Annals of the Institute of Statistical Mathematics (2020) 72:607–626
https://doi.org/10.1007/s10463-018-0703-8

Spatially homogeneous copulas

Fabrizio Durante[1] · Juan Fernández Sánchez[2] · Wolfgang Trutschnig[3]

Received: 24 July 2017 / Revised: 1 November 2018 / Published online: 1 December 2018

Abstract
We consider spatially homogeneous copulas, i.e. copulas whose corresponding measure is invariant under a special transformations of $[0, 1]^2$, and we study their main properties with a view to possible use in stochastic models. Specifically, we express any spatially homogeneous copula in terms of a probability measure on $[0, 1)$ via the Markov kernel representation. Moreover, we prove some symmetry properties and demonstrate how spatially homogeneous copulas can be used in order to construct copulas with surprisingly singular properties. Finally, a generalization of spatially homogeneous copulas to the so-called (m, n)-spatially homogeneous copulas is studied and a characterization of this new family of copulas in terms of the Markov $*$-product is established.

Keywords Copulas · Dependence · Probability measures · Singular measures

1 Introduction

It has been long recognized [see, for example, Brown (1965), Mikusiński and Taylor (2009), Olsen et al. (1996) and the references therein] that Markov operators on $L^1([0, 1])$ are in one-to-one correspondence with copulas or, equivalently, doubly stochastic measures, i.e. probability measures of $[0, 1]^2$ whose marginals coincide

✉ Fabrizio Durante
fabrizio.durante@unisalento.it

Juan Fernández Sánchez
juanfernandez@ual.es

Wolfgang Trutschnig
wolfgang@trutschnig.net

[1] Dipartimento di Scienze dell'Economia, Università del Salento, Campus Ecotekne - Palazzina C, Via Monteroni 165, 73100 Lecce, Italy

[2] Grupo de Investigación de Análisis Matemático, Universidad de Almería, Carretera Sacramento s/n, La Cañada de San Urbano, 04120 Almería, Spain

[3] Department for Mathematics, University of Salzburg, Hellbrunner Strasse 34, 5020 Salzburg, Austria

with the Lebesgue measure. This correspondence has been exploited in various contexts, especially in problems related to convergence and approximation of copulas [see, for example, Mikusiński and Taylor (2010), Trutschnig (2011)]. In Brown (1966), a subclass of the family of all Markov operators on $L^1(\Omega)$ is introduced under the name *spatially homogeneous Markov operators*, which are Markov operators commuting with all rotations. Nevertheless, to the best of the authors' knowledge, the related notion of spatially homogeneous copulas (as directly translated from the Markov operator setting via the isomorphism between the two classes) has not received any attention in the literature yet, despite the fact that it presents strong similarities with related concepts appeared in the study of circular distributions.

The objective of this paper is hence to revisit the concept of spatially homogeneous copulas. In particular, we are interested in their possible use in stochastic modelling. Remarkably, each spatially homogeneous copula can be represented in terms of a unique probability measure on $[0, 1)$, which provides an interesting analogy with other popular classes like Archimedean copulas (induced by survival functions associated with a probability measure on the positive real line) or extreme–value copulas (induced by measures on the unit simplex or, equivalently, Pickands dependence functions). Thanks to the previous different viewpoint, we are hence able to provide a generalization of spatially homogeneous copulas to the so-called (m, n)-spatially homogeneous copulas which, in turn, are shown to be fully characterizable in terms of a Markov product (also called $*$-product), which is used in the study of Markov processes in Darsow et al. (1992). Various properties have been presented to illustrate how spatially homogeneous copulas exhibit several interesting aspects of stochastic dependence. Finally, we sketch a possible multivariate generalization of this concept.

2 Markov kernels of spatially homogeneous copulas

For arbitrary $x \in \mathbb{R}$ and $z \in [0, 1]$, let $R_x : [0, 1] \to [0, 1)$ denote the rotation by x, defined by $R_x(z) = x + z \pmod 1$. Obviously, restricting to $[0, 1)$ we have $R_x^{-1} = R_{1-x}$—in the sequel $R_x^{-1}(F)$ will, however, denote the pre-image of F via R_x. Furthermore, we define the transformation $\oplus : [0, 1]^2 \to [0, 1)^2$ by $(x_1, y_1) \oplus (x_2, y_2) := (x_1 + x_2 \pmod 1), y_1 + y_2 \pmod 1))$.

The symbols $\mathscr{B}([0, 1])$ and $\mathscr{B}([0, 1]^2)$ denote the Borel σ-fields on $[0, 1]$ and $[0, 1]^2$, and λ and λ_2 the Lebesgue measures on $\mathscr{B}([0, 1])$ and $\mathscr{B}([0, 1]^2)$. Moreover, $\mathscr{C}$ denotes the family of all two-dimensional copulas, $K_A(\cdot, \cdot)$ the Markov kernel of $A \in \mathscr{C}$ and μ_A the corresponding doubly stochastic measure. [For background, see Durante and Sempi (2016) and the references therein.]

A copula $A \in \mathscr{C}$ is called completely dependent if there exists a λ-preserving transformation $h : [0, 1] \to [0, 1]$ such that $K(x, F) = \mathbf{1}_F(h(x))$ is a Markov kernel of A; for properties and characterizations of complete dependence, we refer to Trutschnig (2011). In the sequel, we will let $\mathscr{T}$ denote the family of all λ-preserving transformations on $[0, 1]$, $\mathscr{T}_0$ the subclass of all bijective λ-preserving transformations and $\mathscr{C}^{cd}$ the family of all completely dependent copulas. For $h \in \mathscr{T}$, C_h will denote the corresponding completely dependent copula.

As direct application of the results in Lange (1973), the Markov kernel $K_A(\cdot,\cdot)$ of an arbitrary copula $A \in \mathscr{C}$ can be decomposed into the sum of three substochastic kernels $K_A^{\text{abs}}(\cdot,\cdot)$, $K_A^{\text{sing}}(\cdot,\cdot)$, $K_A^{\text{dis}}(\cdot,\cdot)$ from $[0,1]$ to $\mathscr{B}([0,1])$, i.e.

$$K_A(x,E) = K_A^{\text{abs}}(x,E) + K_A^{\text{sing}}(x,E) + K_A^{\text{dis}}(x,E) \tag{1}$$

for every $x \in [0,1]$ and $E \in \mathscr{B}([0,1])$. Therefore, the measure $K_A^{\text{abs}}(x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure λ, the measure $K_A^{\text{sing}}(x,\cdot)$ is singular with respect to λ and has no point masses, and $K_A^{\text{dis}}(x,\cdot)$ is discrete for every $x \in [0,1]$. Letting k_A denote the Radon–Nikodym derivative of the absolutely continuous component of μ_A with respect to λ_2. Then the (almost everywhere) uniqueness of the kernel K_A implies that the measures $K_A^{\text{abs}}(x,\cdot)$ and $E \mapsto \int_E k_A(x,y)\mathrm{d}\lambda(y)$ coincide for almost all $x \in [0,1]$.

In the sequel, we will refer to the induced measures $\mu_A^{\text{abs}}, \mu_A^{\text{sing}}, \mu_A^{\text{dis}}$, given by

$$\mu_A^{\text{abs}}(E\times F) = \int_E K_A^{\text{abs}}(x,F)\mathrm{d}\lambda(x), \qquad \mu_A^{\text{sing}}(E\times F) = \int_E K_A^{\text{sing}}(x,F)\mathrm{d}\lambda(x)$$
$$\mu_A^{\text{dis}}(E\times F) = \int_E K_A^{\text{dis}}(x,F)\mathrm{d}\lambda(x) \tag{2}$$

for all Borel sets $E, F \subseteq [0,1]$ and extended to $\mathscr{B}([0,1]^2)$ in the standard way, simply as *absolutely continuous, discrete and singular components* of μ_A. Notice that the standard definition of a (purely) singular copula as in Durante and Sempi (2016) and Nelsen (2006) translates to $\mu_A^{\text{sing}}([0,1]^2) + \mu_A^{\text{dis}}([0,1]^2) = 1$.

Definition 1 (Brown 1966) $A \in \mathscr{C}$ is called *spatially homogeneous* if

$$\mu_A(x(1,1)\oplus G) = \mu_A(G) \tag{3}$$

holds for every $x \in [0,1]$ and $G \in \mathscr{B}([0,1]^2)$.

In other words, a copula is spatially homogeneous if its associated measure is invariant under the transformation $\Phi_x : (u,v) \mapsto (R_x(u), R_x(v))$ for every $x \in [0,1]$. Roughly speaking, the measure induced by a copula of this type is invariant to some location shifts, assuming the unit square is wrapped around at its edges.

In the sequel, $\mathscr{C}^H$ will denote the class of all spatially homogeneous copulas which includes the comonotonicity copula M_2 and the independence copula Π_2.

Obviously, $A \in \mathscr{C}^H$ if and only if we have

$$\mu_A(R_x(E)\times R_x(F)) = \mu_A(E\times F) \tag{4}$$

for every $x \in [0,1)$ and $E, F \in \mathscr{B}([0,1])$.

Spatially homogeneous copulas can easily be characterized in terms of the corresponding Markov kernel. In fact, suppose that $\vartheta \in \mathscr{P}'([0,1])$, where $\mathscr{P}'([0,1])$

denotes the class of all probability measures on $\mathscr{B}([0,1])$ fulfilling $\vartheta(\{1\}) = 0$. Let ϑ^{R_x} denote the push-forward (i.e. image measure) of ϑ via R_x. Setting

$$K(x, E) := \vartheta^{R_x}(E) \tag{5}$$

for every $x \in [0,1]$ and $E \in \mathscr{B}([0,1])$, the following result holds.

Theorem 1 *The mapping $K(\cdot,\cdot)$ defined according to Eq. (5) is the Markov kernel of a copula $A_\vartheta \in \mathscr{C}^H$.*

Proof It is clear that $E \mapsto K(x, E)$ is a probability measure on $\mathscr{B}([0,1])$ fulfilling $K(x, \{1\}) = 0$ for every $x \in [0,1]$. Moreover, considering

$$R_x^{-1}([0,y]) = \begin{cases} [0, y-x] \cup [1-x, 1] & \text{if } x \le y, \\ [1-x, 1+y-x] & \text{if } x > y, \end{cases} \tag{6}$$

it follows immediately that $x \mapsto K(x, [0,y])$ is measurable in x for every fixed $[0,y] \subseteq [0,1]$. Since the family $\mathscr{D}$ of all Borel sets F for which $x \mapsto K(x, F)$ is measurable forms a Dynkin system containing the family of all intervals of the form $[0,y]$, we conclude that $K(x, E) = \vartheta^{R_x}(E)$ is a Markov kernel. Thus, we only have to prove that $K(\cdot,\cdot)$ is associated with a doubly stochastic measure [i.e. it satisfies Eq. (3.4.8) in Durante and Sempi (2016)]. Using Fubini's theorem and change of coordinates, we get

$$\begin{aligned}\int_{[0,1]} K(x, E)\mathrm{d}\lambda(x) &= \int_{[0,1]}\int_{[0,1]} \mathbf{1}_{R_x^{-1}(E)}(z)\mathrm{d}\vartheta(z)\mathrm{d}\lambda(x) \\ &= \int_{[0,1]}\int_{[0,1]} \mathbf{1}_E(R_x(z))\mathrm{d}\vartheta(z)\mathrm{d}\lambda(x) \\ &= \int_{[0,1]}\int_{[0,1]} \mathbf{1}_E(R_z(x))\mathrm{d}\lambda(x)\mathrm{d}\vartheta(z) \\ &= \int_{[0,1]}\int_{[0,1]} \mathbf{1}_E(y)\mathrm{d}\lambda(y)\mathrm{d}\vartheta(z) \\ &= \lambda(E),\end{aligned}$$

which implies that $K(\cdot,\cdot)$ is the Markov kernel of a copula A_ϑ.

To show that $A_\vartheta \in \mathscr{C}^H$, we can proceed as follows. It is straightforward to verify that for every $F \subseteq [0,1)$ we have $R^{-1}_{R_x(z)}(R_x(F)) = R_z^{-1}(F)$, from which we immediately get

$$K(R_x(z), R_x(F)) = \vartheta\big(R^{-1}_{R_x(z)}(R_x(F))\big) = \vartheta(R_z^{-1}(F)) = K(z, F)$$

for every $F \in \mathscr{B}([0,1])$ with $F \subseteq [0,1)$. Having this, using disintegration theorem [see, for example, Ambrosio et al. (2000), Klenke (2008)] and changing coordinates, for arbitrary $E, F \in \mathscr{B}([0,1])$ with $F \subseteq [0,1)$ the desired equality follows from

$$\mu_{A_\vartheta}(R_x(E) \times R_x(F)) = \int_{R_x(E)} K(z, R_x(F))\mathrm{d}\lambda(z)$$

$$
\begin{aligned}
&= \int_{[0,1]} \mathbf{1}_E(R_{1-x}(z))K(R_x \circ R_{1-x}(z), R_x(F))\, \mathrm{d}\lambda(z) \\
&= \int_{[0,1]} \mathbf{1}_E(y)K(R_x(y), R_x(F))\mathrm{d}\lambda(y) = \int_E K(y, F)\mathrm{d}\lambda(y) \\
&= \mu_{A_\vartheta}(E \times F).
\end{aligned}
$$

Since for $F = \{1\}$ Eq. (4) obviously holds for every $E \in \mathscr{B}([0, 1])$, the proof is complete. □

Remark 1 From the previous construction and the fact that $R_x \circ R_0 = R_x$ holds for every $x \in [0, 1]$, it follows that, if the measures ϑ and $\vartheta^{R_0} \in \mathscr{P}'([0, 1]))$ coincide, then they induce the same spatially homogeneous copula.

Not surprisingly, every spatially homogeneous copula is the result of rotating a probability measure $\vartheta \in \mathscr{P}'([0, 1])$, as the following result shows (compare with (Brown 1966, Theorem 3).

Theorem 2 *Suppose that* $A \in \mathscr{C}^H$. *Then there exists a unique probability measure* $\vartheta \in \mathscr{P}'([0, 1])$ *such that* $A = A_\vartheta$.

Proof Without loss of generality, let $K_A(\cdot, \cdot)$ denote a version of the Markov kernel of A fulfilling $K_A(z, \{1\}) = 0$ for every $z \in [0, 1]$. Expressing Eq. (4) in terms of the corresponding kernel and changing coordinates, we get

$$
\begin{aligned}
\int_E K_A(z, F)\mathrm{d}\lambda(z) &= \int_{R_x(E)} K_A(z, R_x(F))\, \mathrm{d}\lambda(z) \\
&= \int_{[0,1]} \mathbf{1}_E(R_{1-x}(z))K_A(R_x \circ R_{1-x}(z), R_x(F))\, \mathrm{d}\lambda(z) \\
&= \int_{[0,1]} \mathbf{1}_E(y)K_A(R_x(y), R_x(F))\, \mathrm{d}\lambda(y) \\
&= \int_E K_A(R_x(z), R_x(F))\mathrm{d}\lambda(z)
\end{aligned}
$$

for every $x \in [0, 1]$ and $E, F \in \mathscr{B}([0, 1])$. Hence, for every $G \in \mathscr{B}([0, 1])$ we must have

$$
\int_{G \times E} K_A(z, F)\mathrm{d}\lambda_2(z, x) = \int_{G \times E} K_A(R_x(z), R_x(F))\mathrm{d}\lambda_2(z, x).
$$

Considering that E, G were arbitrary, we can find a set $\Omega_F \in \mathscr{B}([0, 1]^2)$, $\lambda_2(\Omega_F) = 1$, such that $K_A(z, F) = K_A(R_x(z), R_x(F))$ holds for every $(z, x) \in \Omega_F$. Repeating the same argument for every set F of the form $F = [0, y]$ with $y \in \mathbb{Q} \cap [0, 1]$, we can find a set $\Omega \in \mathscr{B}([0, 1]^2)$ with $\lambda_2(\Omega) = 1$ such that

$$
K_A(z, [0, y]) = K_A(R_x(z), R_x([0, y])) = K_A(R_x(z), R_{1-x}^{-1}([0, y]))
$$

holds for every $(z, x) \in \Omega$ simultaneously for all $y \in \mathbb{Q} \cap [0, 1]$. As a consequence, for every $(z, x) \in \Omega$ the measure $K_A(z, \cdot)$ and the push-forward $K_A(R_x(z), \cdot)^{R_{1-x}}$ of $K_A(R_x(z), \cdot)$ via R_{1-x} coincide, i.e. we have

$$K_A(z, F) = K_A(R_x(z), R_x(F)) \tag{7}$$

for every $F \in \mathscr{B}([0, 1])$ and every $(z, x) \in \Omega$. Disintegration theorem implies the existence of a set $\Lambda \in \mathscr{B}([0, 1])$ with $\lambda(\Lambda) = 1$ such that $\lambda(\Omega_z) = 1$ for every $z \in \Lambda$. Let $z \in \Lambda$ be arbitrary but fixed and define the probability measure $\vartheta \in \mathscr{P}([0, 1])$ by $\vartheta(F) = K_A(z, R_z(F))$. Then, for every $x \in \Omega_z$ and every $F \in \mathscr{B}([0, 1])$ using Eq. (7) we get

$$\begin{aligned}\vartheta^{R_x}(F) = \vartheta(R_x^{-1}(F)) &= K_A\big(z, R_z \circ R_{1-x}(F)\big)\\ &= K_A\big(R_x(z), R_x \circ R_z \circ R_{1-x}(F)\big)\\ &= K_A\big(R_x(z), R_z(F)\big) = K_A\big(R_z(x), R_z(F)\big) = K_A(x, F).\end{aligned}$$

This last equality completes the proof of the representation (4) since Ω_z has full measure and kernels are only unique up to a set of measure zero. Finally, uniqueness of ϑ is clear. □

Remark 2 According to Brown (1966), $\mathscr{C}^H$ is a convex and compact subset of $\mathscr{C}$ (endowed with the uniform metric). Moreover, its extreme points correspond to extreme points of the class of all probability measures on $\mathscr{P}'([0, 1])$, i.e. to probability measures concentrating their mass on one single point. In other words, a copula $A \in \mathscr{C}^H$ is an extreme point of $\mathscr{C}^H$ if and only if there exists a point $z \in [0, 1)$ such that $A = C_{R_z}$ holds; that is, A is a shuffle of M_2 induced by R_z. Using this fact and applying Choquet's theorem [see Phelps (2001)] can also provide an alternative proof of the one-to-one correspondence between $\mathscr{P}'([0, 1])$ and $\mathscr{C}^H$.

3 Some properties of spatially homogeneous copulas

In this section, we study some symmetry properties of spatially homogeneous copulas and some particular examples underlining their usefulness concerning the construction of copulas with exotic properties.

3.1 Measure-theoretic properties

Suppose that $A_\vartheta \in \mathscr{C}^H$. Obviously $\mu^{\text{sing}}_{A_\vartheta}([0, 1]^2), \mu^{\text{dis}}_{A_\vartheta}([0, 1]^2)$ and $\mu^{\text{abs}}_{A_\vartheta}([0, 1]^2)$ coincide with the masses of the singular, the discrete and the absolutely continuous components of ϑ. Specifically, letting $\vartheta = \vartheta^{\text{sing}} + \vartheta^{\text{dis}} + \vartheta^{\text{abs}}$ the Lebesgue decomposition of ϑ we have

$$\mu^{\text{sing}}_{A_\vartheta}([0, 1]^2) = \vartheta^{\text{sing}}([0, 1]), \quad \mu^{\text{dis}}_{A_\vartheta}([0, 1]^2) = \vartheta^{\text{dis}}([0, 1]), \tag{8}$$

$$\mu^{\text{abs}}_{A_\vartheta}([0, 1]^2) = \vartheta^{\text{abs}}([0, 1]). \tag{9}$$

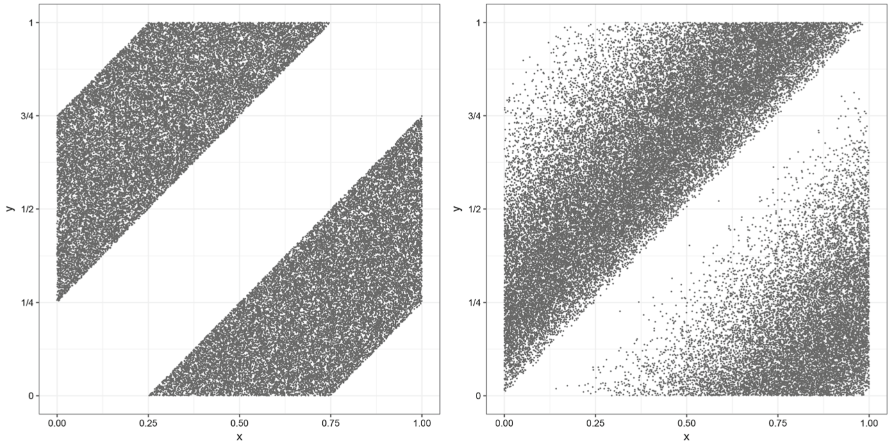

Fig. 1 Samples of size 50,000 of spatially homogeneous, absolutely continuous copulas A_ϑ where ϑ corresponds to a uniform distribution on $[\frac{1}{4}, \frac{3}{4}]$ (left panel) or a beta-distribution $\beta_{2,5}$ with parameters 2 and 5 (right panel)

In particular, in case of $\vartheta^{\text{dis}}([0,1]) = 0$, it follows immediately from Eqs. (5) and (6) that the function $(x, y) \mapsto K(x, [0, y])$ is continuous on $[0, 1]^2$. Notice that, if ϑ is a measure with full support, then A_ϑ has also full support. Moreover, we could obtain singular (respectively, absolutely continuous) spatially homogeneous copulas A_θ with full support by selecting ϑ^{sing} (respectively, ϑ^{abs}) that has full support.

Another aspect of interest is the case of ϑ absolutely continuous with density f. In this case, it is straightforward to verify that (a version of) the density k_ϑ of $A_\vartheta \in \mathscr{C}^H$ is given by $k_\vartheta(x, y) = f(R_{1-x}(y))$. In fact, for every $x, y \in [0, 1]$, we get

$$\begin{aligned}\int_{[0,y]} k_\vartheta(x,s)\,\mathrm{d}\lambda(s) = \vartheta^{R_x}([0,y]) &= \int_{[0,1]} \mathbf{1}_{[0,y]}(R_x(z))f(z)\,\mathrm{d}\lambda(z)\\ &= \int_{[0,1]} \mathbf{1}_{[0,y]}(R_x \circ R_{1-x}(s))f(R_{1-x}(s))\,\mathrm{d}\lambda(s)\\ &= \int_{[0,y]} f(R_{1-x}(s))\,\mathrm{d}\lambda(s).\end{aligned}$$

Thus, the density k_ϑ of $A_\vartheta \in \mathscr{C}^H$

$$k_\vartheta(R_z(x), R_z(y)) = k_\vartheta(x, y)$$

for all $x, y, z \in [0, 1]$. A similar condition has also appeared in Alfonsi and Brigo (2005), where *periodic copulas* have been introduced, and, under a slight modification of the copula domain, in the study of copulas for circular distributions, as investigated, for instance, in Jones and Pewsey (2015).

Figure 1 depicts samples of two spatially homogeneous, absolutely continuous copulas.

3.2 Dependence properties

Now, we are interested in checking whether spatially homogeneous copulas can be related to some measures of association. Since Π_2 and M_2 belong to the convex set $\mathscr{C}^H$, the continuity of Spearman's ρ and Kendall's τ (and, in general, any concordance measure) with respect to the uniform metric d_∞ implies $\rho(\mathscr{C}^H) \supseteq [0, 1]$ as well as $\tau(\mathscr{C}^H) \supseteq [0, 1]$. In other words, spatially homogeneous copulas can describe any degree of concordance in [0, 1].

Furthermore, spatially homogeneous copulas can cover a broader range of concordance values. For Spearman's ρ, it is even possible to determine this exact range, as the following result shows.

Theorem 3 $\rho(\mathscr{C}^H) = [-0.5, 1]$.

Proof For the spatially homogeneous copula A_{δ_z} with $z \in [0, 1)$, in view of the disintegration theorem, we have

$$\begin{aligned}\rho(A_{\delta_z}) &= 12 \int_{[0,1]^2} \Pi_2 \,\mathrm{d}\mu_{A_{\delta_z}} - 3 = 12 \int_{[0,1]} x R_z(x) \,\mathrm{d}\lambda(x) - 3 \\ &= 12 \int_{[0,1-z]} x(x+z) \,\mathrm{d}\lambda(x) + 12 \int_{[1-z,1]} x(x+z-1) \,\mathrm{d}\lambda(x) - 3 \\ &= -6z(1-z) + 1.\end{aligned}$$

The latter expression is minimal for $z = \frac{1}{2}$ and we get $\rho(A_{\delta_z}) \geq -\frac{1}{2}$ with equality if and only if $z = \frac{1}{2}$. Considering that the mapping $A \mapsto \rho(A) = \int_{[0,1]^2} \Pi_2 \,\mathrm{d}\mu_A$ preserves convex combinations, it follows immediately that $\rho(A_\vartheta) \geq -\frac{1}{2}$ holds for every discrete measure ϑ concentrating its mass on finitely many points in [0, 1]. Since every element of $\mathscr{P}'([0, 1])$ is the (weak) limit of a sequence $(\vartheta_n)_{n\in\mathbb{N}}$ of discrete measures of the aforementioned type, the proof is complete. □

Remark 3 Determining the exact range of Kendall's τ seems more challenging. Letting ϑ denote the uniform distribution on $[\frac{1}{4}, \frac{3}{4}]$ (the left panel of Fig. 1 shows samples of this copula), a straightforward calculation yields $\tau(A_\vartheta) = -\frac{1}{6}$. Using continuity of Kendall's τ w.r.t. d_∞, we therefore get $\tau(\mathscr{C}^H) \supseteq [-\frac{1}{6}, 1]$. We conjecture that this interval coincides with the actual range of possible values of Kendall's τ for spatially homogeneous copulas. However, we have not been able to prove formally this conjecture.

We add a final remark about tail dependence properties. By the convexity of $\mathscr{C}^H$, it follows that spatially homogeneous copulas cover all possible tail dependence coefficients (ranging from 0 to 1). In fact, a convex combination of the copulas Π_2 and M_2 is spatially homogeneous. Moreover, by the very construction of the class $\mathscr{C}^H$, it follows that, if $(U, V) \sim A \in \mathscr{C}^H$, then $\mathbb{P}(U \leq u, V \leq u) = \mathbb{P}(U \geq 1-u, V \geq 1-u)$ for every $u \in [0, 1]$, implying that the left lower and the right upper tail dependence coefficients of A coincide.

3.3 Symmetries

Driven by the specific choice of the generating probability measure $\vartheta \in \mathscr{P}'([0,1])$, the class $\mathscr{C}^H$ contains copulas that exhibit different types of symmetries. In order to study these symmetries, we first focus on the following lemma that gathers additional properties of the bijection $\Phi : \mathscr{P}'([0,1]) \to \mathscr{C}^H$, defined by $\Phi(\vartheta) = A_\vartheta$. [For properties of the metric D_1, see Fernández-Sánchez and Trutschnig (2015), Trutschnig (2011).]

Lemma 1 *Suppose that $\vartheta, \nu \in \mathscr{P}'([0,1])$ have distribution function F and G, respectively. Then we have*

$$D_1(A_\vartheta, A_\nu) \leq 2\|F - G\|_\infty. \tag{10}$$

Proof For all $x, y \in [0,1]$, using Eq. (6) and the triangle inequality we get

$$\begin{aligned}
|K_{A_\vartheta}(x,[0,y]) - K_{A_\nu}(x,[0,y])| \\
&\leq \mathbf{1}_{[0,y]}(x)|\vartheta([0,y-x]) - \nu([0,y-x])| \\
&\quad + \mathbf{1}_{[0,y]}(x)|\vartheta([1-x,1]) - \nu([1-x,1])| \\
&\quad + \mathbf{1}_{[y,1]}(x)|\vartheta([1-x,1+y-x]) - \nu([1-x,1+y-x])| \\
&\leq \mathbf{1}_{[0,y]}(x)\, 2\|F-G\|_\infty + \mathbf{1}_{(y,1]}(x)\, 2\|F-G\|_\infty = 2\|F-G\|_\infty
\end{aligned}$$

from which inequality (10) follows immediately by integration. □

Theorem 4 *If $\vartheta \in \mathscr{P}'([0,1])$ fulfils $\vartheta(1-E) = \vartheta(E)$ for every $E \in \mathscr{B}([0,1])$ with $E \subseteq (0,1)$, then A_ϑ is radially symmetric (i.e. A_ϑ coincides with its survival copula $\hat{A}_\vartheta$) and symmetric.*

Proof (i): First suppose that $\vartheta(\{0\}) = 0$. Then, for every $E \in \mathscr{B}([0,1])$ with $E \subseteq (0,1)$ and $x \in [0,1]$, the Markov kernel fulfils $K(1-x, 1-E) = K(x,E)$ since we have

$$\vartheta^{R_{1-x}}(1-E) = \vartheta(1 - R_x^{-1}(E)) = \vartheta(R_x^{-1}(E)) = \vartheta^{R_x}(E).$$

Here, notice that $R_{1-x}^{-1}(1-E)$ and $1 - R_x^{-1}(E)$ need not coincide (consider, for instance, $E = \{1/2\}$ and $x = 1/2$). Using the disintegration of a measure, we get

$$\begin{aligned}
A_\vartheta(x,y) = \mu_{A_\vartheta}([0,x] \times (0,y)) &= \int_{[0,x]} K\big(z,(0,y)\big)\mathrm{d}\lambda(z) \\
&= \int_{[0,x]} K\big(1-z,(1-y,1)\big)\mathrm{d}\lambda(z) \\
&= \mu_{A_\vartheta}([1-x,1] \times (1-y,1)) = \mu_{A_\vartheta}([1-x,1] \times [1-y,1]) \\
&= x + y - 1 + A_\vartheta(1-x,1-y)
\end{aligned}$$

for all $x, y \in [0,1]$, so A_ϑ is radially symmetric. Moreover, for $\vartheta = \delta_0$, we have $A_\vartheta = M \in \mathscr{C}^H$. Thus, the desired result follows by considering that every ϑ can be expressed as convex combination of some $\nu \in \mathscr{P}'([0,1])$, with $\nu(\{0\}) = 0$, and δ_0, and the fact that A_ϑ is rotation symmetric for every $\vartheta \in \mathscr{P}'([0,1])$.

(ii) To prove symmetry of A_ϑ, we proceed in two short steps:

(a) First, assume that the support of ϑ can be written as

$$\mathrm{Supp}(\vartheta) = \{x_1, x_2, \ldots, x_n, 1 - x_n, \ldots, 1 - x_2, 1 - x_1\}$$

for some $n \in \mathbb{N}$ and $0 < x_1 < x_2 \ldots, x_{n-1} < x_n \leq \frac{1}{2}$. Define $\alpha_i = \vartheta(\{x_i, 1 - x_i\})$ and set $\vartheta_i = \frac{1}{\alpha_i}\vartheta \in \mathscr{P}'([0,1])$. Since we obviously have

$$A_{\vartheta_i}(x, y) = \tfrac{1}{2}\big(S_{R_{x_i}}(x, y) + S_{R_{1-x_i}}(x, y)\big) = \tfrac{1}{2}\big(S_{R_{x_i}}(x, y) + S_{R_{x_i}}(y, x)\big)$$

for all $x, y \in [0,1]$, A_{ϑ_i} is symmetric. Considering that the mapping $\Phi : \mathscr{P}'([0,1]) \to \mathscr{C}^H$ preserves convex combinations, A_ϑ is symmetric too.

(b) Since the distribution function F of every $\vartheta \in \mathscr{P}'([0,1])$ with $\vartheta(\{0\}) = 0$ is the uniform limit of discrete distribution functions corresponding to measures considered in (a) and convergence with respect to D_1 implies uniform convergence [see Trutschnig (2011)], it follows that A_ϑ is symmetric too. Finally, the proof can be completed by using the same arguments as in (i) and the fact that M_2 is symmetric. □

Remark 4 An alternative way to prove symmetry of A_ϑ would be to show that for every $\vartheta \in \mathscr{P}'([0,1])$ fulfilling $\vartheta(1 - E) = \vartheta(E)$ for every $E \in \mathscr{B}([0,1])$ with $E \subseteq (0,1)$ the corresponding Markov operator T_{A_ϑ} is self-adjoint (interpreted as operator on $L^2([0,1], \mathscr{B}([0,1]), \lambda)$) and using the fact that $T_{A^t} = (T_A)^{\mathrm{adj}}$ for every copula $A \in \mathscr{C}$ [see Olsen et al. (1996), Trutschnig (2013)].

If we substitute the invariance condition (3) by the condition

$$\mu_A(x(1, -1) \oplus G) = \mu_A(G) \tag{11}$$

for all $x \in [0,1]$ and every $G \in \mathscr{B}([0,1]^2)$, it follows in the same manner as before that such a copula A corresponds to a unique probability measure $\vartheta \in \mathscr{P}'([0,1])$ such that

$$K(x, E) = \vartheta^{R_{1-x}}(E)$$

is a Markov kernel of A. Equivalently, if A_ϑ is spatially homogeneous, then $A' \in \mathscr{C}$, defined by $A'(x, y) = x - A_\vartheta(1 - x, y)$, fulfils condition (11).

The following result holds.

Theorem 5 *The independence copula Π_2 is the unique spatially homogeneous copula satisfying (11) for every $x \in [0,1]$ and every $G \in \mathscr{B}([0,1]^2)$.*

Proof If $A_{\vartheta_1} \in \mathscr{C}^H$ also satisfies (11), then there exists a probability measure $\vartheta_2 \in \mathscr{P}'([0,1])$ such that

$$\vartheta_1^{R_x} = \vartheta_2^{R_{1-x}}$$

holds for every $x \in [0, 1]$. As a direct consequence, we get $\vartheta_1^{R_{2x}} = \vartheta_2$, implying that ϑ_1 is invariant with respect to any rotation $z \mapsto R_x(z)$ with $x \in [0, 1)$, from which $\vartheta_1 = \lambda = \vartheta_2$ follows immediately (uniqueness of Haar measure). This completes the proof since $A_\lambda = \Pi_2 \in \mathscr{C}^H$. □

3.4 Application to the construction of copulas with exotic properties

Theorem 4 can be used to construct various "exotic" copulas, i.e. copulas that do not satisfy standard smoothness and regularity properties. Here, we provide three examples.

Application 1 *We prove the existence of singular copulas $A \in \mathscr{C}$ fulfilling that the partial derivative $\frac{\partial A}{\partial x}$ is continuous on $(0, 1) \times [0, 1]$ and $\frac{\partial A}{\partial y}$ is continuous on $[0, 1] \times (0, 1)$. As shown in Segers (2012), under these regularity conditions weak convergence of the empirical copula process holds.*

Choose $\vartheta \in \mathscr{P}'([0, 1])$ with $\vartheta^{\text{dis}}([0, 1]) = \vartheta^{\text{abs}}([0, 1]) = 0$ and fulfilling $\vartheta(1 - E) = \vartheta(E)$ for every $E \in \mathscr{B}([0, 1])$. Then $\mu_{A_\vartheta}^{\text{sing}}([0, 1]^2) = 1$, so A_ϑ is singular, and from Sect. 3.1 it follows that $(x, y) \mapsto K(x, [0, y]) = \vartheta^{R_x}([0, y])$ is continuous on $[0, 1]^2$. Considering

$$A(x, y) = \int_{[0,x]} K(z, [0, y]) \mathrm{d}\lambda(z)$$

we conclude that $\frac{\partial A}{\partial x}(x, y)$ is continuous on $(0, 1) \times [0, 1]$. Fulfilment of the condition for $\frac{\partial A}{\partial y}$ is now a direct consequence of symmetry of ϑ and Theorem 4. An example of a measure ϑ satisfying the previous condition is given by the measure induced by the Cantor ternary distribution function. See Fig. 2.

Application 2 *Let $\alpha \in (0, 1)$ be arbitrary. Then there exists an absolutely continuous copula A with density k_A fulfilling the following three properties:*

(I) *The set $\Lambda := \{(x, y) \in [0, 1]^2 : k_A(x, y) = 0\}$ fulfils $\lambda_2(\Lambda) > 1 - \alpha$.*
(II) *For $E, F \in \mathscr{B}([0, 1])$, we have $\mu_A(E \times F) > 0$ whenever $\lambda_2(E \times F) > 0$.*
(III) *For $E, F \in \mathscr{B}([0, 1])$, we have $\lambda_2(\Lambda \cap (E \times F)) > 0$ whenever $\lambda_2(E \times F) > 0$.*

We proceed in two steps:
Step 1*: Choose a set $\Omega \in \mathscr{B}([0, 1])$ with $\lambda(\Omega) \in (0, \alpha)$ and $\Omega = 1 - \Omega$ (i.e. Ω is symmetric with respect to $1/2$) such that for every interval $(a, b) \subseteq [0, 1]$ and $a < b$ we have $\lambda(\Omega \cap (a, b)) > 0$ as well as $\lambda(\Omega^c \cap (a, b)) > 0$. Such a set can easily be constructed by slightly modifying the proof of Lemma 3.1 in Fernández-Sánchez and Trutschnig (2016). Without loss of generality, we may assume that each point $\omega \in \Omega$ is a Lebesgue point [see Tao (2011)] of $\mathbf{1}_\Omega$, i.e. for each $\omega \in \Omega$ we have*

$$\lim_{r \to 0+} \frac{1}{2r} \lambda\big(\Omega \cap (\omega - r, \omega + r)\big) = 1.$$

Obviously, $\frac{1}{\lambda(\Omega)} \mathbf{1}_\Omega$ is a probability density on $[0, 1]$. Letting $\vartheta \in \mathscr{P}([0, 1])$ denote the corresponding probability measure, it follows from Theorem 4 and Sect. 3.1 that *the corresponding spatially homogeneous copula A_ϑ is symmetric and absolutely continuous with density*

$$k_\vartheta(x, y) = \frac{1}{\lambda(\Omega)} \mathbf{1}_\Omega \circ R_{1-x}(y).$$

Setting $\Lambda := \{(x, y) \in [0, 1]^2 : k_\vartheta(x, y) = 0\}$, we obviously have $\lambda_2(\Lambda) = 1 - \lambda(\Omega) > 1 - \alpha$, and considering that k_ϑ is zero on Λ by definition, A_ϑ satisfies condition (I). Also notice that for every $\omega \in \Omega$ by symmetry we have $1 - \omega \in \Omega$; hence, $(R_{1-(\omega+x)}, x) \in \Lambda^c$ and $k_\vartheta(R_\omega(x), x) = \frac{1}{\lambda(\Omega)}$ for every $x \in [0, 1]$.

Step 2*: We prove that $\mu_\vartheta := \mu_{A_\vartheta}$ also satisfies condition (II) by showing that for all $E, F \in \mathscr{B}([0, 1])$ with $\lambda_2(E \times F) > 0$ we have $\lambda_2(\Lambda^c \cap (E \times F)) > 0$. By extracting sets of measure zero from E and F if necessary, we may assume that each point $e \in E$ is a Lebesgue point of $\mathbf{1}_E$ and each $f \in F$ is a Lebesgue point of $\mathbf{1}_F$. Furthermore, using symmetry (and intersecting $E \times F$ with small squares of the form $Q_{ij} = [\frac{i-1}{2^n}, \frac{i}{2^n}] \times [\frac{j-1}{2^n}, \frac{j}{2^n}]$ for $i, j \in \{0, \ldots, 2^n\}$ if necessary) we may assume that $E \times F \subset \{(x, y) \in [0, 1]^2 : y \leq x\}$. Fix an arbitrary $\varepsilon \in (0, 1/4)$. Since E and F have positive measure, Steinhaus' theorem [see, for example,* Stromberg (1972)*] implies that $E - F$, defined by*

$$E - F := \{e - f : e \in E, f \in F\} \subseteq [0, 1],$$

contains an open interval I of positive length. Since the construction of Ω implies $\lambda(\Omega \cap I) > 0$, we may choose $\omega \in \Omega \cap I$ and some $r_0 \in (0, 1)$ with $(\omega - r_0, \omega + r_0) \subseteq I \subseteq E - F$ such that, for every $r \in (0, r_0]$, we have

$$\frac{1}{2r} \lambda\big(\Omega \cap (\omega - r, \omega + r)\big) > 1 - \varepsilon. \tag{12}$$

Choose $(e, f) \in E \times F$ with $e - f = \omega$. Then there exists $\Delta_0 > 0$ such that for every $\Delta \leq \Delta_0$

$$\frac{1}{2\Delta} \lambda\big(E \cap (e - \Delta, e + \Delta)\big) > 1 - \varepsilon \ \text{ and } \ \frac{1}{2\Delta} \lambda\big(F \cap (f - \Delta, f + \Delta)\big) > 1 - \varepsilon,$$

hence

$$\lambda_2\Big((E \times F) \cap \big((e - \Delta, e + \Delta) \times (f - \Delta, f + \Delta)\big)\Big) > (1 - \varepsilon)^2 4\, \Delta^2 \tag{13}$$

holds. On the other hand, considering $\delta < \frac{r_0}{2}$, we have

$$(e - \delta, e + \delta) - (f - \delta, f + \delta) = (\omega - 2\delta, \omega + 2\delta) \subseteq (\omega - r_0, \omega + r_0).$$

Hence, using the observation mentioned at the end of Step 1 and inequality (12)*, we get*

$$\lambda_2\Big(\Lambda^c \cap \big((e - \delta, e + \delta) \times (f - \delta, f + \delta)\big)\Big) > (1 - \varepsilon)^2 4\, \delta^2. \tag{14}$$

Considering $\Delta = \delta = \zeta$ *for some* $\zeta < \min\{\frac{r_0}{2}, \Delta_0\}$ *together with the inequalities* (13) *and* (14), $\lambda_2(\Lambda^c \cap (E \times F)) > 0$ *follows, which completes the proof of condition (II). Condition (III) can be proved analogously by working with* Ω^c *instead of* Ω.

Application 3 *In* Fredricks et al. (2005)*, it is shown how iterated function systems with probabilities (IFSP) can be used to construct two-dimensional copulas with fractal support. [for background on IFSP, fractals and Hausdorff dimension, we refer to* Falconer (2014), Kunze et al. (2012)*.] In particular, it is proved that, given an arbitrary* $s \in (1, 2)$ *there exists a copula* A_s *whose support has Hausdorff dimension* s*. [For an extension to the general multivariate setting, we refer to* Trutschnig and Fernández Sánchez (2012)*.]*

Spatially homogeneous copulas allow for an alternative short proof of this result. Let $s \in (0, 1)$ *be arbitrary but fixed, set* $L = \frac{1}{2^{1/s}} \in (0, \frac{1}{2})$ *and consider the (totally disconnected) IFS* $\{[0, 1], (w_i)_{i=1}^2\}$ *with*

$$w_1(x) = Lx, \quad w_2(x) = Lx + 1 - L.$$

Letting $\mathcal{K}([0, 1])$ *the family of all non-empty compact subsets of* $[0, 1]$*, the induced Hutchinson operator* $\mathcal{W} : \mathcal{K}([0, 1]) \to \mathcal{K}([0, 1])$*, defined by* $\mathcal{W}(E) = w_1(E) \cup w_2(E)$ *is easily seen to be a contraction on the complete metric space* $(\mathcal{K}([0, 1]), \delta_H)$*, where* δ_H *denotes the Hausdorff metric. Banach's fixed-point theorem implies the existence of a set* $Z^\star \in \mathcal{K}([0, 1])$ *invariant under* $\mathcal{W}$*, such that the Hausdorff dimension* $dim_H(Z^\star)$ *of* $Z^\star$ *of which is exactly* s*.*
Choosing $p_1 \in (0, 1)$*, setting* $p_2 = 1 - p_1$ *and considering the operator* $\mathcal{V} : \mathcal{P}([0, 1]) \to \mathcal{P}([0, 1])$*, defined by*

$$\mathcal{V}(\vartheta) = p_1 \vartheta^{w_1} + p_2 \vartheta^{w_2} \tag{15}$$

it is straightforward to verify that $\mathcal{V}$ *is a contraction on the complete metric space* $(\mathcal{P}([0, 1]), \delta_K)$ *where* δ_K *denotes the Kantorovich (or Wasserstein) metric on* $\mathcal{P}([0, 1])$*. Banach's fixed-point theorem implies the existence of a probability measure* $\vartheta^\star \in \mathcal{P}([0, 1])$ *(in fact even* $\vartheta^\star \in \mathcal{P}'([0, 1])$*) invariant under* $\mathcal{V}$*, the support of which coincides with* $Z^\star$*. Notice that, for the special case of* $s = \frac{\ln 2}{\ln 3}$ *and* $p_1 = p_2 = \frac{1}{2}$*, the set* $Z^\star$ *is the classical Cantor set and* $\vartheta^\star$ *is the probability measure corresponding to the Cantor staircase function. Figure* 2 *depicts a sample of this homogeneous copula. Obviously, the support* $Supp(A_{\vartheta^\star})$ *of the homogeneous copula* $A_{\vartheta^\star}$ *is given by*

$$Supp(A_{\vartheta^\star}) = \bigcup_{x \in [0,1]} \{x\} \times \overline{R_x(Z^\star)}.$$

Since the latter set has Hausdorff dimension $s + 1 \in (1, 2)$*, the alternative proof of the result is complete.*

Finally notice that, given a fixed $s \in (0, 1)$*, choosing* $\tilde{p_1} \neq p_1$ *and proceeding in the aforementioned manner yields another measure* $\tilde{\vartheta}^\star \in \mathcal{P}'([0, 1])$ *that, on the one hand, has the same support as* $\vartheta^\star$ *but, on the other hand, is singular with respect to*

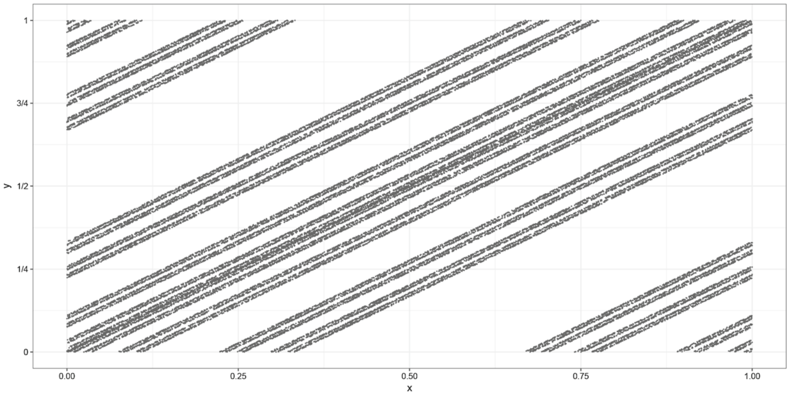

Fig. 2 Sample of size 50,000 of the singular spatially homogeneous copula $A_{\vartheta^\star}$ with $\vartheta^\star$ denoting the probability measure corresponding to the Cantor staircase function

$\vartheta^\star$. *As a consequence, the doubly stochastic measures* $\mu_{A_{\vartheta^\star}}$ *and* $\mu_{A_{\tilde{\vartheta}^\star}}$ *are singular with respect to each other and have the same compact set of Hausdorff dimension* $s+1$ *as support [compare with* Trutschnig and Fernández Sánchez (2014)*].*

4 A generalization of spatially homogeneous copulas

The concept of spatially homogeneous copulas can be extended in a natural way as shown below.

Definition 2 Suppose that $m, n \in \mathbb{N}$. Then $A \in \mathscr{C}$ is called (m, n)-*spatially homogeneous* if

$$\mu_A(x(m,n) \oplus G) = \mu_A(G) \tag{16}$$

holds for every $x \in [0, 1]$ and $G \in \mathscr{B}([0, 1]^2)$.

Obviously, $A \in \mathscr{C}$ is (m, n)-spatially homogeneous if and only if the measure μ_A is invariant under every transformation $\Phi_x^{m,n} : [0, 1]^2 \to [0, 1]^2$, defined by

$$\Phi_x^{m,n}(u, v) = \big(R_{mx}(u), R_{nx}(v)\big),$$

with $x \in [0, 1]$. In other words, $A \in \mathscr{C}$ is (m, n)-spatially homogeneous if and only if we have

$$\mu_A(R_{mx}(E) \times R_{nx}(F)) = \mu_A(E \times F) \tag{17}$$

for all $E, F \in \mathscr{B}([0, 1])$ and $x \in [0, 1]$.

In what follows, let the function $h_j^N : [0, 1] \to [\frac{j-1}{N}, \frac{j}{N}]$ be defined by $h_j^N(x) = \frac{x+(i-1)}{N}$ for every $N \in \mathbb{N}$ and every $j \in 1, \ldots, N$. Suppose that $A \in \mathscr{C}$ is (m, n)-spatially homogeneous and, without loss of generality, assume that m and n are relatively prime. Set $Q_{i,j} = [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{m}, \frac{j}{m}]$ for all $i \in \{1, \ldots, n\}$ and

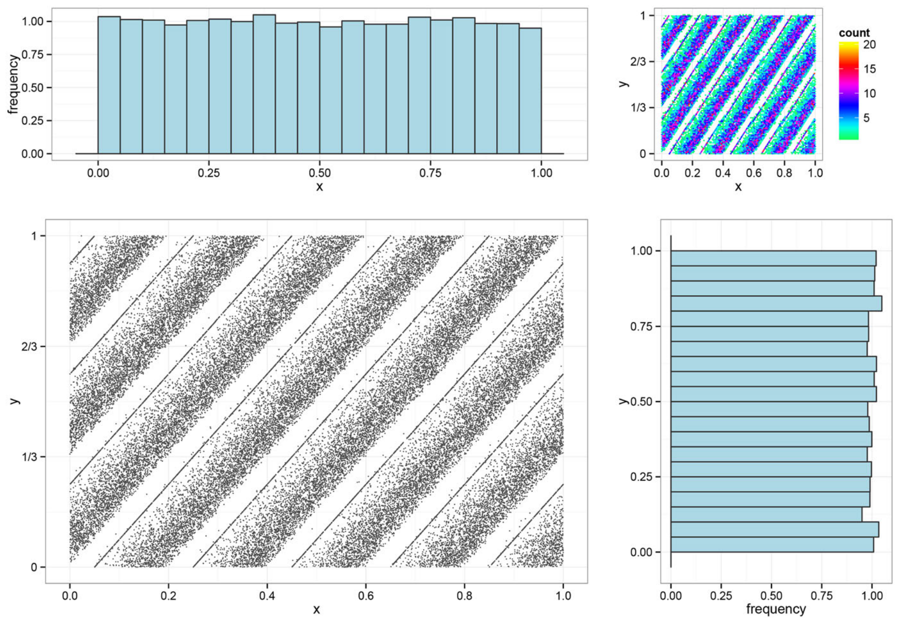

Fig. 3 Sample of size 50,000 of a (3, 5)-homogeneous copula A, its histogram and marginal histograms; therefore, the measure ϑ according to Eq. (20) has been chosen to be of the form $\vartheta = 0.1\,\delta_{3/4} + 0.9\,\beta_{2,5}$ with $\beta_{2,5}$ denoting the beta-distribution with parameters 2, 5

$j \in \{1, \ldots, m\}$ and let $w_{i,j} : [0,1]^2 \to Q_{i,j}$ denote the affine contraction

$$w_{i,j}(x, y) = \left(\frac{x + i - 1}{n}, \frac{y + j - 1}{m}\right) = \big(h_i^n(x), h_j^m(y)\big). \tag{18}$$

Defining $\Psi : [0,1]^2 \to [0,1]^2$ by

$$\Psi(x, y) = \Big(R_{\frac{1}{n}}(x), R_{\frac{1}{m}}(y)\Big),$$

the following property holds: for every pair $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, there exists exactly one $l \in \{0, 1, 2, \ldots, mn - 1\}$ such that $\Psi^l(Q_{1,1}) = Q_{i,j}$. Furthermore, (m, n)-spatial homogeneity yields that, for every $G \in \mathscr{B}([0,1]^2)$ with $G \subseteq Q_{i,j}$, we have

$$\mu_A(G) = \mu_A\big(\Psi^{-l}(G)\big) = \mu_A\big(\Psi^{-l}(G) \cap Q_{1,1}\big).$$

This implies that there exists a copula $B \in \mathscr{C}$ such that

$$\mu_A = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_B^{w_{i,j}}, \tag{19}$$

i.e. A is a τ-draughtboard of B with τ being the $n \times m$-dimensional transformation matrix having all entries equal to $\frac{1}{mn}$. [For the construction of such draughtboard copulas, see Durante et al. (2015).] Furthermore, (m, n)-spatial homogeneity of A implies that B is also spatially homogeneous, so altogether we get that A is a $n \times m$ draughtboard of a spatially homogeneous copula $B \in \mathscr{C}^H$. Since, on the other hand, every copula of the form (19) with $B \in \mathscr{C}^H$ is obviously (m, n)-spatially homogeneous, we have proved the following result.

Theorem 6 *A copula A is (m, n)-spatially homogeneous if and only if there exists $B_\vartheta \in \mathscr{C}^H$ such that the following equality holds:*

$$\mu_A = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_{B_\vartheta}^{w_{i,j}}. \tag{20}$$

In other words, (m, n)-spatially homogeneous copulas can be constructed via suitable push-forwards of a spatially homogeneous copula $B_\vartheta \in \mathscr{C}^H$. Figure 3 depicts a sample of a (3, 5)-homogeneous copula.

We will now derive a simple expression for the Markov operator and the Markov kernel of (m, n)-spatially homogeneous copulas and then characterize (m, n)–spatial homogeneity of a copula A in terms of (i) the Markov product $*$ of copulas and (ii) the Markov operator of A [see Darsow et al. (1992), Olsen et al. (1996), Trutschnig (2011), respectively]. To simplify notation, for ever $\vartheta \in \mathscr{P}'([0, 1])$ and every $m \in \mathbb{N}$, let the measure $\vartheta_m \in \mathscr{P}'([0, 1])$ be defined by

$$\vartheta_m = \frac{1}{m} \sum_{j=1}^{m} \vartheta^{h_j^m}.$$

Theorem 7 *Suppose that $A \in \mathscr{C}$ is (m, n)-spatially homogeneous and let B_ϑ denote the corresponding homogeneous copula according to Eq. (20). Then the Markov operator T_A and the Markov kernel K_A of A are given by*

$$(T_A f)(x) = \int_{[0,1]} f \circ R_{\frac{n}{m}x}(y) \mathrm{d}\vartheta_m(y) = \frac{1}{m} \sum_{j=1}^{m} \int_{[0,1]} f \circ R_{\frac{n}{m}x} \circ h_j^m(z) \mathrm{d}\vartheta(z) \tag{21}$$

and

$$K_A(x, F) = \vartheta_m^{R_{\frac{n}{m}x}}(F) = \frac{1}{m} \sum_{j=1}^{m} \vartheta^{h_j^m \circ R_{\frac{n}{m}x}}(F). \tag{22}$$

Proof Equation (22) is a direct consequence of Eq. (20), Theorem 2 and Eq. (5). Considering that, according to Trutschnig (2011), the Markov operator T_A of a copula A can be expressed in terms of the Markov kernel K_A of A via

$$(T_A f)(x) = \int_{[0,1]} f(y) K_A(x, \mathrm{d}y),$$

Equation (21) follows immediately. □

Translating the composition of Markov operators $T_A, T_B : L^1([0,1]) \to L^1([0,1])$ to the family $\mathscr{C}$ of copulas yields the Markov product of copulas, implicitly defined via $T_{A*B} = T_A \circ T_B$ [see Olsen et al. (1996), Trutschnig (2013)]. Generalizing (Brown 1966, Theorem 3) (m,n)-spatial homogeneity of a copula can be characterized in terms of the Markov product as follows:

Theorem 8 *A copula A is (m,n)-spatially homogeneous if and only if $A * C_{R_{nx}} = C_{R_{mx}} * A$ holds for every $x \in [0,1]$.*

Proof Suppose that T_A is of the form (21) and let $x \in [0,1]$ and $f \in L^1([0,1])$ be arbitrary but fixed. Setting $i = mx + z - R_0(mx+z) = \lfloor mx + z \rfloor \in \{0, 1, \ldots, m-1\}$, it is straightforward to verify that $R_{1-nx} \circ R_{\frac{n}{m} R_{mx}(z)}(y) = R_{\frac{n}{m}z} \circ R_{\frac{i}{m}}(y)$ holds for every $y \in [0,1]$, from which we get

$$\begin{aligned}
\big(T_{C_{R_{mx}}} \circ T_A \circ T^{-1}_{C_{R_{nx}}} f\big)(z) &= \big(T_{C_{R_{mx}}} \circ T_A \circ T_{C_{R_{1-nx}}} f\big)(z) \\
&= \big(T_{C_{R_{mx}}} \circ T_A \circ f \circ R_{1-nx}\big)(z) \\
&= T_{C_{R_{mx}}} \left(\int_{[0,1]} f \circ R_{1-nx} \circ R_{\frac{n}{m}z}(y) \mathrm{d}\vartheta_m(y) \right) \\
&= \int_{[0,1]} f \circ R_{1-nx} \circ R_{\frac{n}{m} R_{mx}(z)}(y) \mathrm{d}\vartheta_m(y) \\
&= \int_{[0,1]} f \circ R_{\frac{n}{m}z} \circ R_{\frac{i}{m}}(y) \mathrm{d}\vartheta_m(y) \\
&= \int_{[0,1]} f \circ R_{\frac{n}{m}z}(w) \mathrm{d}\vartheta_m(w) = (T_A f)(z),
\end{aligned}$$

whereby the penultimate equality follows from change of coordinates and the fact that $\vartheta_m^{R_{\frac{i}{m}}} = \vartheta_m$. We therefore know that $T_{C_{R_{mx}}} \circ T_A = T_A \circ T_{C_{R_{nx}}}$, which, translating to $\mathscr{C}$, reads $A * C_{R_{nx}} = C_{R_{mx}} * A$.

To prove the reverse implication, assume now that $T_{C_{R_{mx}}} \circ T_A = T_A \circ T_{C_{R_{nx}}}$ holds for every $x \in [0,1]$. Considering $f = \mathbf{1}_F \in L^1([0,1])$ for some $F \in \mathscr{B}([0,1])$, we get [again see Trutschnig (2011)]

$$\begin{aligned}
K_A(z, F) = (T_A \mathbf{1}_F)(z) &= \big(T_{C_{R_{mx}}} \circ T_A \circ T^{-1}_{C_{R_{nx}}} \mathbf{1}_F\big)(z) \\
&= \big(T_{C_{R_{mx}}} \circ T_A \mathbf{1}_{R_{nx}(F)}\big)(z) = T_A \mathbf{1}_{R_{nx}(F)}(R_{mx}(z)) \\
&= K_A(R_{mx}(z), R_{nx}(F)),
\end{aligned}$$

from which Eq. (17) easily follows via disintegration. □

5 On multivariate spatially homogeneous copulas

Most naturally the question arises whether the provided class of copulas can be generalized to any dimension $d \geq 3$. The answer is positive—we complete the manuscript by stating the corresponding definition and the representation theorem analogous to the two-dimensional setting and by providing a sketch of the proof.

Definition 3 A d-dimensional copula A is called spatially homogeneous if for every $G \in \mathscr{B}([0,1]^d)$ and every $x \in [0,1]$

$$\mu_A(x(1,\ldots,1) \oplus G) = \mu_A(G)$$

holds.

Theorem 9 *A d-dimensional copula A is spatially homogeneous if and only if there exists a probability measure ϑ on $\mathscr{B}([0,1)^{d-1})$ such that, for every $B \in \mathscr{B}([0,1]^d)$, we have*

$$\mu_A(B) = \int_{[0,1]} \vartheta^{\mathbf{R}_x}(B_x)\mathrm{d}\lambda(x), \tag{23}$$

where $B_x = \{(x_2,\ldots,x_d) \in [0,1]^{d-1} : (x,x_2,\ldots,x_d) \in B\}$ is the x-cut of B and $\mathbf{R}_x$ denotes the rotation of $[0,1]^{d-1}$ defined by

$$\mathbf{R}_x(y_2,\ldots,y_d) = (R_x(y_2),\ldots,R_x(y_d)).$$

Sketch of the proof. First suppose that ϑ is a probability measure on $\mathscr{B}([0,1)^{d-1})$. Then defining μ_A according to Eq. (23) obviously yields a probability measure μ_A on $\mathscr{B}([0,1]^d)$. Letting $\vartheta^\pi \in \mathscr{P}'([0,1])$ denote the push-forward of ϑ via the projection $\pi(y_2,\ldots,y_d) = y_2$ and considering

$$F_2 \times [0,1)^{d-2} \in \mathscr{B}([0,1)^{d-1})$$

we get

$$\vartheta^{\mathbf{R}_x}(F_2 \times [0,1)^{d-2}) = \vartheta^\pi(R_x^{-1}(F_2)),$$

from which (by applying Theorem 1) $\mu_A([0,1] \times F_2 \times [0,1)^{d-2}) = \lambda(F_2)$ follows immediately. Showing that all other univariate marginals of μ_A coincide with the uniform distribution on $[0,1]$ can be done analogously. Finally, the proof that μ_A generates a spatially homogeneous copula can be done as in the two-dimensional setting.

On the other hand, if A is a d-dimensional spatially homogeneous copula, then the existence of a probability measure ϑ on $\mathscr{B}([0,1)^{d-1})$ fulfilling Eq. (23) can be shown by adjusting each of the steps in the proof of Theorem 2 to the multivariate setting. [For properties of Markov kernels of multivariate copulas, see Fernández-Sánchez and Trutschnig (2015).] □

Figure 4 depicts a sample of size 20,000 of a three-dimensional homogeneous copula and its univariate marginals.

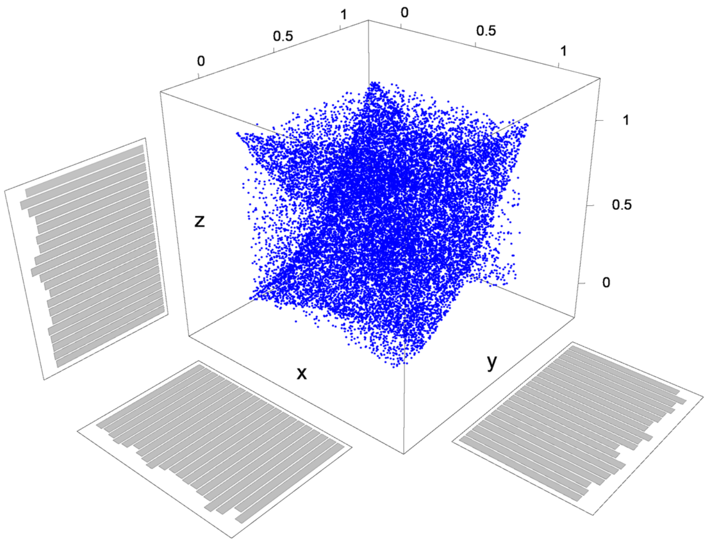

Fig. 4 Sample of size 20,000 of the three-dimensional spatially homogeneous copula with ϑ denoting the doubly stochastic measure corresponding to the two-dimensional Marshall–Olkin copula with parameters $(1, \frac{1}{2})$; histograms of its univariate marginals

6 Concluding remarks

We have focused on spatially homogeneous copulas, and we have shown how they can be helpful in providing novel stochastic models with some special features:

- they are generated by a unique probability measure in $[0, 1)$;
- they cover a broad range of concordance values;
- they include various examples with unusual properties with respect to the smoothness of the copula function and/or the existence of density/singular component.

Most remarkably, spatially homogeneous copulas can exhibit different types of symmetries and/or periodicity in the density. These latter aspects make them appealing in the study of circular data (and copulas for circular distributions); such a link will be the object of future investigations.

Acknowledgements The authors gratefully acknowledge the support of the grant MTM2014-60594-P (partially supported by FEDER) from the Spanish Ministry of Economy and Competitiveness. The first author has been partially supported by INdAM–GNAMPA Project 2017 "Bounds for Risk Functionals in Dependence Models".

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