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SUPPLEMENTARY MATERIAL ON "DETECTING DEVIATIONS FROM SECOND-ORDER STATIONARITY IN LOCALLY STATIONARY FUNCTIONAL TIME SERIES"

Abstract. This supplementary material contains the additional proofs for the main paper. In Appendix A, we provide the remaining proofs for the results in Sections 3.5 and 3.6. Proofs related to Section 4 are provided in Appendix B. Finally, some auxiliary results are collected in Appendix C.

Appendix A. Proofs for Section 3.5 and 3.6

Proof of Proposition 3.11. By the definition of $X_{t,T}$, we can rewrite

$$U_T(u,\tau) = \frac{1}{\sqrt{T}}\Big(\sum_{t=1}^{\lfloor uT\rfloor} Y_{t,T}(\tau) - u\sum_{t=1}^{T} Y_{t,T}(\tau)\Big) - \frac{uT-\lfloor uT\rfloor}{\sqrt{T}}\mu_2(\tau)$$
$$+ \frac{1}{\sqrt{T}}\Big((1-u)\lfloor\lambda T\rfloor \mathbb{1}(\lambda\le u) + (\lfloor uT\rfloor - u\lfloor \lambda T\rfloor)\mathbb{1}(\lambda>u)\Big)\{\mu_1(\tau)-\mu_2(\tau)\}. \quad \text{(A.1)}$$

By Corollary 3.5,

$$\frac{1}{\sqrt{T}}\Big(\sum_{t=1}^{\lfloor uT\rfloor} Y_{t,T}(\tau) - u\sum_{t=1}^{T} Y_{t,T}(\tau)\Big)$$

converges to a centred Gaussian process $\tilde G$. In particular, the norm $\|\cdot\|_{2,2}$ of the previous display is $\mathcal{O}_p(1)$. The norm of the second summand in (A.1) is of order $\mathcal{O}(T^{-1/2})$ and the norm of the last summand diverges to infinity as T tends to infinity. Thus, $\|U_T\|_{2,2}\to\infty$ in probability, and therefore the test statistic $\mathcal{S}_T^{(m)}$ diverges to infinity in probability.

In the proof of Theorem 3.8, we have seen that $\|\hat B_T^{(k)} - \tilde B_T^{(k)}\|_{2,3} = o_{\mathbb{P}}(1)$ as $T\to\infty$ under the assumption of local stationarity, where $\tilde B_T^{(k)}$ is defined in (C.3). The same result can be shown with similar arguments in the setting of a change point. Further,

$$\tilde B_T^{(k)}(u,\tau) - u\tilde B_T^{(k)}(1,\tau)$$
$$= \frac{1}{\sqrt{T}}\Big\{\sum_{i=1}^{\lfloor uT\rfloor}\frac{R_i^{(k)}}{\sqrt{m}}\sum_{t=i}^{(i+m-1)\wedge T}[X_{t,T}(\tau)-\mathbb{E}X_{t,T}(\tau)]$$
$$- u\sum_{i=1}^{T}\frac{R_i^{(k)}}{\sqrt{m}}\sum_{t=i}^{(i+m-1)\wedge T}[X_{t,T}(\tau)-\mathbb{E}X_{t,T}(\tau)]\Big\}$$

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$$= \frac{1}{\sqrt{T}}\Big(\sum_{i=1}^{\lfloor uT\rfloor}\frac{R_i^{(k)}}{\sqrt{m}}\sum_{t=i}^{(i+m-1)\wedge T}Y_{t,T}(\tau) - u\sum_{i=1}^{T}\frac{R_i^{(k)}}{\sqrt{m}}\sum_{t=i}^{(i+m-1)\wedge T}Y_{t,T}(\tau)\Big),$$

where the right-hand side converges according to Theorem C.3 to the process $\tilde{B}$ as well. Thus, $\|\hat{G}_T^{(k)}\|_{2,2} = \mathcal{O}_p(1)$. □

Proof of Lemma 3.13. We begin by proving the formula for the bias. We have

$$\begin{aligned}
\mathbb{E}[\tilde{\sigma}_T(\tau,\varphi)] &= \frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\sum_{t,t'=i}^{(i+m-1)\wedge T}\mathrm{Cov}\big(X_{t,T}(\tau), X_{t',T}(\varphi)\big)\\
&= \frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\sum_{t,t'=i}^{(i+m-1)\wedge T}\mathrm{Cov}\big(X_t^{(i/T)}(\tau), X_{t'}^{(i/T)}(\varphi)\big) + \mathcal{O}(T^{-1})\\
&= \frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\sum_{k=-m+1}^{m-1}(m-|k|)\mathrm{Cov}\big(X_0^{(i/T)}(\tau), X_k^{(i/T)}(\varphi)\big) + \mathcal{O}(T^{-1}) \qquad (A.2)\\
&= \sum_{k=-m+1}^{m-1}\int_0^1 \mathrm{Cov}\big(X_0^{(w)}(\tau), X_k^{(w)}(\varphi)\big)\,\mathrm{d}w\\
&\qquad - \frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\sum_{k=-m+1}^{m-1}|k|\mathrm{Cov}\big(X_0^{(i/T)}(\tau), X_k^{(i/T)}(\varphi)\big) + \mathcal{O}(T^{-1}).
\end{aligned}$$

Further, by Lemma 3.11 in Dehling and Philipp (2002), we can rewrite

$$\begin{aligned}
\sigma_c(\tau,\varphi) &= \sum_{k=-\infty}^{\infty}\int_0^1 \mathrm{Cov}(X_0^{(w)}(\tau), X_k^{(w)}(\varphi))\,\mathrm{d}w\\
&= \sum_{k=-m+1}^{m-1}\int_0^1 \mathrm{Cov}(X_0^{(w)}(\tau), X_k^{(w)}(\varphi))\,\mathrm{d}w + \sum_{|k|\geq m}\int_0^1 \mathrm{Cov}(X_0^{(w)}(\tau), X_k^{(w)}(\varphi))\,\mathrm{d}w\\
&= \sum_{k=-m+1}^{m-1}\int_0^1 \mathrm{Cov}(X_0^{(w)}(\tau), X_k^{(w)}(\varphi))\,\mathrm{d}w + \mathcal{O}(a^{rm}),
\end{aligned}$$

for some $0 < r < 1$. By the previous display, Equation (A.2) and since $\mathcal{O}(a^{rm}) + \mathcal{O}(T^{-1}) = o(m^{-2})$, we obtain that

$$\begin{aligned}
&\int_{[0,1]^2}\big(\mathbb{E}[\tilde{\sigma}_T(\tau,\varphi)] - \sigma_c(\tau,\varphi)\big)^2\,\mathrm{d}(\tau,\varphi)\\
&= \int_{[0,1]^2}\Big\{\frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\sum_{k=-m+1}^{m-1}|k|\mathrm{Cov}\big(X_0^{(i/T)}(\tau), X_k^{(i/T)}(\varphi)\big) + o(m^{-2})\Big\}^2\,\mathrm{d}(\tau,\varphi)\\
&= \int_{[0,1]^2}\Big\{\sum_{k=-m+1}^{m-1}\frac{|k|}{m}\int_0^1 \mathrm{Cov}\big(X_0^{(w)}(\tau), X_k^{(w)}(\varphi)\big)\,\mathrm{d}w\Big\}^2 + o(m^{-2})\,\mathrm{d}(\tau,\varphi)\\
&= \frac{1}{m^2}\Big\|\sum_{k=-\infty}^{\infty}|k|\int_0^1 \mathrm{Cov}(X_0^{(w)}, X_k^{(w)})\,\mathrm{d}w\Big\|_{2,2}^2 + o(m^{-2})
\end{aligned}$$

as asserted.

Next, consider the formula for the variance. Observe that $\mathrm{Var}(\tilde{\sigma}_T(\tau,\varphi)) = \mathbb{E}[\tilde{\sigma}_T^2(\tau,\varphi)] - (\mathbb{E}\tilde{\sigma}_T(\tau,\varphi))^2$. By Theorem 2.3.2 of Brillinger (1981), we can rewrite

$$\mathbb{E}\tilde{\sigma}_T^2(\tau,\varphi) = \frac{1}{T^2}\sum_{i,i'=1}^{T}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_2,t_3,t_4}(\tau,\varphi) + \kappa_{t_1,t_2}(\tau,\varphi)\kappa_{t_3,t_4}(\tau,\varphi) + \kappa_{t_1,t_3}(\tau,\tau)\kappa_{t_2,t_4}(\varphi,\varphi) + \kappa_{t_1,t_4}(\tau,\varphi)\kappa_{t_2,t_3}(\varphi,\tau),$$

where $\kappa_{t_1,t_2,t_3,t_4}(\tau,\varphi) = \mathrm{cum}\left(X_{t_1,T}(\tau), X_{t_2,T}(\varphi), X_{t_3,T}(\tau), X_{t_4,T}(\varphi)\right)$ and $\kappa_{t_1,t_2}(\tau,\varphi) = \mathrm{Cov}(X_{t_1,T}(\tau), X_{t_2,T}(\varphi))$ for any $t_1,t_2,t_3,t_4 \in \{1,\cdots,T\}$ and $\tau,\varphi \in [0,1]$. With this notation, we can further rewrite

$$(\mathbb{E}\tilde{\sigma}_T(\tau,\varphi))^2 = \frac{1}{T^2}\sum_{i,i'=1}^{T}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_2}(\tau,\varphi)\kappa_{t_3,t_4}(\tau,\varphi),$$

thus,

$$\mathrm{Var}(\tilde{\sigma}_T(\tau,\varphi)) = \frac{1}{T^2}\sum_{i,i'=1}^{T}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_2,t_3,t_4}(\tau,\varphi) + \kappa_{t_1,t_3}(\tau,\tau)\kappa_{t_2,t_4}(\varphi,\varphi) + \kappa_{t_1,t_4}(\tau,\varphi)\kappa_{t_2,t_3}(\varphi,\tau).$$

In the following, we investigate the sums B_1, B_2 and B_3 over the three inner summands separately.

First, observe that by the strong mixing condition and Theorem 3 in Statulevicius and Jakimavicius (1988), the sum over the cumulants $\kappa_{t_1,t_2,t_3,t_4}(\tau,\varphi)$ vanishes with rate m^2T^{-2}, i.e.,

$$B_1 = \frac{1}{T^2}\sum_{i,i'=1}^{T}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_2,t_3,t_4}(\tau,\varphi) = \mathcal{O}(m^2T^{-2})$$

To investigate B_2, we split the sum into $B_2 = B_{2,1} + B_{2,2} + B_{2,3}$, where

$$B_{2,1}(\tau,\varphi) = \frac{1}{T^2}\sum_{i=1}^{T}\sum_{|i-i'|\le m}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_3}(\tau,\tau)\kappa_{t_2,t_4}(\varphi,\varphi),$$

$$B_{2,2}(\tau,\varphi) = \frac{1}{T^2}\sum_{i=m+2}^{T}\sum_{i'=1}^{i-m-1}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_3}(\tau,\tau)\kappa_{t_2,t_4}(\varphi,\varphi)$$

and

$$B_{2,3}(\tau,\varphi) = \frac{1}{T^2}\sum_{i=1}^{T-m-1}\sum_{i'=i+m+1}^{T}\frac{1}{m^2}\sum_{t_1,t_2=i}^{(i+m-1)\wedge T}\sum_{t_3,t_4=i'}^{(i'+m-1)\wedge T}\kappa_{t_1,t_3}(\tau,\tau)\kappa_{t_2,t_4}(\varphi,\varphi).$$

In the following, we will see that both $B_{2,2}$ and $B_{2,3}$ are negligible, while $B_{2,1}$ contributes to the claimed limit expression. The covariances κ_{t_1,t_3} and κ_{t_2,t_4} can be bounded by $C\alpha^r(|t_1-t_3|) \le Ca^{r|t_1-t_3|}$ and $C\alpha^r(|t_2-t_4|) \le Ca^{r|t_2-t_4|}$, for some constants $C>0$ and $0<r<1$, respectively. Therefore,

$$\begin{aligned} B_{2,2}(\tau,\varphi) &\le \frac{C}{T^2}\sum_{i=m+2}^{T}\sum_{i'=1}^{i-m-1}\frac{1}{m^2}\sum_{t_1,t_2=i}^{i+m-1}\sum_{t_3,t_4=i'}^{i'+m-1} a^{r|t_1-t_3|}a^{r|t_2-t_4|} \\ &= \frac{C}{T^2}\sum_{i=m+2}^{T}\sum_{i'=1}^{i-m-1}\frac{1}{m^2}\sum_{t_1,\cdots,t_4=1}^{m} a^{r(i-i'+t_1-t_3)}a^{r(i-i'+t_2-t_4)} \\ &= \frac{C}{T^2}\sum_{i=m+2}^{T}\sum_{i'=m+1}^{i-1}\frac{1}{m^2}\sum_{t_1,\cdots,t_4=1}^{m} a^{r(i-i'+m+t_1-t_3)}a^{r(i-i'+m+t_2-t_4)} \\ &= \frac{C}{T^2}\sum_{i=m+2}^{T}\sum_{i'=m+1}^{i-1} a^{2r(i-i')}\frac{1}{m^2}\sum_{t_1,\cdots,t_4=1}^{m} a^{r(m+t_1-t_3)}a^{r(m+t_2-t_4)} = \mathcal{O}(T^{-1}). \end{aligned}$$

Analogously, $B_{2,3}(\tau,\varphi)=\mathcal{O}(T^{-1})$. For $B_{2,1}$ observe that

$$\begin{aligned} \int_{[0,1]^2} B_{2,1}(\tau,\varphi)\,\mathrm{d}(\tau,\varphi) = \int_{[0,1]^2}\frac{1}{T^2}\sum_{i=1}^{T}\sum_{|i-i'|\le m}\frac{1}{m^2}\sum_{t_1,\cdots,t_4=1}^{m}\operatorname{Cov}\big(X_{t_1+i}^{(i/T)}(\tau),X_{t_3+i'}^{(i/T)}(\tau)\big) \\ \times \operatorname{Cov}\big(X_{t_2+i}^{(i/T)}(\varphi),X_{t_4+i'}^{(i/T)}(\varphi)\big)\,\mathrm{d}(\tau,\varphi)+\mathcal{O}(m^4T^{-2}). \end{aligned}$$

The inner sums of the integrand in the above display, can be rewritten as

$$\begin{aligned} &\sum_{i'=i-m}^{i+m}\frac{1}{m^2}\sum_{t_1,\cdots,t_4=1}^{m}\operatorname{Cov}\big(X_{t_1+i}^{(i/T)}(\tau),X_{t_3+i'}^{(i/T)}(\tau)\big)\operatorname{Cov}\big(X_{t_2+i}^{(i/T)}(\varphi),X_{t_4+i'}^{(i/T)}(\varphi)\big) \\ &= \sum_{i'=-m}^{m}\frac{1}{m^2}\sum_{t_1,\cdots,t_4=1}^{m}\operatorname{Cov}\big(X_0^{(i/T)}(\tau),X_{t_3-t_1+i'}^{(i/T)}(\tau)\big)\operatorname{Cov}\big(X_0^{(i/T)}(\varphi),X_{t_4-t_2+i'}^{(i/T)}(\varphi)\big) \\ &= \sum_{i',k_1,k_2=-m}^{m}\Big(1-\frac{|k_1|}{m}\Big)\Big(1-\frac{|k_2|}{m}\Big)\operatorname{Cov}\big(X_0^{(i/T)}(\tau),X_{k_1+i'}^{(i/T)}(\tau)\big)\operatorname{Cov}\big(X_0^{(i/T)}(\varphi),X_{k_2+i'}^{(i/T)}(\varphi)\big) \\ &= \sum_{i'=-m}^{m}\sum_{k_1,k_2=-m}^{m}\Big(1-\frac{|k_1|}{m}\Big)\Big(1-\frac{|k_2|}{m}\Big)\gamma_{k_1+i'}(i/T|\tau)\gamma_{k_2+i'}(i/T|\varphi), \qquad \text{(A.3)} \end{aligned}$$

where $\gamma_k(u|x)=\operatorname{Cov}(X_0^{(u)}(x),X_k^{(u)}(x))$, for any $k\in\mathbb{N}$ and $u,x\in[0,1]$. Let ℓ_m be an increasing sequence in $\mathbb{N}$, with $\ell_m\le m$, $\ell_m^2/m\to 0$ and $m^3a^{r\ell_m}\to 0$ as $m\to\infty$, for some $0<r<1$, as m tends to infinity; for instance, $\ell_m=m^{1/3}$. By the strong mixing property and Lemma 3.11 in Dehling and Philipp (2002), we can rewrite the right-hand side of (A.3) as

$$\begin{aligned} &\sum_{i'=-m}^{m}\sum_{k_1,k_2=-i'-\ell_m}^{-i'+\ell_m}\Big(1-\frac{|k_1|}{m}\Big)\Big(1-\frac{|k_2|}{m}\Big)\gamma_{k_1+i'}(i/T|\tau)\gamma_{k_2+i'}(i/T|\varphi)+\mathcal{O}(m^3a^{r\ell_m}) \\ &= \sum_{i'=-m}^{m}\sum_{k_1,k_2=-\ell_m}^{\ell_m}\Big(1-\frac{|k_1-i'|}{m}\Big)\Big(1-\frac{|k_2-i'|}{m}\Big)\gamma_{k_1}(i/T|\tau)\gamma_{k_2}(i/T|\varphi)+\mathcal{O}(m^3a^{r\ell_m}) \end{aligned}$$

$$= \sum_{k_1,k_2=-\ell_m}^{\ell_m} \gamma_{k_1}(i/T|\tau)\gamma_{k_2}(i/T|\varphi) \sum_{i'=-m}^{m} \Big(1-\frac{|k_1-i'|}{m}\Big)\Big(1-\frac{|k_2-i'|}{m}\Big) + \mathcal{O}(m^3 a^{r\ell_m}). \tag{A.4}$$

A tedious but straight-forward calculation based on splitting the next sum into the three cases $i' < k_1 \wedge k_2$, $i' = k_1 \wedge k_2, \dots, k_1 \vee k_2$ and $i' > k_1 \vee k_2$ implies that

$$\sum_{i'=-m}^{m} \Big(1-\frac{|k_1 \wedge k_2-i'|}{m}\Big)\Big(1-\frac{|k_1 \vee k_2-i'|}{m}\Big) = \frac{2}{3}m + \mathcal{O}(\ell_m^2/m).$$

Plugging this into (A.4) leads, by the dominated convergence theorem and Lipschitz continuity of $\gamma_k(u|\tau)$ in u, to

$$\begin{aligned}\int_{[0,1]^2} B_2(\tau,\varphi)\,\mathrm{d}(\tau,\varphi) &= \int_{[0,1]^2} \frac{1}{T^2}\sum_{i=1}^{T}\sum_{k_1,k_2=-\ell_m}^{\ell_m} \frac{2}{3} m \gamma_{k_1}(i/T|\tau)\gamma_{k_2}(i/T|\varphi)\,\mathrm{d}(\tau,\varphi) + o(m/T)\\ &= \frac{m}{T}\frac{2}{3}\int_0^1 \Big(\sum_{k=-\infty}^{\infty}\int_0^1 \mathrm{Cov}\big(X_0^{(w)}(\tau), X_k^{(w)}(\tau)\big)\,\mathrm{d}\tau\Big)^2 \mathrm{d}w + o(m/T),\end{aligned}$$

since, by the strong mixing property, $\sum_{|k|>\ell_m}\int_0^1 \gamma_k(u|\tau)\,\mathrm{d}\tau$ is of order $\mathcal{O}(a^{r\ell_m})$. By similar arguments, we have

$$\int_{[0,1]^2} B_3(\tau,\varphi)\,\mathrm{d}(\tau,\varphi) = \frac{m}{T}\frac{2}{3}\int_0^1 \Big\| \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_0^{(w)}, X_k^{(w)})\Big\|_{2,2}^2 \mathrm{d}w + o(m/T)$$

and the theorem's statement follows. □

Appendix B. Proofs for Section 4

Proof of Lemma 4.1. Note that $\varepsilon_{t,T} = \sigma(t/T)\tilde{\varepsilon}_t = \sigma(0)\tilde{\varepsilon}_t = \varepsilon_t^{(0)}$, for any $t \le 0$. Further, $\mathbb{E}\varepsilon_t^{(u)} = 0$ and $\mathbb{E}\|\varepsilon_t^{(u)}\|_2^2 = \sigma^2(u)\mathbb{E}\|\tilde{\varepsilon}_t\|_2^2$, which is strictly greater than zero and finite.

Proof of (i): Similar to the proof of Theorem 3.1 of Bosq (2000), yet, with a random operator, we have

$$\|A_u^j(\varepsilon_{u,t-j})\|_{2,\Omega\times[0,1]} = \mathbb{E}\|A_u^j(\varepsilon_{u,t-j})\|_2 \le \mathbb{E}[\|A_u^j\|_{\mathcal{L}}\|\varepsilon_{u,t-j}\|_2] \le Cq^j\mathbb{E}\|\varepsilon_{u,t-j}\|_2 \le Cq^j$$

since

$$\|A_u^j\|_{\mathcal{L}} \le \|A_u^j\|_{\mathcal{S}} \le \|A_u\|_{\mathcal{S}}^j \le q^j$$

by Equation (1.55) of Bosq (2000) and submultiplicativity of the Hilbert-Schmidt norm. We can now follow the proof of Theorem 3.1 of Bosq (2000) to deduce the assertions in (i).

Proof of (ii): Similarly as before, we have

$$\begin{aligned}\Big\|\prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T})\Big\|_{2,\Omega\times[0,1]}^2 &= \mathbb{E}\Big\|\prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T})\Big\|_2^2 \le \mathbb{E}\Big[\Big\|\prod_{i=0}^{j-1} A_{\frac{t-i}{T}}\Big\|_{\mathcal{L}}^2 \|\varepsilon_{t-j,T}\|_2^2\Big]\\ &\le q^{2j}\mathbb{E}\|\varepsilon_{t-j,T}\|_2^2 \le Cq^{2j}\end{aligned}$$

where, by convention, $\prod_{i=0}^{-1} A_{(t-i)/T} = \mathrm{id}_{L^2([0,1])}$. Therefore, for $1 \le m \le m'$,

$$\Delta_m^{m'} = \Big\| \sum_{j=m}^{m'} \prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) \Big\|_{2,\Omega\times[0,1]}^2 \le \Big(\sum_{j=m}^{m'} \Big\| \prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) \Big\|_{2,\Omega\times[0,1]} \Big)^2 \le C \Big(\sum_{j=m}^{m'} q^j \Big)^2,$$

and the right-hand side converges to zero as m and m' tend to infinity. As a consequence, $\tilde{Y}_{t,T} := \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} A_{(t-i)/T}(\varepsilon_{t-j,T})$ converges in $L^2(\Omega \times [0,1], \mathbb{P} \otimes \lambda)$ by the Cauchy criterion. Further,

$$\begin{aligned} \mathbb{E}\|\tilde{Y}_{t,T}\|_2^2 &\le \mathbb{E}\Big[\Big(\sum_{j=0}^{\infty} \| \prod_{i=0}^{j-1} A_{\frac{t-i}{T}} \|_{\mathcal{L}} \|\varepsilon_{t-j,T}\|_2 \Big)^2 \Big] \qquad \text{(B.1)} \\ &\le \mathbb{E}\Big[\Big(\sum_{j=0}^{\infty} q^j \|\varepsilon_{t-j,T}\|_2 \Big)^2 \Big] \\ &\le \Big(\frac{1}{1-q} \Big)^2 \mathbb{E}[\|\tilde{\varepsilon}_0\|_2^2] \sup_{u\in[0,1]} \sigma^2(u) < \infty. \end{aligned}$$

Hence $\sum_{j=0}^{\infty} \| \prod_{i=0}^{j-1} A_{(t-i)/T} \|_{\mathcal{L}} \|\varepsilon_{t-j,T}\|_2 < \infty$ almost surely, which implies almost sure convergence in $L^2([0,1])$ of the series defining $\tilde{Y}_{t,T}$ by the Riesz-Fisher theorem.

Finally, we have

$$\tilde{Y}_{t,T} - A_{t/T}(\tilde{Y}_{t-1,T}) = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) - A_{t/T}\Big(\sum_{j=0}^{\infty} \prod_{i=0}^{j-1} A_{\frac{t-1-i}{T}}(\varepsilon_{t-1-j,T}) \Big) = \varepsilon_{t,T},$$

whence $\tilde{Y}_{t,T}$ is a solution of (18) satisfying $\sup_{t\in\mathbb{Z},T\in\mathbb{N}} \mathbb{E}\|\tilde{Y}_{t,T}\|_2^2 < \infty$ by (B.1) and, as we will show below, is locally stationary of order $\rho = 2$.

Conversely, let $Z_{t,T}$ be a locally stationary solution of (18) of order $\rho = 2$ which satisfies $\sup_{t\in\mathbb{Z},T\in\mathbb{N}} \mathbb{E}\|Z_{t,T}\|_2^2 < \infty$. By induction, we have

$$Z_{t,T} = \sum_{j=0}^{n} \prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) + \prod_{i=0}^{n} A_{\frac{t-i}{T}}(Z_{t-n-1,T}).$$

Thus,

$$\begin{aligned} \mathbb{E}\Big\| Z_{t,T} - \sum_{j=0}^{n} \prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) \Big\|_2^2 &= \mathbb{E}\Big\| \prod_{i=0}^{n} A_{\frac{t-i}{T}}(Z_{t-n-1,T}) \Big\|_2^2 \\ &\le \mathbb{E}\Big[\prod_{i=0}^{n} \|A_{\frac{t-i}{T}}\|_{\mathcal{S}}^2 \|Z_{t-n-1,T}\|_2^2 \Big] \\ &\le q^{2(n+1)} \mathbb{E}\|Z_{t-n-1,T}\|_2^2, \end{aligned}$$

which converges to zero as n tends to infinity.

It remains to show that $\tilde{Y}_{t,T}$ is locally stationary of order $\rho = 2$ with approximating family $\{(Y_t^{(u)})_{t\in\mathbb{Z}} : u \in [0,1]\}$. Note that

$$\prod_{i=1}^{n} B_i - \prod_{i=1}^{n} C_i = \sum_{m=1}^{n} \Big(\prod_{k=1}^{m-1} B_k\Big)(B_m - C_m)\Big(\prod_{k=m+1}^{n} C_k\Big)$$
$$= \sum_{m=1}^{n} \Big(\prod_{k=1}^{n-m} B_k\Big)(B_{n-m+1} - C_{n-m+1})\Big(\prod_{k=n-m+2}^{n} C_k\Big)$$

for all $n \in \mathbb{N}, B_i, C_i \in \mathcal{L}$, the empty product being defined as the identity on $L^2([0,1])$. As a consequence

$$\begin{aligned}
Y_{t,T} - Y_t^{(u)} &= \sum_{j=0}^{\infty}\prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) - \sum_{j=0}^{\infty} A_u^j(\varepsilon_{u,t-j}) \\
&= \sum_{j=0}^{\infty}\prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T}) - \prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{u,t-j}) + \prod_{i=1}^{j} A_{\frac{t-i+1}{T}}(\varepsilon_{u,t-j}) - \prod_{i=1}^{j} A_u(\varepsilon_{u,t-j}) \\
&= \sum_{j=0}^{\infty}\prod_{i=0}^{j-1} A_{\frac{t-i}{T}}(\varepsilon_{t-j,T} - \varepsilon_{u,t-j}) + \sum_{m=1}^{j}\Big(\prod_{i=0}^{j-1-m} A_{\frac{t-i}{T}}\Big)(A_{\frac{t-j+m}{T}} - A_u)A_u^{m-1}(\varepsilon_{u,t-j}).
\end{aligned}$$

Since $\|A_u\|_{\mathcal{S}} \le q$ for any $u \in [0,1]$ (with probability one) and $\|Ax\|_2 \le \|A\|_{\mathcal{S}}\|x\|_2$ for any $A \in \mathcal{L}, x \in L^2([0,1])$, it follows that

$$\begin{aligned}
\|Y_{t,T} - Y_t^{(u)}\|_2 &\le \sum_{j=0}^{\infty}\Big(q^j|\sigma\big(\tfrac{t-j}{T}\big) - \sigma(u)|\|\tilde{\varepsilon}_{t-j}\|_2 + q^j\sum_{m=1}^{j}|a\big(\tfrac{t-j+m}{T}\big) - a(u)|\|\tilde{\varepsilon}_{t-j}\|_2\Big) \\
&\le \sum_{j=0}^{\infty}\Big(q^j\big(\big|\tfrac{t}{T} - u\big| + \tfrac{j}{T}\big)\|\tilde{\varepsilon}_{t-j}\|_2 + q^j\sum_{m=1}^{j}\big(\big|\tfrac{t}{T} - u\big| + \tfrac{j-m}{T}\big)\|\tilde{\varepsilon}_{t-j}\|_2\Big) \\
&\le C\sum_{j=0}^{\infty} q^j\|\tilde{\varepsilon}_{t-j}\|_2\big((j+1)\big|\tfrac{t}{T} - u\big| + \tfrac{j^2}{T}\big) \\
&\le C\big(\big|\tfrac{t}{T} - u\big| + \tfrac{1}{T}\big)\sum_{j=0}^{\infty} q^j(j+1)^2\|\tilde{\varepsilon}_{t-j}\|_2.
\end{aligned}$$

The assertion finally follows from the fact that $P_{t,T}^{(u)} = \sum_{j=0}^{\infty} q^j(j+1)^2\|\tilde{\varepsilon}_{t-j}\|_2$ has a finite second moment. □

Appendix C. Auxiliary results for the proofs in Section 3

Lemma C.1. *$C_{\mathbb{B}}$ is a symmetric, positive trace class operator. As a consequence (Theorem 1.2.5 of Maniglia and Rhandi (2004)), $\mathbb{B}$ is a Gaussian random variable in $\mathcal{H}_{H+2}$.*

Proof. To ensure readability, we will denote the scalar product of $L^2([0,1]^p)$ by $\langle\cdot,\cdot\rangle_p, p = 2,3$ and consider the case $H = 0$ only. The arguments for $H \ge 1$ are the same, yet notationally more involved.

Symmetry: Let (g_1, f_1) and (g_2, f_2) be elements in $\mathcal{H}_2$. Then,

$$\langle C_{\mathbb{B}}(g_1, f_1), (g_2, f_2)\rangle = \big\langle \langle r^{(m)}(*,\cdot), g_1(\cdot)\rangle_2, g_2(*)\big\rangle_2 + \big\langle \langle r_0^{(m,c)}(*,\cdot), f_1(\cdot)\rangle_3, g_2(*)\big\rangle_2 + \big\langle \langle r_0^{(m,c)}(*,\cdot), g_1(*)\rangle_2, f_2(\cdot)\big\rangle_3 + \big\langle \langle r_{0,0}^{(c)}(*,\cdot), f_1(\cdot)\rangle_3, f_2(*)\big\rangle_3. \tag{C.1}$$

Now, writing $u = (u_1, u_2)$ and $v = (v_1, v_2)$,

$$\begin{aligned}
&\big\langle \langle r^{(m)}(*,\cdot), g_1(\cdot)\rangle_2, g_2(*)\big\rangle_2 \\
&= \int_{[0,1]^4} \sum_{k=-\infty}^{\infty} \int_0^{u_1\wedge v_1} \mathrm{Cov}(X_0^{(w)}(u_2), X_k^{(w)}(v_2))\,\mathrm{d}w g_1(v) g_2(u)\,\mathrm{d}(u,v) \\
&= \int_{[0,1]^4} \sum_{k=-\infty}^{\infty} \int_0^{u_1\wedge v_1} \mathrm{Cov}(X_0^{(w)}(u_2), X_k^{(w)}(v_2))\,\mathrm{d}w g_1(u) g_2(v)\,\mathrm{d}(u,v) \\
&= \big\langle g_1(*), \langle r^{(m)}(*,\cdot), g_2(\cdot)\rangle_2\big\rangle_2
\end{aligned}$$

and similarly $\big\langle \langle r_{0,0}^{(c)}(*,\cdot), f_1(\cdot)\rangle_3, f_2(*)\big\rangle_3 = \big\langle f_1(*), \langle r_{0,0}^{(c)}(*,\cdot), f_2(\cdot)\rangle_3\big\rangle_3$. Further, writing $u = (u_1, u_2)$ and $v = (v_1, v_2, v_3)$, we have

$$\begin{aligned}
&\big\langle \langle r_0^{(m,c)}(*,\cdot), f_1(\cdot)\rangle_3, g_2(*)\big\rangle_2 + \big\langle \langle r_0^{(m,c)}(*,\cdot), g_1(*)\rangle_2, f_2(\cdot)\big\rangle_3 \\
&= \int_{[0,1]^5} \sum_{k=-\infty}^{\infty} \int_0^{u_1\wedge v_1} \mathrm{Cov}(X_0^{(w)}(u_2), X_k^{(w)}(v_2) X_{k+h}^{(w)}(v_3)) \\
&\qquad\qquad \times \big(f_1(v) g_2(u) + f_2(v) g_1(u)\big)\,\mathrm{d}w\,\mathrm{d}(u,v) \\
&= \int_{[0,1]^5} \sum_{k=-\infty}^{\infty} \int_0^{u_1\wedge v_1} \mathrm{Cov}(X_0^{(w)}(u_2), X_k^{(w)}(v_2) X_{k+h}^{(w)}(v_3)) \\
&\qquad\qquad \times \big(g_1(u) f_2(v) + g_2(u) f_1(v)\big)\,\mathrm{d}w\,\mathrm{d}(u,v) \\
&= \big\langle g_1(*), \langle r_0^{(m,c)}(*,\cdot), f_2(\cdot)\rangle_3\big\rangle_2 + \big\langle f_1(\cdot), \langle r_0^{(m,c)}(*,\cdot), g_2(*)\rangle_2\big\rangle_3.
\end{aligned}$$

Thus, by (C.1), it follows $\langle C_{\mathbb{B}}(g_1, f_1), (g_2, f_2)\rangle = \langle (g_1, f_2), C_{\mathbb{B}}(g_2, f_2)\rangle$.

Positivity: The positivity of $C_{\mathbb{B}}$ can be seen by similar elementary calculations. Let (g, f) be in $\mathcal{H}_2$ and observe that

$$\begin{aligned}
&\langle C_{\mathbb{B}}(g, f), (g, f)\rangle \\
&= \int_0^1 \sum_{k=-\infty}^{\infty} \Big\{ \int_{[0,1]^4} \mathbb{1}(w \le u_1 \wedge v_1) \mathrm{Cov}(X_0^{(w)}(u_2), X_k^{(w)}(v_2)) g(u) g(v)\,\mathrm{d}(u,v) \\
&\quad + \int_{[0,1]^5} \mathbb{1}(w \le u_1 \wedge v_1) \mathrm{Cov}(X_0^{(w)}(u_2), X_k^{(w)}(v_2) X_{k+h}^{(w)}(v_3)) \big(f(v) g(u) + g(u) f(v)\big)\,\mathrm{d}(u,v) \\
&\quad + \int_{[0,1]^6} \mathbb{1}(w \le u_1 \wedge v_1) \mathrm{Cov}(X_0^{(w)}(u_2) X_h^{(w)}(u_3), X_k^{(w)}(v_2) X_{k+h}^{(w)}(v_3)) f(u) f(v)\,\mathrm{d}(u,v) \Big\}\,\mathrm{d}w \\
&= \int_0^1 \sum_{k=-\infty}^{\infty} \mathbb{E}[Y_0(w) Y_k(w) + 2 Y_0(w) Z_k(w) + Z_0(w) Z_k(w)]\,\mathrm{d}w
\end{aligned}$$

$$= \int_0^1 \sum_{k=-\infty}^{\infty} \mathbb{E}[Y_0(w)\big(Y_k(w) + Z_k(w)\big) + \big(Y_0(w) + Z_0(w)\big)Z_k(w)]\,\mathrm{d}w, \tag{C.2}$$

where, for $k \in \mathbb{Z}$ and $w \in [0,1]$,

$$Y_k(w) = \int_{[0,1]^2} \mathbb{1}(w \le u_1) g(u) (X_k^{(w)}(u_2) - \mathbb{E}[X_k^{(w)}(u_2)])\,\mathrm{d}u$$
$$Z_k(w) = \int_{[0,1]^3} \mathbb{1}(w \le u_1) f(u) (X_k^{(w)}(u_2) X_{k+h}^{(w)}(u_3) - \mathbb{E}[X_k^{(w)}(u_2) X_{k+h}^{(w)}(u_3)])\,\mathrm{d}u.$$

As Y_k and Z_k are defined based on a family of stationary processes, we may write $\mathbb{E}[(Y_0(w) + Z_0(w))Z_k(w)] = \mathbb{E}[(Y_{-k}(w) + Z_{-k}(w))Z_0(w)]$. As the summation runs over all $k \in \mathbb{Z}$, we can rewrite the right-hand side of (C.2) as

$$\begin{aligned}
&\int_0^1 \sum_{k=-\infty}^{\infty} \mathbb{E}[Y_0(w)\big(Y_k(w) + Z_k(w)\big) + \big(Y_k(w) + Z_k(w)\big)Z_0(w)]\,\mathrm{d}w \\
&= \int_0^1 \sum_{k=-\infty}^{\infty} \mathbb{E}[\big(Y_0(w) + Z_0(w)\big)\big(Y_k(w) + Z_k(w)\big)]\,\mathrm{d}w \\
&= \int_0^1 \lim_{n\to\infty} \sum_{k=-n}^{n} \mathbb{E}[\big(Y_0(w) + Z_0(w)\big)\big(Y_k(w) + Z_k(w)\big)]\,\mathrm{d}w,
\end{aligned}$$

which is non-negative since

$$\begin{aligned}
&\sum_{k=-n}^{n} \mathbb{E}[\big(Y_0(w) + Z_0(w)\big)\big(Y_k(w) + Z_k(w)\big)] \\
&= \mathbb{E}\Big[\Big(\frac{1}{\sqrt{n}} \sum_{k=0}^{n} Y_k(w) + Z_k(w)\Big)^2\Big] + \frac{1}{n} \sum_{k=-n}^{n} |k| \mathbb{E}[\big(Y_0(w) + Z_0(w)\big)\big(Y_k(w) + Z_k(w)\big)] \\
&= \operatorname{Var}\Big(\frac{1}{\sqrt{n}} \sum_{k=0}^{n} Y_k(w) + Z_k(w)\Big) + \mathcal{O}(n^{-1})
\end{aligned}$$

by Assumption (A3).

Trace class: Let $(\psi_\ell^{(1)})_{\ell\in\mathbb{N}}$ and $(\psi_\ell^{(2)})_{\ell\in\mathbb{N}}$ be orthonormal bases of $L^2([0,1]^2)$ and $L^2([0,1]^3)$ respectively. Then the union $\{(\psi_\ell^{(1)}, 0)\}_{\ell\in\mathbb{N}} \cup \{(0, \psi_\ell^{(2)})\}_{\ell\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_2$. By the definition of the trace norm, we have

$$\begin{aligned}
\|C_{\mathbb{B}}\|_{\mathcal{N}} &= \sum_{\ell=1}^{\infty} \langle C_{\mathbb{B}}(\psi_\ell^{(1)}, 0), (\psi_\ell^{(1)}, 0)\rangle + \sum_{\ell=1}^{\infty} \langle C_{\mathbb{B}}(0, \psi_\ell^{(2)}), (0, \psi_\ell^{(2)})\rangle \\
&= \sum_{\ell=1}^{\infty} \big\langle \langle r^{(m)}(*,\cdot), \psi_\ell^{(1)}(\cdot)\rangle_2, \psi_\ell^{(1)}(*)\big\rangle_2 + \big\langle \langle r_{0,0}^{(c)}(*,\cdot), \psi_\ell^{(2)}(\cdot)\rangle_3, \psi_\ell^{(2)}(*)\big\rangle_3 \\
&= \|C^{(m)}\|_{\mathcal{N}} + \|C_0^{(c)}\|_{\mathcal{N}},
\end{aligned}$$

where $C^{(m)}$ and $C_0^{(c)}$ are the operators defined by the kernels $r^{(m)}$ and $r_{0,0}^{(c)}$ respectively. By the proof of (D3) in the proof of Proposition C.5, Fatou's lemma and Fubini's theorem,

$$\begin{aligned}
\|C^{(m)}\|_{\mathcal{N}} &= \sum_{\ell=1}^{\infty}\langle C^{(m)}\psi_\ell^{(1)}, \psi_\ell^{(1)}\rangle \\
&= \sum_{\ell=1}^{\infty}\int_{[0,1]^4}\sum_{k=-\infty}^{\infty}\int_0^{u\wedge v}\operatorname{Cov}\big(X_0^{(w)}(\varphi), X_k^{(w)}(\tau)\big)\,\mathrm{d}w\psi_\ell^{(1)}(\varphi,v)\psi_\ell^{(1)}(\tau,u)\,\mathrm{d}(u,v,\tau,\varphi) \\
&= \sum_{\ell=1}^{\infty}\lim_{T\to\infty}\operatorname{Cov}\big(\langle\tilde{B}_T,\psi_\ell^{(1)}\rangle, \langle\tilde{B}_T,\psi_\ell^{(1)}\rangle\big) \\
&= \sum_{\ell=1}^{\infty}\lim_{T\to\infty}\mathbb{E}[\langle\tilde{B}_T,\psi_\ell^{(1)}\rangle^2] \\
&\le \liminf_{T\to\infty}\sum_{\ell=1}^{\infty}\mathbb{E}[\langle\tilde{B}_T,\psi_\ell^{(1)}\rangle^2] \\
&= \liminf_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\langle\tilde{B}_T,\psi_\ell^{(1)}\rangle^2\Big] \\
&= \liminf_{T\to\infty}\mathbb{E}\|\tilde{B}_T\|_{2,2}^2,
\end{aligned}$$

which is finite since for any $T\in\mathbb{N}$ since

$$\begin{aligned}
\mathbb{E}\|\tilde{B}_T\|_{2,2}^2 &= \mathbb{E}\Big[\int_{[0,1]^2}\tilde{B}_T^2(u,\tau)\,\mathrm{d}(u,\tau)\Big] \\
&= \int_{[0,1]^2}\mathbb{E}[\tilde{B}_T^2(u,\tau)]\,\mathrm{d}(u,\tau) \\
&= \int_{[0,1]^2}\mathbb{E}\Big[\frac{1}{T}\sum_{t_1,t_2=1}^{\lfloor uT\rfloor}\big(X_{t_1,T}(\tau)-\mathbb{E}X_{t_1,T}(\tau)\big)\big(X_{t_2,T}(\tau)-\mathbb{E}X_{t_2,T}(\tau)\big)\Big]\,\mathrm{d}(u,\tau) \\
&= \int_{[0,1]^2}\frac{1}{T}\sum_{t_1,t_2=1}^{\lfloor uT\rfloor}\operatorname{Cov}\big(X_{t_1,T}(\tau), X_{t_2,T}(\tau)\big)\,\mathrm{d}(u,\tau) \\
&\le \frac{1}{T}\sum_{t_1,t_2=1}^{T}\int_{[0,1]}|\operatorname{Cov}\big(X_{t_1,T}(\tau), X_{t_2,T}(\tau)\big)|\,\mathrm{d}\tau \\
&\le \frac{1}{T}\sum_{t_1,t_2=1}^{T}\nu_2(t_2-t_1)\le C<\infty.
\end{aligned}$$

By similar arguments, it follows that $\|C_0^{(c)}\|_{\mathcal{N}}\le C$, thus $\|C_{\mathbb{B}}\|_{\mathcal{N}}<\infty$. □

Lemma C.2. *Suppose that $\{X_{t,T}: t=1,\dots,T\}_{T\in\mathbb{N}}$ is a locally stationary time series of order $\rho\ge 1$. Then, for any $1\le p\le\rho$,*

$$\mathbb{E}\big[\|X_t^{(u)}-X_t^{(v)}\|_2^p\big]\le C_p|u-v|^p \qquad \forall\ u,v\in[0,1],$$

where $C_p = 2^{p-1} \sup_{t=1,\dots,T, T\in\mathbb{N}, u\in[0,1]} \mathbb{E}|P^{(u)}_{t,T}|^p$.

Proof. By the triangle inequality and convexity of $x \mapsto |x|^p$,

$$\begin{aligned}\mathbb{E}\Big[\|X_t^{(u)} - X_t^{(v)}\|_2^p\Big] &\le 2^{p-1}\mathbb{E}\Big[\|X_t^{(u)} - X_{\lfloor uT\rfloor,T}\|_2^p + \|X_{\lfloor uT\rfloor,T} - X_t^{(v)}\|_2^p\Big] \\ &\le C_p\big(|u-v| + \tfrac{4}{T}\big)^p,\end{aligned}$$

for any $T \in \mathbb{N}$. □

Recall the notations introduced in Section 3.4. For $k \in \mathbb{N}$, define

$$\tilde{B}_T^{(k)}(u,\tau) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor uT\rfloor} \frac{R_i^{(k)}}{\sqrt{m}} \sum_{t=i}^{(i+m-1)\wedge T} \{X_{t,T}(\tau) - \mu_{t,T}(\tau)\}, \tag{C.3}$$

where $\mu_{t,T}(\tau) = \mathbb{E}[X_{t,T}(\tau)]$ and, for any $0 \le h \le H$, let

$$\tilde{B}_{T,h}^{(k)}(u,\tau_1,\tau_2) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor uT\rfloor\wedge(T-h)} \frac{R_i^{(k)}}{\sqrt{m}} \sum_{t=i}^{(i+m-1)\wedge(T-h)} \{X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) - \mu_{t,T,h}(\tau_1,\tau_2)\}, \tag{C.4}$$

where $\mu_{t,T,h}(\tau_1,\tau_2) = \mathbb{E}[X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)]$. Finally, let

$$\mathbb{B}_T^{(k)} = (\tilde{B}_T^{(k)}, \tilde{B}_{T,0}^{(k)}, \dots, \tilde{B}_{T,H}^{(k)}).$$

We then have the following joint asymptotic behaviour of the primary process $\mathbb{B}_T$ and the non-observable multiplier versions $\mathbb{B}_T^{(k)}$. Note that Theorem 3.4 is an immediate consequence.

Theorem C.3. *Suppose that Assumptions (A1)–(A3) and (B1) and (B3) are met. Then, for any fixed* $K \in \mathbb{N}$,

$$(\mathbb{B}_T, \mathbb{B}_T^{(1)}, \dots, \mathbb{B}_T^{(K)}) \rightsquigarrow (\mathbb{B}, \mathbb{B}^{(1)}, \dots, \mathbb{B}^{(K)})$$

in $\{L^2([0,1]^2) \times (L^2([0,1]^3))^{H+1}\}^{K+1}$*, where* $\mathbb{B}^{(1)}, \dots, \mathbb{B}^{(K)}$ *are independent copies of the centred Gaussian variable* $\mathbb{B}$ *from Theorem 3.4 (see also Lemma C.1).*

Proof of Theorem C.3. We only prove the assertion for $K = 1$; the general case follows by the same arguments but is notationally more involved. The theorem is then an immediate consequence of the fundamental approximation Lemma 6.1, together with Lemma C.4 and C.8. □

Let $\{\psi'_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ be orthonormal bases of $L^2([0,1]^2)$ and $L^2([0,1]^3)$ with

$$\sup_{n\in\mathbb{N}} \sup_{x\in[0,1]^2} |\psi'_n(x)| \le C < \infty \text{ and } \sup_{n\in\mathbb{N}} \sup_{x\in[0,1]^3} |\psi_n(x)| \le C < \infty.$$

Note that such bases can be constructed as tensor products of the orthonormal basis

$$\mathcal{B} = \{\sqrt{2}\cos(2\pi n x), \sqrt{2}\sin(2\pi n x) : n \in \mathbb{N}\} \cup \{1\} \text{ in } L^2([0,1])$$

(c.f. Kadison and Ringrose (1983), Example 2.6.11).

Lemma C.4. *Let assumptions (A1)–(A3) and (B1) and (B3) be satisfied. Then, for any $p \in \mathbb{N}$,*

$$\Big((\langle \tilde{B}_T, \psi_n' \rangle)_{n=1}^p, \{(\langle \tilde{B}_{T,h}, \psi_n \rangle)_{n=1}^p\}_{h=0}^H, (\langle \tilde{B}_T^{(1)}, \psi_n' \rangle)_{n=1}^p, \{(\langle \tilde{B}_{T,h}^{(1)}, \psi_n \rangle)_{n=1}^p\}_{h=0}^H\Big)$$
$$\rightsquigarrow \Big((\langle \tilde{B}, \psi_n' \rangle)_{n=1}^p, \{(\langle \tilde{B}_h, \psi_n \rangle)_{n=1}^p\}_{h=0}^H, (\langle \tilde{B}^{(1)}, \psi_n' \rangle)_{n=1}^p, \{(\langle \tilde{B}_h^{(1)}, \psi_n \rangle)_{n=1}^p\}_{h=0}^H\Big),$$

in $\mathbb{R}^{2(H+2)p}$.

Proof of Lemma C.4. Fix some $p \in \mathbb{N}$. By the Cramér-Wold device, it is sufficient to show that

$$Z_T := \sum_{n=1}^p \Big(c_n \langle \tilde{B}_T, \psi_n' \rangle + d_n \langle \tilde{B}_T^{(1)}, \psi_n' \rangle + \sum_{h=0}^H c_{n,h} \langle \tilde{B}_{T,h}, \psi_n \rangle + d_{n,h} \langle \tilde{B}_{T,h}^{(1)}, \psi_n \rangle \Big) \tag{C.5}$$

converges weakly to

$$Z := \sum_{n=1}^p \Big(c_n \langle \tilde{B}, \psi_n' \rangle + d_n \langle \tilde{B}^{(1)}, \psi_n' \rangle + \sum_{h=0}^H c_{n,h} \langle \tilde{B}_h, \psi_n \rangle + d_{n,h} \langle \tilde{B}_h^{(1)}, \psi_n \rangle \Big), \tag{C.6}$$

for any real numbers $c_n, d_n, c_{n,h}, d_{n,h} \in \mathbb{R}, 1 \leq n \leq p, 0 \leq h \leq H$. By Theorem 30.1 and Example 30.1 of Billingsley (1995), the normal distribution is determined uniquely by its moments. Since there is a one-to-one correspondence between moments and cumulants, this also holds true for the latter ones. The only non-zero cumulants of a normal distribution are the first two, which equal the mean and the variance (Holmquist, 1988).

It is easy to see that $\mathbb{E}Z_T = 0$ since $\tilde{B}_T, \tilde{B}_{T,h}, \tilde{B}_T^{(1)}$ and $\tilde{B}_{T,h}^{(1)}$ are centred, for any $0 \leq h \leq H$. For example, we have by the Fubini-Tonelli theorem

$$\begin{aligned}\mathbb{E}[\langle \tilde{B}_T, \psi_n \rangle] &= \mathbb{E}\Big[\int_{[0,1]^3} \tilde{B}_T(u, \tau_1, \tau_2) \psi_n(u, \tau_1, \tau_2) \, \mathrm{d}(u, \tau_1, \tau_2) \Big] \\ &= \int_{[0,1]^3} \mathbb{E}[\tilde{B}_T(u, \tau_1, \tau_2)] \psi_n(u, \tau_1, \tau_2) \, \mathrm{d}(u, \tau_1, \tau_2) = 0.\end{aligned}$$

The theorem is applicable since, by the moment condition (A2), $\mathbb{E}[|\tilde{B}_T(u, \tau_1, \tau_2)|] < \infty$. From Proposition C.5 follows the convergence of the second moments and by Proposition C.6, the higher-order cumulants vanish. Thus, we can conclude the convergence of Z_T to Z by Theorem 2.22 of van der Vaart (1998). ☐

Proposition C.5. *Let assumptions (A1)–(A3) and (B1) and (B3) be satisfied. Then,*

$$\lim_{T \to \infty} \mathrm{Var}(Z_T) = \mathrm{Var}(Z),$$

with Z_T and Z as defined in (C.5) and (C.6).

Proof of Proposition C.5. Since Z_T is a linear combination of $\langle \tilde{B}_T, \psi_n' \rangle, \langle \tilde{B}_{T,h}, \psi_n \rangle, \langle \tilde{B}_T^{(1)}, \psi_n' \rangle$ and $\langle \tilde{B}_{T,h}^{(1)}, \psi_n \rangle$, for $0 \leq h \leq H$ and $1 \leq n \leq p$, it is sufficient to prove that

$$\lim_{T \to \infty} \mathrm{Cov}(\langle \tilde{B}_T, \psi_n' \rangle, \langle \tilde{B}_T, \psi_\ell' \rangle) = \mathrm{Cov}(\langle \tilde{B}, \psi_n' \rangle, \langle \tilde{B}, \psi_\ell' \rangle), \tag{D1}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_T, \psi_n'\rangle, \langle \tilde B_{T,h}, \psi_\ell\rangle) = \mathrm{Cov}(\langle \tilde B, \psi_n'\rangle, \langle \tilde B_h, \psi_\ell\rangle), \tag{D2}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_{T,h}, \psi_n\rangle, \langle \tilde B_{T,h'}, \psi_\ell\rangle) = \mathrm{Cov}(\langle \tilde B_h, \psi_n\rangle, \langle \tilde B_{h'}, \psi_\ell\rangle), \tag{D3}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_T^{(1)}, \psi_n'\rangle, \langle \tilde B_T^{(1)}, \psi_\ell'\rangle) = \mathrm{Cov}(\langle \tilde B^{(1)}, \psi_n'\rangle, \langle \tilde B^{(1)}, \psi_\ell'\rangle), \tag{D4}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_T^{(1)}, \psi_n'\rangle, \langle \tilde B_{T,h}^{(1)}, \psi_\ell\rangle) = \mathrm{Cov}(\langle \tilde B^{(1)}, \psi_n'\rangle, \langle \tilde B_h^{(1)}, \psi_\ell\rangle), \tag{D5}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_{T,h}^{(1)}, \psi_n\rangle, \langle \tilde B_{T,h'}, \psi_\ell\rangle) = \mathrm{Cov}(\langle \tilde B_h^{(1)}, \psi_n\rangle, \langle \tilde B_{h'}^{(1)}, \psi_\ell\rangle), \tag{D6}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_T, \psi_n'\rangle, \langle \tilde B_T^{(1)}, \psi_\ell'\rangle) = \mathrm{Cov}(\langle \tilde B, \psi_n'\rangle, \langle \tilde B^{(1)}, \psi_\ell'\rangle) = 0, \tag{D7}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_T, \psi_n'\rangle, \langle \tilde B_{T,h}^{(1)}, \psi_\ell\rangle) = \mathrm{Cov}(\langle \tilde B, \psi_n'\rangle, \langle \tilde B_h^{(1)}, \psi_\ell\rangle) = 0, \tag{D8}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_{T,h}, \psi_n\rangle, \langle \tilde B_T^{(1)}, \psi_\ell'\rangle) = \mathrm{Cov}(\langle \tilde B_h, \psi_n\rangle, \langle \tilde B^{(1)}, \psi_\ell'\rangle) = 0, \tag{D9}$$

$$\lim_{T\to\infty} \mathrm{Cov}(\langle \tilde B_{T,h}, \psi_n\rangle, \langle \tilde B_{T,h'}, \psi_\ell\rangle) = \mathrm{Cov}(\langle \tilde B_h, \psi_n\rangle, \langle \tilde B_{h'}^{(1)}, \psi_\ell\rangle) = 0, \tag{D10}$$

for any $h, h' \in \{0, \dots, H\}$ and $n, \ell \in \{1, \dots, p\}$. For that purpose, observe that all scalar products have mean zero.

Proof of (D3). Fix $h, h' \in \{0, \dots, H\}$. We have

$$\begin{aligned}
S_{T,3} &= \mathrm{Cov}\big(\langle \tilde B_{T,h}, \psi_n\rangle, \langle \tilde B_{T,h'}, \psi_\ell\rangle\big) \\
&= \mathbb{E}\Big[\int_{[0,1]^6} \tilde B_{T,h}(u,\tau_1,\tau_2)\psi_n(u,\tau_1,\tau_2)\tilde B_{T,h'}(u',\tau_1',\tau_2')\psi_\ell(u',\tau_1',\tau_2')\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')\Big] \\
&= \frac{1}{T}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \mathbb{E}\Big[\int_{[0,1]^6} \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\big\{X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) - \mu_{t,T,h}(\tau_1,\tau_2)\big\} \\
&\qquad \times \big\{X_{t',T}(\tau_1')X_{t'+h',T}(\tau_2') - \mu_{t',T,h'}(\tau_1',\tau_2')\big\}\mathbb{1}(t \le \lfloor uT\rfloor, t' \le \lfloor u'T\rfloor)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')\Big] \\
&= \frac{1}{T}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \int_{[0,1]^6} \mathrm{Cov}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2), X_{t',T}(\tau_1')X_{t'+h',T}(\tau_2')\big) \\
&\qquad\qquad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\mathbb{1}(t \le \lfloor uT\rfloor, t' \le \lfloor u'T\rfloor)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'),
\end{aligned}$$

where we applied Fubini's theorem in the last equality. Further, we can rewrite

$$\begin{aligned}
\mathrm{Cov}(X_{t,T}\otimes X_{t+h,T}, X_{t',T}\otimes X_{t'+h',T}) &= \mathrm{Cov}\big((X_{t,T} - X_t^{(t/T)})\otimes X_{t+h,T}, X_{t',T}\otimes X_{t'+h',T}\big) \\
&\quad + \mathrm{Cov}\big(X_t^{(t/T)}\otimes (X_{t+h,T} - X_{t+h}^{(t/T)}), X_{t',T}\otimes X_{t'+h',T}\big) \\
&\quad + \mathrm{Cov}\big(X_t^{(t/T)}\otimes X_{t+h}^{(t/T)}, (X_{t',T} - X_{t'}^{(t/T)})\otimes X_{t'+h',T}\big) \\
&\quad + \mathrm{Cov}\big(X_t^{(t/T)}\otimes X_{t+h}^{(t/T)}, X_{t'}^{(t/T)}\otimes (X_{t'+h',T} - X_{t'+h'}^{(t/T)})\big) \\
&\quad + \mathrm{Cov}\big(X_t^{(t/T)}\otimes X_{t+h}^{(t/T)}, X_{t'}^{(t/T)}\otimes X_{t'+h'}^{(t/T)}\big).
\end{aligned}$$

Invoking this decomposition, we can split each integral appearing in $S_{T,3}$ into five summands. By (A3), Proposition C.7 and the Cauchy-Schwarz inequality, the sums over all of this summands are of the order $\mathcal{O}(T^{-1})$, except for the last one. Thus, we obtain that

$$S_{T,3} = \int_{[0,1]^6} \Big\{ \frac{1}{T} \sum_{t=1}^{\lfloor (u\wedge u')T\rfloor} \operatorname{Cov}\big(X_t^{(t/T)}(\tau_1) X_{t+h}^{(t/T)}(\tau_2), X_t^{(t/T)}(\tau_1') X_{t+h'}^{(t/T)}(\tau_2')\big)$$
$$+ \frac{1}{T} \sum_{t=1}^{\lfloor uT\rfloor} \sum_{t'=t+1}^{\lfloor u'T\rfloor} \operatorname{Cov}\big(X_t^{(t/T)}(\tau_1) X_{t+h}^{(t/T)}(\tau_2), X_{t'}^{(t/T)}(\tau_1') X_{t'+h'}^{(t/T)}(\tau_2')\big)$$
$$+ \frac{1}{T} \sum_{t'=1}^{\lfloor u'T\rfloor} \sum_{t=t'+1}^{\lfloor uT\rfloor} \operatorname{Cov}\big(X_t^{(t/T)}(\tau_1) X_{t+h}^{(t/T)}(\tau_2), X_{t'}^{(t/T)}(\tau_1') X_{t'+h'}^{(t/T)}(\tau_2')\big) \Big\}$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2') + \mathcal{O}(T^{-1}). \tag{C.7}$$

The convergence of the integral over the first sum is straightforward:

$$\lim_{T\to\infty} \int_{[0,1]^6} \frac{1}{T} \sum_{t=1}^{\lfloor (u\wedge u')T\rfloor} \operatorname{Cov}\big(X_0^{(t/T)}(\tau_1) X_h^{(t/T)}(\tau_2), X_0^{(t/T)}(\tau_1') X_{h'}^{(t/T)}(\tau_2')\big)$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')$$
$$= \int_{[0,1]^6} \int_0^{u\wedge u'} \operatorname{Cov}\big(X_0^{(w)}(\tau_1) X_h^{(w)}(\tau_2), X_0^{(w)}(\tau_1') X_{h'}^{(w)}(\tau_2')\big) \,\mathrm{d}w$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'),$$

where the limit and the integral can be interchanged, by (A2) and Lebesgue's dominated convergence theorem. The convergence of the remaining two sums is technically more involved, and we only present details for the case $t < t'$. By stationarity of $(X_t^{(u)})_{t\in\mathbb{Z}}$,

$$\int_{[0,1]^6} \Big\{ \frac{1}{T} \sum_{t=1}^{\lfloor uT\rfloor} \sum_{t'=t+1}^{\lfloor u'T\rfloor} \operatorname{Cov}\big(X_t^{(t/T)}(\tau_1) X_{t+h}^{(t/T)}(\tau_2), X_{t'}^{(t/T)}(\tau_1') X_{t'+h'}^{(t/T)}(\tau_2')\big) \Big\}$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')$$
$$= \int_{[0,1]^6} \Big\{ \frac{1}{T} \sum_{t=1}^{\lfloor (u\wedge u')T\rfloor} \sum_{k=1}^{\lfloor u'T\rfloor - t} \operatorname{Cov}\big(X_0^{(t/T)}(\tau_1) X_h^{(t/T)}(\tau_2), X_k^{(t/T)}(\tau_1') X_{k+h'}^{(t/T)}(\tau_2')\big) \Big\}$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')$$
$$= \int_{[0,1]^6} \Big\{ \frac{1}{T} \sum_{k=1}^{\lfloor u'T\rfloor - 1} \sum_{t=1}^{\lfloor uT\rfloor \wedge (\lfloor u'T\rfloor - k)} \operatorname{Cov}\big(X_0^{(t/T)}(\tau_1) X_h^{(t/T)}(\tau_2), X_k^{(t/T)}(\tau_1') X_{k+h'}^{(t/T)}(\tau_2')\big) \Big\}$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2').$$

By Lebesgue's dominated convergence theorem and Lemma C.9, the right-hand side of the latter display converges to

$$\int_{[0,1]^6} \Big\{ \sum_{k=1}^{\infty} \int_0^{u\wedge u'} \operatorname{Cov}\big(X_0^{(w)}(\tau_1) X_h^{(w)}(\tau_2), X_k^{(w)}(\tau_1') X_{k+h'}^{(w)}(\tau_2')\big) \,\mathrm{d}w \Big\}$$
$$\times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2') \,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2').$$

Thus, we have

$$\lim_{T\to\infty} S_{T,3} = \int_{[0,1]^6} \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\mathrm{Cov}\big(\tilde B_h(u,\tau_1,\tau_2), \tilde B_{h'}(u',\tau_1',\tau_2')\big)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')$$
$$= \mathrm{Cov}(\langle \tilde B_h, \psi_n\rangle, \langle \tilde B_{h'}, \psi_\ell\rangle),$$

which proves (D3).

Proof of (D6). By the independence of the standard normally distributed random variables $(R_i)_{i\in\mathbb{N}} = (R_i^{(1)})_{i\in\mathbb{N}}$, we have

$$\begin{aligned}
S_{T,6} &= \mathrm{Cov}(\langle \tilde B^{(1)}_{T,h}, \psi_n\rangle, \langle \tilde B_{T,h'}, \psi_\ell\rangle) \\
&= \frac{1}{T}\sum_{i=1}^{T-h}\sum_{i'=1}^{T-h'}\frac{1}{m}\mathbb{E}\Big[R_iR_{i'}\int_{[0,1]^6}\psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\mathbb{1}(i\le\lfloor uT\rfloor, i'\le\lfloor u'T\rfloor) \\
&\qquad\times\Big(\sum_{t=i}^{(i+m-1)\wedge(T-h)} X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)-\mu_{t,T,h}(\tau_1,\tau_2)\Big) \\
&\qquad\times\Big(\sum_{t'=i'}^{(i'+m-1)\wedge(T-h')} X_{t',T}(\tau_1')X_{t'+h',T}(\tau_2')-\mu_{t',T,h'}(\tau_1',\tau_2')\Big)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')\Big] \\
&= \frac{1}{T}\sum_{i=1}^{T-(h\vee h')}\frac{1}{m}\mathbb{E}\Big[\int_{[0,1]^6}\psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\mathbb{1}(i\le\lfloor uT\rfloor\wedge\lfloor u'T\rfloor) \\
&\qquad\times\Big(\sum_{t=i}^{(i+m-1)\wedge(T-h)} X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)-\mu_{t,T,h}(\tau_1,\tau_2)\Big) \\
&\qquad\times\Big(\sum_{t'=i}^{(i+m-1)\wedge(T-h')} X_{t',T}(\tau_1')X_{t'+h',T}(\tau_2')-\mu_{t',T,h'}(\tau_1',\tau_2')\Big)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2')\Big] \\
&= \frac{1}{T}\sum_{i=1}^{T-(h\vee h')}\frac{1}{m}\int_{[0,1]^6}\psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\mathbb{1}(i\le\lfloor (u\wedge u')T\rfloor) \\
&\qquad\times\Big(\sum_{t=i}^{(i+m-1)\wedge(T-h)}\sum_{t'=i}^{(i+m-1)\wedge(T-h')}\mathrm{Cov}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2), X_{t',T}(\tau_1')X_{t'+h',T}(\tau_2')\big)\Big) \\
&\qquad\qquad\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'),
\end{aligned}$$

by Fubini's theorem. By the same arguments that led to (C.7), we further have

$$\begin{aligned}
S_{T,6} &= \frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\int_{[0,1]^6}\psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\mathbb{1}(i\le\lfloor (u\wedge u')T\rfloor) \\
&\qquad\times\Big(\sum_{t=i}^{i+m-1}\sum_{t'=i}^{i+m-1}\mathrm{Cov}\big(X_t^{(i/T)}(\tau_1)X_{t+h}^{(i/T)}(\tau_2), X_{t'}^{(i/T)}(\tau_1')X_{t'+h'}^{(i/T)}(\tau_2')\big)\Big) \\
&\qquad\qquad\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2') + \mathcal{O}(m^{-1}).
\end{aligned}$$

As before, we split the above sum into three sums L_1, L_2, L_3, for $t = t', t < t'$ and $t > t'$, respectively. For L_1, we have

$$\begin{aligned} L_1 &= \frac{1}{T}\sum_{i=1}^{T}\frac{1}{m}\int_{[0,1]^6}\Big(\sum_{t=i}^{i+m-1}\mathrm{Cov}\big(X_t^{(i/T)}(\tau_1)X_{t+h}^{(i/T)}(\tau_2), X_t^{(i/T)}(\tau_1')X_{t+h'}^{(i/T)}(\tau_2')\big)\Big) \\ &\qquad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u,\tau_1,\tau_2)\mathbb{1}(i \le \lfloor uT\rfloor \wedge \lfloor u'T\rfloor)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2') \\ &= \int_{[0,1]^6}\frac{1}{T}\Big(\sum_{i=1}^{\lfloor (u\wedge u')T\rfloor}\mathrm{Cov}\big(X_0^{(i/T)}(\tau_1)X_h^{(i/T)}(\tau_2), X_0^{(i/T)}(\tau_1')X_{h'}^{(i/T)}(\tau_2')\big)\Big) \\ &\qquad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u,\tau_1,\tau_2)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'), \end{aligned}$$

by stationarity of $(X_t^{(u)})_{t\in\mathbb{Z}}$. The right-hand side converges to

$$\begin{aligned} &\int_{[0,1]^6}\psi_n(u,\tau_1,\tau_2)\psi_\ell(u,\tau_1,\tau_2) \\ &\qquad \times \int_0^{u\wedge u'}\mathrm{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2), X_0^{(w)}(\tau_1')X_h^{(w)}(\tau_2')\big)\,\mathrm{d}w\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'), \end{aligned}$$

as T tends to infinity, by Lebegue's dominated convergence theorem.

The sums L_2 and L_3 can be treated in a similar manner, and we only provide details for L_2. By the same arguments as before and the stationarity of $(X_t^{(u)})_{t\in\mathbb{Z}}$, it follows

$$\begin{aligned} L_2 &= \int_{[0,1]^6}\frac{1}{T}\sum_{i=1}^{\lfloor (u\wedge u')T\rfloor}\frac{1}{m}\sum_{t=i}^{i+m-2}\sum_{t'=t+1}^{i+m-1}\mathrm{Cov}\big(X_t^{(i/T)}(\tau_1)X_{t+h}^{(i/T)}(\tau_2), X_{t'}^{(i/T)}(\tau_1')X_{t'+h'}^{(i/T)}(\tau_2')\big) \\ &\quad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2') \\ &= \int_{[0,1]^6}\frac{1}{T}\sum_{i=1}^{\lfloor (u\wedge u')T\rfloor}\frac{1}{m}\sum_{t=i}^{i+m-2}\sum_{k=1}^{i+m-1-t}\mathrm{Cov}\big(X_0^{(i/T)}(\tau_1)X_h^{(i/T)}(\tau_2), X_k^{(i/T)}(\tau_1')X_{k+h'}^{(i/T)}(\tau_2')\big) \\ &\quad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2') \\ &= \int_{[0,1]^6}\sum_{k=1}^{m-1}\frac{m-k}{m}\frac{1}{T}\sum_{i=1}^{\lfloor (u\wedge u')T\rfloor}\mathrm{Cov}\big(X_0^{(i/T)}(\tau_1)X_h^{(i/T)}(\tau_2), X_k^{(i/T)}(\tau_1')X_{k+h'}^{(i/T)}(\tau_2')\big) \\ &\quad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'). \end{aligned}$$

The right-hand side of the previous display converges to

$$\begin{aligned} &\int_{[0,1]^6}\psi_n(u,\tau_1,\tau_2)\psi_\ell(u',\tau_1',\tau_2')\sum_{k=1}^{\infty}\int_0^{u\wedge u'}\mathrm{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2), X_k^{(w)}(\tau_1')X_{k+h'}^{(w)}(\tau_2')\big)\,\mathrm{d}w \\ &\qquad\qquad \mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2'), \end{aligned}$$

as T tends to infinity, by Lebegue's dominated convergence theorem. Thus, (D6) follows by Fubini's theorem, since

$$\begin{aligned} \lim_{T\to\infty} S_{T,6} &= \int_{[0,1]^6}\sum_{k=-\infty}^{\infty}\int_0^{u\wedge u'}\mathrm{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2), X_k^{(w)}(\tau_1')X_{k+h'}^{(w)}(\tau_2')\big)\,\mathrm{d}w \\ &\quad \times \psi_n(u,\tau_1,\tau_2)\psi_\ell(u,\tau_1,\tau_2)\,\mathrm{d}(u,u',\tau_1,\tau_1',\tau_2,\tau_2') \\ &= \mathbb{E}[\langle\psi_n, \tilde{B}_h^{(1)}\rangle\langle\psi_\ell, \tilde{B}_{h'}^{(1)}\rangle] = \mathrm{Cov}(\langle\psi_n, \tilde{B}_h^{(1)}\rangle, \langle\psi_\ell, \tilde{B}_{h'}^{(1)}\rangle). \end{aligned}$$

Proof of (D7)–(D10). The convergences (D7) to (D10) follows from the fact that the multipliers $R_i = R_i^{(1)}$ are independent from the data and centred. For example, we have

$$\begin{aligned}
&\mathrm{Cov}(\langle \tilde{B}_{T,h}, \psi_n\rangle, \langle \tilde{B}_{T,h}^{(1)}, \psi_\ell\rangle) \\
=&\mathbb{E}[\langle \tilde{B}_{T,h}, \psi_n\rangle \langle \tilde{B}_{T,h}^{(1)}, \psi_\ell\rangle] \\
=&\mathbb{E}\Bigg[\langle \tilde{B}_{T,h}, \psi_n\rangle \sum_{i=1}^{T-h} \frac{R_i}{\sqrt{mT}} \sum_{t=i}^{(i+m-1)\wedge(T-h)} \int_{[0,1]^3} \big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) - \mu_{t,T,h}(\tau_1,\tau_2)\big) \\
&\qquad \times \psi_\ell(u,\tau_1,\tau_2)\mathbb{1}(i \le \lfloor uT \rfloor)\,\mathrm{d}(u,\tau_1,\tau_2)\Bigg] = 0,
\end{aligned}$$

which implies (D10).

Proof of (D1)-(D2), (D4)-(D5). Convergences (D1)-(D2) and (D4)-(D5) can be shown with the same arguments as (D3) and (D6), respectively, but they are technically less involved. □

Proposition C.6. *Let assumptions (A1)–(A3) and (B1) and (B3) be satisfied. Then,*

$$\lim_{T\to\infty} \mathrm{cum}_j(Z_T) = 0,$$

for any $j \ge 3$, where Z_T is defined in (C.5)*.*

Proof. By linearity of cumulants, we have

$$\begin{aligned}
&\mathrm{cum}_j(Z_T) \\
&= \mathrm{cum}_j\left(\sum_{n=1}^{p}\left(c_n\langle \tilde{B}_T, \psi_n'\rangle + d_n\langle \tilde{B}_T^{(1)}, \psi_n'\rangle + \sum_{h=0}^{H} c_{n,h}\langle \tilde{B}_{T,h}, \psi_n\rangle + d_{n,h}\langle \tilde{B}_{T,h}^{(1)}, \psi_n\rangle\right)\right) \\
&= \sum_{n_1,\dots,n_j=1}^{p} \sum_{\substack{w=(w_1,\dots,w_j) \\ w_i\in\{0,1\}\times\{-1,\dots,H\}, 1\le i\le j}} \left(\prod_{i=1}^{j} a_i^{(w_i)}\right) \mathrm{cum}(\langle A_1^{(w_1)}, \psi_{n_1}\rangle, \dots, \langle A_j^{(w_j)}, \psi_{n_j}\rangle),
\end{aligned}$$

where

$$a_i^{(0,-1)} = c_i, A_i^{(0,-1)} = \tilde{B}_T, a_i^{(0,h)} = c_{i,h} \text{ and } A_i^{(0,h)} = \tilde{B}_{T,h},$$

and further,

$$a_i^{(1,-1)} = d_i, A_i^{(1,-1)} = \tilde{B}_T^{(1)}, a_i^{(1,h)} = d_{i,h} \text{ and } A_i^{(1,h)} = \tilde{B}_{T,h}^{(1)},$$

for $h = 0, \dots, H$ and $i = 1, \dots, j$. Fix some integers $n_1, \dots, n_j \in \{1, \dots, p\}$. Further, denote the cumulants in the above sum by

$$\overline{\mathrm{cum}}(w) := \mathrm{cum}(\langle A_1^{(w_1)}, \psi_{n_1}\rangle, \dots, \langle A_j^{(w_j)}, \psi_{n_j}\rangle).$$

In the following, we restrict our attention to the subset $(\{0,1\} \times \{0, \dots, H\})^j$ of the set $(\{0,1\} \times \{-1, \dots, H\})^j$ since the proof for the latter follows the same arguments but is notationally more involved.

First, fix $w = (w_1, \dots, w_j)$ with $w_i \in \{0\} \times \{0, \dots, H\}$. Thus, for any w_i there is a $h_i \in \{0, \dots H\}$ such that $A_i^{(w_i)} = \tilde{B}_{T,h_i}$. By the definition of cumulants and Fubini's theorem, we obtain that

$$\begin{aligned}
&\operatorname{cum}(\langle \tilde{B}_{T,h_1}, \psi_{n_1}\rangle, \dots, \langle \tilde{B}_{T,h_j}, \psi_{n_j}\rangle) \\
&= \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)! \prod_{r=1}^{R} \mathbb{E}\Big[\prod_{i\in\nu_r} \int_{[0,1]^3} \psi_{n_i}\big(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)}\big) \\
&\qquad \times \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor u^{(i)}T\rfloor \wedge (T-h_i)} \big(X_{t,T}(\tau_1^{(i)}) X_{t+h_i,T}(\tau_2^{(i)}) - \mu_{t,T,h_i}(\tau_1^{(i)}, \tau_2^{(i)})\big) \,\mathrm{d}(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)})\Big] \\
&= \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)! \prod_{r=1}^{R} \mathbb{E}\Big[\int_{[0,1]^{3|\nu_r|}} \prod_{i\in\nu_r} \psi_{n_i}\big(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)}\big) \\
&\qquad \times \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor u^{(i)}T\rfloor \wedge (T-h_i)} \big(X_{t,T}(\tau_1^{(i)}) X_{t+h_i,T}(\tau_2^{(i)}) - \mu_{t,T,h_i}(\tau_1^{(i)}, \tau_2^{(i)})\big) \,\mathrm{d}(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)} | i \in \nu_r)\Big] \\
&= \sum_{\{\nu_1,\dots,\nu_R\}} \int_{[0,1]^{3j}} (-1)^{R-1}(R-1)! \prod_{r=1}^{R} \mathbb{E}\Big[\prod_{i\in\nu_r} \psi_{n_i}\big(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)}\big) \\
&\qquad \times \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor u^{(i)}T\rfloor \wedge (T-h_i)} \big(X_{t,T}(\tau_1^{(i)}) X_{t+h_i,T}(\tau_2^{(i)}) - \mu_{t,T,h_i}(\tau_1^{(i)}, \tau_2^{(i)})\big)\Big] \\
&\qquad\qquad \mathrm{d}(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)} | 1 \le i \le j) \\
&= \int_{[0,1]^{3j}} \operatorname{cum}\Big(\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor u^{(1)}T\rfloor \wedge (T-h_1)} X_{t,T}(\tau_1^{(1)}) X_{t+h_1,T}(\tau_2^{(1)}) \psi_n\big(u^{(1)}, \tau_1^{(1)}, \tau_2^{(1)}\big), \dots \\
&\qquad \dots, \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor u^{(j)}T\rfloor \wedge (T-h_j)} X_{t,T}(\tau_1^{(j)}) X_{t+h_j,T}(\tau_2^{(j)}) \psi_n(u^{(j)}, \tau_1^{(j)}, \tau_2^{(j)})\Big) \\
&\qquad\qquad \mathrm{d}(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)} | 1 \le i \le j) \\
&= \int_{[0,1]^{3j}} \frac{1}{T^{j/2}} \sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \operatorname{cum}\big(X_{t_1,T}(\tau_1^{(1)}) X_{t_1+h_1,T}(\tau_2^{(1)}), \dots, X_{t_j,T}(\tau_1^{(j)}) X_{t_j+h_j,T}(\tau_2^{(j)})\big) \\
&\qquad \times \prod_{i=1}^{j} \psi_{n_i}\big(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)}\big) \mathbb{1}(t_i \le \lfloor u^{(i)}T\rfloor) \,\mathrm{d}(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)} | 1 \le i \le j)
\end{aligned}$$

where the summation extends over all partitions $\{\nu_1, \dots, \nu_R\}$ of the set $\{1, \dots, j\}$. The absolute value of this expression is bounded by

$$\begin{aligned}
&\frac{1}{T^{j/2}} \sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \Big(\int_{[0,1]^{3j}} \prod_{i=1}^{j} \psi_{n_i}^2(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)}) \mathbb{1}(t_i \le \lfloor u^{(i)}T\rfloor) \,\mathrm{d}(u^{(i)}, \tau_1^{(i)}, \tau_2^{(i)} | 1 \le i \le j)\Big)^{1/2} \\
&\qquad \times \Big(\int_{[0,1]^{2j}} \operatorname{cum}^2\big(X_{t_1,T}(\tau_1^{(1)}) X_{t_1+h_1,T}(\tau_2^{(1)}), \dots, X_{t_j,T}(\tau_1^{(j)}) X_{t_j+h_j,T}(\tau_2^{(j)})\big)
\end{aligned}$$

$$\mathrm{d}(\tau_1^{(i)}, \tau_2^{(i)} | 1 \le i \le j) \Big)^{1/2}$$

$$\le \frac{C}{T^{j/2}} \sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \| \operatorname{cum}\left(X_{t_1,T} \otimes X_{t_1+h_1,T}, \ldots, X_{t_j,T} \otimes X_{t_j+h,T} \right) \|_{2,2j}$$

since, by assumption, $\psi_n(x) \le C$ uniformly in x and n.

In the following, we will bound the expression

$$\sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \| \operatorname{cum}(X_{t_1,T} \otimes X_{t_1+h_1,T}, \ldots, X_{t_j,T} \otimes X_{t_j+h_j,T}) \|_{2,2j}.$$

For that purpose, consider the table

$$S := \begin{array}{cc} (1,0) & (1,1) \\ \vdots & \vdots \\ (j,0) & (j,1), \end{array}$$

where $j \ge 3$. In the following, the tuple $(i,0)$ will be identified with the index t_i and $(i,1)$ will be identified with $t_i + h_i$. Let $\{\nu_1, \ldots, \nu_q\}$ be a partition of S. Two sets ν_i and ν_ℓ of the partition *hook* if there is an index k such that $(k,0) \in \nu_i$ and $(k,1) \in \nu_\ell$ or vice versa. The sets ν_i and ν_ℓ *communicate* if there is a sequence $\nu_i = \tilde{\nu}_1, \ldots, \tilde{\nu_k} = \nu_\ell$ such that $\tilde{\nu}_{i'}$ and $\tilde{\nu}_{i'+1}$ hook, for any $1 \le i' \le k-1$. The partition $\{\nu_1, \ldots, \nu_q\}$ is *indecomposable* if all pairs of sets communicate. By Theorem 2.3.2 of Brillinger (1981), we can rewrite

$$\operatorname{cum}(X_{t_1,T} X_{t_1+h_1,T}, \ldots, X_{t_j,T} X_{t_j+h,T}) = \sum_{\{\nu_1,\ldots,\nu_q\}} \prod_{k=1}^{q} \operatorname{cum}(X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_k), \tag{C.8}$$

where $s = (s_1, s_2)$, where the summation extends over all indecomposable partitions $\{\nu_1, \ldots, \nu_q\}$ of S and where we omit the arguments $\tau_i^{(k)}$ for the ease of notation. Observe that $X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2} = X_{t_{s_1},T}$ if $s_2 = 0$ and $X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2} = X_{t_{s_1}+h_{s_1},T}$ if $s_2 = 1$.

Clearly, Equation (C.8) leads to the bound

$$\sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \| \operatorname{cum}(X_{t_1,T} \otimes X_{t_1+h_1,T}, \ldots, X_{t_j,T} \otimes X_{t_j+h_j,T}) \|_{2,2j}$$
$$\le \sum_{\{\nu_1,\ldots,\nu_q\}} \sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \prod_{\ell=1}^{q} \left\| \operatorname{cum}\left(X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_\ell \right) \right\|_{2,|\nu_\ell|}$$

Fix an indecomposable partition $\{\nu_1, \ldots, \nu_q\}$ of S. If $q = 1$, the sum

$$\sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \left\| \operatorname{cum}\left(X_{t_1,T}, X_{t_1+h_1,T}, \cdots, X_{t_j,T}, X_{t_j+h_j,T} \right) \right\|_{2,2j}$$

is of order $\mathcal{O}(T)$ by (A3). For $q \geq 2$, there exist $\mu_1, \ldots, \mu_{q-1}$ such that

$$\nu_i \cap \{(\mu_1, 0), \ldots, (\mu_{q-1}, 0), (\mu_1, 1), \ldots, (\mu_{q-1}, 1)\} \neq \varnothing,$$

for any $i = 1, \ldots, q$. Informally speaking, the indices $\mu_1, \ldots, \mu_{q-1}$ 'connect' the sets of the partition. Without loss of generality, we assume $\mu_1 = 1, \ldots, \mu_{q-1} = q-1$, $(\mu_i, 0) = (i, 0) \in \nu_i$ and $(\mu_i, 1) = (i, 1) \in \nu_{i+1}$ for $i = 1, \ldots, q-1$. Then,

$$\begin{aligned}
&\sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \prod_{\ell=1}^{q} \left\| \operatorname{cum}\left(X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_\ell \right) \right\|_{2,|\nu_\ell|} \\
&= \sum_{t_1=1}^{T-h_1} \cdots \sum_{t_j=1}^{T-h_j} \left\| \operatorname{cum}\left(X_{t_1,T}, X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_1 \setminus \{(1,0)\} \right) \right\|_{2,|\nu_1|} \\
&\quad \times \left\| \operatorname{cum}\left(X_{t_2,T}, X_{t_1+h_1,T}, X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_2 \setminus \big\{(2,0),(1,1)\big\} \right) \right\|_{2,|\nu_2|} \\
&\quad \cdots \\
&\quad \times \left\| \operatorname{cum}\left(X_{t_{q-1},T}, X_{t_{q-2}+h_{q-2},T}, X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_{q-1} \setminus \big\{(q-1,0),(q-2,1)\big\} \right) \right\|_{2,|\nu_{q-1}|} \\
&\quad \times \left\| \operatorname{cum}\left(X_{t_{q-1}+h_{q-1},T}, X_{t_{s_1},T}^{1-s_2} X_{t_{s_1}+h_{s_1},T}^{s_2}, s \in \nu_q \setminus \{(q-1,1)\} \right) \right\|_{2,|\nu_q|}.
\end{aligned}$$

Consider the sets $\tilde{\nu}_1 := \nu_1 \setminus \{(1,0)\}, \tilde{\nu}_2 := \nu_2 \setminus \{(2,0),(1,1)\}, \ldots, \tilde{\nu}_{q-1} := \nu_{q-1} \setminus \{(q-1,0),(q-2,1)\}, \tilde{\nu}_q = \nu_q \setminus \{(q-1,1)\}$, and observe that these sets form a partition of the set $\{(q,0), \ldots, (j,0), (q,1), \ldots, (j,1)\}$. Let m_i be the cardinality of $\tilde{\nu}_i$, for $i = 1, \ldots, q$. By adding summands, we can bound the above sum by

$$\begin{aligned}
&\sum_{t_1,\ldots,t_{q-1}=1}^{T} \Bigg(\sum_{t_1^{(1)},\ldots,t_{m_1}^{(1)}=-\infty}^{\infty} \left\| \operatorname{cum}\left(X_{t_1,T}, X_{t_1^{(1)},T}, \ldots, X_{t_{m_1}^{(1)},T} \right) \right\|_{2,m_1+1} \\
&\qquad \times \sum_{t_1^{(2)},\ldots,t_{m_2}^{(2)}=-\infty}^{\infty} \left\| \operatorname{cum}\left(X_{t_2,T}, X_{t_1+h_1,T}, X_{t_1^{(2)},T}, \ldots, X_{t_{m_2}^{(2)},T} \right) \right\|_{2,m_2+2} \\
&\qquad \vdots \\
&\qquad \times \sum_{t_1^{(q-1)},\ldots,t_{m_{q-1}}^{(q-1)}=-\infty}^{\infty} \left\| \operatorname{cum}\left(X_{t_{q-1},T}, X_{t_{q-2}+h_{q-2},T}, X_{t_1^{(q-1)},T}, \ldots, X_{t_{m_{q-1}}^{(q-1)},T} \right) \right\|_{2,m_{q-1}+2} \\
&\qquad \times \sum_{t_1^{(q)},\ldots,t_{m_q}^{(q)}=-\infty}^{\infty} \left\| \operatorname{cum}\left(X_{t_{q-1}+h_{q-1},T}, X_{t_1^{(q)},T}, \ldots, X_{t_{m_q}^{(q)},T} \right) \right\|_{2,m_q+1} \Bigg).
\end{aligned}$$

The last inner sum is bounded by some constant $C_{m_{q+1}}$ by Assumption (A3). The outer sum over the index t_{q-1} can be pulled in front of the last inner sum and we obtain

$$\sum_{t_{q-1}=1}^{T} \sum_{t_1^{(q-1)},\ldots,t_{m_{q-1}}^{(q-1)}=-\infty}^{\infty} \left\| \operatorname{cum}\left(X_{t_{q-1},T}, X_{t_{q-2}+h,T}, X_{t_1^{(q-1)},T}, \ldots, X_{t_{q-1}^{(q-1)},T} \right) \right\|_{2,m_{q-1}+2} \leq C_{m_{q-1}+2}$$

Doing this successively, we have the bound

$$C_{m_q+1}\prod_{i=2}^{q-1}C_{m_i+2}\sum_{t_1=1}^{T}\sum_{t_1^{(1)},\dots,t_{m_1}^{(1)}=-\infty}^{\infty}\left\|\operatorname{cum}\left(X_{t_1,T},X_{t_1^{(1)},T},\dots,X_{t_{m_1}^{(1)},T}\right)\right\|_{2,m_1+1}$$
$$\leq C_{m_1+1}C_{m_q+1}\bigg(\prod_{i=2}^{q-1}C_{m_i+2}\bigg)T=\mathcal{O}(T).$$

We finally obtain that $\operatorname{cum}(\langle\tilde{B}_T,\psi_{n_1}\rangle,\dots,\langle\tilde{B}_T,\psi_{n_j}\rangle)=\mathcal{O}(T^{1-j/2})$, which vanishes as T tends to infinity since $j\geq 3$. Thus, we have proven the statement for any $w=(w_1,\dots,w_j)$ with $w_i\in\{0\}\times\{0,\dots,H\}$.

In the following, we investigate the cumulant $\overline{\operatorname{cum}}(w)$, for $w=(w_1,\dots,w_j)$ with $w_i\in\{1\}\times\{0,\dots,H\}$. The cumulants corresponding to arbitrary $w\in(\{0,1\}\times\{0,\dots,H\})^j$ can be bounded by using the same arguments. By similar arguments as for the case $w_i\in\{0\}\times\{0,\dots,H\}$, we obtain that

$$\begin{aligned}
&|\operatorname{cum}(\langle\tilde{B}^{(1)}_{T,h_1},\psi_{n_1}\rangle,\dots,\langle\tilde{B}^{(1)}_{T,h_j},\psi_{n_j}\rangle)|\\
=&\bigg|\int_{[0,1]^{3j}}\operatorname{cum}\big(\tilde{B}^{(1)}_{T,h_1}(u^{(1)},\tau_1^{(1)},\tau_2^{(1)})\psi_{n_1}(u^{(1)},\tau_1^{(1)},\tau_2^{(1)}),\dots\\
&\qquad\dots,\tilde{B}^{(1)}_{T,h_j}(u^{(j)},\tau_1^{(j)},\tau_2^{(j)})\psi_{n_j}(u^{(j)},\tau_1^{(j)},\tau_2^{(j)})\big)\,\mathrm{d}(u^{(i)},\tau_1^{(i)},\tau_2^{(i)}|1\leq i\leq j)\bigg|\\
=&\bigg|\int_{[0,1]^{3j}}\bigg(\prod_{i=1}^{j}\psi_{n_i}(u^{(i)},\tau_1^{(i)},\tau_2^{(i)})\bigg)\frac{1}{(mT)^{j/2}}\sum_{i_1=1}^{\lfloor u^{(1)}T\rfloor\wedge(T-h_1)}\cdots\sum_{i_j=1}^{\lfloor u^{(j)}T\rfloor\wedge(T-h_j)}\\
&\times\operatorname{cum}\bigg(R_{i_1}\sum_{t=i_1}^{(i_1+m-1)\wedge(T-h_1)}X_{t,T}(\tau_1^{(1)})X_{t+h_1,T}(\tau_2^{(1)}),\dots\\
&\qquad\cdots,R_{i_j}\sum_{t=i_j}^{(i_j+m-1)\wedge(T-h_j)}X_{t,T}(\tau_1^{(j)})X_{t+h_j,T}(\tau_2^{(j)})\bigg)\,\mathrm{d}(u^{(i)},\tau_1^{(i)},\tau_2^{(i)}|1\leq i\leq j)\bigg|\\
\leq&\frac{C}{(mT)^{j/2}}\sum_{i_1=1}^{T-h_1}\cdots\sum_{i_j=1}^{T-h_j}\bigg\|\operatorname{cum}\bigg(R_{i_1}\sum_{t=i_1}^{(i_1+m-1)\wedge(T-h_1)}X_{t,T}\otimes X_{t+h_1,T},\dots\\
&\qquad\dots,R_{i_j}\sum_{t=i_j}^{(i_j+m-1)\wedge(T-h_j)}X_{t,T}\otimes X_{t+h_j,T}\bigg)\bigg\|_{2,2j}.
\end{aligned}\tag{C.9}$$

Most of the cumulants on the right-hand side of the above equation are zero. More specific, if there is an index i_ℓ with $i_\ell\neq i_{\ell'}$ for any $\ell'\neq\ell$, then by Theorem 2.3.1 (iii) and Theorem 2.3.2 of Brillinger (1981), the corresponding cumulant in the above sum equals zero. Thus, we can bound the right-hand side of (C.9) by

$$\frac{C_j}{(mT)^{j/2}}\sum_{k=1}^{\lfloor j/2\rfloor}\sum_{\substack{n_1,\cdots,n_k\geq 2\\ \sum_{i=1}^k n_i=j}}\sum_{i_1,\cdots,i_k=1}^{T}\|\operatorname{cum}(R_{i_1}Y_{i_1},\cdots,R_{i_1}Y_{i_1},\cdots,R_{i_k}Y_{i_k},\cdots,R_{i_k}Y_{i_k})\|_{2,2j},$$

where n_ℓ determines how often the product $R_{i_\ell} Y_{i_\ell}$ occurs in the cumulants and

$$Y_{i_\ell} = \sum_{t=i_\ell}^{(i_\ell+m-1)\wedge(T-h_\ell)} X_{t,T} \otimes X_{t+h_\ell,T},$$

for any $\ell \in \{1,\ldots,k\}$. By Theorem 2.3.2 of Brillinger (1981), we can again rewrite each cumulant in the above sum as a sum over products of cumulants of single random variables, where the sum ranges over all indecomposable partitions of the table

$$\begin{array}{cc} R_{i_1} & Y_{i_1} \\ \vdots & \vdots \\ R_{i_1} & Y_{i_1} \\ \vdots & \vdots \\ R_{i_k} & Y_{i_k} \\ \vdots & \vdots \\ R_{i_k} & Y_{i_k}. \end{array}$$

By making use of the same technique as before, we can use the indecomposability to prove that

$$\sum_{\substack{n_1,\cdots,n_k\geq 2 \\ \sum_{i=1}^k n_i = j}} \sum_{i_1,\cdots,i_k=1}^{T} \| \operatorname{cum}(R_{i_1}Y_{i_1},\cdots,R_{i_1}Y_{i_1},\cdots,R_{i_k}Y_{i_k},\cdots,R_{i_k}Y_{i_k})\|_{2,2j}$$

is of order $\mathcal{O}(m^{j-(k-1)}T)$. Now we can see that the right-hand side of (C.9), and thus, $\operatorname{cum}(\langle \tilde{B}_{T,h_1}, \psi_{n_1}\rangle, \ldots, \langle \tilde{B}_{T,h_j}, \psi_{n_j}\rangle)$ are of order $\mathcal{O}(m^{j/2}T^{1-j/2})$, which vanishes as T tends to infinity. Similar, $\overline{\operatorname{cum}}(w)$ vanishes for any $w \in (\{0,1\} \times \{0,\ldots,H\})^j$, as T tends to infinity and, by this, $\operatorname{cum}_j(Z_T)$ does so as well. □

Proposition C.7. *Let Assumptions (A1)–(A3) be satisfied. Then, for any* $h, h' \in \mathbb{N}_0$,

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{T-h} \sum_{t'=1}^{T-h'} &\|\operatorname{Cov}\big((X_{t,T} - X_t^{(t/T)}) \otimes X_{t+h,T}, X_{t',T} \otimes X_{t'+h',T}\big)\|_{2,4} \\ &+ \|\operatorname{Cov}\big(X_t^{(t/T)} \otimes (X_{t+h,T} - X_{t+h}^{(t/T)}), X_{t',T} \otimes X_{t'+h',T}\big)\|_{2,4} \\ &+ \|\operatorname{Cov}\big(X_t^{(t/T)} \otimes X_{t+h}^{(t/T)}, (X_{t',T} - X_{t'}^{(t/T)}) \otimes X_{t'+h',T}\big)\|_{2,4} \\ &+ \|\operatorname{Cov}\big(X_t^{(t/T)} \otimes X_{t+h}^{(t/T)}, X_{t'}^{(t/T)} \otimes (X_{t'+h',T} - X_{t'+h'}^{(t/T)})\big)\|_{2,4} = \mathcal{O}(T^{-1}). \end{aligned}$$

Proof. To ensure readability, we focus on the sum over the first summand. The other summands can be treated with similar arguments. First, define $Y_{t,T} = X_{t,T} - X_t^{(t/T)}$. From the definition of cumulants, Theorem 2.3.2 of Brillinger (1981) and the triangular inequality, we get the bound

$$\begin{aligned} &\|\operatorname{Cov}\big(Y_{t,T} \otimes X_{t+h,T}, X_{t',T} \otimes X_{t'+h',T}\big)\|_{2,4} \leq \| \operatorname{cum}\big(Y_{t,T}, X_{t+h,T}, X_{t',T}, X_{t'+h',T}\big)\|_{2,4} \\ &\quad + \|\operatorname{cum}(Y_{t,T})\|_2 \|\operatorname{cum}(X_{t+h,T}, X_{t',T}, X_{t'+h',T})\|_{2,3} \\ &\quad + \|\operatorname{cum}(X_{t+h,T})\|_2 \|\operatorname{cum}(Y_{t,T}, X_{t',T}, X_{t'+h',T})\|_{2,3} \end{aligned}$$

$$
\begin{aligned}
&+ \| \operatorname{cum}(X_{t',T})\|_2 \| \operatorname{cum}(Y_{t,T}, X_{t+h,T}, X_{t'+h',T})\|_{2,3} \\
&+ \| \operatorname{cum}(X_{t'+h',T})\|_2 \| \operatorname{cum}(Y_{t,T}, X_{t+h,T}, X_{t',T})\|_{2,3} \\
&+ \| \operatorname{cum}(Y_{t,T})\|_2 \| \operatorname{cum}(X_{t',T})\|_2 \| \operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} \\
&+ \| \operatorname{cum}(X_{t+h,T})\|_2 \| \operatorname{cum}(X_{t',T})\|_2 \| \operatorname{cum}(Y_{t,T}, X_{t'+h',T})\|_{2,2} \\
&+ \| \operatorname{cum}(Y_{t,T})\|_2 \| \operatorname{cum}(X_{t'+h',T})\|_2 \| \operatorname{cum}(X_{t+h,T}, X_{t',T})\|_{2,2} \\
&+ \| \operatorname{cum}(X_{t+h,T})\|_2 \| \operatorname{cum}(X_{t'+h',T})\|_2 \operatorname{cum}(Y_{t,T}, X_{t',T})\|_{2,2} \\
&+ \| \operatorname{cum}(Y_{t,T}, X_{t',T})\|_{2,2} \| \operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} \\
&+ \| \operatorname{cum}(Y_{t,T}, X_{t'+h',T})\|_{2,2} \| \operatorname{cum}(X_{t+h,T}, X_{t',T})\|_{2,2} \\
\le{}& C\Big\{ \| \operatorname{cum}\big(Y_{t,T}, X_{t+h,T}, X_{t',T}, X_{t'+h',T}\big)\|_{2,4} \\
&+ \frac{1}{T}\| \operatorname{cum}(X_{t+h,T}, X_{t',T}, X_{t'+h',T})\|_{2,3} + \| \operatorname{cum}(Y_{t,T}, X_{t',T}, X_{t'+h',T})\|_{2,3} \\
&+ \| \operatorname{cum}(Y_{t,T}, X_{t+h,T}, X_{t'+h',T})\|_{2,3} + \| \operatorname{cum}(Y_{t,T}, X_{t+h,T}, X_{t',T})\|_{2,3} \\
&+ \frac{1}{T}\| \operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} + \| \operatorname{cum}(Y_{t,T}, X_{t'+h',T})\|_{2,2} \\
&+ \frac{1}{T}\| \operatorname{cum}(X_{t+h,T}, X_{t',T})\|_{2,2} + \operatorname{cum}(Y_{t,T}, X_{t',T})\|_{2,2} \\
&+ \| \operatorname{cum}(Y_{t,T}, X_{t',T})\|_{2,2} \| \operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} \\
&+ \| \operatorname{cum}(Y_{t,T}, X_{t'+h',T})\|_{2,2} \| \operatorname{cum}(X_{t+h,T}, X_{t',T})\|_{2,2}\Big\},
\end{aligned}
$$

where we made use of (1) in the second inequality. Now, we can investigate the sums over all summands separately. We focus exemplary on three summands, as the remaining summands can be treated with the same arguments. By (A3), we have

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \| \operatorname{cum}\big(Y_{t,T}, X_{t+h,T}, X_{t',T}, X_{t'+h',T}\big)\|_{2,4} \\
&\qquad \le \frac{1}{T}\sum_{t_1,\cdots,t_4=1}^{T} \| \operatorname{cum}\big(Y_{t_1,T}, X_{t_2,T}, X_{t_3,T}, X_{t_4,T}\big)\|_{2,4} \\
&\qquad \le \frac{1}{T}\sum_{t_1,\cdots,t_4=1}^{T} \frac{1}{T}\eta_4(t_2-t_1, t_3-t_1, t_4-t_1) = \mathcal{O}(T^{-1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \frac{1}{T}\| \operatorname{cum}(X_{t+h,T}, X_{t',T}, X_{t'+h',T})\|_{2,3} \\
&\qquad \le \frac{1}{T^2}\sum_{t_1,t_2,t_3=1}^{T} \| \operatorname{cum}(X_{t_1,T}, X_{t_2,T}, X_{t_3,T})\|_{2,3} \\
&\qquad \le \frac{1}{T^2}\sum_{t_1,t_2,t_3=1}^{T} \eta_3(t_2-t_1, t_3-t_1) = \mathcal{O}(T^{-1}).
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \|\operatorname{cum}(Y_{t,T}, X_{t',T})\|_{2,2}\|\operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} \\
&\quad\le \frac{1}{T}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \|\mathbb{E}[(Y_{t,T})^2]\|_2\|\mathbb{E}[X_{t',T}^2]\|_2\|\operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} \\
&\quad\le \frac{C}{T^2}\sum_{t=1}^{T-h}\sum_{t'=1}^{T-h'} \|\operatorname{cum}(X_{t+h,T}, X_{t'+h',T})\|_{2,2} \\
&\quad\le \frac{C}{T^2}\sum_{t,t'=1}^{T} \eta_2(t-t') = \mathcal{O}(T^{-1}).
\end{aligned}
$$

The proof for the third and fourth summand relies on the summability assumption of $(1+|t_j|)\nu_k(t_1,\dots,t_{k-1})$ rather than $\nu_k(t_1,\dots,t_{k-1})$. □

Lemma C.8. *Let Assumptions (A1)–(A3) and (B1) and (B3) be satisfied. Then,*

$$
\lim_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=n+1}^{\infty}\Big(\langle\tilde{B}_T,\psi'_\ell\rangle^2+\langle\tilde{B}_T^{(1)},\psi'_\ell\rangle^2+\sum_{h=0}^{H}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2+\langle\tilde{B}_{T,h}^{(1)},\psi_\ell\rangle^2\Big)\Big]=0.
$$

Proof of Lemma C.8. By linearity of the expectation, we can prove the property for every process separately. We restrict our attention to the cases

$$
\lim_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=n+1}^{\infty}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\Big]=0 \tag{C.10}
$$

and

$$
\lim_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=n+1}^{\infty}\langle\tilde{B}_{T,h}^{(1)},\psi_\ell\rangle^2\Big]=0; \tag{C.11}
$$

the assertions regarding $\tilde{B}_T$ and $\tilde{B}_T^{(1)}$ follow by similar arguments.

First, by linearity of expectation,

$$
\begin{aligned}
0 &\le \limsup_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=n+1}^{\infty}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\Big] \\
&= \limsup_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2-\sum_{\ell=1}^{n}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\Big] \\
&\le \limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\Big]-\liminf_{n\to\infty}\liminf_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=1}^{n}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\Big] \\
&= \limsup_{T\to\infty}\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\Big]-\sum_{\ell=1}^{\infty}\liminf_{T\to\infty}\mathbb{E}\big[\langle\tilde{B}_{T,h},\psi_\ell\rangle^2\big] \\
&= \limsup_{T\to\infty}\mathbb{E}\|\tilde{B}_{T,h}\|_{2,3}^2-\mathbb{E}\|B_h\|_{2,3}^2,
\end{aligned}
$$

where we used Equation (D3) from the proof of Proposition C.5 in the last step. Thus, it is sufficient to prove $\limsup_{T\to\infty} \mathbb{E}\|\tilde{B}_{T,h}\|_{2,3}^2 \leq \mathbb{E}\|B_h\|_{2,3}^2$. By Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{E}\|\tilde{B}_{T,h}\|_{2,3}^2 \\
&= \frac{1}{T}\sum_{t,t'=1}^{T-h}\int_{[0,1]^3} \mathrm{Cov}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2), X_{t',T}(\tau_1)X_{t'+h,T}(\tau_2)\big)\mathbb{1}(t,t' \leq \lfloor uT\rfloor)\,\mathrm{d}(u,\tau_1,\tau_2).
\end{aligned}$$

As in the proof of (D3) in the proof of Proposition C.5, we split the above sum into three sums $S_{T,1}, S_{T,2}, S_{T,3}$ according to $t = t', t < t'$ and $t > t'$, respectively.

For the convergence of the first sum, we obtain, by stationarity,

$$\begin{aligned}
S_{T,1} &= \frac{1}{T}\sum_{t=1}^{T-h}\int_{[0,1]^3} \mathrm{Var}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)\big)\mathbb{1}(t \leq \lfloor uT\rfloor)\,\mathrm{d}(u,\tau_1,\tau_2) \\
&= \frac{1}{T}\sum_{t=1}^{T-h}\int_{[0,1]^3} \mathrm{Var}\big(X_t^{(t/T)}(\tau_1)X_{t+h}^{(t/T)}(\tau_2)\big)\mathbb{1}(t \leq \lfloor uT\rfloor)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(T^{-1}) \\
&= \int_{[0,1]^3}\frac{1}{T}\sum_{t=1}^{T-h} \mathrm{Var}\big(X_0^{(t/T)}(\tau_1)X_{h}^{(t/T)}(\tau_2)\big)\mathbb{1}(t \leq \lfloor uT\rfloor)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(T^{-1}) \\
&\xrightarrow{T\to\infty} \int_{[0,1]^3}\int_0^u \mathrm{Var}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2)\big)\,\mathrm{d}w(u,\tau_1,\tau_2).
\end{aligned}$$

Next, the double sum involving $t < t'$ can be treated as follows:

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{(T-h)\wedge\lfloor uT\rfloor}\sum_{t'=t+1}^{(T-h)\wedge\lfloor uT\rfloor}\int_{[0,1]^3} \mathrm{Cov}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2), X_{t',T}(\tau_1)X_{t'+h,T}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) \\
&= \frac{1}{T}\int_{[0,1]^3}\sum_{t=1}^{\lfloor uT\rfloor}\sum_{t'=t+1}^{\lfloor uT\rfloor} \mathrm{Cov}\big(X_t^{(t/T)}(\tau_1)X_{t+h}^{(t/T)}(\tau_2), X_{t'}^{(t/T)}(\tau_1)X_{t'+h}^{(t/T)}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(T^{-1}) \\
&= \frac{1}{T}\int_{[0,1]^3}\sum_{t=1}^{\lfloor uT\rfloor}\sum_{k=1}^{\lfloor uT\rfloor-t} \mathrm{Cov}\big(X_t^{(t/T)}(\tau_1)X_{t+h}^{(t/T)}(\tau_2), X_{k+t}^{(t/T)}(\tau_1)X_{k+t+h}^{(t/T)}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(T^{-1}) \\
&= \frac{1}{T}\int_{[0,1]^3}\sum_{t=1}^{\lfloor uT\rfloor}\sum_{k=1}^{\lfloor uT\rfloor-t} \mathrm{Cov}\big(X_0^{(t/T)}(\tau_1)X_{h}^{(t/T)}(\tau_2), X_{k}^{(t/T)}(\tau_1)X_{k+h}^{(t/T)}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(T^{-1}).
\end{aligned}$$

By Lebesgue's dominated convergence theorem, the integral and the limit, as T tends to infinity, are interchangeable in the last equality. Thus, the right-hand side converges according to Lemma C.9 to

$$\int_{[0,1]^3}\sum_{k=1}^{\infty}\int_0^u \mathrm{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2), X_k^{(w)}(\tau_1)X_{k+h}^{(w)}(\tau_2)\big)\,\mathrm{d}w\,\mathrm{d}(u,\tau_1,\tau_2).$$

A similar assertion holds for the double sum involving $t > t'$. Altogether, we obtain that $\mathbb{E}\|\tilde{B}_{T,h}\|_{2,3}^2$ converges to

$$\int_{[0,1]^3} \sum_{k=-\infty}^{\infty} \int_0^u \operatorname{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2), X_k^{(w)}(\tau_1)X_{k+h}^{(w)}(\tau_2)\big)\,\mathrm{d}w\,\mathrm{d}(u,\tau_1,\tau_2)$$
$$= \int_{[0,1]^3} \operatorname{Var}\big(\tilde{B}_h(u,\tau_1,\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) = \mathbb{E}\|\tilde{B}_h\|_{2,3}^2$$

by Fubini's theorem, which proves (C.10).

For the proof of (C.11) observe that

$$0 \le \lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{E}\Big[\sum_{\ell=n+1}^{\infty} \langle \tilde{B}_{T,h}^{(1)}, \psi_\ell\rangle^2\Big] \le \limsup_{T\to\infty} \mathbb{E}\|\tilde{B}_{T,h}^{(1)}\|_{2,3}^2 - \mathbb{E}\|\tilde{B}_h^{(1)}\|_{2,3}^2,$$

as before, and we can conclude the statement by showing $\limsup_{T\to\infty} \mathbb{E}\|\tilde{B}_{T,h}^{(1)}\|_{2,3}^2 \le \mathbb{E}\|\tilde{B}_h^{(1)}\|_{2,3}^2$. Fubini's theorem and the independence of the family $(R_i)_{i\in\mathbb{N}}$ lead to

$$\begin{aligned}
&\mathbb{E}\|\tilde{B}_{T,h}^{(1)}\|_{2,3}^2\\
&= \mathbb{E}\Big[\int_{[0,1]^3} \frac{1}{mT} \sum_{i,i'=1}^{\lfloor uT\rfloor\wedge(T-h)} R_i R_{i'} \sum_{t=i}^{(i+m-1)\wedge(T-h)} \sum_{t'=i'}^{(i'+m-1)\wedge(T-h)} \{X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)\\
&\qquad - \mu_{t,T,h}(\tau_1,\tau_2)\}\{X_{t',T}(\tau_1)X_{t'+h,T}(\tau_2) - \mu_{t',T,h}(\tau_1,\tau_2)\}\,\mathrm{d}(u,\tau_1,\tau_2)\Big]\\
&= S_{T,1} + S_{T,2} + S_{T,3},
\end{aligned}$$

where

$$\begin{aligned}
S_{T,1} &= \int_{[0,1]^3} \frac{1}{T} \sum_{i=1}^{\lfloor uT\rfloor\wedge(T-h)} \frac{1}{m} \sum_{t=i}^{(i+m-1)\wedge(T-h)} \operatorname{Var}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2),\\
S_{T,2} &= \int_{[0,1]^3} \frac{1}{T} \sum_{i=1}^{\lfloor uT\rfloor\wedge(T-h)} \frac{1}{m} \sum_{t=i}^{(i+m-2)\wedge(T-h)} \sum_{t'=t+1}^{(i+m-1)\wedge(T-h)}\\
&\qquad \operatorname{Cov}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2), X_{t',T}(\tau_1)X_{t'+h,T}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2),\\
S_{T,3} &= \int_{[0,1]^3} \frac{1}{T} \sum_{i=1}^{\lfloor uT\rfloor\wedge(T-h)} \frac{1}{m} \sum_{t'=i}^{(i+m-2)\wedge(T-h)} \sum_{t=t'+1}^{(i+m-1)\wedge(T-h)}\\
&\qquad \operatorname{Cov}\big(X_{t,T}(\tau_1)X_{t+h,T}(\tau_2), X_{t',T}(\tau_1)X_{t'+h,T}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2).
\end{aligned}$$

We investigate the three previous terms separately. By the same arguments as in the proof of Proposition C.7 and the stationarity of $(X_t^{(u)})_{t\in\mathbb{Z}}$, we have

$$\begin{aligned}
S_{T,1} &= \int_{[0,1]^3} \frac{1}{T} \sum_{i=1}^{\lfloor uT\rfloor} \frac{1}{m} \sum_{t=i}^{(i+m-1)\wedge(T-h)} \operatorname{Var}\big(X_t^{(t/T)}(\tau_1)X_{t+h}^{(t/T)}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(mT^{-1})\\
&= \int_{[0,1]^3} \frac{1}{T} \sum_{i=1}^{\lfloor uT\rfloor} \frac{1}{m} \sum_{t=i}^{i+m-1} \operatorname{Var}\big(X_0^{(t/T)}(\tau_1)X_h^{(t/T)}(\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2) + \mathcal{O}(mT^{-1}).
\end{aligned}$$

For $u<1$, the previous integrand can be rewritten as

$$\frac{1}{T}\sum_{i=1}^{m-1}\frac{i}{m}\mathrm{Var}\big(X_0^{(t/T)}(\tau_1)X_h^{(t/T)}(\tau_2)\big)+\frac{1}{T}\sum_{i=m}^{\lfloor uT\rfloor}\mathrm{Var}\big(X_0^{(t/T)}(\tau_1)X_h^{(t/T)}(\tau_2)\big)$$
$$+\frac{1}{T}\sum_{i=\lfloor uT\rfloor+1}^{\lfloor uT\rfloor+m-1}\frac{\lfloor uT\rfloor+m-i}{m}\mathrm{Var}\big(X_0^{(t/T)}(\tau_1)X_h^{(t/T)}(\tau_2)\big),$$

which implies that

$$\lim_{T\to\infty}S_{T,1}=\int_{[0,1]^3}\int_0^u\mathrm{Var}\big(X_0^{(w)}(\tau_1),X_h^{(w)}(\tau_2)\big)\,\mathrm{d}w\,\mathrm{d}(u,\tau_1,\tau_2),$$

by Lebesgue's dominated convergence theorem. The sums $S_{T,2}$ and $S_{T,3}$ can be treated similarly, which finally implies that

$$\begin{aligned}&\limsup_{T\to\infty}\mathbb{E}\|\tilde{B}_T^{(1)}\|_{2,3}^2\\&=\int_{[0,1]^3}\sum_{t=-\infty}^{\infty}\int_0^u\mathrm{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2),X_t^{(w)}(\tau_1)X_{t+h}^{(w)}(\tau_2)\big)\,\mathrm{d}w\,\mathrm{d}(u,\tau_1,\tau_2)\\&=\int_{[0,1]^3}\mathrm{Var}\big(\tilde{B}(u,\tau_1,\tau_2)\big)\,\mathrm{d}(u,\tau_1,\tau_2)=\mathbb{E}\|\tilde{B}^{(1)}\|_{2,3}^2.\end{aligned}$$

Thus (C.11) holds true, which proves the lemma. □

Lemma C.9. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of integrable functions on the unit interval $[0,1]$, such that $f_k(x)\le\nu(k)$, for all $x\in[0,1]$, with $\sum_{k=1}^{\infty}\nu(k)<\infty$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of integers with $a_n\to\infty$ as n tends to infinity. Then,*

$$\lim_{n\to\infty}\sum_{k=1}^{a_n}\frac{1}{n}\sum_{\ell=1}^{\lfloor un\rfloor}f_k\big(\tfrac{\ell}{n}\big)=\sum_{k=1}^{\infty}\int_0^u f_k(x)\,\mathrm{d}x$$

for any $u\in[0,1]$.

Proof. The statement is an immediate consequence of Lebesgue's dominated convergence theorem, applied to the sequence of functions $g_n(k,x)=\mathbb{1}(k\le a_n)\sum_{\ell=1}^{\lfloor un\rfloor}f_k(\ell/n)\mathbb{1}(x\in((\ell-1)/n,\ell/n])$. □

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