Annals of the Institute of Statistical Mathematics (2020) 72:645–657
https://doi.org/10.1007/s10463-019-00708-4

On univariate slash distributions, continuous and discrete

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Received: 16 August 2018 / Revised: 16 January 2019 / Published online: 15 February 2019

Abstract
In this article, I explore in a unified manner the structure of uniform slash and α-slash distributions which, in the continuous case, are defined to be the distributions of Y / U and $Y_\alpha/U^{1/\alpha}$ where Y and Y_α follow any distribution on $\mathbb{R}^+$ and, independently, U is uniform on (0, 1). The parallels with the monotone and α-monotone distributions of $Y \times U$ and $Y_\alpha \times U^{1/\alpha}$, respectively, are striking. I also introduce discrete uniform slash and α-slash distributions which arise from a notion of negative binomial thinning/fattening. Their specification, although apparently rather different from the continuous case, seems to be a good one because of the close way in which their properties mimic those of the continuous case.

Keywords Binomial thinning · Monotone density · Negative binomial fattening · Uniform random variable

1 Introduction

The seminal early robustness study of Andrews et al. (1972) introduced the slash distribution, sometimes now called the canonical slash distribution, which is the distribution of Z/U where Z is standard normal and, independently, U is uniform on (0, 1) (Kafadar 2006); henceforth, the latter is written $U \sim \mathrm{U}(0,1)$. The name 'slash' presumably refers to the division sign. In this article, I shall use the term 'uniform slash distribution' to mean the distribution of $X = Y/U$ where Y follows *any* absolutely continuous distribution with density g on $\mathbb{R}^+$ and, independently (and specifically), $U \sim \mathrm{U}(0,1)$. The restriction to $\mathbb{R}^+$ is for convenience only: similar results occur on $\mathbb{R}$, where one might also prefer to replace U by $2U-1$.

I look at the structure of the densities of uniform slash distributions in Sect. 2, along with a pre-existing extension to what I will call 'α-slash' distributions [sometimes called 'standard slash distributions': a uniform slash distribution is a 1-slash

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distribution; α-slash distributions, though not under that name, were considered in the case $Y = Z$ by Rogers and Tukey (1972)]. If it is desired to construct a distribution with at most, say, r (positive integer) moments, then a very good candidate—which can be constructed using a suitable g from a wide palette of possibilities—is an α-slash distribution based on g, with α closely related to r (see Sect. 2.2). Reasons for this desire might include simulation studies or sensitivity analyses; data analysis with heavy-tailed distributions is a potential use of such distributions too. My development is parallel to the equivalent development when the division signs underlying this paragraph are replaced by multiplication signs. The latter, known, work, which centres on monotone densities, is briefly summarised in 'Appendix A'.

There are also versions of the monotonicity results just mentioned for discrete distributions on $\mathbb{N}_0 = 0, 1, \dots$. These, based on Steutel (1988), are summarised in 'Appendix B'. Prior to the current article, there was no such thing as a discrete slash (or discrete α-slash) distribution: these are proposed and investigated in Sect. 3. As (continuous) multiplication by a value between 0 and 1 can be translated to (discrete) binomial thinning (Steutel and van Harn 1979), so (continuous) division by a value between 0 and 1, or multiplication by a value larger than 1, can be translated to what, in Sect. 3, I call 'negative binomial fattening'. And then everything from the continuous case in Sect. 2 has a nice analogue in the discrete case, as given in Sect. 3. In particular, as in the continuous case, discrete α-slash distributions can be constructed to accommodate heavy tails as defined by existence of specified moments, and for the same types of potential application too.

The article is rounded off with some concluding remarks in Sect. 4.

2 The univariate continuous case

2.1 When is a distribution a uniform slash distribution?

Regardless of which g is chosen, the distribution of $X = Y/U$ where $Y \sim g$ is extremely heavy-tailed in the sense that its distribution has no (positive integer) moments. This is because $E(1/U^r)$ does not exist for any $r = 1, 2, \dots$. But is this the defining characteristic of such a uniform slash distribution? Well, not quite. More precisely, the density, f, of X satisfies

$$f(x) = \frac{1}{x^2} \int_0^x y g(y) \mathrm{d}y \tag{1}$$

so that

$$f'(x) = -\frac{2}{x} f(x) + \frac{1}{x} g(x)$$

and hence

$$g(y) = 2f(y) + y f'(y). \tag{2}$$

Validity of g as a density on $\mathbb{R}^+$ therefore requires that the uniform slash density, f, satisfies

$$(\log f)'(x) \geq -\frac{2}{x}, \qquad \text{for all } x > 0, \tag{3}$$

the inequality being strict except where $g(y) = 0$. Indeed, f has a very heavy tail: from (1), even if g has finite mean μ_g, then $f(x) \approx \mu_g/x^2$ as $x \to \infty$, which implies non-existence of moments of f. Infinite μ_g cannot change this state of affairs, of course. Inequality (3) can be utilised to check whether any existing (heavy-tailed) distribution is a uniform slash distribution and, if so, (2) can be used to identify the distribution of Y.

Examples of uniform slash distributions

(a) By way of one example of direct construction, suppose that $g(y) = e^{-y}$, $y > 0$. The uniform slash exponential distribution is then the distribution with density $f(x) = \{1 - (1 + x)e^{-x}\}/x^2$, $x > 0$. This distribution has no (positive integer) moments. Its density can be shown to be monotone decreasing.

(b) The half-Cauchy distribution is a uniform slash distribution on $\mathbb{R}^+$: it has density $f(x) = 2/\{\pi(1 + x^2)\}$, $x > 0$, so that $(\log f)'(x) = -2x/(1 + x^2) = -2/(x + \frac{1}{x}) > -2/x$, and the density of Y is $g(y) = 4/\{\pi(1 + y^2)^2\}$ which is a scaled half-t_3 distribution.

(c) When is the F distribution a uniform slash distribution? The scaled $F_{2a,2b}$ distribution has log-density of the form $\log f(x) = K + (a - 1)\log x - (a + b)\log(1 + x)$ so that $(\log f)'(x) = \{a - 1 - (b + 1)x\}/x(1 + x) \geq -2/x$ if $a + 1 \geq (b - 1)x$ for all $x > 0$, which requires $b \leq 1$.

The cumulative distribution functions (c.d.f.'s) F and G, associated with f and g, respectively, are readily shown to be related by

$$G(x) = F(x) + xf(x).$$

2.2 α-slash distributions

A popular extension of the uniform slash distribution is to replace U by a power of U, that is, to consider the distribution of $X_\alpha = Y_\alpha/U^{1/\alpha}$, $\alpha > 0$, where Y_α follows any distribution with density g_α, say, on $\mathbb{R}^+$ and, independently, $U \sim \mathrm{U}\,(0, 1)$. Equivalently, $X_\alpha = Y_\alpha/B_\alpha$ where Y_α follows a distribution with density g_α and, independently, B_α follows a beta distribution with parameters α and 1; henceforth $B_\alpha \sim \mathrm{Be}(\alpha, 1)$. I shall call the distribution of X_α an α-slash distribution and note that this means that X_α^α follows a uniform slash distribution. A 1-slash distribution is, of course, a uniform slash distribution.

The development of Sect. 2.1 for uniform slash distributions works through in similar fashion for α-slash distributions. The density, f_α, of X_α satisfies

$$f_\alpha(x) = \frac{\alpha}{x^{1+\alpha}} \int_0^x y^\alpha g_\alpha(y)\mathrm{d}y$$

so that

$$\alpha g_\alpha(y) = (1+\alpha) f_\alpha(y) + y f_\alpha'(y).$$

Validity of g_α as a density therefore requires that the α-slash density satisfies

$$(\log f_\alpha)'(x) \geq -\frac{1+\alpha}{x}, \qquad \text{for all } x > 0, \tag{4}$$

strict inequality holding except where $g_\alpha(y) = 0$. Immediately, for $0 < \alpha \leq 1$, an α -slash distribution is also a slash distribution. Also, in an obvious notation,

$$\alpha G_\alpha(x) = \alpha F_\alpha(x) + x f_\alpha(x).$$

This formula and some related ones can be found for the case $G_\alpha = \Phi$, the standard normal c.d.f., in Rogers and Tukey (1972).

Examples of α-slash distributions

(a) Following on from (a) in my examples of uniform slash distributions, the α-slash exponential distribution is the distribution with density $f_\alpha(x) = \alpha\gamma(x;1+\alpha)/x^{1+\alpha}$, $x > 0$, where $\gamma(\cdot;\cdot)$ is the incomplete gamma function. This distribution has $r < \alpha$ moments. The following argument, using the standard inequality $(1+\alpha)\gamma(x;1+\alpha) \geq x^\alpha(1-e^{-x})$ [e.g. (8.10.2) of Olver et al. (2010)], shows that f_α is nonincreasing for all $\alpha > 0$:

$$\begin{aligned} f_\alpha'(x) &= \frac{\alpha}{x^{\alpha+2}}\left\{x^{\alpha+1}e^{-x} - (1+\alpha)\gamma(x;1+\alpha)\right\} \\ &\leq \frac{\alpha}{x^{\alpha+2}}\left\{x^{\alpha+1}e^{-x} - x^\alpha(1-e^{-x})\right\} \\ &= \frac{\alpha}{x^2}\{(1+x)e^{-x} - 1\} \leq \frac{\alpha}{x^2}(e^x e^{-x} - 1) = 0. \end{aligned}$$

(b) When is the half-t_ν distribution an α-slash distribution on $\mathbb{R}^+$? Suitably scaled, its densities are proportional to $(1+x^2)^{-(\nu+1)/2}$, $\nu > 0$, so that $(\log f)'(x) = -(\nu+1)x/(1+x^2) \geq -(1+\alpha)/x$ if $(\alpha-\nu)x^2 \geq -(1+\alpha)$ for all $x > 0$, which requires $\nu \leq \alpha$.

(c) When is the $F_{2a,2b}$ distribution an α-slash distribution? Following (c) in my examples of uniform slash distributions, $(\log f)'(x) = \{a-1-(b+1)x\}/x(1+x) \geq -(1+\alpha)/x$ if $a+\alpha \geq (b-\alpha)x$ for all $x > 0$, which requires $b \leq \alpha$.

The rth moment of $U^{-1/\alpha}$ exists for $r < \alpha$. Since $E(X_\alpha^r) = E(Y_\alpha^r) \times E(U^{-r/\alpha})$, it follows that X_α will also possess $r < \alpha$ moments, provided g_α is chosen to have r or more moments. Heavier-tailed g_α can only make the number of existing moments of f_α fewer, not greater, of course. A related argument supposes that $\lim_{x\to\infty} x^{r+1} f_\alpha(x) = 0$ which is a necessary condition for the existence of the rth moment of f_α. Then, using integration by parts and (4),

$$\int_0^\infty x^r f_\alpha(x)\,\mathrm{d}x = -\frac{1}{1+r}\int_0^\infty x^{r+1} f_\alpha'(x)\,\mathrm{d}x < \frac{1+\alpha}{1+r}\int_0^\infty x^r f_\alpha(x)\,\mathrm{d}x,$$

that is, it must be the case that $r < \alpha$. A third argument with the same outcome is the following. When $E_{g_\alpha}(Y_\alpha)$ is finite, the tail of f_α behaves as $f_\alpha(x) \approx \alpha E_{g_\alpha}(Y_\alpha)/x^{\alpha+1}$, $x \to \infty$, which implies existence of the rth moment of X_α only if $r < \alpha$. And infinite $E_{g_\alpha}(Y_\alpha)$ again changes tailweight in a way that can lead only to fewer moments, not more. Of course, if $r < \alpha$, then $E(U^{-r/\alpha}) = \alpha/(\alpha - r)$.

So, a good way to generate a random variable with precisely r (integer) moments, for the purposes of a simulation study checking out a theory in which the number of assumed moments is important, for example, is to generate a random variable with an α-slash distribution, for any convenient α satisfying $r < \alpha \leq r + 1$; choice of any g_α with r or more moments enables the investigator to be able to try out many such distributions. Random variate generation from an α-slash distribution is itself straightforward by using its defining construction in terms of division of random variables, especially if it is easy to generate from g_α.

3 The univariate discrete case

Transfer of monotonicity properties from the continuous to the discrete case takes place via an appropriate replacement for multiplication by a real number between 0 and 1 while remaining on support $\mathbb{N}_0$. In this section, it will be shown how a similar replacement for division by a real number between 0 and 1 while remaining on support $\mathbb{N}_0$ leads to the transfer of slashness properties from the continuous to the discrete case.

3.1 Discrete uniform slash distributions

To reduce a discrete value m by a factor of $0 < u < 1$ while maintaining support $\mathbb{N}_0$, the key (Steutel and van Harn 1979) is to choose a value from $0, 1, \ldots, m$ ($m = 0, 1, \ldots$) according to the binomial distribution with parameters m and u. Notationally, the binomially thinned quantity $u \circ m$ is defined such that $u \circ m \sim \text{Bi}(m, u)$. (The binomial distribution with parameters 0 and u has mass 1 at value 0.) The parallels between $X = U \times Y$ in the continuous case and $N = U \circ M \sim \text{Bi}(M, U)$ in the discrete case are then compelling (Steutel 1988; Jones 2018; see 'Appendix B'). How, on the other hand, does one increase m by a factor of $1/u > 1$ while maintaining support $\mathbb{N}_0$? Well, $m = 0$ should remain $m = 0$. And I suggest that, for $m = 1, 2, \ldots$, one should choose a value from $m, m + 1, \ldots$ according to the negative binomial distribution with parameters m and u, starting from m. Without adding on m, this is what is no longer so accurately called 'negative binomial thinning' (Ristić et al. 2009), but when m is added back in, it seems clear that it can be called 'negative binomial fattening'.

So, like $u \circ m$, the negative binomially fattened quantity $u \bullet m$ is defined through its distribution: the p.m.f. of $u \bullet m$ is that of $L + m$ where L follows the negative binomial distribution with parameters m and u, starting from zero. (As in the binomial case, the negative binomial distribution with parameters 0 and u has mass 1 at value 0.) The required p.m.f. is therefore

$$\binom{n-1}{m-1}u^m(1-u)^{n-m}, \qquad n=m,m+1,\dots;$$

the p.m.f. of $U \bullet m$ is then, on $n=m,m+1,\dots,$

$$\binom{n-1}{m-1}\int_0^1 u^m(1-u)^{n-m}\mathrm{d}u = \binom{n-1}{m-1}B(m+1,n-m+1)$$
$$\text{(where } B(\cdot,\cdot) \text{ is the beta function)}$$
$$= \frac{m}{n(n+1)} = q_m(n), \text{ say,} \tag{5}$$

where $q_0(0)=1$; and finally, the discrete uniform slash distribution is defined as the distribution of $N = U \bullet M$ where

$$N|M=m \sim q_m(n), \qquad M \sim q. \tag{6}$$

The p.m.f. $q_m(n)$ at (5) can be recognised as the $\rho=1$ special case of that of the (Yule or) Yule–Simon distribution [as in Johnson et al. (2005), Section 6.10.3], truncated to start at m rather than at 1 when $m=2,3,\dots$

Note that, like its $1/U$-based continuous counterpart, the distribution with p.m.f. q_m, and therefore the discrete uniform slash distribution, has no (positive integer) moments. It is easy to see that the p.m.f. of the discrete uniform slash random variable N is

$$p(n) = \begin{cases} q(0) & n=0, \\ \dfrac{1}{n(n+1)}\displaystyle\sum_{m=1}^{n} mq(m), & n=1,2,\dots, \end{cases}$$

and, conversely, that

$$q(m) = (m+1)\,p(m) - (m-1)\,p(m-1), \qquad m=0,1,\dots,$$

where, of course, $p(-1)=0$. Therefore, a p.m.f. p is that of a discrete uniform slash distribution if

$$\frac{p(n+1)}{p(n)} \ge \frac{n}{n+2} \quad \text{for all } n=0,1,\dots,$$

the inequality being strict except where $q(m)=0$. Much as in the continuous case, p has a very heavy tail: if q_m has finite mean μ_q, then $p(n) \approx \mu_q/n^2$ as $n\to\infty$. This ties in, of course, with non-existence of moments.

Examples of discrete uniform slash distributions

(a) The Katz family of distributions (e.g. Johnson et al. 2005, Section 2.3.1) is defined by $p(n+1)/p(n) = (a+bn)/(1+n)$ for suitable a and b, by which

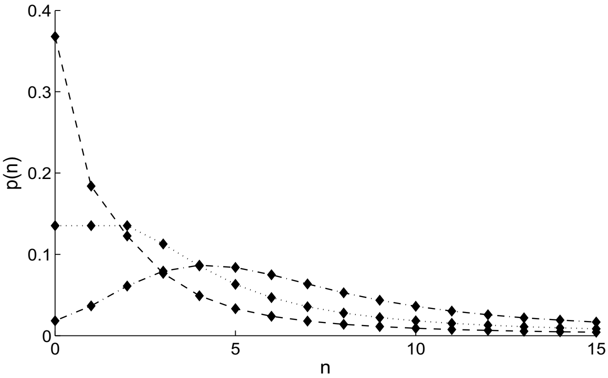

Fig. 1 The diamonds are values of $p(n)$ for uniform slash Poisson distributions, joined by broken lines for clarity: dashed line corresponds to $\lambda = 1$, dotted line to $\lambda = 2$, dot-dashed line to $\lambda = 4$

choice it covers binomial, negative binomial and Poisson distributions. No Katz distribution is a discrete uniform slash distribution, however, because that would require $b > 1$, which is not allowed.

(b) The general Yule–Simon distribution on $n = 0, 1, \ldots,$ has p.m.f. $\rho B(n + 1, \rho + 1)$ for parameter $\rho > 0$ [(6.121) of Johnson et al. (2005)]. When is the Yule–Simon distribution a uniform slash distribution? Well, $p(n + 1)/p(n) = (n + 1)/(n + \rho + 2) \geq n/(n + 2)$ for all $n = 0, 1, \ldots,$ when $\rho \leq 1$.

(c) By way of a novel very heavy-tailed discrete distribution, the uniform slash Poisson distribution has p.m.f.

$$p(n) = \begin{cases} e^{-\lambda} & n = 0, \\ \dfrac{\lambda e^{-\lambda}}{n(n+1)} \displaystyle\sum_{j=0}^{n-1} \frac{\lambda^j}{j!} = \frac{\lambda \Gamma(\lambda; n)}{(n+1)!}, & n = 1, 2, \ldots, \end{cases} \tag{7}$$

where $\Gamma(\cdot;\cdot)$ is the complementary incomplete gamma function. This distribution can be seen to have no (positive integer) moments. Examples of uniform slash Poisson p.m.f.'s are shown in Fig. 1. The shapes of the p.m.f.'s are discussed in 'Appendix C'.

(d) Similarly, the uniform slash geometric distribution has p.m.f.

$$p(n) = \begin{cases} p & n = 0, \\ \dfrac{(1-p)\{1 - (1+np)(1-p)^n\}}{n(n+1)p}, & n = 1, 2, \ldots; \end{cases}$$

again, of course, no (positive integer) moments. The uniform slash geometric p.m.f. is decreasing for every value of its parameter $0 < p < 1$. To see this, first

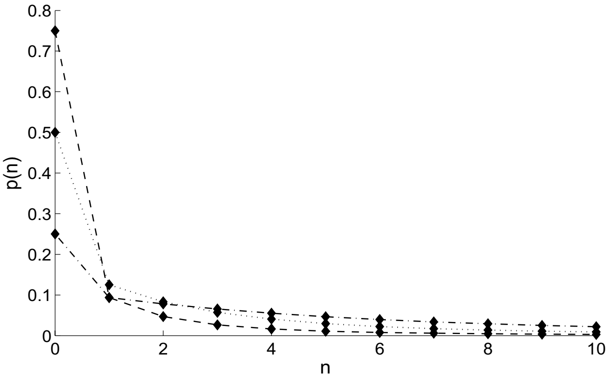

Fig. 2 The diamonds are values of $p(n)$ for uniform slash geometric distributions, joined by broken lines for clarity: dashed line corresponds to $p = 0.75$, dotted line to $p = 0.5$, dot-dashed line to $p = 0.25$

note that $p(0) - p(1) = p(1 + p)/2 > 0$. Then, observe that, for all $n = 1, 2, \dots,$ the difference $p(n) - p(n + 1) \propto 2 - T_p(n)$, where

$$T_p(n) \equiv (1 - p)^n \left\{ n(n + 1)p^2 + 2np + 2 \right\};$$

this quantity can readily be shown to be a decreasing function of p, its supremum being 2, corresponding to $p \downarrow 0$. Examples of uniform slash geometric p.m.f.'s are shown in Fig. 2.

In the discrete case, the c.d.f.'s, defined as, for example, $P(n) = \mathrm{Prob}(N \le n)$, can be shown to be related by

$$Q(n) = P(n) + np(n).$$

3.2 Discrete α-slash distributions

It is now natural to define a discrete α-slash distribution to be that of $N_\alpha \equiv U^{1/\alpha} \bullet M_\alpha$ when $M_\alpha \sim q_\alpha$ on $\mathbb{N}_0$. On $n = m, m + 1, \dots,$ the p.m.f. of $U^{1/\alpha} \bullet m_\alpha$ is

$$\begin{aligned} \alpha \binom{n-1}{m-1} \int_0^1 u^{m+\alpha-1}(1-u)^{n-m} \mathrm{d}u &= \alpha \binom{n-1}{m-1} B(m + \alpha + 1, n - m + 1) \\ &= \alpha \, \frac{(n-1)! \, \Gamma(\alpha + m)}{(m-1)! \, \Gamma(\alpha + n + 1)} \\ &= q_{\alpha,m}(n), \text{ say,} \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function and $q_{\alpha,0}(0) = 1$; $q_{\alpha,m}(n)$ is, by construction, a special form of beta-negative binomial distribution.

It follows that the p.m.f. of the discrete α-slash distribution is

$$p(n) = \begin{cases} q_\alpha(0) & n = 0, \\ \alpha \dfrac{(n-1)!}{\Gamma(\alpha+n+1)} \displaystyle\sum_{m=1}^{n} \dfrac{\Gamma(\alpha+m)\, q_\alpha(m)}{(m-1)!}, & n = 1, 2, \dots . \end{cases}$$

Also,

$$\alpha q_\alpha(m) = (m+\alpha) p_\alpha(m) - (m-1)\, p_\alpha(m-1), \qquad m = 0, 1, \dots , \tag{8}$$

so that a p.m.f. p_α is that of a discrete α-slash distribution if

$$\frac{p_\alpha(n+1)}{p_\alpha(n)} \geq \frac{n}{n+\alpha+1} \qquad \text{for all } n = 0, 1, \dots ,$$

the inequality again being strict except where $q_\alpha(m) = 0$. A discrete α-slash distribution is also a discrete uniform slash distribution when $0 < \alpha \leq 1$.

Examples of discrete *α-slash distributions*

(a) The Yule–Simon distribution on $n = 0, 1, \dots$, is a discrete α-slash distribution whenever $p(n+1)/p(n) = (n+1)/(n+\rho+2) \geq n/(n+\alpha+1)$ for all $n = 0, 1, \dots$. This requires, for any $\alpha > 0$, ρ to satisfy $\rho \leq \alpha$, strict inequality in most places making no difference to this requirement.

(b) The Waring distribution on $n = 0, 1, \dots$, (Johnson et al. 2005, Section 6.10.4) has probability ratios $p(n+1)/p(n) = (n+a)/(n+\rho+2)$, where $a, \rho > 0$; its $a = 1$ special case is the Yule–Simon distribution. The Waring distribution is therefore discrete α-slash if $\rho \leq \alpha + a - 1$.

For discrete α-slash distributions, the c.d.f.'s are related by

$$\alpha Q_\alpha(n) = \alpha P_\alpha(n) + n p_\alpha(n).$$

A direct proof of this fact can be given, but a more immediate verification is provided by its reduction on differencing to the p.m.f. relationship (8).

The result of the multiplication-based arguments given at the end of Sect. 2 in the continuous case concerning the existence of moments continues to apply in the discrete case because, conditional on m and u, all moments of the negative binomial distribution exist, and its rth moment depends on powers of $1/u$ up to $1/u^r$ (e.g. Johnson et al. 2005, Section 5.4). That is, a discrete α-slash distribution has $r < \alpha$ moments, provided q_α has at least r moments. In fact, provided the appropriate factorial-type moment of q_α exists, the tail of p goes as $n^{-(\alpha+1)}$ as $n \to \infty$. So, as in the continuous case, it is this control over tails and hence moments that is the potential practical raison-d'être for discrete α-slash distributions.

As in the continuous case, random variate generation from a discrete α-slash distribution can conveniently be performed by using its defining construction: random variables from uniform, negative binomial and q_α distributions are required. That said, a referee reminds me that the usual direct method for discrete distributions of generating

$U = u$ from U(0, 1) and setting $N_\alpha = n$ if $P_\alpha(n-1) < u \le P_\alpha(n)$ is available (here $P_\alpha(-1) = 0$). This is easily programmed and potentially speedier in many cases.

Finally, a few words on estimation in discrete α-slash distributions should they be fitted to data. The method of moments is inappropriate for obvious reasons, but likelihood and Bayesian methods are available. These can be straightforwardly implemented using standard general approaches and, indeed, software since the distributions in question have just a few parameters, typically two, their precise number depending on the specification of q_α. I have found nothing specifically interesting enough to provide in detail in the paper in terms of explicit estimates, simplified properties of estimators, conjugate prior distributions, etc.; formulae for likelihoods, score equations, information matrices, posterior distributions, etc., can, of course, be written out if and when required.

4 Concluding remarks

The literature is already replete with documents detailing properties and applications of individual special cases of continuous univariate slash distributions (and of continuous multivariate slash distributions, a topic to which I have nothing to contribute here). The current article is unusual in its general investigation of common features of slash distributions. The term 'slash distribution' has also been used for distributions of ratios with other divisors, such as exponential random variables and powers thereof. In this author's view, particular multipliers/divisors of random variables with general distributions are interesting only in as much as they embue the resulting distributions with some worthwhile general property or properties. In this regard, Be$(\alpha, 1)$ multipliers are interesting for the monotonicity (or, on $\mathbb{R}$, unimodality) properties they imply, while Be$(\alpha, 1)$ divisors are interesting for the moment-controlled heavy-tailed properties they give rise to. I have been surprised, however, at the extent to which these apparently quite different properties arise from such closely parallel developments.

Discrete univariate slash (and α-slash) distributions are a new invention, but their specification, although apparently rather different from the continuous case, seems to be a good one because of the close way in which their properties mimic those of the continuous case.

Acknowledgements I am grateful to the reviewers for suggestions that have improved the quality of this article.

Appendix A The continuous case: monotone densities

Consider the density, f, of the distribution of $Z = Y \times U$ where Y follows *any* distribution with density $h > 0$ on $\mathbb{R}^+$ and, independently, $U \sim$ U(0, 1). Then, f satisfies

$$f(z) = \int_z^\infty \frac{1}{y} h(y) \mathrm{d}y$$

so that

$$h(y) = -yf'(y).$$

Validity of h as a density therefore requires that $f'(z) < 0$ for all $z > 0$ and hence that f is a monotone decreasing density (on $\mathbb{R}^+$). This is a version of Khintchine's theorem (Khintchine 1938; Feller 1971). The corresponding c.d.f.'s are related by $H(x) = F(x) - xf(x)$. Relaxing the traditional positivity constraint on h to nonnegativity, gaps in the support of h correspond to constant patches in f.

The distribution of $Z_\alpha = Y_\alpha \times U^{1/\alpha}$ where Y_α follows a distribution with density $h_\alpha \geq 0$ on $\mathbb{R}^+$ and, independently, $U \sim \mathrm{U}(0, 1)$ is that of an α-monotone distribution, in which case Z_α^α has a monotone density (Olshen and Savage 1970; Dharmadhikari and Joag-Dev 1988; Bertin et al. 1997). The α-monotone density f_α satisfies

$$f_\alpha(z) = \alpha z^{\alpha-1} \int_z^\infty \frac{1}{y^\alpha} h_\alpha(y)\mathrm{d}y$$

so that

$$f_\alpha'(z) = \frac{(\alpha-1)}{z} f_\alpha(z) - \frac{\alpha}{z} h_\alpha(z)$$

and

$$\alpha h_\alpha(y) = (\alpha - 1) f_\alpha(y) - y f_\alpha'(y).$$

A density f_α is, therefore, α-monotone iff

$$(\log f_\alpha)'(z) \leq \frac{(\alpha-1)}{z}, \qquad \text{for all } z > 0.$$

Here, the inequality is strict except when $h_\alpha(z) = 0$. Also, $\alpha H_\alpha(x) = \alpha F_\alpha(x) - x f_\alpha(x)$.

Appendix B The discrete case: monotone probability mass functions

On $\mathbb{N}_0$, consider the p.m.f. p, of the distribution of $N = U \circ M$ where M follows any distribution with p.m.f. q and, independently, $U \sim \mathrm{U}(0, 1)$. Recall that, for fixed $0 < u < 1$ and $m = 0, 1, \ldots$, $u \circ m \sim \mathrm{Bi}(m, u)$. It then follows that the distribution of $U \circ m$ is discrete uniform, $\mathrm{U}(0, 1, \ldots, m)$, and, finally, a discrete Khintchine's theorem states that p is monotone nonincreasing iff $N \sim p$ can be written as

$$N | M = m \sim \mathrm{U}(0, 1, \ldots, m), \qquad M \sim q.$$

In fact,

$$p(n) = \sum_{m=n}^\infty \frac{q(m)}{m+1}, \qquad q(m) = (m+1)\{p(m) - p(m+1)\}$$

(Steutel 1988); see also Jones (2018). In terms of c.d.f.'s, $Q(n) = P(n) - (n+1)p(n+1)$.

Steutel (1988) went on to discuss discrete α-monotonicity which corresponds to replacing U by $U^{1/\alpha}$ above. The distribution of $U^{1/\alpha} \circ m_\alpha$ turns out to be the beta-binomial distribution with parameters m_α, α and 1 on $n = 0, 1, \ldots, m_\alpha$. This gives rise to Steutel's (1988) formulae

$$p_\alpha(n) = \alpha \frac{\Gamma(n+\alpha)}{n!} \sum_{m=n}^{\infty} \frac{m!\, q_\alpha(m)}{\Gamma(m+\alpha+1)}$$

and

$$\alpha q_\alpha(m) = (m+\alpha) p_\alpha(m) - (m+1) p_\alpha(m+1).$$

From the latter, it can be concluded that discrete α-monotonicity corresponds to p having the property that

$$\frac{p_\alpha(n+1)}{p_\alpha(n)} \le \frac{n+\alpha}{n+1}, \qquad n = 0, 1, \ldots ,$$

the inequality being strict whenever $q_\alpha(m) > 0$. Also, $\alpha Q_\alpha(n) = \alpha P_\alpha(n) - (n+1) p_\alpha(n+1)$. See Jones (2018) for further discussion.

Appendix C On the shape of the uniform slash Poisson probability mass function

Consider the p.m.f. of the uniform slash Poisson distribution on $n = 0, 1, \ldots,$ given by (7). First, $p(0) - p(1) = e^{-\lambda}(2-\lambda)/2 > (=) < 0$ as $\lambda < (=) > 2$. Second, $p(1) - p(2) = \lambda e^{-\lambda}(2-\lambda)/6 > (=) < 0$ as $\lambda < (=) > 2$ also. Third, $p(2) - p(3) = \lambda e^{-\lambda}\left(2 + 2\lambda - \lambda^2\right)/24 > (=) < 0$ as $\lambda < (=) > 1 + \sqrt{3} \simeq 2.732$. And for $n = 3, 4, \ldots,$

$$\begin{aligned} p(n) - p(n+1) &\propto (n+2) \sum_{j=0}^{n-1} \frac{\lambda^j}{j!} - n \sum_{j=0}^{n} \frac{\lambda^j}{j!} = 2 \sum_{j=0}^{n-1} \frac{\lambda^j}{j!} - \frac{\lambda^n}{(n-1)!} \\ &> \left\{2(n-1) + 2\lambda - \lambda^2\right\} \frac{\lambda^{n-2}}{(n-1)!}. \end{aligned}$$

This is positive whenever $\lambda < 1 + \sqrt{2n-1}$, the upper bound being greater than or equal to $1 + \sqrt{5} \simeq 3.236$. The p.m.f. of the uniform slash Poisson distribution is therefore proven to be decreasing for $0 < \lambda < 2$, to have $p(0) = p(1) = p(2)$ and then to decrease for $\lambda = 2$, and to be unimodal with mode at 2 for $2 < \lambda < 1 + \sqrt{3}$, and with equal modes at 2 and 3 when $\lambda = 1 + \sqrt{3}$.

From numerical evidence, I conjecture but cannot prove that p remains unimodal for all larger values of λ with its mode, occasionally shared over two consecutive values of n, at or a little greater than λ.

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