Supplementary Materials for "Estimation and Hypothesis Test for Partial Linear Single-Index Multiplicative Models" [1]

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1. Appendix

1.1. Assumptions

We begin this section by listing the conditions needed in the proofs of our asymptotic results.

(C1) $E[X_s^4] < \infty$ for $s = 1, \dots, q$, $E[Z_r^4] < \infty$ for $r = 1, \dots, p$, and the matrices $\mathbf{\Gamma}_0$, $\mathbf{\Lambda}_0$ used in Theorem 1 and $\mathbf{\Gamma}_{L,0}$ in Proposition 1 are all positive definite and finite. Moreover, $E[|\ln(Y)|^r] < \infty$ for some $r > 3$.

(C2) $E(\boldsymbol{X}|\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z} = u)$, $E(\boldsymbol{Z}|\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z} = u)$ and the density function $f_{\boldsymbol{\beta}_{\boldsymbol{\phi}}}(u)$ of the random variable $\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}$ are twice continuously differentiable with respect to u. Their second-order derivatives are uniformly Lipschitz continuous on $\mathcal{C} = \{u = \boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z} : \boldsymbol{z} \in \mathcal{Z} \subset \mathbb{R}^p, \boldsymbol{\beta}_{\boldsymbol{\phi}} \in \mathfrak{S}_{n,\boldsymbol{\beta}_{\boldsymbol{\phi}}}\}$, where $\mathcal{Z}$ is a compact support set, and $\mathfrak{S}_{n,\boldsymbol{\beta}_{\boldsymbol{\phi}}} = \{\boldsymbol{\beta}_{\boldsymbol{\phi}} \in \mathcal{B}_{\boldsymbol{\phi}} : \|\boldsymbol{\beta}_{\boldsymbol{\phi}} - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}\| \le c_0 n^{-1/2+c_1}\}$ for some positive constant c_0 and $c_1 \in [0, 0.05)$. Moreover, $g(u)$ has two bounded and continuous derivatives on $u \in \mathcal{C}$ and $\inf_{u\in\mathcal{C}} f_{\boldsymbol{\beta}_{\boldsymbol{\phi}}}(u) > 0$.

(C3) The kernel function $K(\cdot)$ is a symmetric bounded density function supported on $[-A, A]$, satisfying a Lipschitz condition. $K(\cdot)$ also has second-order continuous bounded derivatives, satisfying $K^{(j)}(\pm A) = 0$, $j = 0, 1, 2$, and $\int s^2 K(s)ds \neq 0$.

(C4) As $n \to \infty$, the bandwidth h satisfies $\frac{(\log n)^{2+2s_0}}{nh^2} \to 0$ for some $s_0 > 0$, and $nh^4 \to 0$.

(C5) For all ζ_j $j = 1, \dots, p+q-1$, $\zeta_j \to 0$, $\sqrt{n}\zeta_j \to \infty$ as $n \to \infty$, moreover, $\operatorname{liminf}_{n\to\infty}\operatorname{liminf}_{u\to 0^+} p'_{\zeta_j}(u)/\zeta_j > 0$.

1.2. A Technical Lemma

Lemma 1 Suppose $E(W|\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z} = u) = m(u)$ *and its derivatives up to second order are bounded for all* $\boldsymbol{\beta}_{\boldsymbol{\phi}} \in \mathfrak{S}_{n,\boldsymbol{\beta}_{\boldsymbol{\phi}}}$, *where* $\mathfrak{S}_{n,\boldsymbol{\beta}}$ *is defined in condition (C2), and that* $E|W|^r$ *exists for some* $r > 3$. *Let* $(\boldsymbol{Z}_i, W_i)$, $i = 1, 2, \dots n$ *be an independent and identically distributed (i.i.d.) sample from* $(\boldsymbol{Z}, W)$. *Let* $\tau_{n,h} = \left\{\frac{(\log n)^{1+s_0}}{nh}\right\}^{1/2} + h^3$ *for some* $s_0 > 0$. *Given*

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that $h = n^{-j}$ for some $0 < j < 1$, if conditions (C1)-(C4) hold, we have,

$$\sup_{u\in\mathcal{C}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}-u)\left(\frac{\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{Z}-u}{h}\right)^{d}W_i - f_{\boldsymbol{\beta}_{\boldsymbol{\phi}}}(u)m(u)\mu_{K,d}\right.$$
$$\left.-\{f_{\boldsymbol{\beta}_{\boldsymbol{\phi}}}(u)m(u)\}'\mu_{K,d+1}h-\frac{1}{2}\{f_{\boldsymbol{\beta}_{\boldsymbol{\phi}}}(u)m(u)\}''\mu_{K,d+2}h^2\right| = O(\tau_{n,h}), a.s.,$$

where $\mu_{K,d} = \int K(v)v^d dv$, $d = 0, 1, 2$.

Proof. From condition (C2), we know that $\boldsymbol{\beta}$ satisfies $\|\boldsymbol{\beta}_{\boldsymbol{\phi}} - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}\| \le c_0 n^{-1/2+c_1}$ for some positive constants c_0 and c_1. Then, Lemma 1 can be proved by using similar arguments of Lemma 6.1 of Xia (2006) or Theorem B in Silverman (1986).

1.3. Proof of Theorems 1-2

Proof Define

$$\begin{aligned}&\widetilde{\mathfrak{R}}_n(\boldsymbol{\eta})\\&=\sum_{i=1}^{n}\Big[-Y_i\exp\Big(-\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}_i-\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta})\Big)+Y_i^{-1}\exp\Big(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}_i+\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta})\Big)\Big]\\&\quad\times\begin{pmatrix}\boldsymbol{X}_i+\dfrac{\partial\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta})}{\partial\boldsymbol{\alpha}}\\\dfrac{\partial\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta})}{\partial\boldsymbol{\phi}}\end{pmatrix}.\end{aligned}$$

Note that $\widetilde{\mathfrak{R}}_n(\hat{\boldsymbol{\eta}}) = \mathbf{0}$. Taylor expansion entails that

$$-\frac{1}{\sqrt{n}}\widetilde{\mathfrak{R}}_n(\boldsymbol{\eta}_0)=\left[\frac{1}{n}\frac{\partial\widetilde{\mathfrak{R}}_n(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*}\right]\left[\sqrt{n}\,(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta}_0)\right], \tag{A.1}$$

where $\boldsymbol{\phi}_*$ is between $\hat{\boldsymbol{\phi}}$ and $\boldsymbol{\phi}_0$.

Define $G_{w,\boldsymbol{\eta}}(u) = E\left[[\ln(Y) - \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}]^w\{\boldsymbol{Z} - \boldsymbol{z}\}|\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z} = u\right] f_{\boldsymbol{\beta}_{\boldsymbol{\phi}}}(u)$, $K_h'(u) = \frac{1}{h}K'(u/h)$. Using conditions (C2)-(C4), we have

$$\begin{aligned}&E\left[\frac{\partial}{\partial\boldsymbol{\phi}}T_{n,l_1l_2}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z},\boldsymbol{\eta})\right] &\text{(A.2)}\\&=\frac{1}{n}\sum_{i=1}^{n}E\Big[K_h'(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{z})J_{\boldsymbol{\phi}}^{\mathrm{T}}\left(\frac{\boldsymbol{Z}_i-\boldsymbol{z}}{h}\right)(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{z})^{l_1}\\&\qquad\times[\ln(Y_i)-\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}]^{l_2}\Big]\\&\quad+\frac{1}{n}\sum_{i=1}^{n}E\Big[K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{z})J_{\boldsymbol{\phi}}^{\mathrm{T}}(\boldsymbol{Z}_i-\boldsymbol{z})\,l_1(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{z})^{l_1-1}\\&\qquad\times I\{l_1\ge1\}[\ln(Y_i)-\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}]^{l_2}\Big]\\&=-\sum_{v=0}^{2}\frac{l_1+v}{v!}J_{\boldsymbol{\phi}}^{\mathrm{T}}G_{l_2,\boldsymbol{\eta}}^{(v)}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{z})h^{l_1-1+v}\mu_{K,l_1-1+v}I\{l_1+v\ge1\}\\&\quad+\sum_{v=0}^{2}\frac{l_1}{v!}J_{\boldsymbol{\phi}}^{\mathrm{T}}G_{l_2,\boldsymbol{\eta}}^{(v)}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{z})h^{l_1-1+v}\mu_{K,l_1-1+v}I\{l_1\ge1\}+O(h^{l_1+2}),\end{aligned}$$

where $G_{l_2,\boldsymbol{\eta}}^{(v)}(u) = \frac{\partial^v}{\partial u^v} G_{l_2,\boldsymbol{\eta}}^{(v)}(u)$, $\mu_{K,s} = \int t^s K(t)dt$, and $I\{u\}$ is the indicator function. Similar to the proof of Theorem 3.1 in Fan and Gijbels (1996) and Lemma A.5 in Zhang et al. (2014), together with (A.2) and Lemma 1, we have

$$
\begin{aligned}
&\frac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*} \qquad \text{(A.3)}\\
&= \begin{pmatrix} -\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix} + O_P\left(h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right) \\
&\quad + O_P(\|\boldsymbol{\eta}_* - \boldsymbol{\eta}_0\|),
\end{aligned}
$$

where $\boldsymbol{m}_{\boldsymbol{X}}(u) = E[\boldsymbol{X}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z} = u]$ and $\boldsymbol{m}_{\boldsymbol{Z}}(u) = E[\boldsymbol{Z}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z} = u]$. Moreover, using Lemma 1, and similar to the proof of Theorem 3.1 in Fan and Gijbels (1996), we also have

$$
\begin{aligned}
\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0) &= g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) + \frac{\mu_{K,2} h^2}{2} g''(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \qquad \text{(A.4)}\\
&\quad + \frac{1}{nh f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)} \sum_{j=1}^{n} K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_j - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i}{h}\right) \ln(\epsilon_j) \\
&\quad + o_P\left(h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right).
\end{aligned}
$$

Using (A.3)-(A.4) and $E(\epsilon - \epsilon^{-1}|\boldsymbol{X}, \boldsymbol{Z}) = 0$, as $nh^4 \to 0$ and $\frac{(\log n)^{2+2s_0}}{nh^2} \to 0$, Taylor expansion and the U-statistic (Serfling; 1980) entail that

$$
\begin{aligned}
&\frac{1}{\sqrt{n}} \widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_0) \qquad \text{(A.5)}\\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Bigg\{ \left[-\epsilon_i \exp\left(g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) - \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)\right)\right] \\
&\qquad + \left[\epsilon_i^{-1} \exp\left(\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0) - g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)\right)\right] \Bigg\} \begin{pmatrix} \boldsymbol{X}_i + \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\alpha}_0} \\ \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\phi}_0} \end{pmatrix} \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(\epsilon_i + \epsilon_i^{-1})}{nh f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)} \sum_{j=1}^{n} K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_j - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i}{h}\right) \ln(\epsilon_j) \\
&\qquad \times \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix} + \mathcal{R}_{n,1}, \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E\left\{(\epsilon_i + \epsilon_i^{-1}) \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix} \Bigg| \boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i \right\} \ln(\epsilon_i) \\
&\quad + \mathcal{R}_{n,1} + o_P(1),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{R}_{n,1} \\
&= \frac{\mu_{K,2}h^2}{2\sqrt{n}} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) g''(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix} \\
&\quad + O_P(n^{1/2}) O_P(h^4 + \frac{(\log n)^{1+s_0}}{nh}) \\
&= O_P(n^{1/2}h^2) + O_P(n^{1/2}) O_P\left(h^4 + \frac{(\log n)^{1+s_0}}{nh} \right) = o_P(1).
\end{aligned}
$$

And also,

$$
\frac{1}{n} \frac{\partial \widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_0)}{\partial \boldsymbol{\eta}_0} \stackrel{\mathrm{def}}{=} \mathcal{S}_{n,1} + \mathcal{S}_{n,2},
$$

where

$$
\begin{aligned}
\mathcal{S}_{n,1} &= \frac{1}{n} \sum_{i=1}^{n} \Bigg\{ \epsilon_i \exp\left(g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) - \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0) \right) \\
&\quad + \epsilon_i^{-1} \exp\left(\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0) - g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \right) \Bigg\} \begin{pmatrix} \boldsymbol{X}_i + \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\alpha}_0} \\ \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\phi}_0} \end{pmatrix}^{\otimes 2},
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathcal{S}_{n,2} = \frac{1}{n} \sum_{i=1}^{n} \Bigg\{ & \left[-\epsilon_i \exp\left(g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) - \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0) \right) \right] \\
& + \left[\epsilon_i^{-1} \exp\left(\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0) - g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \right) \right] \Bigg\} \\
& \times \begin{pmatrix} \dfrac{\partial^2 \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\alpha}_0 \partial \boldsymbol{\alpha}_0^{\mathrm{T}}}, \dfrac{\partial^2 \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\alpha}_0 \partial \boldsymbol{\phi}_0^{\mathrm{T}}} \\ \dfrac{\partial^2 \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\phi}_0 \partial \boldsymbol{\alpha}_0^{\mathrm{T}}}, \dfrac{\partial^2 \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i, \boldsymbol{\eta}_0)}{\partial \boldsymbol{\phi}_0 \partial \boldsymbol{\phi}_0^{\mathrm{T}}} \end{pmatrix}.
\end{aligned}
$$

Using Lemma 1 and (A.3), similar to (A.5), as $nh^4 \to 0$ and $\frac{(\log n)^{2+2s_0}}{nh^2} \to 0$, we have

$$
\begin{aligned}
\mathcal{S}_{n,1} &= \frac{1}{n} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix}^{\otimes 2} \qquad \text{(A.6)} \\
&+ \frac{\mu_{K,2}h^2}{2n} \sum_{i=1}^{n} (\epsilon_i^{-1} - \epsilon_i) g''(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \boldsymbol{Z}_i) \end{pmatrix}^{\otimes 2} \\
&\quad + O_P(\tau_{n,h}) = \boldsymbol{\Gamma}_0 + o_P(1).
\end{aligned}
$$

Similar to the analysis of (A.3) and (A.5), using Lemma 1 and $E(\epsilon - \epsilon^{-1} | \boldsymbol{X}, \boldsymbol{Z}) = 0$, we have $\mathcal{S}_{n,2} = o_P(1)$. Together with (A.6), as $nh^4 \to 0$, $\frac{(\log n)^{2+2s_0}}{nh^2} \to 0$ and $\boldsymbol{\eta}_* \xrightarrow{P} \boldsymbol{\eta}_0$, we have $\frac{1}{n}\frac{\partial\widetilde{\mathfrak{N}}_n(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*} \xrightarrow{P} \boldsymbol{\Gamma}_0$. From (A.1), (A.5) and (A.6), we have

$$\begin{aligned}
&\sqrt{n}\left(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta}_0\right) = -\left[\frac{1}{n}\frac{\partial\widetilde{\mathfrak{N}}_n\left(\boldsymbol{\eta}\right)}{\partial\boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*}\right]^{-1}\frac{1}{\sqrt{n}}\widetilde{\mathfrak{N}}_n\left(\boldsymbol{\eta}_0\right) \qquad\qquad (A.7)\\
&= \boldsymbol{\Gamma}_0^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i-\epsilon_i^{-1})\begin{pmatrix} \boldsymbol{X}_i-\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i-\boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}+o_P(1)\\
&-\boldsymbol{\Gamma}_0^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\left\{(\epsilon_i+\epsilon_i^{-1})\begin{pmatrix} \boldsymbol{X}_i-\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i-\boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\Bigg|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right\}\\
&\qquad\times\ln(\epsilon_i)\\
&\xrightarrow{L} N\left(\mathbf{0}_{p+q-1},\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\right).
\end{aligned}$$

According to (A.7), the proof of Theorem 1 has been completed. The proof of Theorem 2 is completed by using the multivariate delta-method. We omit the details.

1.4. Proof of Theorem 3

Proof In this section, we consider to prove Theorem 3. For any $(t_1,t_2)^{\mathrm{T}}\in\mathbb{R}^2$, we define

$$\begin{aligned}
&\widetilde{\mathfrak{A}}_n(s_1,s_2) \qquad\qquad (A.8)\\
&=\sum_{i=1}^{n}K_h(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u)\Bigg\{-Y_i\exp\Big(-\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i-s_1-s_2\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\Big)\\
&\qquad+Y_i^{-1}\exp\Big(\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i+s_1+s_2\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\Big)\Bigg\}\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\end{pmatrix},
\end{aligned}$$

and,

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_n(s_1,s_2) \qquad\qquad (A.9)\\
&=\sum_{i=1}^{n}K_h(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u)\Bigg\{Y_i\exp\Big(-\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i-s_1-s_2\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\Big)\\
&\qquad+Y_i^{-1}\exp\Big(\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i+s_1+s_2\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\Big)\Bigg\}\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\end{pmatrix}^{\otimes 2}.
\end{aligned}$$

Note that $\widetilde{\mathfrak{A}}_n(\hat{g}_L(u,\hat{\boldsymbol{\eta}}),h\widehat{g'}_L(u,\hat{\boldsymbol{\eta}}))=\mathbf{0}$, similar to (A.1), we have

$$\begin{aligned}
-\sqrt{\frac{h}{n}}\widetilde{\mathfrak{A}}_n\left(g(u),hg'(u)\right) &= \left[\frac{1}{n}\mathfrak{B}_n\left(g_*(u),h{g'}_*(u)\right)\right] \qquad\qquad (A.10)\\
&\quad\times\left[\sqrt{nh}\begin{pmatrix}\hat{g}_L(u,\hat{\boldsymbol{\eta}})-g(u)\\ h\widehat{g'}(u,\hat{\boldsymbol{\eta}})-hg'(u)\end{pmatrix}\right],
\end{aligned}$$

where $(g_*(u), hg'_*(u))$ is between $(\hat{g}_L(u,\hat{\boldsymbol{\eta}}), h\widehat{g'}(u,\hat{\boldsymbol{\eta}}))$ and $(g(u), hg'(u))$. We have

$$
\begin{aligned}
&\sqrt{\frac{h}{n}}\widetilde{\mathfrak{A}}_n\left(g(u), hg'(u)\right) \qquad\qquad (A.11)\\
&= \sqrt{\frac{h}{n}}\sum_{i=1}^{n}\Bigg\{-\epsilon_i \exp\Big((\boldsymbol{\alpha}_0-\hat{\boldsymbol{\alpha}})^{\mathrm{T}}\boldsymbol{X}_i + g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) - g(u)\\
&\qquad\qquad -g'(u)(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u)\Big)\\
&\qquad +\epsilon_i^{-1}\exp\Big((\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0)^{\mathrm{T}}\boldsymbol{X}_i + g(u) + g'(u)(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u)\\
&\qquad\qquad -g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\Big)\Bigg\}K_h(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u)\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\end{pmatrix}\\
&= -\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})K\left(\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\right)\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\end{pmatrix}\\
&\quad + R_{n,1} + R_{n,2} + R_{n,3}.
\end{aligned}
$$

Using Taylor expansion, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})K\left(\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\right)\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\end{pmatrix} \qquad\qquad (A.12)\\
&= \frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\right)\begin{pmatrix}1\\ \dfrac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\end{pmatrix} + D_{n,1} + D_{n,2}\\
&\quad + O_P(n^{-1}h^{-2}),
\end{aligned}
$$

where

$$
\begin{aligned}
D_{n,1} &= \frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})K'\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\right)\begin{pmatrix}1\\ \dfrac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\end{pmatrix}\\
&\qquad \times \boldsymbol{Z}_i^{\mathrm{T}}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}} - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}\right)h^{-1} + O_P(n^{-1}h^{-2})\\
&= O_P(n^{-1/2}h^{-1}) + O_P(n^{-1}h^{-2}) = o_P(1),
\end{aligned}
$$

and

$$
\begin{aligned}
D_{n,2} &= \frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\right)\begin{pmatrix}0\\ \dfrac{\boldsymbol{Z}_i^{\mathrm{T}}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}} - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}\right)}{h}\end{pmatrix}\\
&= O_P(n^{-1/2}h^{-1}) = o_P(1).
\end{aligned}
$$

Similar to analysis of (A.12), we have

$$
\begin{aligned}
R_{n,1} &= \frac{1}{nh}\sum_{i=1}^{n}(\epsilon_i + \epsilon_i^{-1})K\left(\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\right)\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i - u}{h}\end{pmatrix} \qquad\qquad (A.13)\\
&\qquad \times\boldsymbol{X}_i^{\mathrm{T}}\sqrt{n}\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0\right)\sqrt{h} = O_P(\sqrt{h}).
\end{aligned}
$$

Similar to (A.12), using Lemma 1 and Taylor expansion, we have

$$\begin{aligned}
&\frac{1}{\sqrt{nh^5}}R_{n,2} \qquad\qquad (A.14)\\
&= -\frac{1}{nh}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\right)\begin{pmatrix}1\\ \dfrac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\end{pmatrix}\\
&\qquad\times\frac{\left(g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)-g(u)-g'(u)(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u)\right)}{h^2}\\
&+\frac{1}{nh}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\right)\begin{pmatrix}1\\ \dfrac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\end{pmatrix}\\
&\qquad\times\frac{g'(u)\boldsymbol{Z}_i^{\mathrm{T}}(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}-\boldsymbol{\beta}_{\boldsymbol{\phi}_0})}{h^2}+O_P(n^{-1}h^{-3})\\
&= -\frac{g''(u)}{2}\begin{pmatrix}E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u)\mu_{K,2}\\ h\dfrac{d}{du}\left(f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u)E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)\right)\int u^4K(u)du\end{pmatrix}\\
&\quad+O_P(h^2+\tau_{n,h})+O_P(n^{-1}h^{-3})+O_P(n^{-1/2}h^{-2}+n^{-1/2}h^{-1}).
\end{aligned}$$

Moreover, similar to the analysis of (A.12)-(A.14), we have

$$\begin{aligned}
R_{n,3} &= -\frac{1}{2\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i-\epsilon_i^{-1})K\left(\frac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\right)\begin{pmatrix}1\\ \dfrac{\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\end{pmatrix} \qquad (A.15)\\
&\qquad\times\left((\boldsymbol{\alpha}_0-\hat{\boldsymbol{\alpha}})^{\mathrm{T}}\boldsymbol{X}_i+g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)-g(u)-g'(u)(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i-u)\right)^2\\
&\quad+o_P(h^4+n^{-1})=O_P(h^4+n^{-1})+O_P(n^{-3/2}h^{-1})+O_P(n^{-1/2}h^3).
\end{aligned}$$

From (A.11)-(A.14), as $nh\to\infty$, we have

$$\begin{aligned}
&\sqrt{\frac{h}{n}}\widetilde{\mathfrak{A}}_n\left(g(u),hg'(u)\right) \qquad\qquad (A.16)\\
&= -\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(\epsilon_i-\epsilon_i^{-1})K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\right)\begin{pmatrix}1\\ \dfrac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\end{pmatrix}\\
&\quad-\sqrt{nh^5}\frac{g''(u)}{2}\begin{pmatrix}E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u)\mu_{K,2}\\ h\dfrac{d}{du}\left(f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u)E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)\right)\int u^4K(u)du\end{pmatrix}\\
&\quad+o_P(1).
\end{aligned}$$

Similar to the analysis of (A.16), we have

$$\begin{aligned}
&\frac{1}{n}\mathfrak{B}_n\left(g_*(u),hg'_*(u)\right) \qquad\qquad (A.17)\\
&=\begin{pmatrix}E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u) & O_P(h)\\ O_P(h) & E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u)\mu_{K,2}\end{pmatrix}\\
&\quad+O_P(h^2+\tau_{n,h}+n^{-1/2}).
\end{aligned}$$

Together with (A.10), (A.16) and (A.17), we obtain that

$$\sqrt{nh}\left(\hat{g}_L(u,\hat{\boldsymbol{\eta}})-g(u)-\frac{g''(u)h^2\mu_{K,2}}{2}\right) \tag{A.18}$$
$$=\frac{\frac{1}{\sqrt{nh}}\sum_{i=1}^n(\epsilon_i-\epsilon_i^{-1})K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-u}{h}\right)}{E(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=u)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(u)}+o_P(1).$$

Directly using (A.18), we have completed the proof of Theorem 3.

1.5. Proof of Propositions 1-2

Proof The proof of Proposition 1 is similar to Cui et al. (2011), we outline the main steps here. Similar to (A.8), we have $\hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)$ and $\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)$ satisfies

$$\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}) \tag{A.19}$$
$$\times\Big\{-Y_i\exp\Big(-\boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X}_i-\hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)-\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\Big)$$
$$+Y_i^{-1}\exp\Big(\boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X}_i+\hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)+\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\Big)\Big\}.$$

Define $\hat{\epsilon}_{L,i}=Y_i\exp\Big(-\boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X}_i-\hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)-\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\Big)$, $i=1,\ldots,n$. Taking derivative with respect to $\boldsymbol{\alpha}_0$ on both side of (A.19), we have

$$\frac{\partial\hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\alpha}_0)}{\partial\boldsymbol{\alpha}_0}\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1}) \tag{A.20}$$
$$=-\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1})\boldsymbol{X}_i$$
$$-\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1})(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\frac{\partial\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\alpha}_0)}{\partial\boldsymbol{\alpha}_0}$$
$$=C_{n,1}+C_{n,2}.$$

The asymptotic expressions of $\hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)$ and $\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)$ are the same as (A.16) and (A.17), thus, we have

$$\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1}) \tag{A.21}$$
$$=E\big(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\big)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})+O_P\left(h^2+\sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right),$$
$$C_{n,1}=-E\big((\epsilon+\epsilon^{-1})\boldsymbol{X}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\big)f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}) \tag{A.22}$$
$$+O_P\left(h^2+\sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right), \tag{A.23}$$
$$C_{n,2}=O_P\left(h^2+h\sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right).$$

Using (A.21)-(A.23), we have

$$\begin{aligned}
&\frac{\partial \hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\alpha}_0)}{\partial \boldsymbol{\alpha}_0} \qquad\qquad (A.24)\\
&= -\frac{E\big((\epsilon+\epsilon^{-1})\boldsymbol{X}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\big)}{E\big(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\big)} + O_P\left(h^2+\sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right).
\end{aligned}$$

Taking derivative with respect to $\boldsymbol{\phi}_0$ on both side of (A.19), we have

$$\begin{aligned}
&\frac{\partial \hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\alpha}_0)}{\partial \boldsymbol{\phi}_0}\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1}) \qquad\qquad (A.25)\\
&= -\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1})\widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\eta}_0)\boldsymbol{J}_{\boldsymbol{\phi}}^{\mathrm{T}}(\boldsymbol{Z}_i-\boldsymbol{z})\\
&\quad -\frac{1}{n}\sum_{i=1}^n K_h(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}+\hat{\epsilon}_{L,i}^{-1})(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\frac{\partial \widehat{g'}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\alpha}_0)}{\partial \boldsymbol{\phi}_0}\\
&\quad -\frac{1}{nh}\sum_{i=1}^n K_h'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i-\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})(\hat{\epsilon}_{L,i}-\hat{\epsilon}_{L,i}^{-1})\boldsymbol{J}_{\boldsymbol{\phi}}^{\mathrm{T}}(\boldsymbol{Z}_i-\boldsymbol{z})\\
&= C_{n,3}+C_{n,4}+C_{n,5}.
\end{aligned}$$

Using (A.16) and (A.17), we have

$$\begin{aligned}
C_{n,3} &= g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\boldsymbol{J}_{\boldsymbol{\phi}}^{\mathrm{T}}\Big\{\boldsymbol{z}E\left[(\epsilon+\epsilon^{-1})|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\right] \qquad\qquad (A.26)\\
&\quad -E\left[(\epsilon+\epsilon^{-1})\boldsymbol{Z}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\right]\Big\} + O_P\left(h^2+\sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right).
\end{aligned}$$

And,

$$C_{n,4} = O_P\left(h^2+h\sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right), C_{n,5} = O_P\left(\sqrt{\frac{(\log n)^{1+s_0}}{nh^2}}\right).$$

Together with (A.21), we have

$$\begin{aligned}
&\frac{\partial \hat{g}_L(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z},\boldsymbol{\phi}_0)}{\partial \boldsymbol{\alpha}_0} \qquad\qquad (A.27)\\
&= \frac{g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})}{E\big(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\big)}\boldsymbol{J}_{\boldsymbol{\phi}}^{\mathrm{T}}\Big\{\boldsymbol{z}E\left[(\epsilon+\epsilon^{-1})|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\right]\\
&\quad - E\left[(\epsilon+\epsilon^{-1})\boldsymbol{Z}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\right]\Big\} + O_P\left(h^2+\sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right)\\
&= g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z})\boldsymbol{J}_{\boldsymbol{\phi}}^{\mathrm{T}}\left\{\boldsymbol{z}-\frac{E\left[(\epsilon+\epsilon^{-1})\boldsymbol{Z}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\right]}{E\big(\epsilon+\epsilon^{-1}|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{z}\big)}\right\}\\
&\quad + O_P\left(h^2+\sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right).
\end{aligned}$$

The proof of Proposition 2 is similar to the proof of Theorem 1 by directly using the asymptotic expressions in Proposition 1. We omit the details.

1.6. Proof of Theorems 4-7

Proof Under the null hypothesis (3.1), we have $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{\eta}_0$. From (3.3), we obtain that

$$\begin{aligned} &\hat{\boldsymbol{\eta}}_{\mathrm{R}} - \boldsymbol{\eta}_0 \\ &= \hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0 - \left[\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right]^{-1} \boldsymbol{A}^{\mathrm{T}} \left[\boldsymbol{A}\left\{\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right\}^{-1} \boldsymbol{A}^{\mathrm{T}}\right]^{-1} (\boldsymbol{A}\hat{\boldsymbol{\eta}} - \boldsymbol{A}\boldsymbol{\eta}_0), \end{aligned} \tag{A.28}$$

From (A.6), we have that $n^{-1}\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}}) \xrightarrow{P} \boldsymbol{\Gamma}_0$. Together with (A.7), we have

$$\begin{aligned} &\sqrt{n}\,(\hat{\boldsymbol{\eta}}_{\mathrm{R}} - \boldsymbol{\eta}_0) \\ &= \left\{\boldsymbol{I}_{p+q-1} - \left[\frac{1}{n}\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right]^{-1} \boldsymbol{A}^{\mathrm{T}} \left[\boldsymbol{A}\left\{\frac{1}{n}\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right\}^{-1} \boldsymbol{A}^{\mathrm{T}}\right]^{-1} \boldsymbol{A}\right\} \\ &\quad \times \sqrt{n}\,(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \\ &= \left\{\boldsymbol{I}_{p+q-1} - \boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}\right\}\sqrt{n}\,(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) + o_P(1) \\ &\xrightarrow{L} N\left(\mathbf{0}_{p+q-1}, \boldsymbol{\Omega}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Omega}_0^{\mathrm{T}}\right). \end{aligned} \tag{A.29}$$

If the model error ϵ is independent of $(\boldsymbol{X}, \boldsymbol{Z})$, the asymptotic variance of (A.29) reduces to $\frac{E[(\epsilon-\epsilon^{-1})^2]}{[E(\epsilon+\epsilon^{-1})]^2}\boldsymbol{\Pi}_0\boldsymbol{\Lambda}_0^{-1}\boldsymbol{\Pi}_0^{\mathrm{T}}$. We have completed the proof of Theorem 4.

From (A.28)-(A.29), we have

$$\begin{aligned} \sqrt{n}\boldsymbol{A}\,(\hat{\boldsymbol{\eta}}_{\mathrm{R}} - \hat{\boldsymbol{\eta}}) &= -\sqrt{n}\boldsymbol{A}\,(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \\ &\xrightarrow{L} N\left(\mathbf{0}_k, \boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right). \end{aligned} \tag{A.30}$$

Under the null hypothesis $\mathcal{H}_0$ and (A.28), we have $\boldsymbol{A}\hat{\boldsymbol{\eta}}_{\mathrm{R}} = \boldsymbol{b}$, together with (A.30), we have

$$\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1/2}\sqrt{n}\,(\boldsymbol{A}\hat{\boldsymbol{\eta}} - \boldsymbol{b}) \xrightarrow{L} N\left(\mathbf{0}_k, \boldsymbol{I}_k\right). \tag{A.31}$$

The continuous mapping theorem entails that

$$n\,(\boldsymbol{A}\hat{\boldsymbol{\eta}} - \boldsymbol{b})^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}(\boldsymbol{A}\hat{\boldsymbol{\eta}} - \boldsymbol{b}) \xrightarrow{L} \chi_k^2. \tag{A.32}$$

Similar to (A.6), we have $n^{-1}\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}}) \xrightarrow{P} \boldsymbol{\Gamma}_0$, $n^{-1}\widetilde{\mathfrak{S}}_n(\hat{\boldsymbol{\eta}}) \xrightarrow{P} \boldsymbol{\Sigma}_0$, then the Slutsky's Theorem with (A.32) entails that $\mathcal{T}_n \xrightarrow{L} \chi_k^2$.

From (A.3), we have

$$\begin{aligned} &\frac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}} \\ &= \begin{pmatrix} -\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \end{pmatrix} + O_P\left(h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right) \\ &\quad + O_P(\|\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0\|), \end{aligned} \tag{A.33}$$

Using (A.33), as $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}_0 + O_P(n^{-1/2})$, we have

$$\widehat{\boldsymbol{\Lambda}} = \frac{1}{n}\sum_{i=1}^{n}\left[\begin{pmatrix} \boldsymbol{X}_i + \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta})}{\partial \boldsymbol{\alpha}} \\ \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta})}{\partial \boldsymbol{\phi}} \end{pmatrix}^{\otimes 2}\Bigg|_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}}\right] \xrightarrow{P} \boldsymbol{\Lambda}_0. \tag{A.34}$$

Using Lemma 1, (A.4) and Theorem 1, we have

$$\hat{g}(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}}) = g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)+\frac{\mu_{K,2}h^2}{2}g''(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) +O_P\left(\sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right)+O_P(\|\hat{\boldsymbol{\eta}}-\boldsymbol{\eta}_0\|). \tag{A.35}$$

Using the model (1.1) and (A.35), as $h\to 0$ and $\frac{(\log n)^{1+s_0}}{nh}\to 0$, we have

$$\begin{aligned} c_{n,1} &= \left\{\frac{1}{n}\sum_{i=1}^n Y_i\exp\left(-\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i-\hat{g}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}}\right)\right)\right. \\ &\quad \left.+\frac{1}{n}\sum_{i=1}^n Y_i^{-1}\exp\left(\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i+\hat{g}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}}\right)\right)\right\}^2 \\ &\xrightarrow{P} \left[E(\epsilon)+E(\epsilon^{-1})\right]^2. \end{aligned} \tag{A.36}$$

And also,

$$\begin{aligned} c_{n,2} &= \frac{1}{n}\sum_{i=1}^n\left\{-Y_i\exp\left(-\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i-\hat{g}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}}\right)\right)\right. \\ &\quad \left.+Y_i^{-1}\exp\left(\hat{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{X}_i+\hat{g}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}}\right)\right)\right\}^2 \\ &\xrightarrow{P} E\left[\left(\epsilon^{-1}-\epsilon\right)^2\right]. \end{aligned} \tag{A.37}$$

Together with (A.36)-(A.37), we have $\hat{\boldsymbol{\kappa}}^{-1}\xrightarrow{P}\frac{[E(\epsilon+\epsilon^{-1})]^2}{E(\epsilon-\epsilon^{-1})^2}$. Using (A.32) and (A.34), we have $\mathcal{T}_{\mathrm{id},n}\xrightarrow{L}\chi_k^2$. We have completed the proof of Theorem 5.

Under the local alternative hypothesis $\mathcal{H}_{1n}$, $\boldsymbol{b}=\boldsymbol{A}\boldsymbol{\eta}_0-n^{-1/2}\boldsymbol{c}$. From (3.3), we have

$$\begin{aligned} \hat{\boldsymbol{\eta}}_{\mathrm{R}}-\boldsymbol{\eta}_0 &= \hat{\boldsymbol{\eta}}-\boldsymbol{\eta}_0 \\ &\quad -\left[\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right]^{-1}\boldsymbol{A}^{\mathrm{T}}\left[\boldsymbol{A}\left\{\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right\}^{-1}\boldsymbol{A}^{\mathrm{T}}\right]^{-1}(\boldsymbol{A}\hat{\boldsymbol{\eta}}-\boldsymbol{A}\boldsymbol{\eta}_0) \\ &\quad -n^{-1/2}\left[\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right]^{-1}\boldsymbol{A}^{\mathrm{T}}\left[\boldsymbol{A}\left\{\widetilde{\mathfrak{M}}_n(\hat{\boldsymbol{\eta}})\right\}^{-1}\boldsymbol{A}^{\mathrm{T}}\right]^{-1}\boldsymbol{c} \\ &= \left\{\boldsymbol{I}_{p+q-1}-\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}\right\}(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta}_0) \\ &\quad -n^{-1/2}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{c}+o_P(n^{-1/2}). \end{aligned} \tag{A.38}$$

The asymptotic results of $\hat{\boldsymbol{\eta}}_{\mathrm{R}}$ under $\mathcal{H}_{1n}$ are obtained by using (A.38). We have completed the proof of Theorem 6.

From (A.38), we have

$$\begin{aligned} \sqrt{n}\boldsymbol{A}\left(\hat{\boldsymbol{\eta}}_{\mathrm{R}}-\hat{\boldsymbol{\eta}}\right) &= -\sqrt{n}\boldsymbol{A}\left(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta}_0\right)-\boldsymbol{c} \\ &\xrightarrow{L} N\left(-\boldsymbol{c},\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right). \end{aligned} \tag{A.39}$$

Under the local alternative hypothesis $\mathcal{H}_{1n}$, together with (A.39), we have

$$\begin{aligned} &\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1/2}\sqrt{n}\boldsymbol{A}\left(\hat{\boldsymbol{\eta}}_{\mathrm{R}}-\hat{\boldsymbol{\eta}}\right) \\ &\xrightarrow{L} N\left(-\left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1/2}\boldsymbol{c},\boldsymbol{I}_k\right). \end{aligned} \tag{A.40}$$

Using (A.40), the continuous mapping theorem and Slutsky's theorem entail that $\mathcal{T}_n \xrightarrow{L} \chi_k^2(\tau_0)$, where $\chi_k^2(\tau_0)$ is a noncentral chi-squared distribution with k degrees of freedom and noncentrality parameter $\tau_0 = \boldsymbol{c}^{\mathrm{T}} \left(\boldsymbol{A}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\boldsymbol{A}^{\mathrm{T}} \right)^{-1} \boldsymbol{c}$. The asymptotic distribution of $\mathcal{T}_{\mathrm{id},n}$ under the local alternative hypothesis is obtained similarly, we omit the details. We have completed the proof of Theorem 7.

1.7. Proof of Theorem 8

Proof Step 1. In this step, we establish the asymptotic order of minimizer estimator $\hat{\boldsymbol{\phi}}_P$. Define

$$\begin{aligned}
\mathcal{L}_P(\boldsymbol{\eta}) &= \sum_{i=1}^{n} \Big\{ Y_i \exp\left(-\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}_i - \hat{g}\left(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta} \right) \right) \\
&\quad + Y_i^{-1} \exp\left(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}_i + \hat{g}\left(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta} \right) \right) \Big\} + n \sum_{s=1}^{p+q-1} p_{\zeta_s}(|\eta_s|).
\end{aligned}$$

Let

$$a_n^* = \max\left\{ \max_{1\le j\le q} \{p'_{\zeta_j}(|\alpha_{0,j}|), \alpha_{0,j} \neq 0\}, \max_{1\le j\le p-1} \{p'_{\zeta_{q+j}}(|\phi_{0,j}|), \phi_{0,j} \neq 0\} \right\}.$$

Let $\delta_n = n^{-1/2} + a_n^*$, $\boldsymbol{s}_1 = (s_1, \ldots, s_q)^{\mathrm{T}}$, $\boldsymbol{s}_2 = (s_{q+1}, \ldots, s_{p+q-1})^{\mathrm{T}}$, $\boldsymbol{s} = (s_1, \ldots, s_{q+p-1})^{\mathrm{T}}$ with $\|\boldsymbol{s}\| = C_0$. Furthermore, define $\boldsymbol{\alpha}(n) = \boldsymbol{\alpha}_0 + \delta_n \boldsymbol{s}_1$, $\boldsymbol{\phi}(n) = \boldsymbol{\phi}_0 + \delta_n \boldsymbol{s}_2$, $\boldsymbol{\eta}(n) = ((\boldsymbol{\alpha}(n))^{\mathrm{T}}, (\boldsymbol{\phi}(n))^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{\beta}_{\boldsymbol{\phi}(n)} = \left(\sqrt{1 - \|\boldsymbol{\phi}(n)\|^2}, (\boldsymbol{\phi}(n))^{\mathrm{T}} \right)^{\mathrm{T}}$, and

$$\begin{aligned}
\mathcal{D}_{n,1} &= \sum_{i=1}^{n} \Big\{ Y_i \exp\left(-(\boldsymbol{\alpha}(n))^{\mathrm{T}}\boldsymbol{X}_i - \hat{g}\left(\boldsymbol{\beta}_{\boldsymbol{\phi}(n)}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta}(n) \right) \right) \\
&\quad + Y_i^{-1} \exp\left((\boldsymbol{\alpha}(n))^{\mathrm{T}}\boldsymbol{X}_i + \hat{g}\left(\boldsymbol{\beta}_{\boldsymbol{\phi}(n)}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta}(n) \right) \right) \Big\} \\
&\quad - \sum_{i=1}^{n} \Big\{ Y_i \exp\left(-\boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X}_i - \hat{g}\left(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta}_0 \right) \right) \\
&\quad + Y_i^{-1} \exp\left(\boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X}_i + \hat{g}\left(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta}_0 \right) \right) \Big\}, \\
\mathcal{D}_{n,2} &= -n \sum_{j=1}^{q_0} \{p_{\zeta_j}(|\alpha_{0,j} + \delta_n s_j|) - p_{\zeta_j}(|\alpha_{0,j}|)\} \\
&\quad - n \sum_{j=1}^{p_0-1} \{p_{\zeta_{q+j}}(|\phi_{0,j} + \delta_n s_{q+j}|) - p_{\zeta_{q+j}}(|\phi_{0,j}|)\}.
\end{aligned}$$

Using (A.4), (A.33) and Taylor expansion, we have

$$\begin{aligned}
\mathcal{D}_{n,1} &= -\delta_n \sum_{i=1}^{n} (\epsilon_i - \epsilon_i^{-1}) \boldsymbol{s}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \end{pmatrix} \qquad \text{(A.41)} \\
&\quad + \frac{\delta_n^2}{2} \sum_{i=1}^{n} (\epsilon_i + \epsilon_i^{-1}) \boldsymbol{s}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}} \left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \end{pmatrix}^{\otimes 2} \boldsymbol{s} \\
&\quad + O_P(\sqrt{nh^4})\|\boldsymbol{s}\|.
\end{aligned}$$

As $a_n^* = O_P(n^{-1/2})$, we have $\delta_n = O_P(n^{-1/2})$ and the asymptotic expression (A.41) entails that the first argument of $\mathcal{D}_{n,1}$ is $O_P(1)C_0$ and dominated by the second argument of $\frac{n}{2}\delta_n^2 C_0^2$ in probability. Taylor expansion and Cauchy-Schawz inequality entail that

$$|\mathcal{D}_{n,2}| \leq n\sqrt{p_0+q_0}\delta_n a_n^* \|\boldsymbol{s}\| + n\delta_n^2 a_n^{**}\|\boldsymbol{s}\|^2 \leq C_0 n\delta_n^2\big\{\sqrt{p_0+q_0} + a_n^{**}C_0\big\}.$$

where

$$a_n^{**} = \max\left\{\max_{1\leq j\leq q}\{p''_{\zeta_j}(|\alpha_{0,j}|), \alpha_{0,j}\neq 0\}, \max_{1\leq j\leq p-1}\{p''_{\zeta_{q+j}}(|\phi_{0,j}|), \phi_{0,j}\neq 0\}\right\}.$$

Furthermore, $\mathcal{D}_{n,2}$ is bounded by $n\delta_n^2 C_0^2$ in probability. Thus, as a_n^{**} tends to 0 and C_0 sufficiently large, $\mathcal{D}_{n,1}$ dominates $\mathcal{D}_{n,2}$. As a consequence, for any given $0 < \xi < 1$, there exists a large constant C_0 such that

$$P\left\{\inf_{\mathcal{S}} \mathcal{L}_P(\boldsymbol{\eta}(n)) > \mathcal{L}_P(\boldsymbol{\eta}_0)\right\} \geq 1-\xi,$$

where $\mathcal{S} = \{\boldsymbol{s} : \|\boldsymbol{s}\| = C_0\}$. We conclude that $\hat{\boldsymbol{\eta}}_P$ is $O_P(n^{-1/2})$.

Step 2. Let $\boldsymbol{\eta}_{\mathbf{1}}^*$ satisfies $\|\boldsymbol{\eta}_{\mathbf{1}}^* - \boldsymbol{\eta}_{0,\mathbf{1}}\| = O_P(n^{-1/2})$. Similar to the proof of Lemma 1 in Fan and Li (2001), we can show that

$$\mathcal{L}_P\left((\boldsymbol{\eta}_{\mathbf{1}}^{*\mathrm{T}}, \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}\right) = \min_{\mathcal{D}^*} \mathcal{L}_P\left((\boldsymbol{\eta}_{\mathbf{1}}^{*\mathrm{T}}, \boldsymbol{\eta}_{\mathbf{2}}^{*\mathrm{T}})^{\mathrm{T}}\right), \tag{A.42}$$

where, $\mathcal{D}^* = \{\|\boldsymbol{\eta}_{\mathbf{2}}^*\| \leq D^* n^{-1/2}\}$ and D^* is a positive constant. We omit the details for the proof in this step.

Step 3. Denote that $\hat{\boldsymbol{\eta}}_{P,\mathbf{1}}$ is the penalized least squares estimator of $\boldsymbol{\eta}_{\mathbf{1}}$. In addition, we denote that $\boldsymbol{X}_{\mathbf{1}i}$ consists of the first q_0 components of $\boldsymbol{X}_i$ corresponding to $\boldsymbol{\alpha}_{0,\mathbf{1}}$, and $\boldsymbol{Z}_{\mathbf{1}i}$ consists of the first p_0 components of $\boldsymbol{X}_i$ corresponding to $(\beta_{0,1}, \boldsymbol{\phi}_{0,\mathbf{1}})$. Define $\mathcal{L}_P^*(\boldsymbol{\eta}_{\mathbf{1}}) = \mathcal{L}_P\left((\boldsymbol{\eta}_{\mathbf{1}}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}\right)$. Taylor expansion entails that

$$\begin{aligned} \mathbf{0} &= \left.\frac{\partial \mathcal{L}_P^*(\boldsymbol{\eta}_{\mathbf{1}})}{\partial \boldsymbol{\eta}_{\mathbf{1}}}\right|_{\boldsymbol{\eta}_{\mathbf{1}} = \hat{\boldsymbol{\eta}}_{P,\mathbf{1}}} \qquad\qquad (A.43) \\ &= \widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_{0,\mathbf{1}}) + n\mathcal{R}_{\boldsymbol{\zeta}_{\mathbf{1}}} + \left(\frac{\partial \widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_{0,\mathbf{1}})}{\partial \boldsymbol{\eta}_{0,\mathbf{1}}} + n\boldsymbol{\Sigma}_{\boldsymbol{\zeta}_{\mathbf{1}}}\right)(\hat{\boldsymbol{\eta}}_{P,\mathbf{1}} - \boldsymbol{\eta}_{0,\mathbf{1}}) + O_P(\gamma_n), \end{aligned}$$

where $\gamma_n = n\|\hat{\boldsymbol{\eta}}_{P,\mathbf{1}} - \boldsymbol{\eta}_{0,\mathbf{1}}\|^2$ and

$$\begin{aligned} \widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_{0,\mathbf{1}}) &= \sum_{i=1}^n \Bigg[-Y_i \exp\left(-\boldsymbol{\alpha}_{0,\mathbf{1}}^{\mathrm{T}}\boldsymbol{X}_{\mathbf{1}i} - \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_{0,\mathbf{1}}}^{\mathrm{T}}\boldsymbol{Z}_{\mathbf{1}i}, \boldsymbol{\eta}_{0,\mathbf{1}})\right) \\ &\qquad + Y_i^{-1}\exp\left(\boldsymbol{\alpha}_{0,\mathbf{1}}^{\mathrm{T}}\boldsymbol{X}_{\mathbf{1}i} + \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_{0,\mathbf{1}}}^{\mathrm{T}}\boldsymbol{Z}_{\mathbf{1}i}, \boldsymbol{\eta}_{0,\mathbf{1}})\right)\Bigg] \\ &\qquad \times \begin{pmatrix} \boldsymbol{X}_{\mathbf{1}i} + \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_{0,\mathbf{1}}}^{\mathrm{T}}\boldsymbol{Z}_{\mathbf{1}i}, \boldsymbol{\eta}_{0,\mathbf{1}})}{\partial \boldsymbol{\alpha}} \\ \dfrac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_{0,\mathbf{1}}}^{\mathrm{T}}\boldsymbol{Z}_{\mathbf{1}i}, \boldsymbol{\eta}_{0,\mathbf{1}})}{\partial \boldsymbol{\phi}_{0,\mathbf{1}}} \end{pmatrix}. \end{aligned}$$

The asymptotic results of Theorem 8(b) has been completed by using (A.43) and the analysis of (A.1). We have completed the proof of Theorem 8.

1.8. Proof of Theorems 9-10

Proof As $nh^8 \to 0$ and $\frac{(\log n)^{2+2s_0}}{nh^2} \to 0$, we have

$$\begin{aligned} \mathcal{R}_n(u) &= \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} \\ &\quad -\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i + \epsilon_i^{-1})\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0\right)^{\mathrm{T}}\boldsymbol{X}_i I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} \\ &\quad -\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i + \epsilon_i^{-1})\left[\hat{g}\left(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i, \hat{\boldsymbol{\eta}}\right) - g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right] I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} \\ &\quad +O_P\left(n^{-1} + h^4 + \frac{(\log n)^{1+s_0}}{nh}\right) \\ &= \mathcal{R}_{n,1}^*(u) - \mathcal{R}_{n,2}^*(u) - \mathcal{R}_{n,3}^*(u) + o_P(n^{-1/2}). \end{aligned}$$

Using Theorem 1 and $E(\epsilon - \epsilon^{-1}|\boldsymbol{X}, \boldsymbol{Z}) = 0$, we have

$$\mathcal{R}_{n,2}^*(u) = E\left[(\epsilon + \epsilon^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S} \le u\}\boldsymbol{X}^{\mathrm{T}}\right](\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) + o_P(n^{-1/2}). \tag{A.44}$$

Using (A.33) and (A.35), we have

$$\begin{aligned} &\mathcal{R}_{n,3}^*(u) \qquad\qquad (\text{A.45}) \\ &= \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i + \epsilon_i^{-1})g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)[\boldsymbol{Z}_i - m_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)]^{\mathrm{T}}\boldsymbol{J}_{\boldsymbol{\phi}_0}\left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}_0\right) I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S} \le u\} \\ &\quad - \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i + \epsilon_i^{-1})[m_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)]^{\mathrm{T}}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) \\ &\quad + \frac{h^2\mu_{K,2}}{2n}\sum_{i=1}^{n}(\epsilon_i + \epsilon_i^{-1})g''(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S} \le u\} \\ &\quad + \frac{1}{n^2h}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(\epsilon_i + \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\}}{f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)}K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_j - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i}{h}\right)\ln(\epsilon_j) + o_P(n^{-1/2}). \end{aligned}$$

Together with (A.44) and (A.45), using U-statistic (Serfling; 1980), as $nh^4 \to 0$ and $\frac{(\log n)^{2+2s_0}}{nh^2} \to 0$, we have

$$\begin{aligned} \mathcal{R}_n(u) &= \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} \qquad (\text{A.46}) \\ &\quad -E\left[(\epsilon + \epsilon^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S} \le u\}\{\boldsymbol{X} - m_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\}^{\mathrm{T}}\right](\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) \\ &\quad -E\left[(\epsilon + \epsilon^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S} \le u\}g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z})\{\boldsymbol{Z} - m_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z})\}^{\mathrm{T}}\right]\boldsymbol{J}_{\boldsymbol{\phi}_0} \\ &\qquad \times\left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}_0\right) \\ &\quad -\frac{1}{n}\sum_{i=1}^{n}E\left[(\epsilon_i + \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\}\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right]\ln(\epsilon_i) + o_P(n^{-1/2}). \end{aligned}$$

From (A.46), we have

$$\begin{aligned} \mathcal{R}_n(u) &= \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} - [\boldsymbol{\Delta}(u)]^{\mathrm{T}}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \qquad (\text{A.47}) \\ &\quad -\frac{1}{n}\sum_{i=1}^{n}E\left[(\epsilon_i + \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\}\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right]\ln(\epsilon_i) + o_P(n^{-1/2}), \end{aligned}$$

where $\boldsymbol{\Delta}(u)$ is defined in Theorem 9. Together with (A.7) and (A.48), we have completed the proof of Theorem 9.

Next, we consider the local alternative hypothesis $\mathcal{H}_{1n}^*$. In the following, we define $M(\boldsymbol{X}_i, \boldsymbol{Z}_i) = m(\boldsymbol{X}_i, \boldsymbol{Z}_i) - \boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X}_i - g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i), i = 1, \ldots, n$ and $M(\boldsymbol{X}, \boldsymbol{Z}) = m(\boldsymbol{X}, \boldsymbol{Z}) - \boldsymbol{\alpha}_0^{\mathrm{T}}\boldsymbol{X} - g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z})$. Similar to the analysis of (A.2) and (A.3), using Taylor expansion for $\ln(Y_i)$ under $\mathcal{H}_{1n}^*$, we have

$$
\begin{aligned}
&\frac{\partial \hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*} \qquad \text{(A.48)}\\
&= \begin{pmatrix} -\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right] g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) + n^{-1/2}\boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{X}_i, \boldsymbol{Z}_i) \end{pmatrix}\\
&\quad + O_P\left(h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh^3}}\right) + O_P(\|\boldsymbol{\eta}_* - \boldsymbol{\eta}_0\|) + O_P(n^{-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
&\boldsymbol{Q}(\boldsymbol{X}_i, \boldsymbol{Z}_i)\\
&= \left(\frac{d}{dt}E\{\exp(M(\boldsymbol{X}, \boldsymbol{Z}))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z} = t\}\Big|_{t=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i}\right)\boldsymbol{Z}_i\\
&- \left(\frac{d}{dt}E\{\boldsymbol{Z}\exp(M(\boldsymbol{X}, \boldsymbol{Z}))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z} = t\}\Big|_{t=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i}\right)\\
&+\Bigg\{\left[\boldsymbol{Z}_i E\{\exp(M(\boldsymbol{X}_i, \boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\} - E\{\boldsymbol{Z}_i\exp(M(\boldsymbol{X}_i, \boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\}\right]\\
&\quad - \left[\{\boldsymbol{Z}_i - m_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\}E\{\exp(M(\boldsymbol{X}_i, \boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\}\right]\Bigg\}\frac{f'_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)}{f_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)}\\
&+E\{\exp(M(\boldsymbol{X}_i, \boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\}m'_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i, \boldsymbol{\eta}_0) \qquad \text{(A.49)}\\
&= g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) + n^{-1/2}E\{\exp(M(\boldsymbol{X}_i, \boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\} + \frac{\mu_{K,2}h^2}{2}g''(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\\
&\quad + \frac{\mu_{K,2}h^2 n^{-1/2}}{2}\left[E\{\exp(M(\boldsymbol{X}, \boldsymbol{Z}))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z} = t\}\right]''|_{t=\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i}\\
&\quad + \frac{1}{nhf_{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)}\sum_{j=1}^n K\left(\frac{\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_j - \boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i}{h}\right)\ln(\epsilon_j)\\
&\quad + o_P\left(h^2 + \sqrt{\frac{(\log n)^{1+s_0}}{nh}}\right).
\end{aligned}
$$

Similar to (A.5), under $\mathcal{H}_{1n}^*$, as $nh^4 \to 0$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_0) \qquad\qquad (A.50)\\
&= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})\begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\left\{(\epsilon_i + \epsilon_i^{-1})\begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right\}\\
&\qquad\times\ln(\epsilon_i)\\
&-\digamma_0 + o_P(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\digamma_0 \;=\; & E\Bigg\{(\epsilon + \epsilon^{-1})\begin{pmatrix} \boldsymbol{X} - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z} - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z})\end{pmatrix}\\
&\times\left\{\exp(\boldsymbol{M}(\boldsymbol{X},\boldsymbol{Z})) - E\left[\exp(\boldsymbol{M}(\boldsymbol{X},\boldsymbol{Z}))\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}\right]\right\}\Bigg\}.
\end{aligned}
$$

Similar to (A.6), under $\mathcal{H}_{1n}^*$, we have $\frac{1}{n}\frac{\partial\widetilde{\mathfrak{N}}_n(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*} \xrightarrow{P} \boldsymbol{\Gamma}_0$. From (A.50), we have

$$
\begin{aligned}
&\sqrt{n}\left(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0\right) = -\left[\frac{1}{n}\frac{\partial\widetilde{\mathfrak{N}}_n(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}}\Big|_{\boldsymbol{\eta}=\boldsymbol{\eta}_*}\right]^{-1}\frac{1}{\sqrt{n}}\widetilde{\mathfrak{N}}_n(\boldsymbol{\eta}_0) \qquad (A.51)\\
&\xrightarrow{L} N\left(\boldsymbol{\Gamma}_0^{-1}\digamma_0, \boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}_0^{-1}\right).
\end{aligned}
$$

Next, we consider the asymptotic expression of $\mathcal{R}_n(u)$ under the local alternative hypothesis.

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\hat{\epsilon}_i - \hat{\epsilon}_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} \qquad (A.52)\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} + H_{n,1} + H_{n,2} + H_{n,3}\\
&\quad + O_P\left(n^{1/2}h^4 + \frac{(\log n)^{1+s_0}}{\sqrt{n}h} + n^{-1/2}\right).
\end{aligned}
$$

Using Taylor expansion and (A.51), we have

$$
\begin{aligned}
H_{n,1} &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\boldsymbol{X}_i^{\mathrm{T}}(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0) \qquad \text{(A.53)}\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\left(\hat{g}(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}})-\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta}_0)\right)\\
&= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\left(\boldsymbol{X}_i+\frac{\partial\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta}_0)}{\partial\boldsymbol{\alpha}_0}\right)^{\mathrm{T}}(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0)\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\left(\frac{\partial\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta}_0)}{\partial\boldsymbol{\phi}_0}\right)^{\mathrm{T}}(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}_0)\\
&\quad + O_P(n^{-1/2})\\
&= -[\boldsymbol{\Delta}(u)]^{\mathrm{T}}\boldsymbol{\Gamma}_0^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i-\epsilon_i^{-1})\begin{pmatrix}\boldsymbol{X}_i-\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i-\boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\\
&\quad +[\boldsymbol{\Delta}(u)]^{\mathrm{T}}\boldsymbol{\Gamma}_0^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\left\{\begin{pmatrix}\boldsymbol{X}_i-\boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i-\boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\right.\\
&\qquad\qquad \left.\times(\epsilon_i+\epsilon_i^{-1})\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right\}\ln(\epsilon_i)\\
&\quad -[\boldsymbol{\Delta}(u)]^{\mathrm{T}}\boldsymbol{\Gamma}_0^{-1}F_0+o_P(1).
\end{aligned}
$$

Define $\boldsymbol{\varpi}(u)=E\left((\epsilon+\epsilon^{-1})\exp(M(\boldsymbol{X},\boldsymbol{Z}))I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}\le u\}\right)$. As $nh^4\to 0$, we have

$$
\begin{aligned}
H_{n,2} &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\left(\hat{g}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i,\boldsymbol{\eta}_0)-g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right)\\
&= -\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}E\{\exp(M(\boldsymbol{X}_i,\boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\}\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\left\{(\epsilon_i+\epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right\}\ln(\epsilon_i)+o_P(1)\\
H_{n,3} &= \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i+\epsilon_i^{-1})\exp(\boldsymbol{M}(\boldsymbol{X}_i,\boldsymbol{Z}_i))I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i\le u\}\\
&\quad\times\left\{\exp\left((\boldsymbol{\alpha}_0-\hat{\boldsymbol{\alpha}})^{\mathrm{T}}\boldsymbol{X}_i+g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)-\hat{g}(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}})\right)\right.\\
&\qquad \left.+\exp\left(-(\boldsymbol{\alpha}_0-\hat{\boldsymbol{\alpha}})^{\mathrm{T}}\boldsymbol{X}_i-g(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)+\hat{g}(\hat{\boldsymbol{\beta}}_{\hat{\boldsymbol{\phi}}}^{\mathrm{T}}\boldsymbol{Z}_i,\hat{\boldsymbol{\eta}})\right)\right\}\\
&\xrightarrow{P} \boldsymbol{\varpi}(u).
\end{aligned}
$$

Under $\mathcal{H}_{1n}^*$, using (A.52)-(A.53), as $nh^4 \to 0$, we have

$$
\begin{aligned}
\sqrt{n}\mathcal{R}_n(u) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\} \qquad \text{(A.54)}\\
&-[\boldsymbol{\Delta}(u)]^{\mathrm{T}}\boldsymbol{\Gamma}_0^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i - \epsilon_i^{-1})\begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\\
&+[\boldsymbol{\Delta}(u)]^{\mathrm{T}}\boldsymbol{\Gamma}_0^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\left\{\begin{pmatrix} \boldsymbol{X}_i - \boldsymbol{m}_{\boldsymbol{X}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i) \\ \boldsymbol{J}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\left[\boldsymbol{Z}_i - \boldsymbol{m}_{\boldsymbol{Z}}(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\right]g'(\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i)\end{pmatrix}\right.\\
&\qquad\qquad\left.\times(\epsilon_i + \epsilon_i^{-1})\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right\}\ln(\epsilon_i)\\
&-E\left\{(\epsilon_i + \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\}E\{\exp(M(\boldsymbol{X}_i, \boldsymbol{Z}_i))|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\}\right\}\\
&-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\left\{(\epsilon_i + \epsilon_i^{-1})I\{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{S}_i \le u\}\middle|\boldsymbol{\beta}_{\boldsymbol{\phi}_0}^{\mathrm{T}}\boldsymbol{Z}_i\right\}\ln(\epsilon_i)\\
&+\boldsymbol{\varpi}(u) - [\boldsymbol{\Delta}(u)]^{\mathrm{T}}\boldsymbol{\Gamma}_0^{-1}\digamma_0 + o_P(1).
\end{aligned}
$$

We have completed the proof of Theorem 10.

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