Supplementary Materials for “Estimation and Hypothesis Test for Partial Linear Single-Index Multiplicative Models”

Jun Zhang1*, Xia Cui2 and Heng Peng3

1 College of Mathematics and Statistics, Shenzhen-Hong Kong Joint Research Center for Applied Statistical Sciences, Institute of Statistical Sciences, Shenzhen University, 518060, Shenzhen, China.
2 School of Economics and Statistics, Guangzhou University, 510006, Guangzhou, China.
3 Department of Mathematics, The Hong Kong Baptist University, Kowloon Tong, Hong Kong, China.

1. APPENDIX

1.1. Assumptions

We begin this section by listing the conditions needed in the proofs of our asymptotic results.

(C1) $E[|X_s^2|] < \infty$ for $s = 1, \ldots, q$, $E[Z_r^2] < \infty$ for $r = 1, \ldots, p$, and the matrices $\Gamma_0$, $\Lambda_0$ used in Theorem 1 and $\Gamma_{L,0}$ in Proposition 1 are all positive definite and finite. Moreover, $E[|\ln(Y)|^r] < \infty$ for some $r > 3$.

(C2) $E(X|\beta^T\phi Z = u)$, $E(Z|\beta^T\phi Z = u)$ and the density function $f_{\beta,\phi}(u)$ of the random variable $\beta^T\phi Z$ are twice continuously differentiable with respect to $u$. Their second-order derivatives are uniformly Lipschitz continuous on $\mathcal{C} = \{u = \beta^T\phi Z : z \in Z \subset \mathbb{R}^p, \beta, \phi \in \mathcal{S}_{n,\beta,\phi}\}$, where $Z$ is a compact support set, and $\mathcal{S}_{n,\beta,\phi} = \{\beta, \phi \in \mathcal{S} : \|eta - \beta_0\| \leq c_0n^{-1/2+c_1}\}$ for some positive constant $c_0$ and $c_1 \in [0, 0.05)$. Moreover, $g(u)$ has two bounded and continuous derivatives on $u \in \mathcal{C}$ and $\inf_{u \in \mathcal{C}} f_{\beta,\phi}(u) > 0$.

(C3) The kernel function $K(\cdot)$ is a symmetric bounded density function supported on $[-A, A]$, satisfying a Lipschitz condition. $K(\cdot)$ also has second-order continuous bounded derivatives, satisfying $K^{(j)}(\pm A) = 0$, $j = 0, 1, 2$, and $\int s^2K(s)ds \neq 0$.

(C4) As $n \to \infty$, the bandwidth $h$ satisfies $\frac{(\log n)^{2+2\alpha}}{nh^2} \to 0$ for some $s_0 > 0$, and $nh^4 \to 0$.

(C5) For all $\zeta_j, j = 1, \ldots, q$, $\zeta_j \to 0$, $\sqrt{n}\zeta_j \to \infty$ as $n \to \infty$, moreover, $\liminf_{n \to \infty} \liminf_{u \to 0} p_j(u)/\zeta_j > 0$.

1.2. A Technical Lemma

Lemma 1 Suppose $E(W|\beta^T\phi Z = u) = n(u)$ and its derivatives up to second order are bounded for all $\beta, \phi \in \mathcal{S}_{n,\beta,\phi}$, where $\mathcal{S}_{n,\beta}$ is defined in condition (C2), and that $E|W|^r$ exists for some $r > 3$. Let $(Z_i, W_i, i = 1, 2, \ldots, n)$ be an independent and identically distributed (i.i.d.) sample from $(Z, W)$. Let $\tau_{n,h} = \left\{\frac{(\log n)^{1+s_0}}{nh}\right\}^{1/2} + h^3$ for some $s_0 > 0$. Given

*Corresponding Author: Jun Zhang, email: zhangjunstat@gmail.com.
that \( h = n^{-j} \) for some \( 0 < j < 1 \), if conditions (C1)-(C4) hold, we have,

\[
\sup_{u \in \mathcal{E}} \left| \sum_{i=1}^{n} K_{h}(\beta_{\phi}^{T}Z - u) \left( \frac{\beta_{\phi}^{T}Z - u}{h} \right)^{d} W_{i} - f_{\beta_{0}}(u)m(u)\mu_{K,d} \right|
\]

\[
- \{f_{\beta_{0}}(u)m(u)\}'\mu_{K,d+1}h - \frac{1}{2}\{f_{\beta_{0}}(u)m(u)\}''\mu_{K,d+2}h^{2} \right\} = O(\tau_{n,h}), \text{a.s.,}
\]

where \( \mu_{K,d} = \int K(v)v^{d}dv \), \( d = 0, 1, 2 \).

**Proof.** From condition (C2), we know that \( \beta \) satisfies \( \|\beta_{\phi} - \beta_{\phi_{0}}\| \leq c_{0}n^{-1/2+c_{1}} \) for some positive constants \( c_{0} \) and \( c_{1} \). Then, Lemma 1 can be proved by using similar arguments of Lemma 6.1 of Xia (2006) or Theorem B in Silverman (1986).

### 1.3. Proof of Theorems 1-2

**Proof Define**

\[
\tilde{n}_{n}(\eta) = \sum_{i=1}^{n} \left[-Y_{i}\exp\left(-\alpha^{T}X_{i} - \tilde{g}(\beta_{\phi}^{T}Z_{i}, \eta)\right) + Y_{i}^{-1}\exp\left(\alpha^{T}X_{i} + \tilde{g}(\beta_{\phi}^{T}Z_{i}, \eta)\right)\right]
\]

\[
\times \left( X_{i} + \frac{\partial \tilde{g}(\beta_{\phi}^{T}Z_{i}, \eta)}{\partial \alpha} + \frac{\partial \tilde{g}(\beta_{\phi}^{T}Z_{i}, \eta)}{\partial \phi} \right).
\]

Note that \( \tilde{n}_{n}(\eta_0) = 0 \). Taylor expansion entails that

\[
- \frac{1}{\sqrt{n}} \tilde{n}_{n}(\eta_0) = \left[ 1 \frac{\partial \tilde{n}_{n}(\eta)}{\partial \eta} \right]_{\eta = \eta_0} \left[ \sqrt{n}(\tilde{\eta}_{n} - \eta_0) \right],
\]

(A.1)

where \( \phi_{\ast} \) is between \( \tilde{\phi} \) and \( \phi_{0} \).

Define \( G_{w, \eta}(u) = E \left[ \ln(Y) - \alpha^{T}X \right| w(Y - z) = u \] \( f_{\beta_{0}}(u), K_{h}'(u) = \frac{1}{h}K'(u/h). \)

Using conditions (C2)-(C4), we have

\[
E \left[ \frac{\partial}{\partial \phi} T_{n, t_{1}t_{2}}(\beta_{\phi}^{T}Z, \eta) \right] = \frac{1}{n} \sum_{i=1}^{n} E \left[ K_{h}(\beta_{\phi}^{T}Z_{i} - \beta_{\phi_{0}}^{T}z)J_{\phi}^{T} \left( \frac{Z_{i} - z}{h} \right) (\beta_{\phi}^{T}Z_{i} - \beta_{\phi_{0}}^{T}z)^{l_{1}} \right.
\]

\[
\times \left[ \ln(Y_{i}) - \alpha^{T}X \right]^{t_{2}}\right]
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} E \left[ K_{h}(\beta_{\phi}^{T}Z_{i} - \beta_{\phi_{0}}^{T}z)J_{\phi}^{T} \left( \frac{Z_{i} - z}{h} \right) l_{1}(\beta_{\phi}^{T}Z_{i} - \beta_{\phi_{0}}^{T}z)^{l_{1}-1} \right.
\]

\[
\times \left[ I \left( l_{1} \geq 1 \right) \left[ \ln(Y_{i}) - \alpha^{T}X \right]^{t_{2}} \right]
\]

\[
= - \sum_{v=0}^{2} \frac{l_{1} + v}{v!} J_{\phi}^{T} G_{i_{2}, \eta}(\beta_{\phi_{0}}^{T}z)h^{l_{1}-1+v}\mu_{K,l_{1}-1+v}I \{ l_{1} + v \geq 1 \}
\]

\[
+ \sum_{v=0}^{2} \frac{l_{1}}{v!} J_{\phi}^{T} G_{i_{2}, \eta}(\beta_{\phi_{0}}^{T}z)h^{l_{1}-1+v}\mu_{K,l_{1}-1+v}I \{ l_{1} \geq 1 \} + O(h^{l_{1}+2}),
\]

(A.2)
where $C_{1u}(u) = \frac{\partial}{\partial u} C_{1u}(u), \mu_{K,s} = \int t^s K(t)dt$, and $I\{u\}$ is the indicator function.

Similar to the proof of Theorem 3.1 in Fan and Gijbels (1996) and Lemma A.5 in Zhang et al. (2014), together with (A.2) and Lemma 1, we have

$$
\frac{\partial \hat{g}(\beta_{\phi_0}^T Z_i, \eta)}{\partial \eta}|_{\eta = \eta_0} = 
\begin{pmatrix}
-m_X(\beta_{\phi_0}^T Z_i) \\
J_{\phi_0}^T [Z_i - m_Z(\beta_{\phi_0}^T Z_i)] g'(\beta_{\phi_0}^T Z_i)
\end{pmatrix}
+ O_P\left(h^2 + \frac{(\log n)^1+\mbox{s}_0}{nh^d}\right)
$$

where $m_X(u) = E[X|\beta_{\phi_0}^T Z = u]$ and $m_Z(u) = E[Z|\beta_{\phi_0}^T Z = u]$. Moreover, using Lemma 1, and similar to the proof of Theorem 3.1 in Fan and Gijbels (1996), we also have

$$
\tilde{g}(\beta_{\phi_0}^T Z_i, \eta_0) =
\begin{pmatrix}
g(\beta_{\phi_0}^T Z_i) \\
g(\beta_{\phi_0}^T Z_i) + \frac{\mu_{K,2} h^2}{2} g''(\beta_{\phi_0}^T Z_i)
\end{pmatrix}
+ \frac{1}{nhf_{\phi_0}} \sum_{j=1}^{n} K \left(\beta_{\phi_0}^T Z_j - \beta_{\phi_0}^T Z_i\right) \ln(\epsilon_j)
+ o_P\left(h^2 + \frac{(\log n)^1+\mbox{s}_0}{nh^d}\right).
$$

Using (A.3)-(A.4) and $E(\epsilon - c^{-1}|X, Z) = 0$, as $nh^d \to 0$ and $\frac{(\log n)^1+\mbox{s}_0}{nh^d} \to 0$, Taylor expansion and the U-statistic (Serfling, 1980) entail that

$$
\frac{1}{\sqrt{n}} \hat{\eta}_n(\eta_0)
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[-\epsilon_i \exp\left(g(\beta_{\phi_0}^T Z_i) - \hat{g}(\beta_{\phi_0}^T Z_i, \eta_0)\right)\right]
+ \left[\epsilon_i^{-1} \exp\left(\hat{g}(\beta_{\phi_0}^T Z_i, \eta_0) - g(\beta_{\phi_0}^T Z_i)\right)\right]
\begin{pmatrix}
X_i + \frac{\partial \hat{g}(\beta_{\phi_0}^T Z_i, \eta_0)}{\partial \alpha_0} \\
\frac{\partial \hat{g}(\beta_{\phi_0}^T Z_i, \eta_0)}{\partial \phi_0}
\end{pmatrix}
$$

$$
= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) \left(J_{\phi_0}^T [Z_i - m_X(\beta_{\phi_0}^T Z_i)] g'(\beta_{\phi_0}^T Z_i)\right)
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(\epsilon_i + \epsilon_i^{-1})}{nhf_{\phi_0}} \sum_{j=1}^{n} K \left(\beta_{\phi_0}^T Z_j - \beta_{\phi_0}^T Z_i\right) \ln(\epsilon_j)
\times \left(J_{\phi_0}^T [Z_i - m_Z(\beta_{\phi_0}^T Z_i)] g'(\beta_{\phi_0}^T Z_i)\right)
+ o_P(1).
$$
where
\[
\mathcal{R}_{n,1} = \frac{\mu K_2 h^2}{2\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) g''(\beta_{\phi_0}^T Z_i) \left( J_{\phi_0}^T \left[ X_i - m_X(\beta_{\phi_0}^T Z_i) \right] g'(\beta_{\phi_0}^T Z_i) \right) + O_P(n^{1/2}h^4) \left( \log n \right)^{1+\alpha_0} \frac{h}{n} = O_P(n^{1/2}h^2) + O_P(n^{1/2}) \left( \log n \right)^{1+\alpha_0} \frac{h}{n} = \alpha_P(1).
\]

And also,
\[
\frac{1}{n} \frac{\partial \tilde{n}_n}{\partial \eta_0} \overset{\text{def}}{=} S_{n,1} + S_{n,2},
\]
where
\[
S_{n,1} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \epsilon_i \exp \left( g(\beta_{\phi_0}^T Z_i) - g(\beta_{\phi_0}^T Z_i, \eta_0) \right) + \epsilon_i^{-1} \exp \left( g(\beta_{\phi_0}^T Z_i, \eta_0) - g(\beta_{\phi_0}^T Z_i) \right) \right\} \left( X_i - m_X(\beta_{\phi_0}^T Z_i) \right) \left( g'(\beta_{\phi_0}^T Z_i) \right)^{\otimes 2},
\]
\[
S_{n,2} = \frac{1}{n} \sum_{i=1}^{n} \left\{ -\epsilon_i \exp \left( g(\beta_{\phi_0}^T Z_i) - g(\beta_{\phi_0}^T Z_i, \eta_0) \right) + \epsilon_i^{-1} \exp \left( g(\beta_{\phi_0}^T Z_i, \eta_0) - g(\beta_{\phi_0}^T Z_i) \right) \right\} \left( \frac{\partial^2 g(\beta_{\phi_0}^T Z_i, \eta_0)}{\partial \alpha_0^2} \right)^{\otimes 2} + O_P(\tau_{n,1}^2) = \Gamma_0 + o_P(1).
\]

Using Lemma 1 and (A.3), similar to (A.5), as \( nh^4 \to 0 \) and \( \frac{(\log n)^{1+\alpha_0}}{h} \to 0 \), we have
\[
S_{n,1} = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) \left( J_{\phi_0}^T \left[ X_i - m_X(\beta_{\phi_0}^T Z_i) \right] g'(\beta_{\phi_0}^T Z_i) \right) + O_P(\tau_{n,1}^2) = \Gamma_0 + o_P(1).
\]

Using the analysis of (A.3) and (A.5), using Lemma 1 and \( E(\epsilon - \epsilon^{-1} | X, Z) = 0 \), we have \( S_{n,2} = o_P(1) \). Together with (A.6), as \( nh^4 \to 0 \), \( \frac{(\log n)^{1+\alpha_0}}{n^{1/2}h^2} \to 0 \) and \( \eta_0 \overset{P}{\to} \eta_0 \), we
Note that and, Proof completed by using the multivariate delta-method. We omit the details.

\[
\sqrt{n} (\hat{\eta} - \eta_0) = - \left[ \frac{1}{n} \frac{\partial \tilde{g}_n (\eta)}{\partial \eta} \right]_{\eta = \eta_0} ^{-1} \frac{1}{\sqrt{n}} \tilde{g}_n (\eta_0)
\]

(A.7)

\[
\text{and,}
\]

\[
\sqrt{n} \left( \frac{\partial \hat{g}_n (u, \hat{\eta})}{\partial \eta} \bigg|_{\eta = \eta_0} \right) \rightarrow P \Gamma_0. \text{ From (A.1), (A.5) and (A.6), we have}
\]

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \varepsilon_i - \epsilon_i^{-1} \right) \left( \begin{array}{c} X_i - m_X (\beta_{\phi_\alpha}^T Z_i) \\ Z_i - m_Z (\beta_{\phi_\alpha}^T Z_i) \end{array} \right) \bigg| \begin{array}{c} \beta_{\phi_\alpha}^T Z_i \\ \beta_{\phi_\alpha}^T Z_i \end{array} \right) + o_P(1)
\]

L \rightarrow N \left( \mathbf{0}_{p+q-1}, \Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} \right).

According to (A.7), the proof of Theorem 1 has been completed. The proof of Theorem 2 is completed by using the multivariate delta-method. We omit the details.

1.4. Proof of Theorem 3

\textbf{Proof} In this section, we consider to prove Theorem 3. For any \((t_1, t_2)^T \in \mathbb{R}^2\), we define

\[
\tilde{\mathcal{A}}_n (s_1, s_2)
\]

\[
= \sum_{i=1}^{n} K_h (\beta_{\phi_\alpha}^T Z_i - u) \left\{ - Y_i \exp \left( - \alpha^T X_i - s_1 - s_2 \frac{\beta_{\phi_\alpha}^T Z_i - u}{h} \right) 
\right.
\]

\[
+ \frac{1}{Y_i} \exp \left( \frac{\alpha^T X_i + s_1 + s_2 \beta_{\phi_\alpha}^T Z_i - u}{h} \right) \left( \frac{1}{h} \right) \left( \frac{\beta_{\phi_\alpha}^T Z_i - u}{h} \right)
\]

and,

\[
\tilde{\mathcal{B}}_n (s_1, s_2)
\]

\[
= \sum_{i=1}^{n} K_h (\beta_{\phi_\alpha}^T Z_i - u) \left\{ Y_i \exp \left( - \alpha^T X_i - s_1 - s_2 \frac{\beta_{\phi_\alpha}^T Z_i - u}{h} \right) 
\right.
\]

\[
+ \frac{1}{Y_i} \exp \left( \frac{\alpha^T X_i + s_1 + s_2 \beta_{\phi_\alpha}^T Z_i - u}{h} \right) \left( \frac{1}{h} \right) \left( \frac{\beta_{\phi_\alpha}^T Z_i - u}{h} \right) \right)^{\otimes 2}
\]

Note that \(\tilde{\mathcal{A}}_n (\hat{g}_L (u, \hat{\eta}), h \hat{g}'_L (u, \hat{\eta})) = 0\), similar to (A.1), we have

\[
- \sqrt{\frac{h}{n}} \tilde{\mathcal{A}}_n (g(u), h g'(u)) = \left[ \frac{1}{n} \tilde{\mathcal{B}}_n (g(u), h g'(u)) \right] \times \left[ \sqrt{n h} \left( \hat{g}_L (u, \hat{\eta}) - g(u) \right) \right.
\]

\[
\left. \left( h \hat{g}'_L (u, \hat{\eta}) - h g'(u) \right) \right]
\]

(A.10)
where \( (g_{-}(u), hg'_{-}(u)) \) is between \( (\hat{g}_{L}(u, \bar{\eta}), hg'(u, \bar{\eta})) \) and \( (g(u), hg'(u)) \). We have
\[
\sqrt{\frac{h}{n}} \tilde{\xi}_{n}(g(u), hg'(u)) \quad (A.11)
\]
\[
= \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} \left( -\epsilon_i \exp \left( (\alpha_0 - \hat{\alpha})^T \mathbf{X}_i + g(\beta_{\phi_0}^T \mathbf{Z}_i) - g(u) \right) \right.
\]
\[
- g'(u) (\beta_{\phi}^T \mathbf{Z}_i - u) \bigg) + \epsilon_i^{-1} \exp \left( (\hat{\alpha} - \alpha_0)^T \mathbf{X}_i + g(u) + g'(u) (\hat{\beta}_{\phi}^T \mathbf{Z}_i - u) \right)
\]
\[
- g(\beta_{\phi_0}^T \mathbf{Z}_i) \bigg) K_h (\beta_{\phi}^T \mathbf{Z}_i - u) \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right)
\]
\[
= - \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \]\n\[
+ R_{n,1} + R_{n,2} + R_{n,3}.
\]

Using Taylor expansion, we have
\[
\frac{1}{\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \quad (A.12)
\]
\[
= \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) + D_{n,1} + D_{n,2}
\]
\[
+ O_P(n^{-1} h^{-2}),
\]

where
\[
D_{n,1} = \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K' \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right)
\]
\[
\times \mathbf{Z}_i^T \left( \beta_{\phi} - \beta_{\phi_0} \right) h^{-1} + O_P(n^{-1} h^{-2})
\]
\[
= O_P(n^{-1/2} h^{-1}) + O_P(n^{-1} h^{-2}) = o_P(1),
\]

and
\[
D_{n,2} = \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \left( \frac{0}{h} \right)
\]
\[
\times \mathbf{Z}_i^T \left( \beta_{\phi} - \beta_{\phi_0} \right) \quad (A.13)
\]
\[
= O_P(n^{-1/2} h^{-1}) = o_P(1).
\]

Similar to analysis of (A.12), we have
\[
R_{n,1} = \frac{1}{nh} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) K \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right) \left( \frac{1}{\beta_{\phi}^T \mathbf{Z}_i - u} \right)
\]
\[
\times \mathbf{X}_i^T \sqrt{n} (\hat{\alpha} - \alpha_0) \sqrt{n} = O_P(\sqrt{n}).
\]
Similar to (A.12), using Lemma 1 and Taylor expansion, we have

\[
\frac{1}{\sqrt{nh^2}} R_{n,2} = -\frac{1}{nh} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) K \left( \frac{\beta_{\phi_0}^T Z_i - u}{h} \right) \left( \frac{1}{\beta_{\phi_0}^T Z_i - u} \right) \\
\times \left( g(\beta_{\phi_0}^T Z_i) - g(u) - g'(u)(\beta_{\phi_0}^T Z_i - u) \right)
\]

Moreover, similar to the analysis of (A.12)-(A.14), we have

\[
\frac{1}{nh} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) K \left( \frac{\beta_{\phi_0}^T Z_i - u}{h} \right) \left( \frac{1}{\beta_{\phi_0}^T Z_i - u} \right) \\
\times \left( g'(u) Z_i^T (\beta_{\phi_0} - \beta_{\phi_n}) \right) + O_P(n^{-1}h^{-3})
\]

\[
= -\frac{g''(u)}{2} \left( \int_{\mu_{\phi_n}(u)}^u d\mu_{\phi_n}(u) E(\epsilon + \epsilon^{-1}|\beta_{\phi_0}^T Z = u) f_{\beta_{\phi_0}}(u) \mu_{K,2} \right) \]
\[
+ O_P(h^2 + \tau_{n,h}) + O_P(n^{-1}h^{-3}) + O_P(n^{-1/2}h^{-2} + n^{-1/2}h^{-1}).
\]

Moreover, similar to the analysis of (A.12)-(A.14), we have

\[
R_{n,3} = -\frac{1}{2\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{\beta_{\phi_0}^T Z_i - u}{h} \right) \left( \frac{1}{\beta_{\phi_0}^T Z_i - u} \right) \\
\times \left( (\alpha_0 - \alpha_0)^T X_i + g(\beta_{\phi_0}^T Z_i) - g'(u)(\beta_{\phi_0}^T Z_i - u) \right)^2 \\
+ O_P(h^4 + n^{-1}) = O_P(h^4 + n^{-1}) + O_P(n^{-3/2}h^{-1}) + O_P(n^{-1/2}h^3).
\]

From (A.11)-(A.14), as \( nh \to \infty \), we have

\[
\sqrt{\frac{h}{n}} G_n (g(u), h g'(u)) \quad (A.16)
\]

\[
= -\frac{1}{\sqrt{nh}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{\beta_{\phi_0}^T Z_i - u}{h} \right) \left( \frac{1}{\beta_{\phi_0}^T Z_i - u} \right) \\
- \sqrt{nh} \frac{g''(u)}{2} \left( \int_{\mu_{\phi_n}(u)}^u d\mu_{\phi_n}(u) E(\epsilon + \epsilon^{-1}|\beta_{\phi_0}^T Z = u) f_{\beta_{\phi_0}}(u) \mu_{K,2} \right) \]
\[
+ O_P(1).
\]

Similar to the analysis of (A.16), we have

\[
\frac{1}{n} \mathfrak{B}_n (g_*(u), h g'_*(u)) \quad (A.17)
\]

\[
= \begin{pmatrix}
E(\epsilon + \epsilon^{-1}|\beta_{\phi_0}^T Z = u) f_{\beta_{\phi_0}}(u) & O_P(h) \\
O_P(h) & E(\epsilon + \epsilon^{-1}|\beta_{\phi_0}^T Z = u) f_{\beta_{\phi_0}}(u) \mu_{K,2}
\end{pmatrix}
\]
\[
+ O_P(h^2 + \tau_{n,h} + n^{-1/2}).
\]
Together with (A.10), (A.16) and (A.17), we obtain that
\[
\sqrt{n\hbar} \left( \hat{g}_L(u, \eta) - g(u) - \frac{g''(u)\hbar^2\mu K \nu}{2} \right)
\]
(A.18)
\[
= \frac{1}{\sqrt{n\hbar}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) K \left( \frac{\beta_{\phi_0}^T Z_i - u}{\hbar} \right) E(\epsilon + \epsilon^{-1}|\beta_{\phi_0}^T Z = u) f_{\beta_{\phi_0}}(u) + o_P(1).
\]

Directly using (A.18), we have completed the proof of Theorem 3.

1.5. Proof of Propositions 1-2

Proof The proof of Proposition 1 is similar to Cui et al. (2011), we outline the main steps here. Similar to (A.8), we have \( \hat{g}_L(\beta_{\phi_0}^T z, \eta_0) \) and \( \hat{g}'_L(\beta_{\phi_0}^T z, \eta_0) \) satisfies
\[
\sum_{i=1}^{n} K_h(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)
\]
(A.19)
\[
\times \left\{ -Y_i \exp \left(-\alpha_0^T X_i - \hat{g}_L(\beta_{\phi_0}^T z, \eta_0) - \hat{g}'_L(\beta_{\phi_0}^T z, \eta_0)(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z) \right) 
\]
\[
+ Y_i^{-1} \exp \left(\alpha_0^T X_i + \hat{g}_L(\beta_{\phi_0}^T z, \eta_0) + \hat{g}'_L(\beta_{\phi_0}^T z, \eta_0)(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z) \right) \right\}.
\]

Define \( \hat{\epsilon}_{L,i} = Y_i \exp \left(-\alpha_0^T X_i - \hat{g}_L(\beta_{\phi_0}^T z, \eta_0) - \hat{g}'_L(\beta_{\phi_0}^T z, \eta_0)(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z) \right), i = 1, \ldots, n. \) Taking derivative with respect to \( \alpha_0 \) on both side of (A.19), we have
\[
\frac{\partial \hat{g}_L(\beta_{\phi_0}^T z, \alpha_0)}{\partial \alpha_0} \frac{1}{n} \sum_{i=1}^{n} K_h(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)(\hat{\epsilon}_{L,i} + \hat{\epsilon}_{L,i}^{-1})
\]
(A.20)
\[
= -\frac{1}{n} \sum_{i=1}^{n} K_h(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)(\hat{\epsilon}_{L,i} + \hat{\epsilon}_{L,i}^{-1}) X_i
\]
\[
= -\frac{1}{n} \sum_{i=1}^{n} K_h(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)(\hat{\epsilon}_{L,i} + \hat{\epsilon}_{L,i}^{-1}) (\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z) \frac{\partial \hat{g}'_L(\beta_{\phi_0}^T z, \alpha_0)}{\partial \alpha_0}
\]
\[
= C_{n,1} + C_{n,2}.
\]

The asymptotic expressions of \( \hat{g}_L(\beta_{\phi_0}^T z, \eta_0) \) and \( \hat{g}'_L(\beta_{\phi_0}^T z, \eta_0) \) are the same as (A.16) and (A.17), thus, we have
\[
\frac{1}{n} \sum_{i=1}^{n} K_h(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)(\hat{\epsilon}_{L,i} + \hat{\epsilon}_{L,i}^{-1})
\]
(A.21)
\[
= E(\epsilon + \epsilon^{-1}|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z) f_{\beta_{\phi_0}}(\beta_{\phi_0}^T z) + o_P \left( h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh}} \right),
\]
\[
C_{n,1} = -E((\epsilon + \epsilon^{-1}) X|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z) f_{\beta_{\phi_0}}(\beta_{\phi_0}^T z)
\]
(A.22)
\[
+ o_P \left( h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh}} \right),
\]
\[
C_{n,2} = o_P \left( h^2 + h \sqrt{\frac{(\log n)^{1+s_0}}{nh}} \right).
\]
Using (A.21)-(A.23), we have
\[ \frac{\partial g_L}{\partial \alpha_0} = \frac{\partial g_L}{\partial \beta_{\phi_0}^T} \frac{\partial \beta_{\phi_0}^T}{\partial \alpha_0} \]
\[ = -E((\epsilon + \epsilon')X|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z) + O_P \left( h^2 + \sqrt{\frac{(\log n)^{1+\gamma_0}}{nh^3}} \right). \]

Taking derivative with respect to $\phi_0$ on both side of (A.19), we have
\[ \frac{\partial g_L}{\partial \phi_0} = \frac{\partial g_L}{\partial \phi_0 n} + \frac{\partial g_L}{\partial \phi_0 C_n} \]
\[ = -\frac{1}{n} \sum_{i=1}^{n} \left[ K_i(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)(\epsilon_{L,i} + \epsilon_{L,i}^{-1}) \right] \]
\[ = -\frac{1}{n} \sum_{i=1}^{n} \left[ K_i(\beta_{\phi_0}^T Z_i - \beta_{\phi_0}^T z)(\epsilon_{L,i} + \epsilon_{L,i}^{-1}) \right] \frac{\partial g_L}{\partial \phi_0} \]
\[ = C_{n,3} + C_{n,4} + C_{n,5}. \]

Using (A.16) and (A.17), we have
\[ C_{n,3} = g'_{\phi_0}(\beta_{\phi_0}^T Z) \frac{\partial g_L}{\partial \phi_0} \left\{ zE[(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z] \right\} \]
\[ -E[(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z] + O_P \left( h^2 + \sqrt{\frac{(\log n)^{1+\gamma_0}}{nh^3}} \right). \]

And,
\[ C_{n,4} = O_P \left( h^2 + h \sqrt{\frac{(\log n)^{1+\gamma_0}}{nh^3}} \right), C_{n,5} = O_P \left( \frac{(\log n)^{1+\gamma_0}}{nh^2} \right). \]

Together with (A.21), we have
\[ \frac{\partial g_L}{\partial \phi_0} = \frac{g'_{\phi_0}(\beta_{\phi_0}^T Z)}{E(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z)} \]
\[ \frac{\partial g_L}{\partial \phi_0} \left\{ zE[(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z] \right\} \]
\[ -E[(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z] + O_P \left( h^2 + \sqrt{\frac{(\log n)^{1+\gamma_0}}{nh^3}} \right) \]
\[ = g'_{\phi_0}(\beta_{\phi_0}^T Z) \frac{\partial g_L}{\partial \phi_0} \left\{ z - \frac{E[(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z]}{E(\epsilon + \epsilon')|\beta_{\phi_0}^T Z = \beta_{\phi_0}^T z)} \right\} \]
\[ + O_P \left( h^2 + \sqrt{\frac{(\log n)^{1+\gamma_0}}{nh^3}} \right). \]

The proof of Proposition 2 is similar to the proof of Theorem 1 by directly using the asymptotic expressions in Proposition 1. We omit the details.
1.6. Proof of Theorems 4-7

**Proof** Under the null hypothesis (3.1), we have \( b = A \eta_0 \). From (3.3), we obtain that

\[
\hat{\eta}_R - \eta_0 = \hat{\eta} - \eta_0 = \left( \frac{\bar{\Omega}_n(\hat{\eta})}{n} \right)^{-1} A^T \left[ A \left( \frac{\bar{\Omega}_n(\hat{\eta})}{n} \right)^{-1} A^T \right]^{-1} (A \hat{\eta} - A \eta_0),
\]  

(A.28)

From (A.6), we have that \( n^{-1} \bar{\Omega}_n(\hat{\eta}) \overset{P}{\rightarrow} \Gamma_0 \). Together with (A.7), we have

\[
\sqrt{n} \left( \hat{\eta}_R - \eta_0 \right) = \sqrt{n} \left( \hat{\eta} - \eta_0 \right) = \left\{ I_{p+q-1} - \left[ \frac{1}{n} \bar{\Omega}_n(\hat{\eta}) \right]^{-1} A^T \left[ A \left( \frac{1}{n} \bar{\Omega}_n(\hat{\eta}) \right)^{-1} A^T \right]^{-1} A \right\}
\]

\[
\times \sqrt{n} \left( \hat{\eta} - \eta_0 \right) = \left\{ I_{p+q-1} - \Gamma_0^{-1} A^T \left( A \Gamma_0^{-1} A^T \right)^{-1} A \right\} \sqrt{n} \left( \hat{\eta} - \eta_0 \right) + o_P(1)
\]

\[
\xrightarrow{L} N \left( 0_k, \Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} A^T \right).
\]

(A.29)

If the model error \( \epsilon \) is independent of \((X, Z)\), the asymptotic variance of (A.29) reduces to \( \frac{Var(\epsilon)}{\sum_{i=1}^{N} \Omega_i(\alpha)} \). We have completed the proof of Theorem 4.

From (A.28)-(A.29), we have

\[
\sqrt{n} A (\hat{\eta}_R - \hat{\eta}) = -\sqrt{n} A (\hat{\eta} - \eta_0) \xrightarrow{L} N \left( 0_k, \Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} A^T \right).
\]

Under the null hypothesis \( \mathcal{H}_0 \) and (A.28), we have \( A \hat{\eta}_R = b \), together with (A.30), we have

\[
\left( A \Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} A^T \right)^{-1/2} \sqrt{n} (A \hat{\eta} - b) \xrightarrow{L} N (0_k, I_k).
\]

(A.31)

The continuous mapping theorem entails that

\[
n (A \hat{\eta} - b)^T \left( A \Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} A^T \right)^{-1} (A \hat{\eta} - b) \xrightarrow{L} \chi^2_k.
\]

(A.32)

Similar to (A.6), we have \( n^{-1} \bar{\Omega}_n(\hat{\eta}) \overset{P}{\rightarrow} \Gamma_0 \), \( n^{-1} \bar{\Omega}_n(\hat{\eta}) \overset{P}{\rightarrow} \Sigma_0 \), then the Slutsky’s Theorem with (A.32) entails that \( \mathcal{T}_n \xrightarrow{L} \chi^2_k \).

From (A.3), we have

\[
\frac{\partial \hat{\eta}(\beta_{\phi_0}^T Z_i, \eta)}{\partial \eta} \bigg|_{\eta = \eta_0} = \left( J_{\phi_0}^T \left[ Z_i - m_Z(\beta_{\phi_0}^T Z_i) \right] g'(\beta_{\phi_0}^T Z_i) \right) + O_P \left( h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{n h^3}} \right)
\]

\[
+ O_P (||\hat{\eta} - \eta_0||),
\]

(A.33)

Using (A.33), as \( \hat{\eta} = \eta_0 + O_P(n^{-1/2}) \), we have

\[
\hat{\Lambda} = \frac{1}{n} \sum_{i=1}^{n} \left[ \left( X_i + \frac{\partial \hat{\eta}(\beta_{\phi_0}^T Z_i, \eta)}{\partial \eta} \right) \bigg|_{\eta = \eta_0} \right] \xrightarrow{P} \Lambda_0.
\]

(A.34)
Using Lemma 1, (A.4) and Theorem 1, we have
\[
\hat{g}(\hat{\beta}_0^T Z_i, \hat{\eta}) = g(\hat{\beta}_0^T Z_i) + \frac{\mu K^2 h^2}{2} g''(\hat{\beta}_0^T Z_i) + O_P\left(\frac{(\log n)^{1+\rho}}{nh}\right) + O_P(\|\hat{\eta} - \eta_0\|).
\] (A.35)

Using the model (1.1) and (A.35), as $h \to 0$ and $\frac{(\log n)^{1+\rho}}{nh} \to 0$, we have
\[
c_{n,1} = \left\{ \frac{1}{n} \sum_{i=1}^n Y_i \exp \left(-\hat{\alpha}^T X_i - \hat{g}(\hat{\beta}_0^T Z_i, \hat{\eta})\right) \right\}^2 \xrightarrow{P} \left[E(\epsilon) + E(\epsilon^{-1})\right]^2.
\] (A.36)

And also,
\[
c_{n,2} = \frac{1}{n} \sum_{i=1}^n \left\{ -Y_i \exp \left(-\hat{\alpha}^T X_i - \hat{g}(\hat{\beta}_0^T Z_i, \hat{\eta})\right) + Y_i^{-1} \exp \left(\hat{\alpha}^T X_i + \hat{g}(\hat{\beta}_0^T Z_i, \hat{\eta})\right) \right\}^2 \xrightarrow{P} E\left[(\epsilon^{-1} - \epsilon)^2\right].
\] (A.37)

Together with (A.36)-(A.37), we have $\kappa^{-1} \xrightarrow{P} \frac{E(\epsilon + \epsilon^{-1})^2}{E(\epsilon^{-1} - \epsilon)^2}$. Using (A.32) and (A.34), we have $\tau_{id,n} \xrightarrow{L} \chi^2_k$. We have completed the proof of Theorem 5.

Under the local alternative hypothesis $H_{1,n}$, $b = A\eta_0 - n^{-1/2}c$. From (3.3), we have
\[
\eta_R - \eta_0 = \hat{\eta} - \eta_0 - \left[\hat{M}_n(\hat{\eta})\right]^{-1} A^T \left[\hat{M}_n(\hat{\eta})\right]^{-1} A^T (A\hat{\eta} - A\eta_0)
\] (A.38)
\[
- n^{-1/2} \left[\hat{M}_n(\hat{\eta})\right]^{-1} A^T \left[\hat{M}_n(\hat{\eta})\right]^{-1} A^T c
\]
\[
= \left\{ I_{p+q-1} - \Gamma_0^{-1} A^T (A\Gamma_0^{-1} A^T)^{-1} A \right\} (\hat{\eta} - \eta_0)
\]
\[
- n^{-1/2} \Gamma_0^{-1} A^T (A\Gamma_0^{-1} A^T)^{-1} c + o_P(n^{-1/2}).
\]

The asymptotic results of $\eta_R$ under $H_{1,n}$ are obtained by using (A.38). We have completed the proof of Theorem 6.

From (A.38), we have
\[
\sqrt{n} A (\eta_R - \hat{\eta}) \xrightarrow{L} N\left(-\sqrt{n} A (\hat{\eta} - \eta_0) - c, A\Gamma_0^{-1} A^T\right).
\] (A.39)

Under the local alternative hypothesis $H_{1,n}$, together with (A.39), we have
\[
\left(\Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} A^T\right)^{-1/2} \sqrt{n} A (\eta_R - \hat{\eta}) \xrightarrow{L} N\left(-\left(\Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} A^T\right)^{-1/2} c, I_h\right).
\] (A.40)
Using (A.40), the continuous mapping theorem and Slutsky’s theorem entail that $T_n \xrightarrow{L} \chi^2_k(\tau_0)$, where $\chi^2_k(\tau_0)$ is a noncentral chi-squared distribution with $k$ degrees of freedom and noncentrality parameter $\tau_0 = c^T \left( A \Sigma^{-1} \Sigma_0 \Sigma^{-1} A^T \right)^{-1} c$. The asymptotic distribution of $T_{id,n}$ under the local alternative hypothesis is obtained similarly, we omit the details. We have completed the proof of Theorem 7.

1.7. Proof of Theorem 8

**Proof Step 1.** In this step, we establish the asymptotic order of minimizer estimator $\hat{\phi}_p$.

Define

$$\mathcal{L}_p(\eta) = \sum_{i=1}^{n} \left\{ Y_i \exp \left( -\alpha^T X_i - \tilde{g} \left( \beta^T \phi Z_i, \eta \right) \right) 
+ Y_i^{-1} \exp \left( \alpha^T X_i + \tilde{g} \left( \beta^T \phi Z_i, \eta \right) \right) \right\} + n \sum_{s=1}^{p+q-1} p_c(|\eta_s|).
$$

Let

$$a_n^* = \max \left\{ \max_{1 \leq j \leq q} \{ p'_{c_j}(|\alpha_{0,j}|), \alpha_{0,j} \neq 0 \}, \max_{1 \leq j \leq p} \{ p'_{c_{q+j}}(|\phi_{0,j}|), \phi_{0,j} \neq 0 \} \right\}.$$

Let $\delta_n = n^{-1/2} + a_n^*$, $s_1 = (s_1, \ldots, s_q)^T$, $s_2 = (s_{q+1}, \ldots, s_{p+q-1})^T$, $s = (s_1, \ldots, s_{q+p-1})^T$ with $\|s\| = C_0$. Furthermore, define $\alpha(n) = \alpha_0 + \delta_n s_1$, $\phi(n) = \phi_0 + \delta_n s_2$, $\eta(n) = ((\alpha(n))^T, (\phi(n))^T)^T, \beta_{\phi(n)} = \left( \sqrt{1 - \|\phi(n)\|^2}, (\phi(n))^T \right)^T$, and

$$\mathcal{D}_{n,1} = \sum_{i=1}^{n} \left\{ Y_i \exp \left( -(\alpha(n))^T X_i - \tilde{g} \left( \beta_{\phi(n)}^T Z_i, \eta(n) \right) \right) 
+ Y_i^{-1} \exp \left( (\alpha(n))^T X_i + \tilde{g} \left( \beta_{\phi(n)}^T Z_i, \eta(n) \right) \right) \right\}$$

$$- \sum_{i=1}^{n} \left\{ Y_i \exp \left( -\alpha_0^T X_i - \tilde{g} \left( \beta_{\phi_0}^T Z_i, \eta_0 \right) \right) 
+ Y_i^{-1} \exp \left( \alpha_0^T X_i + \tilde{g} \left( \beta_{\phi_0}^T Z_i, \eta_0 \right) \right) \right\},$$

$$\mathcal{D}_{n,2} = -n \sum_{j=1}^{q_0} \{ p_{c_j}(|\alpha_{0,j} + \delta_n s_j|) - p_{c_j}(|\alpha_{0,j}|) \}$$

$$- n \sum_{j=1}^{p_0-1} \{ p_{c_{q+j}}(|\phi_{0,j} + \delta_n s_{q+j}|) - p_{c_{q+j}}(|\phi_{0,j}|) \}.$$
As \( a_n^* = O_P(n^{-1/2}) \), we have \( \delta_n = O_P(n^{-1/2}) \) and the asymptotic expression (A.41) entails that the first argument of \( \mathcal{D}_{n,1} \) is \( O_P(1)C_0 \) and dominated by the second argument of \( \frac{1}{2} \delta_n^2 C_0^2 \) in probability. Taylor expansion and Cauchy-Schwarz inequality entail that

\[
|\mathcal{D}_{n,2}| \leq n\sqrt{p_0 + q_0 \delta_n a_n^*} ||s|| + n\delta_n^2 a_n^{**} ||s||^2 \leq C_0 n\delta_n^2 \left\{ \sqrt{p_0 + q_0 + a_n^*} C_0 \right\}.
\]

where

\[
a_n^{**} = \max \left\{ \max_{1 \leq j \leq q} \{ p_{ij}^\gamma (\alpha_{0,j}), \alpha_{0,j} \neq 0 \}, \max_{1 \leq j \leq p - 1} \{ p_{ij}^\gamma (\phi_{0,j}), \phi_{0,j} \neq 0 \} \right\}.
\]

Furthermore, \( \mathcal{D}_{n,2} \) is bounded by \( n\delta_n^2 C_0^2 \) in probability. Thus, as \( a_n^{**} \) tends to 0 and \( C_0 \) sufficiently large, \( \mathcal{D}_{n,1} \) dominates \( \mathcal{D}_{n,2} \). As a consequence, for any given 0 < \( \xi < 1 \), there exists a large constant \( C_0 \) such that

\[
P\left\{ \inf_{\bar{S}} \mathcal{L}_P(\eta(n)) > \mathcal{L}_P(\eta_0) \right\} \geq 1 - \xi,
\]

where \( \bar{S} = \{ s : ||s|| = C_0 \} \). We conclude that \( \eta_P \) is \( O_P(n^{-1/2}) \).

**Step 2.** Let \( \eta_1^* \) satisfies \( ||\eta_1^* - \eta_{0,1}|| = O_P(n^{-1/2}) \). Similar to the proof of Lemma 1 in Fan and Li (2001), we can show that

\[
\mathcal{L}_P((\eta_1^*^T, 0^T)^T) = \min_{\mathcal{D}^*} \mathcal{L}_P((\eta_1^T, \eta_2^*^T)^T),
\]

(A.42)

where, \( \mathcal{D}^* = \{ ||\eta_2^*|| \leq D^* n^{-1/2} \} \) and \( D^* \) is a positive constant. We omit the details for the proof in this step.

**Step 3.** Denote that \( \eta_{P,1} \) is the penalized least squares estimator of \( \eta_1 \). In addition, we denote that \( X_1 \) consists of the first \( q_0 \) components of \( X_i \) corresponding to \( \alpha_{0,1} \), and \( Z_{1i} \) consists of the first \( p_0 \) components of \( X_i \) corresponding to \( (\beta_{0,1}, \phi_{0,1}) \). Define \( \mathcal{L}_P(\eta_1) = \mathcal{L}_P((\eta_1^T, 0^T)^T) \). Taylor expansion entails that

\[
0 = \frac{\partial \mathcal{L}_P(\eta_1)}{\partial \eta_1} \bigg|_{\eta_1 = \eta_{P,1}} \tag{A.43}
\]

\[
= \tilde{\eta}_{n} (\eta_{0,1}) + n\mathbb{R}_{\xi_1} + \left( \frac{\partial \tilde{\eta}_{n} (\eta_{0,1})}{\partial \eta_{0,1}} + n\Sigma_{\xi_1} \right) (\eta_{P,1} - \eta_{0,1}) + O_P(\gamma_n),
\]

where \( \gamma_n = n||\eta_{P,1} - \eta_{0,1}||^2 \) and

\[
\tilde{\eta}_{n} (\eta_{0,1}) = \sum_{i=1}^{n} \left[ -Y_i \exp \left( -\alpha_{0,1}^T X_{1i} - \hat{g}(\beta_{0,1}^T Z_{1i}, \eta_{0,1}) \right) \right.
\]

\[
+ Y_i^{-1} \exp \left( \alpha_{0,1}^T X_{1i} + \hat{g}(\beta_{0,1}^T Z_{1i}, \eta_{0,1}) \right) \left. \right]
\]

\[
\times \left( \frac{\partial \hat{g}(\beta_{0,1}^T Z_{1i}, \eta_{0,1})}{\partial \alpha_{0,1}} \right) \left( \frac{\partial \hat{g}(\beta_{0,1}^T Z_{1i}, \eta_{0,1})}{\partial \phi_{0,1}} \right).
\]

The asymptotic results of Theorem 8(b) has been completed by using (A.43) and the analysis of (A.1). We have completed the proof of Theorem 8.
1.8. Proof of Theorems 9-10

Proof As \( nh^6 \to 0 \) and \( \frac{(\log n)^{2+2\alpha}}{nh^6} \to 0 \), we have

\[
\mathcal{R}_n(u) = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \\
- \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) (\hat{\alpha} - \alpha_0)^T X_i I\{\delta^T S_i \leq u\} \\
- \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) \left[ \hat{\gamma}(\beta_{\phi,0}^T Z_i, \hat{\eta}) - g(\beta_{\phi,0}^T Z_i) \right] I\{\delta^T S_i \leq u\} \\
+ Op\left( n^{-1} + h^4 + \frac{(\log n)^{1+s_0}}{nh} \right).
\]

Using Theorem 1 and \( E(\epsilon^{-1}|X, Z) = 0 \), we have

\[
\mathcal{R}_{n,3}(u) = E \left[ (\epsilon + \epsilon^{-1}) I\{\delta^T S \leq u\} X^T \right] (\hat{\alpha} - \alpha_0) + op(n^{-1/2}). \tag{A.44}
\]

Using (A.33) and (A.35), we have

\[
\mathcal{R}_{n,3}(u) = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) g^\prime(\beta_{\phi,0}^T Z_i) [Z_i - m_Z(\beta_{\phi,0}^T Z_i)]^T J_{\phi_0} (\hat{\phi} - \phi_0) I\{\delta^T S \leq u\} \\
- \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) [m_X(\beta_{\phi,0}^T Z_i)]^T (\hat{\alpha} - \alpha_0) \\
+ \frac{h^2 \mu K^2}{2n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) g''(\beta_{\phi,0}^T Z_i) I\{\delta^T S \leq u\} \\
+ \frac{1}{n^2 h} \sum_{i=1}^{n} \sum_{j=1}^{n} 3 (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \left[ \beta_{\phi,0}^T Z_j - \beta_{\phi,0}^T Z_i \right] K \left( \beta_{\phi,0}^T Z_j - \beta_{\phi,0}^T Z_i \right) \ln(\epsilon_j) + op(n^{-1/2}).
\]

Together with (A.44) and (A.45), using \( U \)-statistic (Serfling; 1980), as \( nh^4 \to 0 \) and \( \frac{(\log n)^{2+2\alpha}}{nh^6} \to 0 \), we have

\[
\mathcal{R}_n(u) = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \tag{A.46} \\
- E \left[ (\epsilon + \epsilon^{-1}) I\{\delta^T S \leq u\} \right] X - m_X(\beta_{\phi,0}^T Z_i)]^T (\hat{\alpha} - \alpha_0) \\
- E \left[ (\epsilon + \epsilon^{-1}) I\{\delta^T S \leq u\} g^\prime(\beta_{\phi,0}^T Z) [Z - m_Z(\beta_{\phi,0}^T Z)]^T J_{\phi_0} \right. \\
- \left. \hat{\phi} - \phi_0 \right) \\
- \frac{1}{n} \sum_{i=1}^{n} E \left[ (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \right] \beta_{\phi,0}^T Z_i \ln(\epsilon_i) + op(n^{-1/2}).
\]

From (A.46), we have

\[
\mathcal{R}_n(u) = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} - [\Delta(u)]^T (\hat{\eta} - \eta_0) \tag{A.47} \\
- \frac{1}{n} \sum_{i=1}^{n} E \left[ (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \right] \beta_{\phi,0}^T Z_i \ln(\epsilon_i) + op(n^{-1/2}),
\]

14
where \( \Delta(u) \) is defined in Theorem 9. Together with (A.7) and (A.48), we have completed the proof of Theorem 9.

Next, we consider the local alternative hypothesis \( \mathcal{H}^*_1 \). In the following, we define \( M(X_i, Z_i) = m(X_i, Z_i) - \alpha_i^T X_i - g(\beta_{\phi_0}^T Z_i), i = 1, \ldots, n \) and \( M(X, Z) = m(X, Z) - \alpha_i^T X - g(\beta_{\phi_0}^T Z) \). Similar to the analysis of (A.2) and (A.3), using Taylor expansion for \( \ln(Y_i) \) under \( \mathcal{H}^*_1 \), we have

\[
\frac{\partial \hat{g}(\beta_{\phi_0}^T Z_i, \eta)}{\partial \eta} \bigg|_{\eta = \eta_i} = \left(J_{\phi_0}^T \left[ Z_i - m_Z(\beta_{\phi_0}^T Z_i) \right] g'(\beta_{\phi_0}^T Z_i) + n^{-1/2} J_{\phi_0}^T Q(X_i, Z_i) \right) + O_P \left(h^2 + \sqrt{\frac{(\log n)^{1+\alpha_0}}{n h^2}}\right) + O_P(||\eta_i - \eta_0||) + O_P(n^{-1}),
\]

where

\[
Q(X_i, Z_i) = \left(\frac{d}{dt} E\{\exp(M(X, Z))|\beta_{\phi_0}^T Z = t\} \bigg|_{t = \beta_{\phi_0}^T Z_i} \right) Z_i
\]

Moreover,

\[
\hat{g}(\beta_{\phi_0}^T Z_i, \eta_0) = g(\beta_{\phi_0}^T Z_i) + n^{-1/2} E\{\exp(M(X_i, Z_i))|\beta_{\phi_0}^T Z_i\} + \frac{\mu K_2 h^2}{2} g''(\beta_{\phi_0}^T Z_i)
\]

\[
+ \frac{1}{nh f_{\phi_0}^T (\beta_{\phi_0}^T Z_i)} \sum_{j=1}^n K \left( \frac{\beta_{\phi_0}^T Z_j - \beta_{\phi_0}^T Z_i}{h} \right) \ln(\epsilon_j)
\]

\[
+ o_P \left(h^2 + \sqrt{\frac{(\log n)^{1+\alpha_0}}{n h}}\right).
\]
Similar to (A.5), under $\mathcal{H}_n^*$, as $nh^4 \to 0$, we have

\begin{equation}
\frac{1}{\sqrt{n}} \tilde{\eta}_n(\eta_0) \tag{A.50}
\end{equation}

\begin{align*}
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) \left( J_{\phi_0}^T \left[ X_i - m_X(\beta_{\phi_0}^T Z_i) \right] \right) \\
&\quad \times \ln(\epsilon_i) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( (\epsilon_i + \epsilon_i^{-1}) \left( J_{\phi_0}^T \left[ X_i - m_X(\beta_{\phi_0}^T Z_i) \right] \right) \right) \left( \frac{\partial}{\partial \eta} \right)_{\eta=\eta_0} \\
&\quad - F_0 + o_P(1),
\end{align*}

where

\begin{align*}
F_0 &= E \left\{ (\epsilon + \epsilon^{-1}) \left( J_{\phi_0}^T \left[ X - m_X(\beta_{\phi_0}^T Z) \right] \right) \right. \\
&\quad \left. \times \left\{ \exp(M(X, Z)) - E \left[ \exp(M(X, Z)) \right] \beta_{\phi_0}^T Z \right\} \right\}.
\end{align*}

Similar to (A.6), under $\mathcal{H}_n^*$, we have \( \frac{1}{n} \left( \frac{\partial \tilde{\eta}_n(\eta)}{\partial \eta} \right)_{\eta=\eta_0} \xrightarrow{p} \Gamma_0 \). From (A.50), we have

\begin{equation}
\sqrt{n} (\hat{\eta} - \eta_0) = - \left[ \frac{1}{n} \left( \frac{\partial \tilde{\eta}_n(\eta)}{\partial \eta} \right)_{\eta=\eta_0} \right]^{-1} \frac{1}{\sqrt{n}} \tilde{\eta}_n(\eta_0) \tag{A.51}
\end{equation}

\[ \xrightarrow{L} N \left( \Gamma_0^{-1} F_0, \Gamma_0^{-1} \Sigma_0 \Gamma_0^{-1} \right). \]

Next, we consider the asymptotic expression of $R_n(u)$ under the local alternative hypothesis.

\begin{equation}
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \tag{A.52}
\end{equation}

\begin{align*}
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} + H_{n,1} + H_{n,2} + H_{n,3} \\
&\quad + O_P \left( n^{1/2} h^4 + \frac{(\log n)^{1+s_0}}{\sqrt{n} h} + n^{-1/2} \right).
\end{align*}
Using Taylor expansion and (A.51), we have

\[
H_{n,1} = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \mathbf{X}_i^T (\mathbf{\alpha} - \mathbf{\alpha}_0) \tag{A.53}
\]

\[
- \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \left( \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \hat{\eta}) - \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \eta_0) \right)
\]

\[
= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \left( \mathbf{X}_i + \frac{\partial \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \eta_0)}{\partial \mathbf{\alpha}_0} \right)^T (\mathbf{\alpha} - \mathbf{\alpha}_0)
\]

\[
- \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \left( \frac{\partial \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \eta_0)}{\partial \phi_0} \right)^T (\phi - \phi_0)
\]

\[+ O_P(n^{-1/2})
\]

\[
= [\Delta(u)]^T \Gamma_0^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) \left( J_{\phi_0} \left[ X_i - m_X(\beta_{\phi_0}^T \mathbf{Z}_i) \right] \right)
\]

\[
+ [\Delta(u)]^T \Gamma_0^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E \left\{ \left( J_{\phi_0} \left[ X_i - m_X(\beta_{\phi_0}^T \mathbf{Z}_i) \right] g'(\beta_{\phi_0}^T \mathbf{Z}_i) \right) \times (\epsilon_i + \epsilon_i^{-1}) \right\} \ln(\epsilon_i)
\]

\[= [\Delta(u)]^T \Gamma_0^{-1} f_0 + o_P(1).
\]

Define \(\varpi(u) = E \left( (\epsilon + \epsilon^{-1}) \exp(M(X, Z)) I\{\delta^T S \leq u\} \right)\). As \(nh^2 \to 0\), we have

\[
H_{n,2} = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \left( \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \eta_0) - g(\beta_{\phi_i}^T \mathbf{Z}_i) \right)
\]

\[
= -\frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} E\{\exp(M(X_i, Z_i))|\beta_{\phi_i}^T \mathbf{Z}_i\}
\]

\[
- \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E \left\{ (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \right\} \ln(\epsilon_i) + o_P(1)
\]

\[
H_{n,3} = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) \exp(M(X_i, Z_i)) I\{\delta^T S_i \leq u\}
\]

\[
\times \left\{ \exp \left( (\alpha_0 - \hat{\alpha})^T X_i + g(\beta_{\phi_i}^T \mathbf{Z}_i) - \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \hat{\eta}) \right)
\]

\[
+ \exp \left( -(\alpha_0 - \hat{\alpha})^T X_i - g(\beta_{\phi_i}^T \mathbf{Z}_i) + \hat{g}(\beta_{\phi_i}^T \mathbf{Z}_i, \hat{\eta}) \right) \right\}
\]

\[\xrightarrow{P} \varpi(u).
\]

17
Under \( H_{1n} \), using (A.52)-(A.53), as \( nh^4 \to 0 \), we have

\[
\sqrt{n} R_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} \tag{A.54}
\]

\[-[\Delta(u)]^T \Gamma_0^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \epsilon_i - \epsilon_i^{-1} \right) \left( \begin{array}{c} X_i - m_X(\beta_{\phi_0}^T Z_i) \\ J_{\phi_0}^T Z_i - m_Z(\beta_{\phi_0}^T Z_i) \end{array} \right) g'(\beta_{\phi_0}^T Z_i) \]

\[+ [\Delta(u)]^T \Gamma_0^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E \left\{ \left( \begin{array}{c} X_i - m_X(\beta_{\phi_0}^T Z_i) \\ J_{\phi_0}^T Z_i - m_Z(\beta_{\phi_0}^T Z_i) \end{array} \right) g'(\beta_{\phi_0}^T Z_i) \right\} \times (\epsilon_i + \epsilon_i^{-1}) \right| \beta_{\phi_0}^T Z_i \right) \ln(\epsilon_i) \]

\[-E \left\{ (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} E\{\exp(M(X_i, Z_i))|\beta_{\phi_0}^T Z_i\} \right\} \]

\[-\frac{1}{\sqrt{n}} \sum_{i=1}^{n} E \left\{ (\epsilon_i + \epsilon_i^{-1}) I\{\delta^T S_i \leq u\} |\beta_{\phi_0}^T Z_i\right\} \ln(\epsilon_i) \]

\[+ \varpi(u) - [\Delta(u)]^T \Gamma_0^{-1} F_0 + o_P(1). \]

We have completed the proof of Theorem 10.

REFERENCES


18