Supplementary Material: Theoretical properties of bandwidth selectors for kernel density estimation on the circle

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Appendix B

We will derive the conditional expectation γ_i. If t is odd, then we obtain that the term $\int_{-\pi}^{\pi}\gamma(y)y^t dy = 0$, because the function $\gamma(y)$ is symmetry. By the binomial theorem, $\int_{-\pi}^{\pi}\gamma(y)y^{2t}dy$ is reduced to

$$\begin{aligned}\int_{-\pi}^{\pi}\gamma(y)y^{2t}dy &= \int K_\kappa(w)\int K_\kappa(s)(s-w)^{2t}dwds - 2\alpha_{2t}(K_\kappa)\\ &= \sum_{m=0}^{2t}(-1)^m{}_{2t}C_m\alpha_m(K_\kappa)\alpha_{2t-m}(K_\kappa) - 2\alpha_{2t}(K_\kappa). \qquad (58)\end{aligned}$$

Recalling that the kernel K_κ is second-order, by combining (58) and Lemma 3 , it is derived that

$$\int_{-\pi}^{\pi}\gamma(y)y^{2t}dy = \begin{cases}-1, & t=0,\\ 0, & t=1,\\ 24\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-2}+O(\kappa^{-3}), & t=2,\\ O(\kappa^{-3}), & t=3.\end{cases} \qquad (59)$$

Noting $\gamma(y)$ is a symmetric function, from (59), the conditional expectation γ_i is given by

$$\begin{aligned}\gamma_i &= \int_{-\pi}^{\pi}\gamma(\Theta_i-\theta_j)f(\theta_j)d\theta_j\\ &= \int_{-\pi}^{\pi}\gamma(y)f(\Theta_i+y)dy\\ &= \sum_{t=0}^{2}\frac{f^{(2t)}(\Theta_i)}{(2t)!}\int_{-\pi}^{\pi}\gamma(y)y^{2t}dy + O\left(\int\gamma(y)y^6dy\right)\\ &= -f(\Theta_i)+f^{(4)}(\Theta_i)\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-2}+O(\kappa^{-3}).\end{aligned}$$

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Appendix C

We derive each term of the variance $\mathrm{Var}_f[\overline{\mathrm{CV}}(\kappa)]$. We present the expectation $\mathrm{E}_f[\gamma_{ij}^2]$ as

$$\begin{aligned}\mathrm{E}_f[\gamma_{ij}^2] &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\gamma^2(\theta_i-\theta_j)f(\theta_i)f(\theta_j)d\theta_i d\theta_j \\ &= \int_{-\pi}^{\pi} f(\theta_j)\int_{-\pi}^{\pi}\gamma^2(u)f(\theta_j+u)dud\theta_j \\ &= \int f(\theta_j)\int_{-\pi}^{\pi}\gamma^2(u)[f(\theta_j)+O(u^2)]dud\theta_j \\ &= R(f)R(\gamma)+O(R(\gamma(y)y). \end{aligned} \tag{60}$$

We produce the following lemma regarding $R(\gamma(y)y^t)$.

Lemma 5. *We set* $Q_{2t}(L) := \int_{-\infty}^{\infty}\left\{2^{-1}\mu_0^{-2}(L)\eta(z) - 2^{1/2}\mu_0^{-1}(L)L(z^2/2)\right\}^2 z^{2t}dz$. *Then, the term* $R(\gamma(y)y^t)$ *is given by*

$$R(\gamma(y)y)^t = \kappa^{-(2t-1)/2}[Q_{2t}(L)+o(1)], \quad t=0,1.$$

Proof of Lemma 5. Let $y=\kappa^{-1/2}z$. Then, applying $\cos(\kappa^{-1/2}z) = 1 - z^2/(2\kappa) + O(\kappa^{-2})$, the Taylor expansion of $L_\kappa(\kappa^{-1/2}z)$ is given by

$$\begin{aligned} L_\kappa(\kappa^{-1/2}z) &= L(\kappa[1-\{1-z^2/(2\kappa)+O(\kappa^{-2})\}]) \\ &= L(z^2/2)+O(\kappa^{-1}). \end{aligned} \tag{61}$$

It follows from (61) that

$$\begin{aligned} \int_{-\pi}^{\pi} L_\kappa(w)L_\kappa(w+\kappa^{-1/2}z)dw &= \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} L_\kappa(\kappa^{-1/2}t)L_\kappa(\kappa^{-1/2}(t+z))\kappa^{-1/2}dt \\ &= \kappa^{-1/2}\int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} L(t^2/2)L((t+z)^2/2)dt + O(\kappa^{-3/2}) \\ &= \kappa^{-1/2}[\eta(z)+o(1)]. \end{aligned} \tag{62}$$

We put $Q_{\kappa^{1/2},2t}(L) := \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi}\left\{2^{-1}\mu_0^{-2}(L)\eta(z) - 2^{1/2}\mu_0^{-1}(L)L(z^2/2)\right\}^2 z^{2t}dz$. Then it holds from (b) and (e) that $Q_{\kappa^{1/2},2t}(L) = Q_{2t}(L)+o(1)$ for $t=0,1$. By combining (62) and Lemma 2, the term $R(\gamma(y)y^t)$ is given by

$$
\begin{aligned}
R(\gamma(y)y^t) &= \int_{-\pi}^{\pi} \left\{ \int_{-\pi}^{\pi} K_\kappa(w)K_\kappa(w+y)dw - 2K_\kappa(y) \right\}^2 y^{2t} dy \\
&= \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} \left\{ \int_{-\pi}^{\pi} K_\kappa(w)K_\kappa(w+\kappa^{-1/2}z)dw - 2K_\kappa(\kappa^{-1/2}z) \right\}^2 (\kappa^{-1/2}z)^{2t}\kappa^{-1/2}dz \\
&= \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} \left\{ C_\kappa^{-2}(L) \int_{-\pi}^{\pi} L_\kappa(w)L_\kappa(w+\kappa^{-1/2}z)dw - 2K_\kappa(\kappa^{-1/2}z) \right\}^2 (\kappa^{-1/2}z)^{2t}\kappa^{-1/2}dz \\
&= \kappa^{-(2t+1)/2} \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} \left\{ C_\kappa^{-2}(L)\kappa^{-1/2}[\eta(z)+o(1)] - 2C_\kappa^{-1}(L)[L(z^2/2)+O(\kappa^{-1})] \right\}^2 z^{2t}dz \\
&= \kappa^{-(2t+1)/2} \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} \Big\{ (\kappa^{-1/2}2^{1/2}\mu_0(L)+O(\kappa^{-3/2}))^{-2}\kappa^{-1/2}[\eta(z)+o(1)] \\
&\quad - 2(\kappa^{-1/2}2^{1/2}\mu_0(L)+O(\kappa^{-3/2}))^{-1}[L(z^2/2)+O(\kappa^{-1})] \Big\}^2 z^{2t}dz \\
&= \kappa^{-(2t+1)/2} \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} \left[\kappa^{1/2} \left\{ 2^{-1}\mu_0^{-2}(L)\eta(z) - 2^{1/2}\mu_0^{-1}(L)L(z^2/2) + o(1) \right\} \right]^2 z^{2t}dz \\
&= \kappa^{-(2t-1)/2}\left[Q_{\kappa,2t}(L)+o(1)\right] \\
&= \kappa^{-(2t-1)/2}[Q_{2t}(L)+o(1)].
\end{aligned}
$$

□

Noting that $Q_0(L) = Q(L)$, from combining (60) and Lemma 5, the expectation $\mathrm{E}_f[\gamma_{ij}^2]$ is given by

$$\mathrm{E}_f[\gamma_{ij}^2] = \kappa^{1/2}[Q(L)R(f)+o(1)]. \tag{63}$$

From combining (18) and (63), it follows that $\mathrm{Var}_f[\gamma_{ij}]$ is equivalent to (20).

Noting that $\gamma_i = \int_{-\pi}^{\pi} \gamma(\theta_i-\theta_j)f(\theta_j)d\theta_j$, then, from (17) we derive $\mathrm{E}_f[\gamma_{ij}\gamma_{ik}]$. That is,

$$
\begin{aligned}
\mathrm{E}_f[\gamma_{ij}\gamma_{ik}] &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \gamma(\theta_i-\theta_j)\gamma(\theta_i-\theta_k)f(\theta_i)f(\theta_j)f(\theta_k)d\theta_i d\theta_j d\theta_k \\
&= \int_{-\pi}^{\pi} f(\theta_i)\left[\int_{-\pi}^{\pi}\gamma(\theta_i-\theta_j)f(\theta_j)d\theta_j\right]^2 d\theta_i \\
&= R(f^{3/2}) - 2R((f^{(4)})^{1/2}f)\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-2} + o(\kappa^{-2}).
\end{aligned} \tag{64}
$$

By combining (18) and (64), we obtain that $\mathrm{Cov}_f[\gamma_{ij}, \gamma_{ik}]$ is equivalent to (22).

From (18), we derive that $\mathrm{E}_f[\gamma_{ij}f(\Theta_i)]$ is given by

$$
\begin{aligned}
\mathrm{E}_f[\gamma_{ij}f(\Theta_i)] &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \gamma(\theta_i-\theta_j)f(\theta_i)f(\theta_j)f(\theta_i)d\theta_i d\theta_j \\
&= -R(f^{3/2}) + R((f^{(4)})^{1/2}f)\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-2} + o(\kappa^{-2}).
\end{aligned} \tag{65}
$$

From combining (18) and (65), we derive that $\mathrm{Cov}_f[\gamma_{ij}, f(\Theta_i)]$ is given by (23).

The variance $\mathrm{Var}_f[f(\Theta_i)]$ is equivalent to

$$\begin{aligned}\mathrm{Var}_f[f(\Theta_i)] &= \mathrm{E}_f[f^2(\Theta_i)] - \mathrm{E}_f[f(\Theta_i)]^2 \\ &= R(f^{3/2}) - R(f)^2 \\ &= I_3.\end{aligned}$$

Appendix D

we derive the expectation $\mathrm{E}_f[U_{ij}^{2m}]$. That is,

$$\begin{aligned}\mathrm{E}_f[U_{ij}^{2m}] &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} T_g^{(4)}(\theta_i-\theta_j)^{2m} f(\theta_i)f(\theta_j)d\theta_i d\theta_j \\ &= \int_{-\pi}^{\pi} f(\theta_j)\int_{-\pi}^{\pi} T_g^{(4)}(u)^{2m} f(\theta_j+u)dud\theta_j \\ &= \int_{-\pi}^{\pi} f(\theta_j)\int_{-\pi}^{\pi} T_g^{(4)}(u)^{2m}[f(\theta_j)+O(u^2)]dud\theta_j \\ &= \psi_0 R(\{T^{(4)}\}_g^{2m}) + O(R(\{T_g^{(4)}(u)\}^{2m}u). \end{aligned} \tag{66}$$

Lemma 6. *The term $R(\{T_g^{(4)}(\theta)\}^m\theta^t)$ is given by*

$$R(\{T_g^{(4)}(\theta)\}^m\theta^t) = g^{(10m-2t-1)/2}\left\{G_{m,t}(S_4) + o(1)\right\}, \tag{67}$$

for $t = 0, 1$ and $m = 0, 1$.

Proof of Lemma 6. The Taylor expansions of $\cos(g^{-1/2}z)$ and $\sin g^{-1/2}z$ are reduced to

$$\cos(g^{-1/2}z) = 1 - z^2/(2g) + O(g^{-2}), \tag{68}$$

and,

$$\sin(g^{-1/2}z) = g^{-1/2}z + O(g^{-3/2}), \tag{69}$$

respectively. From considering (9), (61), (68), and (69), the approximation of $S_g^{(4)}(g^{-1/2}z)$ is given by

$$\begin{aligned}S_g^{(4)}(g^{-1/2}z) &= g^2\{S^{(2)}(z^2/2) + 6z^2S^{(3)}(z^2/2) + z^4S^{(4)}(z^2/2) + o(1)\} \\ &= g^2\{S_4(z^2/2) + o(1)\}.\end{aligned} \tag{70}$$

We set $\delta_{g^{1/2},t}(S_4^m) := \int_{-g^{1/2}\pi}^{g^{1/2}\pi} S_4^{2m}(z^2/2)z^{2t}dz$. Then, it holds from (g) that $\delta_{g^{1/2},t}(S_4^m) = \delta_t(S_4^m) + o(1)$ for $t = 0, 1$, and $m = 1, 2$. By combining Lemma 1 and (70), The term $R(\{T_g^{(4)}(\theta)\}^m\theta^t)$ is reduced to

$$
\begin{aligned}
R(\{T_g^{(4)}(\theta)\}^m\theta^t) &= C_g^{-2m}(S)\int_{-\pi}^{\pi}\left\{S_g^{(4)}(\theta)^m\theta^t\right\}^2 d\theta \\
&= C_g^{-2m}(S)\int_{-g^{1/2}\pi}^{g^{1/2}\pi}\left\{S_g^{(4)}(g^{-1/2}z)^m(g^{-1/2}z)^t\right\}^2 g^{-1/2}dz \\
&= C_g^{-2m}(S)g^{-(2t+1)/2}\int_{-g^{1/2}\pi}^{g^{1/2}\pi}[g^2\{S_4(z^2/2)+o(1)\}]^{2m}z^{2t}dz \\
&= \{2^{1/2}\mu_0^{-1}(S)g^{-1/2}+O(g^{-(p+1)/2})\}^{-2m}g^{(8m-2t+1)/2}\left\{\delta_{g^{1/2},t}(S_4^m)+o(1)\right\} \\
&= 2^{-m}\mu_0^{2m}(S)g^{(10m-2t-1)/2}\left\{\delta_t(S_4^m)+o(1)\right\} \\
&= g^{(10m-2t-1)/2}\left\{G_{m,t}(S_4)+o(1)\right\}.
\end{aligned}
$$

□

From combining (66) and Lemma 6, the expectation $\mathrm{E}_f[U_{ij}^{2m}]$ is given by

$$
\mathrm{E}_f[U_{ij}^{2m}] = g^{(10m-1)/2}[\psi_0 G_{m,0}(S_4)+o(1)]. \tag{71}
$$

It follows from (33) that

$$
\begin{aligned}
\mathrm{E}_f[U_i^2] &= \mathrm{E}_f[\{f^{(4)}(\Theta_i)+o(1)\}^2] \\
&= \mathrm{E}_f[f^{(4)}(\Theta_i)^2]+o(1).
\end{aligned}
$$

Appendix E

We calculate $\frac{d}{d\kappa}\gamma(y_{ij})$. We derive

$$
\frac{d}{d\kappa}L_\kappa(w)L_\kappa(w+y) = L_\kappa'(w)L_\kappa(w+y)\{1-\cos(w)\}+L_\kappa(w)L_\kappa'(w+y)\{1-\cos(w+y)\}. \tag{72}
$$

We set $\frac{d}{d\kappa}C_\kappa(L)=C_\kappa'(L)$ and $\alpha_t(\phi_\kappa) := \int_{-\pi}^{\pi}\phi_\kappa(y)y^t dy$. It follows that

$$
\begin{aligned}
\kappa C_\kappa^{-1}(L)C_\kappa'(L) &= \kappa C_\kappa^{-1}(L)\int_{-\pi}^{\pi}\frac{d}{d\kappa}L_\kappa(\theta)d\theta \\
&= \alpha_0(\phi_\kappa).
\end{aligned} \tag{73}
$$

We provide the following lemma regarding $\alpha_t(\phi_\kappa)$.

Lemma 7. *The term $\alpha_\kappa(\phi_\kappa)$ is given by*

$$
\alpha_t(\phi_\kappa) = \begin{cases} -\frac{1}{2}-\frac{3}{8}\mu_0^{-1}(L)\mu_2(L)\kappa^{-1}+O(\kappa^{-2}), & t=0, \\ -3\mu_0^{-1}(L)\mu_2(L)\kappa^{-1}+O(\kappa^{-2}), & t=2, \\ 24\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-2}=O(\kappa^{-2}), & t=4. \end{cases}
$$

Proof of Lemma 7. From (b), the partial integration of $\mu_{\kappa,l}(L') := \int_0^\kappa L(r) r^{(l-1)/2} dr$ for $l \le 4$ is

$$\begin{aligned}\mu_{\kappa,l}(L') &= [L(r)r^{(l-1)/2}]_0^\kappa - \frac{l-1}{2}\int_0^\kappa L(r)r^{(l-3)/2}dr \\ &= -\frac{l-1}{2}\mu_{\kappa,l-2}(L) + O(\kappa^{-3}). \end{aligned} \tag{74}$$

The term $\alpha_{2t}(\phi_\kappa)$ is divided into the following two terms. That is,

$$\alpha_{2t}(\phi_\kappa) = 2\int_0^{\pi/2} \phi_\kappa(\theta)\theta^{2t}d\theta + 2\int_{\pi/2}^{\pi} \phi_\kappa(\theta)\theta^{2t}d\theta. \tag{75}$$

Recalling that we chose the second-order kernel for LSCV, the second term of (75) is ignored from combining (d), (74), and Lemma 2. That is,

$$\begin{aligned} 2\int_{\pi/2}^{\pi} \phi_\kappa(\theta)\theta^{2t}d\theta &\le 2\pi^{2t}\int_{\pi/2}^{\pi}\phi_\kappa(\theta)d\theta \\ &\le 2\pi^{2t}C_\kappa^{-1}(L)\int_{\pi/2}^{\pi} L'(\kappa\{1-\cos(\theta)\})\kappa\{1-\cos(\theta)\}d\theta \\ &= 2\pi^{2t}C_\kappa^{-1}(L)\int_\kappa^{2\kappa} L'(r)r\{r\kappa(2-r/\kappa)\}^{-1/2}dr \\ &= 2\pi^{2t}C_\kappa^{-1}(L)\kappa^{-1/2}\int_\kappa^{2\kappa} L'(r)r^{1/2}dr\{2^{-1/2}+O(\kappa^{-1})\} \\ &= O(\kappa^{-3}). \end{aligned} \tag{76}$$

By considering (d), (74), and (75),we derive the terms $\alpha_0(\phi_\kappa)$, $\alpha_2(\phi_\kappa)$, and $\alpha_4(\phi_\kappa)$. That is,

$$\begin{aligned} \alpha_0(\phi_\kappa) &= 2\int_0^{\pi/2}\phi_\kappa(\theta)d\theta + O(\kappa^{-3}) \\ &= 2C_\kappa^{-1}(L)\int_0^\kappa L'(r)r\{r\kappa(2-r\kappa)\}^{-1/2}dr + O(\kappa^{-3}) \\ &= 2C_\kappa^{-1}(L)\kappa^{-1/2}\int_0^\kappa L'(r)r^{1/2}[2^{-1/2} - 2^{-5/2}r/\kappa + O(\kappa^{-2})]dr + O(\kappa^{-3}) \\ &= 2C_\kappa^{-1}(L)\kappa^{-1/2}[2^{-1/2}\mu_{2,\kappa}(L') - 2^{-5/2}\kappa^{-1}\mu_{4,\kappa}(L') + O(\kappa^{-2})] + O(\kappa^{-3}) \\ &= -\frac{1}{2} - \frac{3}{8}\mu_0^{-1}(L)\mu_2(L)\kappa^{-1} + O(\kappa^{-2}), \end{aligned} \tag{77}$$

$$\begin{aligned} \alpha_2(\phi_\kappa) &= 2C_\kappa^{-1}(L)\int_0^{\pi/2} L'(\kappa\{1-\cos(\theta)\})\kappa\{1-\cos(\theta)\}\theta^2 d\theta + O(\kappa^{-3}) \\ &= 2C_\kappa^{-1}(L)\int_0^\kappa L'(r)r[r/\kappa(2-r/\kappa)+O(\kappa^{-2})]\{r\kappa(2-r/\kappa)\}^{-1/2}dr + O(\kappa^{-3}) \\ &= 2C_\kappa^{-1}(L)\kappa^{-3/2}\int_0^\kappa L'(r)r^{3/2}(2-r/\kappa)^{1/2}dr + O(\kappa^{-2}) \\ &= 2C_\kappa^{-1}(L)\kappa^{-3/2}\mu_{4,\kappa}(L')\{2^{1/2}+O(\kappa^{-1})\} + O(\kappa^{-2}) \\ &= 2\mu_0^{-1}(L)(-3\mu_2(L)/2)\kappa^{-1} + O(\kappa^{-2}) \\ &= -3\mu_0^{-1}(L)\mu_2(L)\kappa^{-1} + O(\kappa^{-2}), \end{aligned}$$

and

$$
\begin{aligned}
\alpha_4(\phi_\kappa) &= 2\int_0^{\pi/2} \phi_\kappa(\theta)\theta^4 d\theta + O(\kappa^{-3}) \\
&= 2C_\kappa^{-1}(L)\int_0^{\kappa} L'(r)r[\{r/\kappa(2-r/\kappa)\}^2 + O(\kappa^{-3})]\{r\kappa(2-r/\kappa)\}^{-1/2}dr + O(\kappa^{-3}) \\
&= O(\kappa^{-2}). \qquad (78)
\end{aligned}
$$

□

Then, by combining (72), (73), and Lemma 7, it follows that

$$
\begin{aligned}
\frac{d\gamma(y_{ij})}{d\kappa} &= \frac{d}{d\kappa}\left\{C_\kappa^{-2}(L)\int_{-\pi}^{\pi} L_\kappa(w)L_\kappa(w+y_{ij})dw - 2C_\kappa^{-1}(L)L_\kappa(y_{ij})\right\} \\
&= -2C_\kappa^{-3}(L)C_\kappa'(L)\int_{-\pi}^{\pi} L_\kappa(w)L_\kappa(w+y_{ij})dw \\
&+ C_\kappa^{-2}(L)\int_{-\pi}^{\pi} \frac{d}{d\kappa}\{L_\kappa(w)L_\kappa(w+y_{ij})\}dw \\
&+ 2C_\kappa^{-2}(L)C_\kappa'(L)L_\kappa(y_{ij}) - 2C_\kappa^{-1}(L)\frac{d}{d\kappa}L_\kappa(y_{ij}) \\
&= \kappa^{-1}\Bigg[-2\alpha_0(\phi_\kappa)\int_{-\pi}^{\pi} K_\kappa(w)K_\kappa(w+y_{ij})dw \\
&+ \int_{-\pi}^{\pi}\{\phi_\kappa(w)K_\kappa(w+y_{ij}) + K_\kappa(w)\phi_\kappa(w+y_{ij})\}dw \\
&+ 2\alpha_0(\phi_\kappa)K_\kappa(y_{ij}) - 2\phi_\kappa(y_{ij})\Bigg] \\
&= \kappa^{-1}\Bigg[\int_{-\pi}^{\pi} K_\kappa(w)K_\kappa(w+y_{ij})dw - 2K_\kappa(y_{ij}) \\
&+ K_\kappa(y_{ij}) + \int_{-\pi}^{\pi}\{\phi_\kappa(w)K_\kappa(w+y_{ij}) + K_\kappa(w)\phi_\kappa(w+y_{ij})\}dw - 2\phi_\kappa(y_{ij}) \\
&+ \frac{3}{4}\mu_0^{-1}(L)\mu_2(L)\kappa^{-1}\left\{\int_{-\pi}^{\pi} K_\kappa(w)K_\kappa(w+y_{ij})dw - K_\kappa(y_{ij})\right\}\Bigg] \\
&= \kappa^{-1}[\gamma(y_{ij}) + \rho(y_{ij}) + 3/4\mu_0^{-1}(L)\mu_2(L)\kappa^{-1}\tau(y_{ij})] \\
&= \kappa^{-1/2}V_{ij}. \qquad (79)
\end{aligned}
$$

We obtain (40) from (39) and (79).

Appendix F

Let $\rho_i := \mathrm{E}_f[\rho_{ij}|\Theta_i]$ and $\tau_i := \mathrm{E}_f[\tau_{ij}|\Theta_i]$. Then, the conditional expectation V_i is presented as the following linear combination of the three conditional expectations: γ_i, ρ_i, and τ_i.

$$
V_i = \kappa^{-1/2}\left[\gamma_i + \rho_i + \frac{3}{4}\mu_0^{-1}(L)\mu_2(L)\kappa^{-1}\tau_i\right]. \qquad (80)
$$

We present the following lemma regarding ρ_i.

Lemma 8. *The conditional expectation ρ_i is given by*

$$\rho_i = f(\Theta_i) - 3\left[\frac{f^{(2)}(\Theta_i)}{4} + f^{(4)}(\Theta_i)\right]\mu_0^{-2}\mu_2^2(L)\kappa^{-2} + O(\kappa^{-3}). \tag{81}$$

Proof of Lemma 8. The term $\alpha_{2t}(\rho) = \int_{-\pi}^{\pi} \rho(y)y^{2t}dy$ is given by

$$\begin{aligned}\alpha_{2t}(\rho) &= \alpha_{2t}(K_\kappa) \\ &\quad + \kappa C_\kappa^{-2}(L)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{\phi_\kappa(w)K_\kappa(w+y) + K_\kappa(w)\phi_\kappa(w+y)\}y^{2t}dwdy - 2\alpha_{2t}(\phi_\kappa).\end{aligned} \tag{82}$$

The second term of (82) is reduced to

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{\phi_\kappa(w)K_\kappa(w+y) + K_\kappa(w)\phi_\kappa(w+y)\}y^{2t}dwdy \\ &\quad = 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_\kappa(w)\phi_\kappa(s)\}(s-w)^{2t}dwdy \\ &\quad = 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_\kappa(w)\phi_\kappa(s)\left[\sum_{m=0}^{2t}(-1)^m {}_{2t}C_m w^m s^{2t-m}\right]dwds \\ &\quad = 2\sum_{m=0}^{2t}(-1)^m {}_{2t}C_m \alpha_m(K_\kappa)\alpha_{2t-m}(\phi_\kappa).\end{aligned} \tag{83}$$

It follows from (82) and (83) that

$$\alpha_0(\rho) = \alpha_0(K_\kappa) = 1, \tag{84}$$

and

$$\alpha_{2t}(\rho) = \alpha_{2t}(K_\kappa) + 2\sum_{m=1}^{2t}(-1)^m {}_{2t}C_m \alpha_m(K_\kappa)\alpha_{2t-m}(\phi_\kappa) \quad t \geq 1. \tag{85}$$

From combining Lemmas 3 and 7, and (85), it follows that

$$\alpha_2(\rho) = -\frac{3}{2}\mu_0^{-2}\mu_2^2(L)\kappa^{-2} + O(\kappa^{-3}), \tag{86}$$

$$\alpha_4(\rho) = -72\mu_0^{-2}\mu_2^2(L)\kappa^{-2} + O(\kappa^{-3}), \tag{87}$$

and,

$$\alpha_6(\rho) = O(\kappa^{-3}). \tag{88}$$

By combining (84), (86), (87), and (88), we obtain the conditional expectation ρ_i. That is,

$$\begin{aligned}\rho_i &= \int_{-\pi}^{\pi} \rho(\theta_j - \Theta_i) f(\theta_j) d\theta_j \\ &= \int_{-\pi}^{\pi} \rho(y) f(\Theta_i + y) dy \\ &= \sum_{t=0}^{2} \frac{f^{(2t)}(\Theta_i)}{(2t)!} \alpha_{2t}(\rho) + O(\alpha_6(\rho)) \\ &= f(\Theta_i) - 3\left[\frac{f^{(2)}(\Theta_i)}{4} + f^{(4)}(\Theta_i)\right] \mu_0^{-2} \mu_2^2(L) \kappa^{-2} + O(\kappa^{-3}). \end{aligned} \tag{89}$$

□

We present the following lemma regarding τ_i.

Lemma 9. *The conditional expectation τ_i is given by*

$$\tau_i = f^{(2)}(\Theta_i) \mu_0^{-1} \mu_2(L) \kappa^{-1} + O(\kappa^{-2}). \tag{90}$$

Proof of Lemma 9. We set $\alpha_t(\tau) := \int_{-\pi}^{\pi} \tau(y) y^t dy$. Then, it follows that

$$\alpha_{2t}(\tau) = \sum_{m=0}^{2t} (-1)^m {}_{2t}C_m \alpha_m(K_\kappa) \alpha_{2t-m}(K_\kappa) - \alpha_{2t}(K_\kappa). \tag{91}$$

From combining Lemma 3 and (91) It follows that the terms $\alpha_0(\tau)$, $\alpha_2(\tau)$ and $\alpha_4(\tau)$ are equal to,

$$\alpha_0(\tau) = 0, \tag{92}$$
$$\alpha_2(\tau) = 2\mu_0^{-1}(L)\mu_2(L)\kappa^{-1} + O(\kappa^{-1}), \tag{93}$$

and,

$$\alpha_4(\tau) = O(\kappa^{-2}), \tag{94}$$

respectively. It is shown from (92), (93) and (94) that

$$\begin{aligned}\tau_i &= \int_{-\pi}^{\pi} \tau(\theta_j - \Theta_i) f(\theta_j) d\theta_j \\ &= \int_{-\pi}^{\pi} \tau(y) f(\Theta_i + y) dy \\ &= f(\Theta_i)\alpha_0(\tau) + \frac{f^{(2)}(\Theta_i)}{2} \alpha_2(\tau) + O(\alpha_4(\tau)) \\ &= f^{(2)}(\Theta_i) \mu_0^{-1}(L) \mu_2(L) \kappa^{-1} + O(\kappa^{-2}).\end{aligned}$$

By combining (17), (80), and Lemmas 8 and 9, The conditional expectation V_i is reduced to

$$V_i = \kappa^{-1/2}\left[\gamma_i + \rho_i + \frac{3}{4}\mu_0^{-1}(L)\mu_2(L)\kappa^{-1}\tau_i\right]$$
$$= -2f^{(4)}(\Theta_i)\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-5/2} + o(\kappa^{-5/2}). \quad (95)$$

The expectations of V_i and V_i^2 are given by

$$\mathrm{E}_f[V_i] = -2R(f'')\mu_0^{-2}(L)\mu_2^2(L)\kappa^{-5/2} + o(\kappa^{-5/2}), \quad (96)$$

and

$$\mathrm{E}_f[V_i^2] = 4[R(f^{(4)}f^{1/2})]\mu_0^{-4}(L)\mu_2^4(L)\kappa^{-5} + o(\kappa^{-5}), \quad (97)$$

respectively. We obtain the variance of X_i from (96) and (97). That is,

$$\mathrm{Var}_f[X_i] = 4[R(f^{(4)}f^{1/2}) - R(f'')^2]\mu_0^{-4}(L)\mu_2^4(L)\kappa^{-5} + o(\kappa^{-5}). \quad (98)$$

From (98), we show that the variance $\mathrm{Var}_f[X_i]$ is finite. Thus, we obtain (42) from the central limit theorem.

Appendix G

We derive the expectation $\mathrm{E}_f[V_{ij}^{2m}]$. That is,

$$\mathrm{E}_f[V_{ij}^{2m}] = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[\kappa^{-1/2}\{\gamma(\theta_i-\theta_j) + \rho(\theta_i-\theta_j) + O(\kappa^{-1})\}]^{2m} f(\theta_i)f(\theta_j)d\theta_i d\theta_j$$
$$= \kappa^{-m}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{\gamma(\theta_i-\theta_j) + \rho(\theta_i-\theta_j)\}^{2m} f(\theta_i)f(\theta_j)d\theta_i d\theta_j\right]\{1+o(1)\}$$
$$= \kappa^{-m}\left[R((\gamma+\rho)^m)R(f) + O(R((\gamma+\rho)^m y))\right]. \quad (99)$$

Lemma 10. *The term $R((\gamma+\rho)^m y^t)$ is given by*

$$R((\gamma+\rho)^m y^t) = \kappa^{(2m-2t-1)/2}[M_{m,t}(L) + o(1)].$$

Proof of Lemma 10. We set

$$\psi(y) = \int_{-\pi}^{\pi} L'(\kappa\{1-\cos(w)\})\kappa\{1-\cos(w)\}L(\kappa\{1-\cos(w+y)\})dw.$$

Then, the term $\int_{-\pi}^{\pi}\frac{d}{d\kappa}\{L_\kappa(w)L_\kappa(w+y)\}dw$ reduces to

$$\kappa\int_{-\pi}^{\pi}\frac{d}{d\kappa}\{L_\kappa(w)L_\kappa(w+y)\}dw = \psi(y) + \psi(-y). \quad (100)$$

We set $\lambda_{\kappa^{1/2}}(L) := \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} L'(t^2/2)L((t+z)^2/2)t^2/2dt$. Then, it holds that $\lambda_{\kappa^{1/2}}(L) = \lambda(L)+o(1)$ from (f). Thus, it follows that

$$\begin{aligned}
\psi(\kappa^{-1/2}z) &= \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} L'(\kappa\{1-\cos(\kappa^{-1/2}t)\})\kappa\{1-\cos(\kappa^{-1/2}t)\}L(\kappa\{1-\cos(\kappa^{-1/2}(t+z))\})\kappa^{-1/2}dt \\
&= \kappa^{-1/2}\left[\int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi} L((t+z)^2/2)L'(t^2/2)t^2/2dt + O(\kappa^{-1})\right] \\
&= \kappa^{-1/2}\left[\lambda_{\kappa^{1/2}}(z) + O(\kappa^{-1})\right] \\
&= \kappa^{-1/2}\left[\{\lambda(z)+o(1)\} + O(\kappa^{-1})\right] \\
&= \kappa^{-1/2}[\lambda(z)+o(1)]. \qquad (101)
\end{aligned}$$

We set $M_{\kappa,m,t}(L) := \int_{-\kappa\pi}^{\kappa\pi} m(L)^{2m}z^{2t}dz$. Then, it holds from combining (b), (e), and (f) that $M_{\kappa,m,t}(L) = M_{m,t}(L)+o(1)$. From considering this, (62), and (101), The term $R(\{\gamma+\rho\}^m y^t)$ is reduced to

$$\begin{aligned}
R((\gamma+\rho)^m y^t) &= \int_{-\pi}^{\pi}\{\gamma(y)+\rho(y)\}^{2m}y^{2t}dy \\
&= \int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi}\Bigg[C_\kappa^{-2}(L)\left\{\int_{-\pi}^{\pi}L_\kappa(w)L_\kappa(w+\kappa^{-1/2}z)dw + \psi(\kappa^{-1/2}z)+\psi(-\kappa^{-1/2}z)\right\} \\
&\quad - C_\kappa^{-1}(L)\{L_\kappa(\kappa^{-1/2}z) + 2L'(\kappa\{1-\cos(\kappa^{-1/2}z)\})\kappa\{1-\cos(\kappa^{-1/2}z)\}\}\Bigg]^{2m} \\
&\quad \times (\kappa^{-1/2}z)^{2t}\kappa^{-1/2}dz \\
&= \kappa^{-(2t+1)/2}\int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi}\Bigg[C_\kappa^{-2}(L)\kappa^{-1/2}\{\eta(z)+\lambda(z)+\lambda(-z)+o(1)\} \\
&\quad - C_\kappa^{-1}(L)\{L(z^2/2)+L'(z^2/2)z^2+O(\kappa^{-1})\}\Bigg]^{2m} z^{2t}dz \\
&= \kappa^{-(2t+1)/2}\int_{-\kappa^{1/2}\pi}^{\kappa^{1/2}\pi}\left[\kappa^{1/2}\left\{\frac{\eta(z)+\lambda(z)+\lambda(-z)}{2\mu_0^2(L)} - \frac{L(z^2/2)+L'(z^2/2)z^2}{2^{1/2}\mu_0(L)} + o(1)\right\}\right]^{2m} \\
&\quad \times z^{2t}dz \\
&= \kappa^{(2m-2t-1)/2}[M_{\kappa,m,t}(L)+o(1)] \\
&= \kappa^{(2m-2t-1)/2}[M_{m,t}(L)+o(1)].
\end{aligned}$$

□

From combining (99) and Lemma 10, it follows that

$$\mathrm{E}_f[V_{ij}^{2m}] = \kappa^{-1/2}[M_{m,0}(L)R(f)+o(1)]. \qquad (102)$$

From (95), it follows that $V_i = O(\kappa^{-5/2})$. Then, The expectation $\mathrm{E}_f[H_{ij}^{2m}]$ is reduced to

$$\begin{aligned}
\mathrm{E}_f[H_{ij}^{2m}] &= \mathrm{E}_f[\{V_{ij}-V_i-V_j+\mathrm{E}_f[V_{ij}]\}^{2m}] \\
&= \mathrm{E}_f[V_{ij}^{2m}]\{1+o(1)\} \\
&= \kappa^{-1/2}[M_{m,0}(L)R(f)+o(1)]. \qquad (103)
\end{aligned}$$

Noting that V_{ii} is a constant, it follows that

$$\begin{aligned} G_{ij} &:= \mathrm{E}_f[H_{ii}H_{ij}] \\ &= \mathrm{E}_f[\{V_{ii} + \mathrm{E}_f[V_i] - 2V_i\}\{V_{ij} - V_i - V_j + \mathrm{E}_f[V_i]\}] \\ &= 0 - 2\mathrm{E}_f[V_i^2] + 2\mathrm{E}_f[V_i^2] + 2\mathrm{E}_f[V_i]^2 - 2\mathrm{E}_f[V_i]^2 \\ &= 0. \end{aligned} \tag{104}$$

From (103) and (104), it follows that the U-statistic H_{ij} satisfies (43). That is,

$$\begin{aligned} \frac{\mathrm{E}_f[G_{ij}^2] + n^{-1}\mathrm{E}_f[H_{ij}^4]}{\mathrm{E}_f[H_{ij}^2]^2} &= \frac{n^{-1}[\kappa^{-1/2}[M_{2,0}(L)R(f) + o(1)]}{[\kappa^{-1/2}M_{1,0}(L)R(f) + o(1)]^2} \\ &= o(1). \end{aligned} \tag{105}$$

We obtain the asymptotic normality for (44) from (105).

Appendix H

Let $g^{-9/4}W_{ij} = Q_{ij}$. From (38), the expectation $\mathrm{E}_f[Q_{ij}^2]$ is given by

$$\begin{aligned} \mathrm{E}_f[Q_{ij}^2] &= g^{-9/2}\mathrm{E}_f[W_{ij}^2] \\ &= G_{1,0}(S_4)\psi_0 + o(1). \end{aligned} \tag{106}$$

From combining (33), (34), and Lemma 6, it follows that

$$\begin{aligned} \mathrm{E}_f[Q_{ij}^4] &= g^{-9}\mathrm{E}_f[W_{ij}^4] \\ &= g^{-9}\mathrm{E}_f[U_{ij}^4]\{1 + o(1)\} \\ &= g^{1/2}\{G_{2,0}(S_4)\psi_0 + o(1)\}. \end{aligned} \tag{107}$$

By combining $G_{ij} = 0$, (106), and (107), it follows that that

$$\begin{aligned} \frac{\mathrm{E}_f[G_{ij}^2] + n^{-1}\mathrm{E}_f[Q_{ij}^4]}{\mathrm{E}_f[Q_{ij}^2]^2} &= \frac{0 + n^{-1}[G_{2,0}(S_4)\psi_0 g^{1/2} + o(g^{1/2})]}{[G_{1,0}(S_4)\psi_0 + o(1)]^2} \\ &= o(1). \end{aligned} \tag{108}$$

the d-generate U statistic Q_{ij} satisfies Lemma 4 by (108). Therefore, as $n \to \infty$, it holds that

$$\sum_{i<j} Q_{ij} \xrightarrow{d} N(0, n^2 G_{1,0}(S_4)\psi_0/2). \tag{109}$$

We obtain the asymptotic normality for (53) from (109).

Table 3: The means (its standard error) of the integrated squared errors ISE $\times 10^3$ (in the number of repetitions $N = 10000$) of the three selectors in models 1–8 (and sample sizes n are 50, 100, 200, 500, and 1000) in the simulation in section 5.

n	Selecter	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8
50	DPI.vM	9.845(0.073)	8.280(0.062)	14.157(0.093)	11.219(0.081)	10.044(0.074)	14.405(0.092)	10.177(0.065)	17.896(0.097)
	DPI.wC	12.717(0.095)	10.335(0.081)	16.804(0.113)	15.169(0.109)	13.460(0.102)	17.637(0.115)	11.755(0.086)	17.280(0.106)
	LSCV	15.734(0.160)	14.053(0.155)	20.123(0.165)	17.437(0.167)	16.089(0.163)	20.416(0.165)	15.902(0.152)	21.184(0.155)
100	DPI.vM	5.783(0.042)	4.794(0.034)	8.633(0.055)	6.609(0.045)	5.832(0.040)	8.800(0.055)	6.221(0.035)	10.927(0.059)
	DPI.wC	7.036(0.048)	5.621(0.039)	9.465(0.056)	8.312(0.052)	7.300(0.048)	9.933(0.058)	6.651(0.041)	10.051(0.058)
	LSCV	8.552(0.080)	7.300(0.073)	11.278(0.082)	9.510(0.080)	8.608(0.077)	11.623(0.083)	8.779(0.070)	12.014(0.078)
200	DPI.vM	3.424(0.023)	2.791(0.019)	5.174(0.032)	3.944(0.026)	3.427(0.022)	5.283(0.032)	3.922(0.020)	6.384(0.035)
	DPI.wC	3.979(0.025)	3.163(0.020)	5.394(0.030)	4.713(0.028)	4.103(0.025)	5.675(0.032)	4.011(0.022)	5.792(0.033)
	LSCV	4.646(0.038)	3.969(0.036)	6.303(0.041)	5.301(0.041)	4.736(0.039)	6.508(0.043)	5.103(0.035)	6.709(0.041)
500	DPI.vM	1.701(0.011)	1.370(0.008)	2.605(0.015)	1.958(0.012)	1.713(0.010)	2.670(0.015)	2.064(0.010)	3.090(0.016)
	DPI.wC	1.902(0.011)	1.506(0.009)	2.628(0.014)	2.222(0.012)	1.964(0.011)	2.758(0.014)	2.042(0.010)	2.848(0.015)
	LSCV	2.148(0.016)	1.777(0.014)	2.982(0.018)	2.432(0.017)	2.176(0.016)	3.078(0.018)	2.468(0.014)	3.180(0.017)
1000	DPI.vM	0.996(0.006)	0.799(0.005)	1.542(0.009)	1.157(0.007)	1.006(0.006)	1.591(0.009)	1.266(0.006)	1.763(0.009)
	DPI.wC	1.090(0.006)	0.864(0.005)	1.537(0.008)	1.288(0.007)	1.125(0.006)	1.612(0.008)	1.236(0.006)	1.655(0.008)
	LSCV	1.208(0.008)	0.991(0.007)	1.706(0.009)	1.380(0.009)	1.224(0.008)	1.771(0.010)	1.441(0.007)	1.809(0.009)