frac{\lambda_j\left(t - \tau_j(\vartheta_u)\right)\left|\frac{t-\tau_j(\vartheta_u)}{\delta}\right|^\kappa + \lambda_0}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}$$
$$- 1 - \ln\left(\frac{\lambda_j\left(t - \tau_j(\vartheta_u)\right)\left|\frac{t-\tau_j(\vartheta_u)}{\delta}\right|^\kappa + \lambda_0}{\lambda_j\left(t - \tau_j\right)\left|\frac{t-\tau_j}{\delta}\right|^\kappa + \lambda_0}\right)$$
$$= \frac{\lambda_j^2}{2\lambda_0^2\delta^{2\kappa}}\left(\left|t - \tau_j(\vartheta_u)\right|^\kappa - \left|t - \tau_j\right|^\kappa\right)^2(1 + o(1)).$$

These relations allow us to write

$$I_{j,n} = \frac{n\lambda_j^2}{2\lambda_0\delta^{2\kappa}} \int_{\tau_j}^{\tau_j(\vartheta_u)} \left|t - \tau_j\right|^{2\kappa} \mathrm{d}t$$
$$+ \frac{n\lambda_j^2}{2\lambda_0\delta^{2\kappa}} \int_{\tau_j(\vartheta_u)}^{\tau_j+\delta}\left(\left|t - \tau_j(\vartheta_u)\right|^{2\kappa} - \left|t - \tau_j\right|^{2\kappa}\right)\mathrm{d}t + o(1)$$
$$= \frac{n\varphi_n^{2\kappa+1}\lambda_j^2}{2\lambda_0\delta^{2\kappa}} \int_0^{-\langle m_j, u\rangle} |s|^{2\kappa} \, \mathrm{d}s$$
$$+ \frac{n\lambda_j^2\varphi_n^{2\kappa+1}}{2\lambda_0\delta^{2\kappa}} \int_{-\langle m_j, u\rangle}^{\frac{\delta}{\varphi_n}} \left(\left|s + \langle m_j, u\rangle\right|^\kappa - |s|^\kappa\right)^2 \mathrm{d}s + o(1)$$
$$= \frac{\lambda_j^2\left|\langle m_j, u\rangle\right|^{2\kappa+1}}{2\lambda_0\delta^{2\kappa}} \int_0^1 |s|^{2\kappa} \, \mathrm{d}s$$

$$+\frac{n\lambda_j^2\varphi_n^{2\kappa+1}}{2\lambda_0\delta^{2\kappa}}\int_1^{\tau_j-\tau_j(\vartheta_u)\frac{\delta}{\varphi_n}}\left(|v-1|^\kappa-|v|^\kappa\right)^2\mathrm{d}s+o\left(1\right)$$
$$\to\frac{\gamma_j^2}{2}\int_0^\infty\left[|s|^{2\kappa}\,\mathrm{1\!I}_{\{s<-\langle m_j,u\rangle\}}+\left(\left|s+\langle m_j,u\rangle\right|^\kappa-|s|^\kappa\right)^2\mathrm{1\!I}_{\{s\geq-\langle m_j,u\rangle\}}\right]\mathrm{d}s.$$

Note that all convergences mentioned above are uniform on compacts $\mathbb{K}\subset\Theta$. □

Lemma 2 *Let the conditions $\mathscr{C}_2$ be fulfilled, then there exists a constant $C>0$, which does not depend on n such that for any $R>0$*

$$\sup_{\vartheta_0\in\Theta}\sup_{\|u_1\|+\|u_2\|\leq R}\|u_1-u_2\|^{-2\kappa-1}\,\mathbf{E}_{\vartheta_0}\left|Z_n^{\frac{1}{2}}\left(u_1\right)-Z_n^{\frac{1}{2}}\left(u_2\right)\right|^2\leq C\left(1+R\right).$$

Proof We have the estimate (see, e.g., Kutoyants 1998)

$$\mathbf{E}_{\vartheta_0}\left|Z_n^{\frac{1}{2}}\left(u_1\right)-Z_n^{\frac{1}{2}}\left(u_2\right)\right|^2\leq\sum_{j=1}^k\int_0^T\left[\sqrt{\lambda_{j,n}\left(\vartheta_{u_2},t\right)}-\sqrt{\lambda_{j,n}\left(\vartheta_{u_1},t\right)}\right]^2\mathrm{d}t$$
$$=\sum_{j=1}^k\int_0^T\frac{n^2\left[S_j\left(t-\tau_j\left(\vartheta_{u_2}\right)\right)-S_j\left(t-\tau_j\left(\vartheta_{u_1}\right)\right)\right]^2}{\left[\sqrt{\lambda_{j,n}\left(\vartheta_{u_2},t\right)}+\sqrt{\lambda_{j,n}\left(\vartheta_{u_1},t\right)}\right]^2}\mathrm{d}t$$
$$\leq\frac{n}{4\lambda_0}\sum_{j=1}^k\int_0^T\left[S_j\left(t-\tau_j\left(\vartheta_{u_2}\right)\right)-S_j\left(t-\tau_j\left(\vartheta_{u_1}\right)\right)\right]^2\mathrm{d}t,$$

where we used the estimate $\lambda_{j,n}\left(\vartheta_u,t\right)\geq n\lambda_0$. Suppose that $\tau_j(\vartheta_{u_1})<\tau_j(\vartheta_{u_2})$ and denote $\Delta_t=\sqrt{n}\left[S_j\left(t-\tau_j\left(\vartheta_{u_2}\right)\right)-S_j\left(t-\tau_j\left(\vartheta_{u_1}\right)\right)\right]$. Then,

$$\int_0^T\Delta_t^2\mathrm{d}t=\int_0^{\tau_j(\vartheta_{u_1})}\Delta_t^2\mathrm{d}t+\int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})}\Delta_t^2\mathrm{d}t+\int_{\tau_j(\vartheta_{u_2})}^T\Delta_t^2\mathrm{d}t$$
$$=\int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})}\Delta_t^2\mathrm{d}t+\int_{\tau_j(\vartheta_{u_2})}^T\Delta_t^2\mathrm{d}t.$$

Remark that the function $\Delta_t=0$ on the interval $\left[0,\tau_j(\vartheta_{u_1})\right]$ and $\Delta_t=nS_j\left(\vartheta_{u_1},t\right)$ on the interval $\left[\tau_j(\vartheta_{u_1}),\tau_j(\vartheta_{u_2})\right]$. Therefore,

$$\int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})}\Delta_t^2\mathrm{d}t=n\int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})}\lambda_j\left(t-\tau_j(\vartheta_{u_1})\right)^2\left|\frac{t-\tau_j(\vartheta_{u_1})}{\delta}\right|^{2\kappa}\mathrm{d}t$$
$$\leq Cn\int_{\tau_j(\vartheta_{u_1})}^{\tau_j(\vartheta_{u_2})}\left|\frac{t-\tau_j(\vartheta_{u_1})}{\delta}\right|^{2\kappa}\mathrm{d}t\leq Cn\left|\frac{\tau_j(\vartheta_{u_2})-\tau_j(\vartheta_{u_1})}{\delta}\right|^{2\kappa+1}$$
$$\leq Cn\varphi_n^{2\kappa+1}\left\|u_2-u_1\right\|^{2\kappa+1}=C\left\|u_2-u_1\right\|^{2\kappa+1}.$$

Further,

$$\int_{\tau_j(\vartheta_{u_2})}^{T} \Delta_t^2 \mathrm{d}t = \int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_1})+\delta} \Delta_t^2 \mathrm{d}t + \int_{\tau_j(\vartheta_{u_1})+\delta}^{\tau_j(\vartheta_{u_2})+\delta} \Delta_t^2 \mathrm{d}t + \int_{\tau_j(\vartheta_{u_2})+\delta}^{T} \Delta_t^2 \mathrm{d}t. \quad (8)$$

Using the estimate

$$\left|\lambda_j\left(t-\tau_j(\vartheta_{u_2})\right)-\lambda_j\left(t-\tau_j(\vartheta_{u_1})\right)\right|^2 \le C\varphi_n^2 \left\|u_2-u_1\right\|^2$$

we obtain for the first integral

$$\begin{aligned}
\int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_1})+\delta} \Delta_t^2 \mathrm{d}t &= n\int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_1})+\delta} \left[\lambda_j\left(t-\tau_j(\vartheta_{u_2})\right)\left|\frac{t-\tau_j(\vartheta_{u_2})}{\delta}\right|^\kappa\right. \\
&\quad \left. -\lambda_j\left(t-\tau_j(\vartheta_{u_1})\right)\left|\frac{t-\tau_j(\vartheta_{u_1})}{\delta}\right|^\kappa\right]^2 \mathrm{d}t \\
&\le Cn\varphi_n^2 \left\|u_2-u_1\right\|^2 + Cn\int_{\tau_j(\vartheta_{u_2})}^{\tau_j(\vartheta_{u_1})+\delta} \left[\left|t-\tau_j(\vartheta_{u_2})\right|^\kappa-\left|t-\tau_j(\vartheta_{u_1})\right|^\kappa\right]^2 \mathrm{d}t \\
&\le C\varphi_n^{1-2\kappa} \left\|u_2-u_1\right\|^2 \\
&\quad + Cn\varphi_n^{2\kappa+1}\int_0^{\frac{\tau_j(\vartheta_{u_1})-\tau_j(\vartheta_{u_2})+\delta}{\varphi_n}} \left[|s|^\kappa-\left|s-\frac{\tau_j(\vartheta_{u_1})-\tau_j(\vartheta_{u_2})}{\varphi_n}\right|^\kappa\right]^2 \mathrm{d}s \\
&\le C\varphi_n^{1-2\kappa} \left\|u_2-u_1\right\|^2 + C\left\|u_2-u_1\right\|^{2\kappa+1},
\end{aligned}$$

where we used the relations

$$\begin{aligned}
&\left|\frac{\tau_j(\vartheta_{u_1})-\tau_j(\vartheta_{u_2})}{\varphi_n}+\langle m_j,u_1\rangle-\langle m_j,u_2\rangle\right| \le C\varphi_n \left\|u_2-u_1\right\|^2, \\
&\int_0^\infty \left[|s|^\kappa-\left|s-\langle m_j,u_1-u_2\rangle\right|^\kappa\right]^2 \mathrm{d}s \\
&\quad \le \left\|u_2-u_1\right\|^{2\kappa+1}\int_0^\infty \left[|v|^\kappa-\left|v-\langle m_j,e\rangle\right|^\kappa\right]^2 \mathrm{d}v \le C\left\|u_2-u_1\right\|^{2\kappa+1}.
\end{aligned}$$

Here, we set $s=v\left\|u_2-u_1\right\|$ and $e=\left\|u_2-u_1\right\|^{-1}(u_2-u_1)$.

As on the interval $\left[\tau_j(\vartheta_{u_2})+\delta,T\right]$, the function $S(t)$ has a bounded derivative $S'(t)$, we can write

$$\int_{\tau_j(\vartheta_{u_2})+\delta}^{T} \Delta_t^2 \mathrm{d}t \le Cn\varphi_n^2 \left\|u_2-u_1\right\|^2 \le C(1+R)\left\|u_2-u_1\right\|^{2\kappa+1}.$$

The other cases can be estimated by a similar way. □

Lemma 3 *Let the conditions $\mathscr{C}$ be fulfilled, then there exists a constant $\kappa > 0$, which does not depend on n such that*

$$\sup_{\vartheta_0\in\Theta} \mathbf{E}_{\vartheta_0} Z_n^{\frac{1}{2}}(u) \leq e^{-\kappa\|u\|^{\frac{2}{2\kappa+1}}}. \tag{9}$$

Proof Let us denote $\theta_u = \vartheta_0 + \nu\varphi_n u$ and put

$$Z_{j,n}(u) = \exp\left\{\int_0^T \ln\left(\frac{\lambda_{j,n}(\theta_u,t)}{\lambda_{j,n}(\vartheta_0,t)}\right) \mathrm{d}X_j(t) - \int_0^T \left[\lambda_{j,n}(\theta_u,t) - \lambda_{j,n}(\vartheta_0,t)\right]\mathrm{d}t\right\}.$$

By Lemma 2.2 of Kutoyants (1998), we can write

$$\mathbf{E}_{\vartheta_0} Z_{j,n}^{\frac{1}{2}}(u) = \exp\left\{-\frac{1}{2}\int_0^T \left[\sqrt{\lambda_{j,n}(\theta_u,t)} - \sqrt{\lambda_{j,n}(\vartheta_0,t)}\right]^2 \mathrm{d}t\right\}.$$

Hence,

$$\begin{aligned}\mathbf{E}_{\vartheta_0} Z_n^{\frac{1}{2}}(u) &= \prod_{j=1}^k \mathbf{E}_{\vartheta_0} Z_{j,n}^{\frac{1}{2}}(u)\\ &= \exp\left\{-\frac{1}{2}\sum_{j=1}^k\int_0^T \left[\sqrt{\lambda_{j,n}(\theta_u,t)} - \sqrt{\lambda_{j,n}(\vartheta_0,t)}\right]^2 \mathrm{d}t\right\}.\end{aligned} \tag{10}$$

First for simplicity of calculation, we write

$$\begin{aligned}&\int_0^T \left[\sqrt{\lambda_{j,n}(\vartheta,t)} - \sqrt{\lambda_{j,n}(\vartheta_0,t)}\right]^2 \mathrm{d}t\\ &\quad= \int_0^T \frac{\left[\lambda_{j,n}(\vartheta,t) - \lambda_{j,n}(\vartheta_0,t)\right]^2}{\left[\sqrt{\lambda_{j,n}(\vartheta,t)} + \sqrt{\lambda_{j,n}(\vartheta_0,t)}\right]^2}\mathrm{d}t\\ &\quad\geq c_j n\int_0^T \left[S_j\left(t-\tau_j(\vartheta)\right) - S_j\left(t-\tau_j\right)\right]^2 \mathrm{d}t,\end{aligned} \tag{11}$$

where $c_j = (4\lambda_M)^{-1} > 0$ and $\lambda_M = \lambda_0 + \max_{t\in\mathcal{T}_j} S_j(t)$. Therefore, it is sufficient to study the integral

$$\begin{aligned}I_j(\vartheta) &= \int_0^T \left[S_j\left(t-\tau_j(\vartheta)\right) - S_j\left(t-\tau_j\right)\right]^2 \mathrm{d}t\\ &= \int_{\tau_j(\vartheta)\wedge\tau_j}^T \left[S_j\left(t-\tau_j(\vartheta)\right) - S_j\left(t-\tau_j\right)\right]^2 \mathrm{d}t.\end{aligned}$$

We evaluate these integrals on two sets $\mathbb{A} = \{\vartheta : \|\vartheta - \vartheta_0\| \le h\}$ and $\mathbb{A}^c$. Here, $h > 0$ is some small number. Recall that we denoted $\tau_j = \tau_j(\vartheta_0)$.

Let $\vartheta \in \mathbb{A} \cap \mathbb{B}$, where $\mathbb{B} = \left\{\vartheta \in \mathbb{A} : \tau_j(\vartheta) > \tau_j(\vartheta_0)\right\}$. Moreover, $\tau_j(\vartheta) - \tau_j(\vartheta_0) < \delta$. Then,

$$
\begin{aligned}
I_j(\vartheta) &\ge \int_{\tau_j}^{\tau_j(\vartheta)} S_j\left(t-\tau_j\right)^2 \mathrm{d}t + \int_{\tau_j(\vartheta)}^{\tau_j+\delta} \left[S_j\left(t-\tau_j(\vartheta)\right) - S_j\left(t-\tau_j\right)\right]^2 \mathrm{d}t \\
&= \int_0^{\tau_j(\vartheta)-\tau_j} S_j(s)^2 \mathrm{d}s + \int_0^{\tau_j-\tau_j(\vartheta)+\delta} \left[S_j(s) - S_j\left(s-\Delta\tau_j\right)\right]^2 \mathrm{d}s,
\end{aligned}
$$

where $\Delta\left(\tau_j\right) = \tau_j - \tau_j(\vartheta)$. Further, (below $\lambda_m = \min_{t\in\mathcal{T}_j} \lambda_j(t) > 0$)

$$
\int_0^{\tau_j(\vartheta)-\tau_j} \lambda_j(s)^2 \left(\frac{s}{\delta}\right)^{2\kappa} \mathrm{d}s \ge \frac{\lambda_m^2}{\delta^{2\kappa}} \int_0^{\tau_j(\vartheta)-\tau_j} s^{2\kappa} \mathrm{d}s = \frac{\lambda_m^2 \left|\tau_j(\vartheta) - \tau_j\right|^{2\kappa+1}}{\delta^{2\kappa}(2\kappa+1)}.
$$

Recall that

$$
\tau_j(\vartheta) - \tau_j = \langle m_j, \vartheta - \vartheta_0\rangle + O\left(h^2\right) = \langle m_j, e\rangle \|\vartheta - \vartheta_0\| + O\left(h^2\right),
$$

where the unit vector $e = (\vartheta - \vartheta_0)\|\vartheta - \vartheta_0\|^{-1}$. Therefore,

$$
\begin{aligned}
&\int_0^{\tau_j(\vartheta)-\tau_j} \lambda_j(s)^2 \left(\frac{s}{\delta}\right)^{2\kappa} \mathrm{d}s \\
&\quad \ge \frac{\lambda_m^2 \left|\langle m_j, e\rangle\right|^{2\kappa+1}}{\delta^{2\kappa}(2\kappa+1)} \|\vartheta - \vartheta_0\|^{2\kappa+1} \left(1 + o\left(\|\vartheta - \vartheta_0\|\right)\right)
\end{aligned}
$$

and we can take such h that

$$
\int_0^{\tau_j(\vartheta)-\tau_j} \lambda_j(s)^2 \left(\frac{s}{\delta}\right)^{2\kappa} \mathrm{d}s \ge \frac{\lambda_m^2 \left|\langle m_j, e\rangle\right|^{2\kappa+1}}{2\delta^{2\kappa}(2\kappa+1)} \|\vartheta - \vartheta_0\|^{2\kappa+1}.
$$

For the second integral, we have ($\delta_* = \tau_j - \tau_j(\vartheta) + \delta > 0$)

$$
\begin{aligned}
&\int_0^{\delta_*} \left[S_j(s) - S_j\left(s - \Delta\left(\tau_j\right)\right)\right]^2 \mathrm{d}s \\
&\quad = \frac{1}{\delta^{2\kappa}} \int_0^{\delta_*} \left[\lambda_j(s) s^\kappa - \lambda_j\left(s - \Delta\left(\tau_j\right)\right) \left|s - \Delta\left(\tau_j\right)\right|^\kappa\right]^2 \mathrm{d}s \\
&\quad \ge \frac{\lambda_m^2}{\delta^{2\kappa}} \int_0^{\delta_*} \left[s^\kappa - \left|s - \Delta\left(\tau_j\right)\right|^\kappa\right]^2 \mathrm{d}s - C\|\vartheta - \vartheta_0\|^2 \\
&\quad \ge \frac{\lambda_m^2}{\delta^{2\kappa}} \int_0^{\frac{\delta_*}{\Delta(\tau_j)}} \left[v^\kappa - |v-1|^\kappa\right]^2 \mathrm{d}v \, \Delta\left(\tau_j\right)^{2\kappa+1} - C\|\vartheta - \vartheta_0\|^2
\end{aligned}
$$

$$\geq \frac{\lambda_m^2}{\delta^{2\kappa}} \int_0^{\frac{\delta_*}{ch}} \left[v^\kappa - |v-1|^\kappa\right]^2 \mathrm{d}v \left|\langle m_j, e\rangle\right|^{2\kappa+1} \|\vartheta - \vartheta_0\|^{2\kappa+1}$$
$$- C \|\vartheta - \vartheta_0\|^2,$$

where we used the relation $\lambda_j\left(s - \Delta\left(\tau_j\right)\right) = \lambda_j(s) + O\left(\Delta\left(\tau_j\right)\right)$ and set $s = v\Delta\left(\tau_j\right)$.

These estimates from below of the integral allow us to write

$$\sum_{j=1}^k I_j(\vartheta) \geq \gamma \sum_{j=1}^k \left|\langle m_j, e\rangle\right|^{2\kappa+1} \|\vartheta - \vartheta_0\|^{2\kappa+1} - C \|\vartheta - \vartheta_0\|^2.$$

As $k \geq 3$, we have

$$Q(e) = \sum_{j=1}^k \left|\langle m_j, e\rangle\right|^{2\kappa+1}, \qquad \inf_{\|e\|=1} Q(e) = q_1 > 0.$$

Indeed, if $q_1 = 0$, then there exists a vector e_* such that $Q(e_*) = 0$ and this vector is orthogonal to all m_j, $j = 1, \ldots, k$. Of course, this is impossible. Therefore, we can take such sufficiently small h that for $\vartheta \in \mathbb{A} \cap \mathbb{B}$, we obtain the estimate

$$\sum_{j=1}^k \int_0^T \left[S_j\left(t - \tau_j(\vartheta)\right) - S_j\left(t - \tau_j(\vartheta_0)\right)\right]^2 \mathrm{d}t \geq \gamma_1 \|\vartheta - \vartheta_0\|^{2\kappa+1} \tag{12}$$

with some positive γ_1. For the other values of $\vartheta \in \mathbb{A}$, we have the similar estimates.

Let us consider these integrals for the values $\vartheta \in \mathbb{A}^c$. According to (11), we have to study the function

$$g(h) = \inf_{\vartheta_0 \in \Theta} \inf_{\|\vartheta - \vartheta_0\| > h} \sum_{j=1}^k \int_0^T \left[S_j\left(t - \tau_j(\vartheta)\right) - S_j\left(t - \tau_j(\vartheta_0)\right)\right]^2 \mathrm{d}t,$$

and show that $g(h) > 0$.

Suppose that $g(h) = 0$, then this implies that there exists at least one point $\vartheta^* \in \Theta$ such that $\|\vartheta^* - \vartheta_0\| \geq h$ and for all $j = 1, \ldots, k$, we have

$$\int_0^T \left[S_j\left(t - \tau_j\left(\vartheta^*\right)\right) - S_j\left(t - \tau_j(\vartheta_0)\right)\right]^2 \mathrm{d}t = 0.$$

Let $\tau_j(\vartheta^*) > \tau_j$. Then for all $t \in \left[\tau_j, \tau_j(\vartheta^*)\right]$, we have

$$\lambda_j\left(t - \tau_j\right)\left|t - \tau_j\right|^\kappa = 0$$

and for $t \in \left[\tau_j\left(\vartheta^*\right), \tau_j + \delta\right]$

$$\lambda_j\left(t - \tau_j\left(\vartheta^*\right)\right)\left|t - \tau_j\left(\vartheta^*\right)\right|^\kappa = \lambda_j\left(t - \tau_j\right)\left|t - \tau_j\right|^\kappa .$$

Of course, we can have these two equalities if and only if $\tau_j\left(\vartheta^*\right) = \tau_j\left(\vartheta_0\right)$ for all $j = 1, \ldots, k$. Recall that $\lambda_j\left(t\right)$ are strictly positive functions. From the geometry of the model, it follows that it is impossible to have two different points such that the distances from these points and $k \geq 3$ detectors coincide.

Therefore, for $\vartheta \in \mathbb{A}^c$

$$\sum_{j=1}^{k} \int_0^T \left[S_j(t - \tau_j\left(\vartheta\right)) - S_j(t - \tau_j\left(\vartheta_0\right))\right]^2 \mathrm{d}t \geq g\left(h\right)$$
$$\geq \frac{g\left(h\right)\left\|\vartheta - \vartheta_0\right\|^{2\kappa+1}}{D^{2\kappa+1}} \geq \gamma_2 \left\|\vartheta - \vartheta_0\right\|^{2\kappa+1}, \qquad (13)$$

where $D = \sup_{\vartheta_1, \vartheta_2 \in \Theta} \left\|\vartheta_1 - \vartheta_2\right\|$.

From estimates (12) and (13), it follows that if we put $\vartheta = \vartheta_0 + \nu\varphi_n u$, then

$$\sum_{j=1}^{k} \int_0^T \left[\sqrt{\lambda_{j,n}\left(\theta_u, t\right)} - \sqrt{\lambda_{j,n}\left(\vartheta_0, t\right)}\right]^2 \mathrm{d}t \geq \gamma n \left\|\vartheta - \vartheta_0\right\|^{2\kappa+1}$$
$$= \gamma \nu^{2\kappa+1} \left\|u\right\|^{2\kappa+1} .$$

This estimates and (10) proves (9).

The properties of the likelihood ratio field $Z_n\left(\cdot\right)$ established in Lemmas 1–3 are sufficient conditions for Theorem 1.10.2 of Ibragimov and Khasminskii (1981). Therefore, Theorem 1 is proved. □

3 Discussion

There are several problems which naturally arise for this model of observations. Note that the properties of the MLE $\hat{\vartheta}_n$ can be studied too. This requires a special modification of Lemma 2 to verify the tightness of the corresponding family of measures. If this modification is proved, then we obtain the convergence

$$n^{\frac{1}{2\kappa+1}}\left(\hat{\vartheta}_n - \vartheta_0\right) \Longrightarrow \hat{u}, \qquad Z\left(\hat{u}\right) = \sup_{u \in \mathcal{R}^2} Z\left(u\right)$$

and convergence of moments.

If the position ϑ_0 is known and we have to estimate unknown $\kappa \in (0, 1/2)$, then we have regular statistical experiment, and the MLE and BE are asymptotically normal and asymptotically efficient with regular rate $\sqrt{n}$.

The joint estimation of ϑ_0 and $\kappa \in (0, 1/2)$ is essentially much more difficult problem because the rate of convergence of one estimator (ϑ) depends on another unknown parameter κ. The general method applied in this article does not work in such cases.

For simplicity of the exposition, we supposed that the noise level in all detectors is known and is the same. It is possible to consider the problem of the joint estimation of ϑ and λ_0. To do it, we have to add one variable else in the normalized likelihood ratio process

$$Z_n(u,v)=\frac{L\left(\vartheta_0+\nu\varphi_n u,\lambda_0+n^{-1/2}v,X^n\right)}{L\left(\vartheta_0,\lambda_0,X^n\right)},\quad u\in\mathbb{U}_n,\quad v\in\mathbb{V}_n$$

and verify the convergence of this random function to the limit random process

$$Z(u,v)=Z(u)\;Z_\lambda(v),\quad Z_\lambda(v)=\exp\left\{u\eta-\frac{v^2}{2}\mathrm{I}_\lambda\right\},\quad v\in\mathcal{R}.$$

Here, $Z(\cdot)$ is the same random function as in Sect. 2, η is Gaussian random variable independent of $Z(\cdot)$ and I_λ is the corresponding Fisher information. As a result, we will obtain the joint convergence of $\tilde\vartheta_n$ (see Theorem 1) and $\sqrt{n}\left(\tilde\lambda_n-\lambda_0\right)\Rightarrow\mathcal{N}\left(0,\mathrm{I}_\lambda^{-1}\right)$.

In this work, we supposed that the source starts emission at the instant $t=0$. It is interesting to consider the more general statement with unknown start of the emission.

The limit distribution of the BE is described in this work with the help of the random field $Z(u)$, $u\in\mathcal{R}^2$ and it will be interesting to have some pictures obtained by numerical simulations for the densities of $\tilde\zeta$. In one-dimensional models with cusp-type singularity, such densities were numerically calculated in Kordzakhia et al. (2018).

Note that it is possible to construct a consistent estimator of ϑ_0 in two steps as it was proposed in Chernoyarov and Kutoyants (2019). First, we estimate k moments $\tau=(\tau_1,\ldots,\tau_k)$ of arriving signals in detectors, say, $\tilde\tau_{1,n},\ldots,\tilde\tau_{k,n}$. Recall that

$$\tilde\xi_{j,n}=n^{\frac{1}{2\kappa+1}}\left(\tilde\tau_{j,n}-\tau_j\right)\Longrightarrow\tilde\xi_j,\quad j=1,\ldots,k$$

where $\tilde\xi_j$ are independent random variables (see Dachian 2003). Hence, we have

$$\nu^2\tilde\tau_{j,n}=\nu^2\tau_j^2+2\varphi_n\tau_j\tilde\xi_{j,n}+\varphi_n^2\tilde\xi_{j,n}^2=\rho_j^2+2\nu\rho_j\tilde\xi_{j,n}\varphi_n+O\left(\varphi_n^2\right).$$

Then, we write the equations

$$\left(x_j-x_0^*\right)^2+\left(y_j-y_0^*\right)^2=\nu^2\tilde\tau_{j,n}=\rho_j^2+2\nu\rho_j\tilde\xi_{j,n}\varphi_n+O\left(\varphi_n^2\right),\quad j=1,\ldots,k,$$

and obtain the least squares estimator ϑ_n^*, which is consistent and has the same rate of convergence as the BE $\tilde\vartheta_n$ studied in this work. See details in Section 3 of Chernoyarov and Kutoyants (2019).

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