Supplemental Appendix to Semiparametric quantile regression with random censoring

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August 2018

Abstract

This supplemental appendix additional results and all the proofs of the results presented in Semiparametric quantile regression with random censoring

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1 Two-step semiparametric quantile regression estimators

Two-step semiparametric estimators are important in the econometric literature, where it is often the case that the nonparametric component is given by a latent variable, such as consumer preferences or expected demands or supplies, which can be represented as a conditional expectation, see for example Ahn & Manski (1993).

Consider the same semiparametric quantile regression model

$$Q_{Y|X}(\tau|X) = \inf(t : \Pr(Y \leq t|X) \geq \tau) = X_1^T \beta_{0\tau} + \theta_{0\tau}(X_2),$$

as that given in (2.1) in the main paper, and suppose that there is a preliminary consistent estimator $\widetilde{\theta}(\cdot)$ for $\theta_0(\cdot)$, assumed to have the following linear representation

$$(nb_2)^{1/2}\left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right) = \frac{1}{(nb_2)^{1/2}} \sum_{j=1}^{n} \eta(X_j) L_{2b_2}(X_{2j} - X_{2i}) + o_p(1), \tag{1.1}$$

where $L_{2b_2}(\cdot) = L_2(\cdot/b_2)$ is a kernel function, b_2 is a bandwidth and $\eta(\cdot)$ is a known real valued function that may also depend on additional covariates. Then the two-step semiparametric quantile estimator for $\beta_{0\tau}$ and its resampled analog are defined as

$$\widehat{\beta}_\tau = \arg\min_{\beta_\tau} \sum_{i=1}^{n} \frac{\delta_i}{\widehat{G}(\cdot)} \rho_\tau\left(Z_i - X_{1i}^T \beta_\tau - \widetilde{\theta}(X_{2i})\right) \tag{1.2}$$

and

$$\widetilde{\beta}_\tau^* = \arg\min_{\beta_\tau} \sum_{i=1}^{n} \frac{\delta_i \xi_i}{\widehat{G}_\xi(\cdot)} \rho_\tau\left(Z_i - X_{1i}^T \beta_\tau - \widetilde{\theta}_\tau(X_{2i})\right),$$

The following two theorems are the two-step analog of Theorems 3-6 in the main paper.

Theorem 13 *Under the assumptions of Theorems 3-5 in the main paper, with $K(\cdot)$ replaced by $L_2(\cdot)$ and h replaced by b_2*

$$n^{1/2}\left(\widetilde{\beta}_\tau - \beta_{0\tau}\right) \xrightarrow{d} N\left(0, \Sigma_2^{-1}\Sigma_{3*}\Sigma_2^{-1}\right)$$

where Σ_{3*} is Σ_{3km} or Σ_{3p} or Σ_{3np} and

$$\Sigma_{3km} = E\left[\left(X_1\rho'_\tau(\varepsilon) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X) | X_2\right]\right)^{\otimes 2}\right] + E\left[\int_0^L \left(X_1\rho'_\tau(\varepsilon) - \frac{E\left[\left(X_1\rho'_\tau(\varepsilon) - E\left[f_{\varepsilon|X}(0|X) X_1\eta(X)|X_2\right]\right) I(Z \geq u)\right]}{S(u)} I(Z > u)\right)^{\otimes 2} \frac{\lambda_0(u)}{G_0(u)} du\right],$$

$$\Sigma_{3p} = E\left[\frac{\left(X_1\rho_\tau'\left(\varepsilon\right) - E\left[f_{\varepsilon|X}\left(0|X\right)X_1\eta\left(X\right)|X_2\right]\right)}{G_0\left(Z|X\right)} - \right.$$
$$\left. E\left(\frac{X_{12}\rho_\tau'\left(\varepsilon_2\right) - E\left[f_{\varepsilon|X}\left(0|X\right)X_{12}\eta\left(X\right)|X_{22}\right]\psi_{\gamma_0}\left(W_1, X_{12}, X_{22}\right)}{G_0\left(Z_1|X_{12}, X_{22}\right)}|W_1\right)\right]^{\otimes 2}$$

$$\Sigma_{3np} = E\left[\frac{\left(X_1\rho_\tau'\left(\varepsilon\right) - E\left[f_{\varepsilon|X}\left(0|X\right)X_1\eta\left(X\right)|X_2\right]\right)}{G_0\left(Z|X_2\right)} - \right.$$
$$\left. E\left[f_{X_2}\left(X_2\right)\frac{\psi\left(Z,\delta,Y,X_2\right)\left(X_1\rho_\tau'\left(\varepsilon\right) - E\left[f_{\varepsilon|X}\left(0|X\right)X_1\eta\left(X\right)|X_2\right]\right)}{G_0\left(Z|X_2\right)}|X_2\right]\right]^{\otimes 2}$$

Theorem 14 *Under the same assumptions of Theorem 6 in the main paper, conditionally on* $\left(Z_i, \delta_i, X_i^T\right)_{i=1}^n$

$$n^{1/2}\left(\widetilde{\beta}_\tau^* - \widetilde{\beta}_\tau\right) \xrightarrow{d} N\left(0, \Sigma_2^{-1}\Sigma_{3*}\Sigma_2^{-1}\right).$$

Similarly, we can define two-step analogs for the estimators of the parametric component in the semiparametric quantile partially linear varying coefficient model given in (3.2) in the main paper, that is

$$\widehat{\beta}_\tau = \arg\min_{\beta_\tau}\sum_{i=1}^n \frac{\delta_i}{\widehat{G}\left(\cdot\right)}\rho_\tau\left(Z_i - X_{1i}^T\beta_\tau - X_{3i}^T\widetilde{\theta}\left(X_{2i}\right)\right)$$

and

$$\widetilde{\beta}_\tau^* = \arg\min_{\beta_\tau}\sum_{i=1}^n \frac{\delta_i\xi_i}{\widehat{G}_\xi\left(\cdot\right)}\rho_\tau\left(Z_i - X_{1i}^T\beta_\tau - X_{3i}^T\widetilde{\theta}\left(X_{2i}\right)\right),$$

The following two theorems are the two-step analog of Theorems 8 and 9 in the main paper.

Theorem 15 *Under the assumptions of Theorem 8 in the main paper, with* $K\left(\cdot\right)$ *replaced by* $L_2\left(\cdot\right)$ *and* h *replaced by* b_2

$$n^{1/2}\left(\widetilde{\beta}_\tau - \beta_{0\tau}\right) \xrightarrow{d} N\left(0, \Sigma_2^{-1}\Omega_{3*}\Sigma_2^{-1}\right)$$

where Ω_{3*} *is* Ω_{3km} *or* Ω_{3p} *or* Ω_{3np} *and*

$$\Omega_{3km} = E\left[\left(X_1\rho_\tau'\left(\varepsilon\right) - E\left[f_{\varepsilon|X}\left(0|X\right)X_1X_3^T\eta\left(X\right)|X_2\right]\right)^{\otimes 2}\right] + E\left[\int_0^L\left(X_1\rho_\tau'\left(\varepsilon\right) - \right.\right.$$
$$\left.\left.\frac{E\left[X_1\rho_\tau'\left(\varepsilon\right)E\left[f_{\varepsilon|X}\left(0|X\right)X_1X_3^T\eta\left(X\right)|X_2\right]I\left(Z \geq u\right)\right]}{S\left(u\right)}I\left(Z > u\right)\right)^{\otimes 2}\frac{\lambda_0\left(u\right)}{G_0\left(u\right)}du\right],$$

$$\Omega_{3p} = E\left[\frac{\left(X_1\rho'_\tau(\varepsilon) - E\left[f_{\varepsilon|X}(0|X)\, X_1 X_3^T \eta(X)\,|X_2\right]\right)}{G_0(Z|X)} - \right.$$
$$\left. E\left(\frac{X_{12}\rho'_\tau(\varepsilon_2) - E\left[f_{\varepsilon|X}(0|X)\, X_{12} X_{32}^T \eta(X)\,|X_{22}\right]\psi_{\gamma_0}(W_1, X_{12}, X_{22})}{G_0(Z_1|X_{12}, X_{22})}|W_1\right)\right]^{\otimes 2},$$

$$\Omega_{3np} = E\left[\frac{\left(X_1\rho'_\tau(\varepsilon) - E\left[f(0|X)\, X_1 X_3^T \eta(X)\,|X_2\right]\right)}{G_0(Z|X_2)} - \right.$$
$$\left. E\left[f_{X_2}(X_2)\,\frac{\psi(Z, \delta, Y, X_2)\left(X_1\rho'_\tau(\varepsilon) - E\left[f_{\varepsilon|X}(0|X)\, X_1 X_3^T \eta(X)\,|X_2\right]\right)}{G_0(Z|X_2)}|X_2\right]\right]^{\otimes 2}.$$

Theorem 16 *Under the same assumptions of Theorem 8 in the main paper, conditionally on* $\left(Z_i, \delta_i, X_i^T\right)_{i=1}^n$

$$n^{1/2}\left(\widetilde{\beta}_\tau^* - \widetilde{\beta}_\tau\right) \xrightarrow{d} N\left(0, \Sigma_2^{-1}\Omega_{3*}\Sigma_2^{-1}\right).$$

2 Proofs

Throughout this section we use the following abbreviations: "CLT", "CMT" and "LNN" denote, respectively, central limit theorem, continuous mapping theorem and (possibly uniform) law of large numbers. We also use "CL" and "QAL" to denote, respectively, the convexity lemma (Pollard 1991) and the "quadratic approximation lemma" (Fan & Gijbels 1994). Finally we use the following notation

$$\begin{aligned}
W_i &= \left[X_{1i}^T, 1, (X_{2i} - x_2)/h\right]^T, \\
Z_i^* &= Z_i - X_{1i}^T\beta_{0\tau} - a_\tau - b\tau(X_{2i} - x_2), \\
\gamma_\tau &= (nh)^{1/2}\left[(\beta_\tau - \beta_{0\tau})^T, a - \theta_{0\tau}(x_2), h(b - \theta'_{0\tau}(x_2))\right]^T, \\
\gamma_{\beta_\tau} &= n^{1/2}(\beta_\tau - \beta_{0\tau})^T,
\end{aligned}$$

and the following identity (Knight 1999)

$$\rho_\tau(x - y) - \rho_\tau(x) = -y(\tau - I(x < 0)) + \int_0^y (I(x \le t) - I(x \le 0))\,dt. \tag{2.1}$$

Proof of Theorem 1. We first consider the independent censoring case[1]. Let

$$R_n\left(\gamma_\tau, \widehat{G}, x_2\right) = \sum_{i=1}^n \frac{\delta_i}{\widehat{G}(Z_i)}\left[\rho_\tau\left(Z_i^* - \frac{W_i^T\gamma_\tau}{(nh)^{1/2}}\right) - \rho_\tau(Z_i^*)\right] K_h(X_{2i} - x_2)$$

[1]Note that the asymptotic theory for Z^L is the same as that for Z, since the quantile restriction on Y is the same as that for Y^L, hence in what follows we omit the superscript L in $(Z_i, Y_i, \delta_i)_{i=1}^n$.

denote the normalized local objective function. Note that for the Kaplan-Meier estimator

$$\sup_{0\leq z\leq L}\left|\widehat{G}\left(z\right)-G_0\left(z\right)\right|=O_p\left(\frac{1}{n^{1/2}}\right)$$

(Zhou 1991, Foldes & Rejto 1981), hence

$$\left|R_n\left(\gamma_\tau,\widehat{G},x_2\right)-R_n\left(\gamma_\tau,G_0,x_2\right)\right|\leq\sup_{0\leq Z_i\leq L}\left|\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)\right|\times$$
$$\left|\sum_{i=1}^n\frac{\delta_i}{G\left(Z_i\right)^2}\left[\rho_\tau\left(Z_i^*-\frac{W_i^T\gamma_\tau}{\left(nh\right)^{1/2}}\right)-\rho_\tau\left(Z_i^*\right)\right]K_h\left(X_{2i}-x_2\right)\right|+o_p\left(1\right)=$$
$$O_p\left(n^{-1/2}\right)O_p\left(\left(nh\right)^{1/2}\right)=o_p\left(1\right).$$

By (2.1) we have

$$R_n\left(\gamma_\tau,G_0,x_2\right)=\frac{\gamma_\tau^T}{\left(nh\right)^{1/2}}\sum_{i=1}^n\frac{\delta_i}{G_0\left(Z_i\right)}W_i\rho_\tau'\left(Z_i^*\right)K_h\left(X_{2i}-x_2\right)+S_n\left(\gamma_\tau,G_0\right)$$

with $\rho_\tau'\left(\cdot\right)=\left(\tau-I\left(\cdot<0\right)\right)$ and

$$S_n\left(\gamma_\tau,G_0,x_2\right)=\sum_{i=1}^n\frac{\delta_i}{G_0\left(Z_i\right)}\int_0^{\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}}\left(I\left(Z_i^*\leq t\right)-I\left(Z_i^*\leq 0\right)\right)K_h\left(X_{2i}-x_2\right)dt.$$

Note that $E\left(\delta/G_0\left(Z\right)\right)=1$, hence by the uniform consistency results for kernel estimators of Masry (1996)

$$S_n\left(\gamma_\tau,G_0,x_2\right)=E\left[S_n\left(\gamma_\tau,G_0,x_2\right)\right]+O_p\left(\left(\frac{\log n}{nh}\right)^{1/2}\right)\tag{2.2}$$

uniformly for $x_2\in\mathcal{X}_2$. Let $\varsigma_{i\tau}\left(x_2\right)=\theta_{0\tau}\left(X_{2i}\right)-a_\tau-b_\tau\left(X_{2i}-x_2\right)$; by iterated expectations $E\left[S_n\left(\gamma_\tau,G_0,x_2\right)\right]=EE\left[S_n\left(\gamma_\tau,G_0,x_2\right)|X_i\right]$, so using a mean value expansion on $t=0$, we have

$$E\left[S_n\left(\gamma_\tau,G,x_2\right)|X_i\right]=\sum_{i=1}^n\int_0^{\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}}\left(F_{\varepsilon|X}\left(\varsigma_\tau\left(x_2\right)+t|X\right)-F_{\varepsilon|X}\left(\varsigma_{i\tau}\left(x_2\right)|X_i\right)\right)\times$$
$$K_h\left(X_{2i}-x_2\right)dt=\sum_{i=1}^n\int_0^{\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}}f_{\varepsilon|X}\left(\overline{\varsigma_{i\tau}}\left(x_2\right)|X_i\right)tK_h\left(X_{2i}-x_2\right)dt,$$

where $\overline{\varsigma_{i\tau}}\left(x_2\right)$ is the mean value between 0 and $\varsigma_\tau\left(x_2\right)+t$. Adding and subtracting

$$\sum_{i=1}^n\int_0^{\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}}f_{\varepsilon|X}\left(0|X_i\right)tK_h\left(X_{2i}-x_2\right)dt$$

$$\left|\sum_{i=1}^{n}\int_{0}^{\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}} f_{\varepsilon|X}\left(\overline{\varsigma_{i\tau}}\left(x_2\right)|X_i\right)tK_h\left(X_{2i}-x_2\right)dt-\sum_{i=1}^{n}\int_{0}^{\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}} f_{\varepsilon|X}\left(0|X_i\right)tK_h\left(X_{2i}-x_2\right)dt\right|\leq$$

$$\sup_{x_2\in\mathcal{X}_2}\frac{C}{2}\frac{\gamma_\tau^T}{nh}\sum_{i=1}^{n}\left|\overline{\varsigma_{i\tau}}\left(x_2\right)\right|W_i^{\otimes 2}\gamma_\tau K_h\left(X_{2i}-x_2\right)=O_p\left(h^2\right),$$

for some $C>0$, hence

$$E\left[S_n\left(\gamma_\tau,G,x_2\right)|X_i\right]=\frac{1}{2}\frac{\gamma_\tau^T}{nh}\sum_{i=1}^{n}f_{\varepsilon|X}\left(0|X_i\right)W_i^{\otimes 2}\gamma_\tau K_h\left(X_{2i}-x_2\right)+o_p\left(1\right). \tag{2.3}$$

Combining (2.2), (2.3) and a standard kernel calculation, we have that

$$R_n\left(\gamma_\tau,G_0,x_2\right)=\frac{\gamma_\tau^T}{(nh)^{1/2}}\sum_{i=1}^{n}\frac{\delta_i}{G_0\left(Z_i\right)}W_i\rho_\tau'\left(Z_i^*\right)K_h\left(X_{2i}-x_2\right)+\frac{1}{2}f_{X_2}\left(x_2\right)\gamma_\tau^T\Sigma\left(x_2\right)\gamma_\tau+$$
$$O_p\left(\left(\frac{\log n}{nh}\right)^{1/2}+h^2\right)$$

uniformly in $x_2\in\mathcal{X}_2$, where

$$\Sigma\left(x_2\right)=E\left\{f_{\varepsilon|X}\left(0|X\right)\begin{bmatrix}X_1^{\otimes 2} & X_1 & 0\\ X_1^T & 1 & 0\\ 0 & 0 & \kappa_2\end{bmatrix}|X_2=x_2\right\}. \tag{2.4}$$

Since $R_n\left(\gamma_\tau,G_0,x_2\right)$ is convex in γ_τ, by CL and QAL the minimizer $\widehat{\gamma}_\tau$ of $R_n\left(\gamma_\tau,G_0,x_2\right)$ is

$$\widehat{\gamma}_\tau=-\left(f_{X_2}\left(x_2\right)\Sigma\left(x_2\right)\right)^{-1}\frac{1}{(nh)^{1/2}}\sum_{i=1}^{n}\frac{\delta_i}{G_0\left(Z_i\right)}W_i\rho_\tau'\left(Z_i^*\right)K_h\left(X_{2i}-x_2\right)+ \tag{2.5}$$
$$O_p\left(\left(\frac{\log n}{nh}\right)^{1/2}+h^2\right).$$

Using iterated expectations and the fact that

$$\theta_{0\tau}\left(X_2\right)-\theta_{0\tau}\left(x_2\right)-\theta_{0\tau}'\left(x_2\right)\left(X_2-x_2\right)=\theta_{0\tau}''\left(x_2\right)\left(X_2-x_2\right)^2/2,$$

a Taylor expansion and a standard kernel calculation show that

$$E\left[\frac{\delta}{G_0\left(Z\right)}W\rho_\tau'\left(Z^*\right)K_h\left(X_2-x_2\right)\right]=E\left\{E\left[\left(W\left(F_{\varepsilon|X}\left(0|X\right)-F_{\varepsilon|X}\left(\varsigma_\tau\left(x_2\right)\right)\right)|X\right)\times\right.\right.$$
$$\left.\left.K_h\left(X_2-x_2\right)|X_2\right]\right\}=\frac{h^2}{2}f_{X_2}\left(x_2\right)\theta_{0\tau}''\left(x_2\right)E\left\{f_{\varepsilon|X}\left(0|X\right)\begin{bmatrix}X_1\kappa_2\\ \kappa_2\\ 0\end{bmatrix}|X_2=x_2\right\}+o\left(1\right).$$

Similarly, it can be showed that

$$Var\left[\frac{\delta}{G_0(Z)}W\rho_\tau'\left(\varepsilon+\varsigma_\tau(x_2)\right)K_h(X_2-x_2)-\right.$$
$$\left.\frac{\delta}{G_0(Z)}W\rho_\tau'(\varepsilon)K_h(X_2-x_2)\right]=O\left(h^2\right),$$

where $\varepsilon = Z - X_1^T\beta_{0\tau} - \theta_{0\tau}(X_2)$, and by iterated expectations and a standard kernel calculation

$$Var\left[\frac{\delta}{G_0(Z)}W\rho_\tau'(\varepsilon)K_h(X_2-x_2)\right]=f_{X_2}(x_2)E\left\{\frac{\tau(1-\tau)}{G_0(Z)}\begin{bmatrix} v_0X_1^{\otimes 2} & v_0X_1 & 0\\ v_0X_1^T & v_0 & 0\\ 0 & 0 & v_2\end{bmatrix}|X_2=x_2\right\}+$$
$$o(1),$$

hence the conclusion follows by CLT and CMT.

For the case of dependent censoring with the censoring distribution estimated parametrically, note that by a mean value expansion

$$\sup_i\left|G_{\widehat{\gamma}}(Z_i|X_i)-G_0(Z_i|X_i)\right|=\sup_i\left|g_{\overline{\gamma}}(Z_i|X_i)\right|\left\|\widehat{\gamma}-\gamma_0\right\|$$
$$=O_p\left(n^{1/\alpha}\right)O_p\left(n^{-1/2}\right)=o_p(1),$$

where $\overline{\gamma}$ is the mean value and the last equality follows by A5(ii) and the LLN, since

$$\sup_i\left|g_{\overline{\gamma}}(Z_i|X_i)\right|\le n^{1/\alpha}\left(\sum_{i=1}^n\sup_{\gamma\in\Gamma}\left|g_\gamma(Z_i|X_i)\right|/n\right)^{1/\alpha}=O_p\left(n^{1/\alpha}\right).$$

Finally, for the dependent censoring case with the censoring distribution estimated nonparametrically, note that for the local Kaplan-Meier estimator

$$\sup_{z\le\mathcal{G},x_2\in\mathcal{X}_2}\left|\widehat{G}(z|x_2)-G_0(z|x_2)\right|=O_p\left(\left(\frac{\log n}{nb}\right)^{1/2}+b^2\right), \quad (2.6)$$

where $\mathcal{G} = \inf\left(z : G(z|x_2) = 1, \forall x_2 \in \mathcal{X}_2\right)$ (Gonzales-Manteiga & Cadarso-Suarez 1994), hence by (2.6) and the assumption on the bandwidth

$$R_n\left(\gamma_\tau,\widehat{G},x_2\right)\le\sum_{i=1}^n\frac{\delta_i}{G_0(Z_i|X_{2i})}\left[\rho_\tau\left(Z_i^*-\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}\right)-\rho_\tau(Z_i^*)\right]K_h(X_{2i}-x_2)+$$
$$\sup_{z\le\mathcal{G},x_2\in\mathcal{X}_2}\left|\left(G_0(Z_i|X_{2i})-\widehat{G}(Z_i|X_{2i})\right)\right|\sum_{i=1}^n\frac{\delta_i}{\widehat{G}(Z_i|X_{2i})G_0(Z_i|X_{2i})}\left[\rho_\tau\left(Z_i^*-\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}\right)-\rho_\tau(Z_i^*)\right]\times$$
$$K_h(X_{2i}-x_2)=$$
$$\sum_{i=1}^n\frac{\delta_i}{G_0(Z_i|X_{2i})}\left[\rho_\tau\left(Z_i^*-\frac{W_i^T\gamma_\tau}{(nh)^{1/2}}\right)-\rho_\tau(Z_i^*)\right]K_h(X_{2i}-x_2)+o_p(1)$$

and the conclusion follows by CLT and CMT. ■

Proof of Theorem 2. By the $n^{1/2}$ consistency of $\widehat{\beta}_\tau$, we can assume it is known, hence the conclusion follows by the same arguments as those used in the proof of Theorem 1. ■

Proof of Theorem 3. Let

$$R_n\left(\gamma_{\beta_\tau}, \widehat{G}\right) = \sum_{i=1}^{n} \frac{\delta_i}{\widehat{G}\left(Z_i\right)}\left[\rho_\tau\left(\widehat{Z}_i^{l*} - \frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widehat{Z}_i^{l*}\right)\right]$$

denote the normalized objective function, where $\widehat{Z}_i^{l*} = Z_i - X_{1i}^T\beta_\tau - \widehat{\theta}_\tau^l\left(X_{2i}\right)$. As in the proof of Theorem 1

$$\begin{aligned} R_n\left(\gamma_{\beta_\tau}, \widehat{G}\right) = & \sum_{i=1}^{n} \frac{\delta_i}{G_0\left(Z_i\right)}\left[\rho_\tau\left(\varepsilon_i - \left(\widehat{\theta}_\tau^l\left(X_{2i}\right) - \theta_{0\tau}\left(X_{2i}\right)\right) - \frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right) - \right. \\ & \left.\rho_\tau\left(\varepsilon_i - \left(\widehat{\theta}_\tau^l\left(X_{2i}\right) - \theta_{0\tau}\left(X_{2i}\right)\right)\right)\right] + \\ & \sum_{i=1}^{n} \frac{\delta_i\left(G_0\left(Z_i\right) - \widehat{G}\left(Z_i\right)\right)}{\widehat{G}\left(Z_i\right)G_0\left(Z_i\right)}\left[\rho_\tau\left(\varepsilon_i - \left(\widehat{\theta}_\tau^l\left(X_{2i}\right) - \theta_{0\tau}\left(X_{2i}\right)\right) - \frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right) - \right. \\ & \left.\rho_\tau\left(\varepsilon_i - \left(\widehat{\theta}_\tau^l\left(X_{2i}\right) - \theta_{0\tau}\left(X_{2i}\right)\right)\right)\right] \\ & := R_{1n}\left(\gamma_{\beta_\tau}, G_0\right) + R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right), \end{aligned}$$

where $\varepsilon_i = Z_i - X_{1i}^T\beta_{0\tau} - \theta_{0\tau}\left(X_{2i}\right)$. By (2.1)

$$R_{1n}\left(\gamma_{\beta_\tau}, G_0\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^{n} \frac{\delta_i}{G_0\left(Z_i\right)}X_{1i}\rho_\tau'\left(\varepsilon_i\right) + S_{1n}\left(\gamma_{\beta_\tau}, G_0\right) \tag{2.7}$$

where

$$S_{1n}\left(\gamma_{\beta_\tau}, G_0\right) = \sum_{i=1}^{n} \int_{\widehat{\theta}_\tau^l\left(X_{2i}\right)-\theta_{0\tau}\left(X_{2i}\right)}^{\widehat{\theta}_\tau^l\left(X_{2i}\right)-\theta_{0\tau}\left(X_{2i}\right)+\frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}} \frac{\delta_i}{G_0\left(Z_i\right)}\left(I\left(\varepsilon_i \le t\right) - I\left(\varepsilon_i \le 0\right)\right)dt.$$

Similarly to (2.3) we can show that

$$E\left[S_{1n}\left(\gamma_{\beta_\tau}, G_0\right)\right] = \frac{1}{2}\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^{n} f_{\varepsilon|X}\left(0|X_i\right)X_{1i}^{\otimes 2}\gamma_{\beta_\tau} - \frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^{n} f_{\varepsilon|X}\left(0|X_i\right)X_{1i}\left(\widehat{\theta}_\tau^l\left(X_{2i}\right) - \theta_{\tau 0}\left(X_{2i}\right)\right),$$

so that (2.7) can be written as

$$\begin{aligned} R_{1n}\left(\gamma_\beta, G_0\right) = & \frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^{n} \frac{\delta_i}{G_0\left(Z_i\right)}X_{1i}\rho_\tau'\left(\varepsilon_i\right) + \frac{1}{2}\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^{n} f_{\varepsilon|X}\left(0|X_i\right)X_{1i}^{\otimes 2}\gamma_{\beta_\tau} - \\ & \frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^{n} f_{\varepsilon|X}\left(0|X_i\right)X_{1i}\left(\widehat{\theta}_\tau^l\left(X_{2i}\right) - \theta_{0\tau}\left(X_{2i}\right)\right) + T_{1n}\left(\gamma_{\beta_\tau}, G_0\right), \end{aligned}$$

where

$$\left|T_{1n}\left(\gamma_{\beta_\tau},G_0\right)\right|=\left|S_{1n}\left(\gamma_{\beta_\tau},G_0\right)-E\left[S_{1n}\left(\gamma_{\beta_\tau},G_0\right)\right]\right|=o_p\left(1\right),$$

since

$$\begin{aligned}
&E\left[T_{1n}\left(\gamma_{\beta_\tau},G_0\right)^2\right]\leq nES_{1i}\left(\gamma_{\beta_\tau},G_0\right)^2= \\
&nE\left[\int_{\widehat{\theta}^l_\tau(X_{2i})-\theta_{0\tau}(X_{2i})}^{\widehat{\theta}^l_\tau(X_{2i})-\theta_{0\tau}(X_{2i})+\frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}}\Pr\left(0\leq\left|\varepsilon_i\right|\leq\max\left(\left|t\right|,\left|u\right|\right)\left|X\right)dtdu\right] \\
&\leq nE\left[\Pr\left(0\leq\left|\varepsilon_i\right|\leq\left|\widehat{\theta}^l_\tau\left(X_{2i}\right)-\theta_{0\tau}\left(X_{2i}\right)\right|+\left|\frac{X_1^T\gamma_{\beta_\tau}}{n^{1/2}}\right|\left|X\right)\frac{\gamma_{\beta_\tau}^TX_1^{\otimes 2}\gamma_{\beta_\tau}}{n}\right] \\
&=o\left(1\right)
\end{aligned}\tag{2.8}$$

as both $\left|X_1^T\gamma_{\beta_\tau}/n^{1/2}\right|$ and $\left|\widehat{\theta}^l_\tau\left(X_2\right)-\theta_{0\tau}\left(X_2\right)\right|$ are $o_p\left(1\right)$. Then using (2.5), we have

$$\begin{aligned}
R_{1n}\left(\gamma_{\beta_\tau},G_0\right)=&\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\frac{\delta_i}{G_0\left(Z_i\right)}X_{1i}\rho_\tau'\left(\varepsilon_i\right)+\frac{1}{2}\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^nf_{\varepsilon|X}\left(0|X_i\right)X_{1i}^{\otimes 2}\gamma_\beta- \\
&\frac{\gamma_{\beta_\tau}^T}{n^{3/2}}\sum_{i=1}^n\sum_{j=1}^nf_{\varepsilon|X}\left(0|X_i\right)X_{1i}\left[0^T,1,0\right]\left(f_{X_2}\left(X_{2i}\right)\Sigma\left(X_{2i}\right)\right)^{-1}\times \\
&\frac{\delta_j}{G_0\left(Z_j\right)}\left[X_{1j}^T,1,0\right]^T\rho_\tau'\left(\varepsilon_j\right)K_h\left(X_{2j}-X_{2i}\right)+O_p\left(n^{1/2}h^2+\left(\frac{\log n}{nh^2}\right)^2\right),
\end{aligned}\tag{2.9}$$

which by LLN and a U-statistic projection argument simplifies to

$$R_{1n}\left(\gamma_{\beta_\tau},G_0\right)=\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\frac{\delta_i}{G_0\left(Z_i\right)}\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right)-\varphi\left(X_i\right)\rho_\tau'\left(\varepsilon_i\right)\right)+\gamma_{\beta_\tau}^T\Sigma_2\gamma_{\beta_\tau},\tag{2.10}$$

where

$$\varphi\left(X_i\right)=E\left[f_{\varepsilon|X}\left(0|X\right)X_1\left[0^T,1,0\right]\left|X_2=X_{2i}\right.\right]\Sigma\left(X_{2i}\right)^{-1}\left[X_{1i}^T,1,0\right]^T.$$

Then as in (2.9)

$$\begin{aligned}
R_{2n}\left(\gamma_{\beta_\tau},\widehat{G}\right)=&\sum_{i=1}^n\frac{\left(G_0\left(Z_i\right)-\widehat{G}\left(Z_i\right)\right)}{\widehat{G}\left(Z_i\right)}\left[\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\frac{\delta_i}{G_0\left(Z_i\right)}\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)+\right. \\
&\left.\frac{1}{2}\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^nf_{\varepsilon|X}\left(0|X_i\right)X_{1i}^{\otimes 2}\gamma_{\beta_\tau}+T_{1i}\left(\gamma_{\beta_\tau},G_0\right)\right]+o_p\left(1\right) \\
=&\sum_{i=1}^n\frac{\left(G_0\left(Z_i\right)-\widehat{G}\left(Z_i\right)\right)}{\widehat{G}\left(Z_i\right)}\left[\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\frac{\delta_i}{G_0\left(Z_i\right)}\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)\right]+o_p\left(1\right),
\end{aligned}$$

where $T_{1i}\left(\gamma_{\beta_\tau}, G_0\right)$

$$T_{1i}\left(\gamma_{\beta_\tau}, G_0\right)=\int_{\widehat{\theta}_\tau^l(X_{2i})-\theta_{0\tau}(X_{2i})}^{\widehat{\theta}_\tau^l(X_{2i})-\theta_{0\tau}(X_{2i})+\frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}} \frac{\delta_i}{G_0\left(Z_i\right)} I\left(\varepsilon_i \leq t\right)-I\left(\varepsilon_i \leq 0\right) dt- E\left[\int_{\widehat{\theta}_\tau^l(X_{2i})-\theta_{0\tau}(X_{2i})}^{\widehat{\theta}_\tau^l(X_{2i})-\theta_{0\tau}(X_{2i})+\frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}} \frac{\delta_i}{G_0\left(Z_i\right)} I\left(\varepsilon_i \leq t\right)-I\left(\varepsilon_i \leq 0\right) dt\right],$$

since by the consistency of the Kaplan-Meier estimator and (2.8)

$$\left|\sum_{i=1}^n \frac{\left(G_0\left(Z_i\right)-\widehat{G}\left(Z_i\right)\right)}{\widehat{G}\left(Z_i\right)} \frac{\gamma_{\beta_\tau}^T}{n} f_{\varepsilon \mid X}\left(0 \mid X_i\right) X_{1i}^{\otimes 2} \gamma_{\beta_\tau}\right| \leq \sup_{0 \leq Z_i \leq L}\left|\frac{\left(\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)\right)}{\widehat{G}\left(Z_i\right)}\right|\left|\sum_{i=1}^n \frac{\gamma_{\beta_\tau}^T}{n} f_{\varepsilon \mid X}\left(0 \mid X_i\right) X_{1i}^{\otimes 2} \gamma_{\beta_\tau}\right|=o_p(1) O_p(1)$$

and

$$\left|\sum_{i=1}^n \frac{\delta_i\left(\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)\right)}{G_0\left(Z_i\right)} T_{1i}\left(\gamma_{\beta_\tau}, G_0\right)\right| \leq\left(\sum_{i=1}^n \frac{\delta_i\left(\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)\right)^2}{G_0\left(Z_i\right)^2}\right)^{1/2}\left\|\sum_{i=1}^n T_{1i}\left(\gamma_{\beta_\tau}, G_0\right)\right\|_2 = O_p(1) o_p(1),$$

by the Cauchy-Schwarz inequality. Thus by CL and QAL, we have that $\widehat{\gamma}_{\beta_\tau}=\Sigma_2^{-1} \zeta_{kn}+o_p(1)$, where

$$\zeta_{kn}=\frac{1}{n^{1/2}} \sum_{i=1}^n \frac{\delta_i}{G_0\left(Z_i\right)}\left(1+\frac{\left(G_0\left(Z_i\right)-\widehat{G}\left(Z_i\right)\right)}{\widehat{G}\left(Z_i\right)}\right)\left(X_{1i}-\varphi\left(X_i\right)\right) \rho_\tau^{\prime}\left(\varepsilon_i\right)+o_p(1).$$

Let

$$M_i(u)=N_i(u)-\int_0^u \lambda(t) Y_i(t) dt,$$

where $N_i(u)=I\left(Z_i \leq u, \delta_i=0\right)$ and $Y_i(u)=I\left(Z_i \geq u\right)$; we use the following identity (Robins & Rotnitzky 1992)

$$\frac{\delta_i}{G_0\left(Z_i\right)}=1-\int_0^{\infty} \frac{dM_i(u)}{G_0(u)} \tag{2.11}$$

and the well-known martingale integral representation (Gill 1980)

$$\frac{\widehat{G}(u)-G_0(u)}{G_0(u)}=-\int_0^t \frac{\widehat{G}\left(u^{-}\right)}{G_0(u)} \frac{dM_n(u)}{Y_n(u)}, \tag{2.12}$$

where $\widehat{G}(u^-)$ is the left continuous version of the Kaplan-Meier estimator, $dM_n(\cdot) = \sum_{i=1}^n dM_i(\cdot)$, $Y_n(u) = \sum_{i=1}^n Y_i(u)$, and note also that $Y_n(u)/n = \widehat{G}(u^-)\widehat{S}(u^-)$, where $\widehat{S}(u)$ is the Kaplan-Meier estimator for $S(u) = \Pr(Z > u)$. By (2.11), (2.12) and LLN, we have that

$$\zeta_{kn} = \frac{1}{n^{1/2}} \sum_{i=1}^n \Bigg\{ (X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) - \int_0^\infty \frac{dM_i(u)}{G_0(u)} \Bigg[(X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) - \sum_{j=1}^n \int_0^L \frac{1}{n} \frac{\delta_i}{\widehat{S}(u)} \frac{(X_{1j} - \varphi(X_j)) \rho'_\tau(\varepsilon_j) I(Z_j \geq u)}{G_0(Z_j)} \Bigg] \Bigg\} + o_p(1)$$

$$= \frac{1}{n^{1/2}} \sum_{i=1}^n \Bigg\{ (X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) - \int_0^\infty \frac{dM_i(u)}{G_0(u)} \Bigg[(X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) - \int_0^L \frac{E[(X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) I(Z_i \geq u)]}{S(u)} \Bigg] \Bigg\} + o_p(1).$$

Thus the conclusion follows by CLT and CMT, noting that

$$Var(\zeta_{kn}) = E\left[((X_1 - \varphi(X)) \rho'_\tau(\varepsilon))^{\otimes 2} \right] + E\left[\left(\int_0^L (X_1 - \varphi(X)) \rho'_\tau(\varepsilon) - \frac{E[(X_1 - \varphi(X)) \rho'_\tau(\varepsilon) I(Z \geq u)]}{S(u)} I(Z > u) \right)^{\otimes 2} \frac{\lambda(u)}{G(u)} du \right].$$

■

Proof of Theorem 4. By the same arguments as those of Theorem 3

$$R_n\left(\gamma_{\beta_\tau}, \widehat{G}_{\widehat{\gamma}}\right) = R_{1n}(\gamma_{\beta_\tau}, G_0) + R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}_{\widehat{\gamma}}\right),$$

where $R_{1n}(\gamma_{\beta_\tau}, G_0)$ is as that given in (2.10), whereas by the linear representation (3.1) in the main paper, it follows that

$$R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}_{\widehat{\gamma}}\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \sum_{j=1}^n \frac{\delta_i}{G_0(Z_i|X_i)} \psi_{\gamma_0}(W_j, X_i) (X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) + o_p(1). \quad (2.13)$$

Thus by CL and QAL, we have that $\widehat{\gamma}_{\beta_\tau} = -\Sigma_2^{-1} \zeta_{\psi pn}(W_i) + o_p(1)$, where

$$\zeta_{\psi pn}(W_i) = \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{\delta_i}{G_0(Z_i|X_i)} \left[(X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) - \sum_{j=1}^n \frac{\psi_{\gamma_0}(W_j, X_i)}{G_0(Z_j|X_i)} (X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) \right].$$

Let

$$h(W_i, W_j) = \frac{\delta_i}{G_0(Z_i|X_i)} (X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) + \frac{\delta_j}{G_0(Z_j|X_j)} (X_{1j} - \varphi(X_j)) \rho'_\tau(\varepsilon j) - \frac{\delta_i}{G_0(Z_i|X_i)} (X_{1i} - \varphi(X_i)) \rho'_\tau(\varepsilon_i) \frac{\psi_{\gamma_0}(W_j, X_i)}{G_0(Z_j|X_i)} - \frac{\delta_j}{G_0(Z_j|X_j)} (X_{1j} - \varphi(X_j)) \rho'_\tau(\varepsilon_j) \frac{\psi_{\gamma_0}(W_i, X_j)}{G_0(Z_i|X_j)};$$

then

$$\zeta'_{\psi pn}(W_i) = \frac{1}{n^2}\sum_{i<j=1}^{n} h(W_i, W_j) + r_{hn},$$

where $r_{hn} = n^{-3/2}\sum_{i=1}^{n}(X_{1i} + \varphi(X_i))\rho'_\tau(\varepsilon_i)$. Clearly $\zeta_{\psi pn}(W_i) = n^2 2\zeta'_{\psi n}(W_i)/n(n-1)$ is a U statistic with $E[h(W_i, W_j)] = 0$,

$$h_1(W_1) := E[h(W_1, W_2)|W_1] = \frac{\delta_1}{G_0(Z_1|X_1)}(X_{11} - \varphi(X_1))\rho'_\tau(\varepsilon_1) -$$
$$E\left[(X_{12} - \varphi(X_{12}))\rho'_\tau(\varepsilon_2)\frac{\psi_{\gamma 0}(W_1, X_{12}, X_{22})}{G_0(Z_1|X_{12}, X_{22})}|W_1\right]$$

and

$$Var(h_1(W_1)) := \Sigma_{2p} = E\left\{\frac{(X_{11} - \varphi(X_1))\rho'_\tau(\varepsilon_1)}{G_0(Z_1|X_1)} - \right.$$
$$\left. E\left[(X_{12} - \varphi(X_{12}, X_{22}))\rho'_\tau(\varepsilon_2)\frac{\psi_{\gamma 0}(W_1, X_{12}, X_{22})}{G_0(Z_1|X_{12}, X_{22})}|W_1\right]\right\}^{\otimes 2}.$$

The conclusion follows by a standard CLT for U statistics and CMT, since by the Cauchy-Schwarz inequality and LLN

$$|r_{hn}| \le \frac{(\tau(1-\tau))^{1/2}}{n^{1/2}}\left(E(X_{1i} + \varphi(X_i))^2\right)^{1/2} \to 0.$$

■

Proof of Theorem 5. By the same arguments as those of Theorem 3 and the consistency of the local Kaplan-Meier estimator (2.6), we have that

$$R_n\left(\gamma_{\beta_\tau}, \widehat{G}\right) = R_{1n}(\gamma_{\beta_\tau}, G_0) + R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right),$$

where $R_{1n}(\gamma_{\beta_\tau}, G_0)$ is as that given in (2.10) and

$$R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^{n}\frac{\delta_i G_0(Z_i|X_{2i}) - \widehat{G}(Z_i|X_{2i})}{\widehat{G}(Z_i|X_{2i})G_0(Z_i|X_{2i})}(X_{1i} + \varphi(X_i))\rho'_\tau(\varepsilon_i) + o_p(1).$$

By conditioning on $\widehat{G}$ and using the linear representation of the local Kaplan-Meier estimator (Gonzales-Manteiga & Cadarso-Suarez 1994)

$$\widehat{G}(Z_i|x_2) - G_0(Z_i|x_2) = \frac{1}{nb_2}\sum_{i=1}^{n}G_0(Z_i|x_2)L_b(X_{2i} - x_2)\psi(Z_i, \delta_i, Y_i, x_2) + O_p\left(\left(\frac{\log n}{nb}\right)^{3/4} + b^2\right),$$

where

$$\psi(Z, \delta, t, u) = \int_0^{\min(Z,t)} - \frac{g_0(s|u)\,ds}{G_0(s|u)^2(1 - F(s|u))} + \frac{(1-\delta)I(Z \le t)}{G_0(Z|u)(1 - F(Z|u))},$$

we have by a standard kernel calculation

$$\frac{1}{n^{1/2}}\sum_{i=1}^{n}\frac{\delta_i G_0\left(Z_i|X_{2i}\right)-\widehat{G}\left(Z_i|X_{2i}\right)}{\widehat{G}\left(Z_i|X_{2i}\right)G_0\left(Z_i|X_{2i}\right)}\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)=$$

$$\frac{1}{n^{1/2}}\sum_{i=1}^{n}E\left(\frac{G_0\left(Z_i|X_{2i}\right)-\widehat{G}\left(Z_i|X_{2i}\right)}{G_0\left(Z_i|X_{2i}\right)}\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)|\widehat{G}\right)+o_p\left(1\right)=$$

$$-\frac{1}{n^{1/2}}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(f_{X_2}\left(X_{2i}\right)E\left[\frac{1}{nb}\frac{L_b\left(X_{2j}-X_{2i}\right)\psi\left(Z_j,\delta_j,Y_j,X_{2i}\right)\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)}{G_0\left(Z_i|X_{2i}\right)}|X_{2i}\right]\right)+$$

$$O_p\left(n^{1/2}\left(\frac{\log n}{nb}\right)^{3/4}+n^{1/2}b^2\right)+o_p\left(1\right)=$$

$$-\frac{1}{n^{1/2}}\sum_{i=1}^{n}f_{X_2}\left(X_{2i}\right)E\left[\frac{\psi\left(Z_i,\delta_i,Y_i,X_{2i}\right)\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)}{G_0\left(Z_i|X_{2i}\right)}|X_{2i}\right]+o_p\left(1\right),$$

hence

$$R_{2n}\left(\gamma_{\beta_\tau},\widehat{G}\right)=-\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^{n}f_{X_2}\left(X_{2i}\right)E\left[\frac{\psi\left(Z_i,\delta_i,Y_i,X_{2i}\right)\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)}{G_0\left(Z_i|X_{2i}\right)}|X_{2i}\right]+o_p\left(1\right).$$

By CL and QAL, we have that $\widehat{\gamma}_{\beta_\tau}=-\Sigma_2^{-1}\zeta_{\psi npn}\left(W_i\right)+o_p\left(1\right)$, where

$$\zeta_{\psi npn}\left(W_i\right)=\frac{1}{n^{1/2}}\sum_{i=1}^{n}\frac{\delta_i}{G_0\left(Z_i|X_{2i}\right)}\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right)-\varphi\left(X_i\right)\rho_\tau'\left(\varepsilon_i\right)\right)-$$
$$\sum_{i=1}^{n}f_{X_2}\left(X_{2i}\right)E\left[\frac{\psi\left(Z_i,\delta_i,Y_i,X_{2i}\right)\left(X_{1i}-\varphi\left(X_i\right)\right)\rho_\tau'\left(\varepsilon_i\right)}{G_0\left(Z_i|X_{2i}\right)}|X_{2i}\right],$$

and the conclusion follows by CLT and CMT. ■

Proof of Theorem 6. By the same arguments as those used in the proof of Theorem 3 and using

$$\frac{\delta_i\xi_i}{G_0\left(Z_i\right)}=\xi_i-\int_0^\infty\frac{dM_{\xi i}\left(u\right)}{G_0\left(u\right)},$$

$$\frac{\widehat{G}_\xi\left(u\right)-G_0\left(u\right)}{G_0\left(u\right)}=-\int_0^t\frac{\widehat{G}_\xi\left(u^-\right)}{G_0\left(u\right)}\frac{dM_{\xi n}\left(u\right)}{Y_\xi\left(u\right)},$$

where

$$M_{\xi n}\left(u\right)=\sum_{i=1}^{n}\xi_i\left(I\left(Z_i\leq u,\delta_i=0\right)-\int_0^u\lambda\left(t\right)\xi_iY_i\left(t\right)dt\right),$$

we have that, conditionally on $\left(Z_i, \delta_i, X_i^T\right)_{i=1}^n$

$$
\begin{aligned}
R_{\xi n}\left(\gamma_{\beta_\tau}, \widehat{G}_\xi\right) = & \sum_{i=1}^n \frac{\delta_i \xi_i}{G_0\left(Z_i\right)}\left[\rho_\tau\left(\widehat{Z}_i^* - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widehat{Z}_i^*\right)\right] + \\
& \sum_{i=1}^n \frac{\delta_i^L \xi_i\left(G_0\left(Z_i\right) - \widehat{G}_\xi\left(Z_i\right)\right)}{\widehat{G}_\xi\left(Z_i\right) G_0\left(Z_i\right)}\left[\rho_\tau\left(\widehat{Z}_i^* - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widehat{Z}_i^*\right)\right] \\
& := R_{\xi 1n}\left(\gamma_{\beta_\tau}, G_0\right) + R_{\xi 2n}\left(\gamma_{\beta_\tau}, \widehat{G}_\xi\right) + o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
R_{\xi 1n}\left(\gamma_{\beta_\tau}, G_0\right) &= \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \frac{\delta_i \xi_i}{G_0\left(Z_i\right)} X_{1i} \rho_\tau'\left(\varepsilon_i\right) + S_{\xi 1n}\left(\gamma_\beta, G_0\right), \\
R_{\xi 2n}\left(\gamma_\beta, \widehat{G}_\xi\right) &= \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \int_0^\infty \frac{dM_{\xi i}(u)}{G_0(u)}\left[\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) - \right. \\
& \qquad \left. \frac{E\left[\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) I\left(Z_i \geq u\right)\right]}{S(u)}\right] + o_p(1).
\end{aligned}
$$

Thus by CL and QAL, we have that $\widehat{\gamma}_{\xi \beta_\tau} = \Sigma_2^{-1} \zeta_{\xi kn}$, where

$$
\begin{aligned}
\zeta_{\xi kn} = & \left\{\frac{1}{n^{1/2}} \sum_{i=1}^n \xi_i\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) - \frac{1}{n^{1/2}} \sum_{i=1}^n \int_0^L \xi_i\left[\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) - \right.\right. \\
& \left.\left. \frac{E\left[\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) I\left(Z_i \geq u\right)\right]}{S(u)}\right] dM_i(u)\right\} + o_p(1),
\end{aligned}
$$

hence

$$
\begin{aligned}
& n^{1/2}\left(\widehat{\beta}_\tau^* - \widehat{\beta}_\tau\right) = \Sigma_2^{-1} \frac{1}{n^{1/2}} \sum_{i=1}^n \left(\xi_i - 1\right)\left\{\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) - \right. \\
& \left. \int_0^L \left[\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) - \frac{E\left[\left(X_{1i} - \varphi\left(X_i\right)\right) \rho_\tau'\left(\varepsilon_i\right) I\left(Z_i \geq u\right)\right]}{S(u)}\right] dM_i(u)\right\} + o_p(1),
\end{aligned}
$$

and the conclusion follows by CMT and Lemma 2.9.5 of van der Vaart & Wellner (1996). For the dependent censoring case with the parametric specification, we have

$$
\frac{1}{n} \sum_{i=1}^n \xi_i \psi\left(W_i, x\right) + r_{\xi n \psi}(x)
$$

with $\sup_x \left|r_{\xi n \psi}(x)\right| = o_p\left(n^{-1/2}\right)$ conditionally on $\left(Z_i, \delta_i, X_i^T\right)_{i=1}^n$. Then for

$$
\zeta_{\xi \psi n}\left(W_i\right) = \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{\xi_i \delta_i}{G_0\left(Z_i | X_i\right)}\left[\left(X_{1i} - \varphi\left(X_i\right)\right) - \sum_{j=1}^n \frac{\xi_i \psi\left(W_j, X_i\right)}{G_0\left(Z_j | X_i\right)}\left(X_{1i} - \varphi\left(X_i\right)\right)\right] \rho_\tau'\left(\varepsilon_i\right),
$$

and

$$h_{\xi}(W_i, W_j) = \frac{\delta_i \xi_i}{G_0(Z_i|X_i)}(X_{1i} - \varphi(X_i))\rho'_\tau(\varepsilon_i) + \frac{\delta_j \xi_j}{G_0(Z_j|X_j)}(X_{1j} - \varphi(X_j))\rho'_\tau(\varepsilon_j) -$$
$$\frac{\delta_i \xi_j}{G_0(Z_i|X_i)}(X_{1i} - \varphi(X_i))\rho'_\tau(\varepsilon_i)\frac{\psi(W_j, X_i)}{G_0(Z_j|X_i)} - \frac{\delta_j \xi_i}{G_0(Z_j|X_j)}(X_{1j} - \varphi(X_j))\rho'_\tau(\varepsilon_j)\frac{\psi(W_i, X_j)}{G_0(Z_i|X_j)},$$

we have that $\zeta_{\xi\psi n}(W_i) = n^2 2\zeta'_{\xi\psi n}(W_i)/n(n-1)$ is a U statistic with $E[h_\xi(W_i, W_j)] = 0$ and

$$\zeta'_{\xi\psi n}(W_i) = \frac{1}{n^2}\sum_{i<j=1}^{n} h_\xi(W_i, W_j) + r_{\xi hn}$$

with $r_{\xi hn} = n^{-3/2}\sum_{i=1}^{n}\xi_i(X_{1i} + \varphi(X_i))\rho'_\tau(\varepsilon_i)$. Clearly

$$h_{\xi 1}(W_1) := E[h_\xi(W_1, W_2)|W_1] = \frac{\delta_1\xi_1}{G_0(Z_1|X_{11}, X_{21})}(X_{11} - \varphi(X_{11}))\rho'_\tau(\varepsilon_1) +$$
$$\xi_1 E\left[(X_{12} - \varphi(X_{12}))\rho'_\tau(\varepsilon_2)\frac{\psi(W_1, X_{12}, X_{22})}{G_0(Z_1|X_{12}, X_{22})}|W_1\right],$$

hence

$$n^{1/2}\left(\widehat{\beta}^*_\tau - \widehat{\beta}_\tau\right) = \Sigma_2^{-1}\frac{1}{n^{1/2}}\sum_{i=1}^{n}(\xi_i - 1)h_1(W_i) + o_p(1)$$

and the conclusion follows by CMT and Lemma 2.9.5 of van der Vaart & Wellner (1996). Finally for the dependent censoring case with nonparametric estimation note that, given the independence of ξ, the same arguments of Gonzales-Manteiga & Cadarso-Suarez (1994) can be used to show that

$$\sup_{z\leq\mathcal{G}, x_2\in\mathcal{X}_2}\left|\widehat{G}_\xi(z|x_2) - G_0(z|x_2)\right| = O_p\left(\left(\frac{\log n}{nb}\right)^{1/2} + b^2\right).$$

Then using the linear representation

$$\widehat{G}_\xi(Z_i|x_2) - G_0(Z_i|x_2) = \frac{1}{nb_2}\sum_{i=1}^{n}\xi_i G_0(Z_i|x_2)L_b(X_{2i} - x_2)\psi(Z_i, \delta_i, Y_i, x_2) + O_p\left(\left(\frac{\log n}{nb}\right)^{3/4} + b^2\right),$$

we have by CL and QAL and the same arguments as those used in the proof of Theorem 5 that $\widehat{\gamma}_{\xi\beta} = -\Sigma_2^{-1}\zeta_{\xi\psi npn}(W_i)$, where

$$\zeta_{\xi\psi npn}(W_i) = \frac{1}{n^{1/2}}\sum_{i=1}^{n}\frac{\xi_i\delta_i}{G_0(Z_i|X_{2i})}(X_{1i} - \varphi(X_i))\rho'_\tau(\varepsilon_i) -$$
$$\sum_{i=1}^{n}\xi_i f_{X_2}(X_{2i})E\left[\frac{\psi(Z_i, \delta_i, Y_i, X_{2i})(X_{1i} - \varphi(X_i))\rho(\varepsilon_i)'}{G_0(Z_i|X_{2i})}|X_{2i}\right].$$

Hence

$$n^{1/2}\left(\widehat{\beta}_{\tau}^{*}-\widehat{\beta}_{\tau}\right)=\Sigma_{2}^{-1}\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left(\xi_{i}-1\right)\left\{\frac{\delta_{i}}{G_{0}\left(Z_{i}|X_{2i}\right)}\left(X_{1i}-\varphi\left(X_{i}\right)\right)\rho_{\tau}^{\prime}\left(\varepsilon_{i}\right)-\right.$$
$$\left.\sum_{i=1}^{n}f_{X_{2}}\left(X_{2i}\right)E\left[\frac{\psi\left(Z_{i},\delta_{i},Y_{i},X_{2i}\right)\left(X_{1i}-\varphi\left(X_{i}\right)\right)\rho\left(\varepsilon_{i}\right)^{\prime}}{G_{0}\left(Z_{i}|X_{2i}\right)}|X_{2i}\right]\right\},$$

and the conclusion follows by CMT and Lemma 2.9.5 of van der Vaart & Wellner (1996). ■

Proof of Theorem 7. The proof is similar to that of Theorem 1, so we just sketch it. Let

$$W_{i}=\left[X_{1i}^{T},X_{3i}^{T},X_{3i}^{T}\left(X_{2i}-x_{2}\right)/h\right]^{T},$$
$$Z_{i}^{*}=Z_{i}-X_{1i}^{T}\beta_{0\tau}-X_{3i}^{T}\left(a_{\tau}+b_{\tau}\left(X_{2i}-x_{2}\right)\right),$$
$$\gamma_{\tau}=(nh)^{1/2}\left[\left(\beta_{\tau}-\beta_{0\tau}\right)^{T},\left(a_{\tau}-\theta_{0\tau}\left(x_{2}\right)\right)^{T},h\left(b_{\tau}-\theta_{0\tau}^{\prime}\left(x_{2}\right)\right)^{T}\right]^{T}.$$

Then, the same arguments used in the proof of Theorem 1 show that for the three different estimators $\widehat{G}(\cdot)$ of $G_0(\cdot)$

$$R_{n}\left(\gamma_{\tau},\widehat{G},x_{2}\right)=\sum_{i=1}^{n}\frac{\delta_{i}}{\widehat{G}\left(Z_{i}\right)}\left[\rho_{\tau}\left(Z_{i}^{*}-\frac{W_{i}^{T}\gamma_{\tau}}{(nh)^{1/2}}\right)-\rho_{\tau}\left(Z_{i}^{*}\right)\right]K_{h}\left(X_{2i}-x_{2}\right)$$
$$=\frac{\gamma_{\tau}^{T}}{(nh)^{1/2}}\sum_{i=1}^{n}\frac{\delta_{i}}{G_{0}\left(Z_{i}\right)}W_{i}\rho_{\tau}^{\prime}\left(Z_{i}^{*}\right)K_{h}\left(X_{2i}-x_{2}\right)+\frac{1}{2}f_{X_{2}}\left(x_{2}\right)\gamma_{\tau}^{T}\Omega\left(x_{2}\right)\gamma_{\tau}+$$
$$O_{p}\left(\left(\frac{\log n}{nh}\right)^{1/2}+h^{2}\right),$$

uniformly in $x_2 \in \mathcal{X}_2$, where

$$\Omega\left(x_{2}\right)=E\left\{f_{\varepsilon|X}(0|X)\left[\begin{array}{ccc}X_{1}^{\otimes 2} & X_{1}X_{3}^{T} & 0\\ X_{1}X_{3}^{T} & X_{3}^{\otimes 2} & 0\\ 0 & 0 & \kappa_{2}X_{3}^{\otimes 2}\end{array}\right]|X_{2}=x_{2}\right\}. \tag{2.14}$$

hence by CL and QAL, the minimizer $\widehat{\gamma}_{\tau}$ of $R_n(\gamma_\tau, G_0, x_2)$ is

$$\widehat{\gamma}_{\tau}=-\left(f_{X_{2}}\left(x_{2}\right)\Omega\left(x_{2}\right)\right)^{-1}\frac{1}{(nh)^{1/2}}\sum_{i=1}^{n}\frac{\delta_{i}}{G_{0}\left(Z_{i}\right)}W_{i}\rho_{\tau}^{\prime}\left(Z_{i}^{*}\right)K_{h}\left(X_{2i}-x_{2}\right)+$$
$$O_{p}\left(\left(\frac{\log n}{nh}\right)^{1/2}+h^{2}\right)$$

and the conclusion follows by CLT and CMT noting that

$$E\left[\frac{\delta}{G_{0}(Z)}W\rho_{\tau}^{\prime}\left(\varepsilon+\varsigma_{i\tau}\left(x_{2}\right)\right)K_{h}\left(X_{2}-x_{2}\right)\right]=$$
$$\frac{h^{2}}{2}f_{X_{2}}\left(x_{2}\right)E\left\{f_{\varepsilon|X}(0|X)\left[\begin{array}{c}X_{1}X_{3}^{T}\kappa_{2}\\ X_{3}^{\otimes 2}\kappa_{2}\\ 0\end{array}\right]|X_{2}=x_{2}\right\}\theta_{0\tau}^{\prime\prime}\left(x_{2}\right)+o(1)$$

and

$$Var\left[\frac{\delta}{G_0(Z)}W\rho_\tau'(\varepsilon+\varsigma_{i\tau}(x_2))K_h(X_2-x_2)\right]=$$

$$f_{X_2}(x_2)E\left\{\frac{\tau(1-\tau)}{G_0(Z)}\left[\begin{array}{ccc} v_0X_1^{\otimes 2} & v_0X_1X_3^T & 0 \\ v_0X_3X_1^T & v_0X_3^{\otimes 2} & 0 \\ 0 & 0 & v_2X_3^{\otimes 2}\end{array}\right]|X_2=x_2\right\}+o(1),$$

where $\varepsilon=Z-X_1^T\beta_{0\tau}-X_3^T\theta_{0\tau}(X_2)$. ■

Proof of Theorem 8. Let

$$\widehat{Z}_i^*=Z_i-X_{1i}^T\beta_\tau-X_{3i}^T\widehat{\theta}_\tau(X_{2i}),$$
$$\gamma_{\beta_\tau}=n^{1/2}(\beta_\tau-\beta_{0\tau})^T.$$

For the independent censoring case, the same arguments as those used in the proof of Theorem 3 show that

$$R_n\left(\gamma_{\beta_\tau},\widehat{G}\right)=\sum_{i=1}^n\frac{\delta_i}{G_0(Z_i)}\left[\rho_\tau\left(\varepsilon_i-X_{3i}^T\left(\widehat{\theta}_\tau^l(X_{2i})-\theta_{0\tau}(X_{2i})\right)-\frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right)-\right.$$
$$\left.\rho_\tau\left(\varepsilon_i-X_{3i}^T\left(\widehat{\theta}_\tau^l(X_{2i})-\theta_{0\tau}(X_{2i})\right)\right)\right]+\sum_{i=1}^n\frac{\left(G_0(Z_i)-\widehat{G}(Z_i)\right)}{\widehat{G}(Z_i)}\times$$
$$\left[\rho_\tau\left(\varepsilon_i-X_{3i}^T\left(\widehat{\theta}_\tau^l-\theta_0(X_{2i})\right)-\frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right)-\right.$$
$$\left.\rho_\tau\left(\varepsilon_i-X_{3i}^T\left(\widehat{\theta}_\tau^l-\theta_{0\tau}(X_{2i})\right)\right)\right]=$$
$$\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\frac{\delta_i}{G_0(Z_i)}X_{1i}\rho_\tau'(\varepsilon_i)+\frac{1}{2}\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^n f_{\varepsilon|X}(0|X_i)X_{1i}^{\otimes 2}\gamma_{\beta_\tau}-$$
$$\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^n f_{\varepsilon|X}(0|X_i)X_{1i}X_{3i}^T\left(\widehat{\theta}_\tau^l-\theta_{0\tau}(X_{2i})\right)+$$
$$\sum_{i=1}^n\frac{\left(G_0(Z_i)-\widehat{G}(Z_i)\right)}{\widehat{G}(Z_i)}\left[\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\frac{\delta_i}{G_0(Z_i)}(X_{1i}-\xi(X_i))\rho_\tau'(\varepsilon_i)\right]+o_p(1)=$$
$$\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\frac{\delta_i}{G_0(Z_i)}X_{1i}\rho_\tau'(\varepsilon_i)+\frac{1}{2}\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^n f_{\varepsilon|X}(0|X_i)X_{1i}^{\otimes 2}\gamma_{\beta_\tau}-$$
$$\frac{\gamma_{\beta_\tau}^T}{n}\sum_{i=1}^n\sum_{j=1}^n f_{\varepsilon|X}(0|X_i)X_{1i}X_{3i}^TS\Omega(X_{2i})^{-1}\left[X_{1i}^T,X_{3i}^T,0^T\right]^T\rho_\tau'(\varepsilon_j)K_h(X_{2j}-X_{2i})+$$
$$\sum_{i=1}^n\frac{\left(G_0(Z_i)-\widehat{G}(Z_i)\right)}{\widehat{G}(Z_i)}\left[\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\frac{\delta_i}{G_0(Z_i)}(X_{1i}-\xi(X_i))\rho_\tau'(\varepsilon_i)\right]+O_p\left(n^{1/2}h^2+\left(\frac{\log n}{nh^2}\right)^2\right).$$

Then using CL, QAL, (2.11), (2.12) and LLN, we have that $\widehat{\gamma}_{\beta_\tau} = -\Omega_2^{-1}\zeta_{\psi pn}(W_i) + o_p(1)$, where

$$\zeta_{kn} = \frac{1}{n^{1/2}}\sum_{i=1}^{n}\left\{(X_{1i}-\xi(X_i))\,\rho_\tau'(\varepsilon_i) - \int_0^\infty \frac{dM_i(u)}{G_0(u)}\left[(X_{1i}-\xi(X_i))\,\rho_\tau'(\varepsilon_i) - \right.\right.$$
$$\left.\left.\int_0^L \frac{E\left[(X_{1i}-\xi(X_i))\,\rho_\tau'(\varepsilon_i)\,I(Z_i\geq u)\right]}{S(u)}\right]\right\} + o_p(1)$$

and the conclusion follows by the same arguments as those used in the proof of Theorem 3. For the dependent censoring case with $G_0(\cdot)$ estimated parametrically, using CL, QAL and the linear representation (3.1) in the main paper, we have that

$$\zeta_{\psi pn}(W_i) = \frac{1}{n^{1/2}}\sum_{i=1}^{n}\frac{\delta_i}{G_0(Z_i|X_i)}\left[(X_{1i}-\xi(X_i)) - \sum_{j=1}^{n}\frac{\psi_{\gamma_0}(W_j,X_i)}{G_0(Z_j|X_i)}(X_{1i}-\xi(X_i))\right]$$

and the conclusion follows by the same arguments as those used in Theorem 4. Finally for the dependent censoring case with the censoring distribution estimated nonparametrically, using the same arguments as those used in the proof of Theorem 5, we have that

$$\zeta_{\psi npn}(W_i) = \frac{1}{n^{1/2}}\sum_{i=1}^{n}\frac{\delta_i}{G_0(Z_i|X_i)}\left\{(X_{1i}-\xi(X_i)) - \right.$$
$$\left. f_{X_2}(X_{2i})\,E\left[\frac{\psi(Z_i,\delta_i,Y_i,X_{2i})\,(X_{1i}-\xi(X_i))\,\rho_\tau'(\varepsilon_i)}{G_0(Z_i|X_{2i})}|X_{2i}\right]\right\}$$

and the conclusion follows by CMT and CMT. ■

Proof of Theorem 9. The proof is similar to that of Theorem 6, hence omitted. ■

Proof of Proposition 10. The uniform consistency results for kernel estimators of Masry (1996), the uniform consistency of $\widehat{G}(\cdot)$ (see the proof of Theorem 1) and the triangle inequality show that

$$\left\|\widehat{\Omega}_1(x_2^*) - \Omega_1(x_2^*)\right\| \leq \sup_{X_i\in\mathcal{X}}\left|\widehat{f}_{\widehat{\varepsilon}|X}(0|X_i) - f_{\varepsilon|X}(0|X_i)\right| \times$$
$$\left\|f_{X_2}(x_2^*)\frac{1}{nh}\sum_{i=1}^{n}\frac{\delta_i}{\widehat{G}(\cdot)}\begin{bmatrix}X_{1i}\\X_{3i}\end{bmatrix}^{\otimes 2}K_h(X_{2i}-x_2^*)\right\| + \sup\left|\widehat{G}(\cdot) - G_0(\cdot)\right| \times$$
$$\left\|f_{X_2}(x_2^*)\frac{1}{nh}\sum_{i=1}^{n}\frac{\delta_i}{\widehat{G}(\cdot)\,G_0(\cdot)}\begin{bmatrix}X_{1i}\\X_{3i}\end{bmatrix}^{\otimes 2}K_h(X_{2i}-x_2^*)\right\| +$$
$$\left\|f_{X_2}(x_2^*)\frac{1}{nh}\sum_{i=1}^{n}f_{\varepsilon|X}(0|X_i)\begin{bmatrix}X_{1i}\\X_{3i}\end{bmatrix}^{\otimes 2}K_h(X_{2i}-x_2^*) - \Omega_1(x_2^*)\right\| + o_p(1) =$$
$$o_p(1)\,O_p(1) + o_p(1)\,O_p(1) + o_p(1) = o_p(1).$$

Similarly, we have that

$$\left\|\widehat{\Omega}_{1\widehat{G}}\left(x_2^*\right)-\Omega_{1G}\left(x_2^*\right)\right\| \leq \sup\left|\widehat{G}\left(\cdot\right)-G_0\left(\cdot\right)\right|\left|\frac{\tau\left(1-\tau\right)v_0}{G\left(\cdot\right)^2}\right|$$
$$\left\|f_{X_2}\left(x_2^*\right)\frac{1}{nh}\sum_{i=1}^{n}\left[\begin{array}{c}X_{1i}\\X_{3i}\end{array}\right]^{\otimes 2}K_h\left(X_{2i}-x_2^*\right)\right\|+$$
$$\left\|f_{X_2}\left(x_2^*\right)\frac{1}{nh}\sum_{i=1}^{n}\frac{\tau\left(1-\tau\right)v_0}{G_0\left(\cdot\right)}\left[\begin{array}{c}X_{1i}\\X_{3i}\end{array}\right]^{\otimes 2}K_h\left(X_{2i}-x_2^*\right)-\Omega_{1G}\left(x_2^*\right)\right\|+o_p\left(1\right)=$$
$$o_p\left(1\right)O_p\left(1\right)+o_p\left(1\right)=o_p\left(1\right),$$

hence by CMT $\left\|\widehat{\Omega}_{1\widehat{G}\theta_\tau}\left(x_2^*\right)-\Omega_{1G\theta_\tau}\left(x_2^*\right)\right\|=o_p\left(1\right)$. Under the local hypothesis (4.1) given in the main paper, the same arguments as those used in Theorem 7 and CMT show that

$$\left(nh\right)^{1/2}R\left(\widehat{\theta}_\tau\left(x_2^*\right)-\theta_{0\tau}\right)\xrightarrow{d}N\left(\gamma_\tau\left(x_2^*\right),R\Omega_{1G\theta_\tau}\left(x_2^*\right)R^T\right)\quad j=1,2,3,$$

hence the first conclusion follows by standard results on quadratic forms in nonzero mean normal random vectors. The consistency of $W_l\left(x_2^*\right)$ under the assumption that $\left(nh\right)^{1/2}\gamma_{\tau n}\left(x_2^*\right)\to\infty$ is a direct consequence of the previous conclusion. ■

Proof of Proposition 11. Note that by A1(i) and Theorem 1

$$Cov\left(\left(nh\right)^{1/2}\left(\widehat{\theta}_\tau\left(x_{2j}^*\right)-\theta_{0\tau j}\right),\left(nh\right)^{1/2}\left(\widehat{\theta}_\tau\left(x_{2k}^*\right)-\theta_{0\tau k}\right)\right)=$$
$$E\left[\frac{1}{h}S_1\left(f_{X_2}\left(x_j\right)\Omega_1\left(x_{2j}\right)\right)^{-1}\frac{\delta_i^2}{G_0^2\left(\cdot\right)}W_i^{\otimes 2}\rho_\tau'\left(\varepsilon_i\right)^2K_h\left(X_{2i}-x_{2j}\right)\times\right.$$
$$\left.K_h\left(X_{2i}-x_{2k}\right)\left(f_{X_2}\left(x_k\right)\Omega_1\left(x_{2k}\right)\right)^{-1}S_1^T\right]+o\left(1\right),$$

where $S_1=\left[O_{pk},I_p\right]$. By iterated expecations and a standard kernel calculation, we have for any $1\leq j\neq k\leq m$

$$Cov\left(\left(nh\right)^{1/2}\left(\widehat{\theta}_\tau\left(x_{2j}^*\right)-\theta_{0\tau j}\right),\left(nh\right)^{1/2}\left(\widehat{\theta}_\tau\left(x_{2k}^*\right)-\theta_{0\tau k}\right)\right)=$$
$$S_1\left(f_{X_2}\left(x_j\right)\Omega_1\left(x_{2j}\right)\right)^{-1}E\left[\frac{\tau\left(1-\tau\right)}{G_0\left(\cdot\right)}\left[\begin{array}{c}X_1\\X_3\end{array}\right]^{\otimes 2}|X_2\right]\int K_h\left(X_2-x_{2j}\right)K_h\left(X_2-x_{2k}\right)f\left(X_2\right)dX_2\times$$
$$\left(f_{X_2}\left(x_k\right)\Omega_1\left(x_{2k}\right)\right)^{-1}S_1^T,$$

and

$$\int K_h(X_2 - x_{2j}) K_h(X_2 - x_{2k}) f(X_2) dX_2 = \int K(u) K\left(u + \frac{x_{2j} - x_{2k}}{h}\right) f(x_{2j} + hu) du =$$

$$\int_{|u| < \frac{x_{2j} - x_{2k}}{h}} K(u) K\left(u + \frac{x_{2j} - x_{2k}}{h}\right) f(x_{2j} + hu) du +$$

$$\int_{|u| \geq \frac{x_{2j} - x_{2k}}{2h}} K(u) K\left(u + \frac{x_{2j} - x_{2k}}{h}\right) f(x_{2j} + hu) du \leq$$

$$\sup_{|v| > \frac{x_{2j} - x_{2k}}{2h}} |K(v)| \sup_v |f(v)| \int |K(u)| du + \sup_{|u| \geq \frac{x_{2j} - x_{2k}}{2h}} |K(u)| \sup_u |f(u)| \int |K(u)| du,$$

where $v := u + (x_{2j} - x_{2k})/h > (x_{2j} - x_{2k})/2h$ and $\sup_{|\cdot| > (x_{2j} - x_{2k})/2h} |K(v)|$, for $\cdot = u$ or v, both tend to 0 as $h \to 0$ by A3, hence

$$Cov\left((nh)^{1/2}\left(\widehat{\theta}_\tau(x_{2j}^*) - \theta_{0\tau j}\right), (nh)^{1/2}\left(\widehat{\theta}_\tau(x_{2k}^*) - \theta_{0\tau k}\right)\right) = o(1).$$

Therefore by the Cramer-Wold device and CLT we have

$$(nh)^{1/2}\begin{bmatrix} \left(\widehat{\theta}_\tau(x_{21}^*) - \theta_{0\tau 1}\right) \\ \vdots \\ \left(\widehat{\theta}_\tau(x_{2m}^*) - \theta_{0\tau m}\right) \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} \gamma_\tau(x_{21}^*) \\ \vdots \\ \gamma_\tau(x_{2m}^*) \end{bmatrix}, diag\left[\Omega_{1G\theta_\tau}(x_{21}^*), ..., \Omega_{1G\theta_\tau}(x_{2m}^*)\right]\right),$$

and the conclusion follows by the same arguments as those of Proposition 10. ■

Proof of Proposition 12. Under the alternative hypothesis, the same arguments as those used in Theorem 8 show that

$$n^{1/2} R\left(\widehat{\beta}_\tau - r_\tau\right) \xrightarrow{d} N\left(\gamma_\tau, R\widehat{\Omega}_{2*\beta_\tau} R^T\right)$$

and the conclusions follow by the same arguments as those of Proposition 10. ■

Proof of Theorem 13. Let

$$\begin{aligned} R_n\left(\gamma_{\beta_\tau}, \widehat{G}\right) &= \sum_{i=1}^n \frac{\delta_i}{\widehat{G}(\cdot)}\left[\rho_\tau\left(\widetilde{Z}_i^* - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widetilde{Z}_i^*\right)\right] = \\ &\sum_{i=1}^n \frac{\delta_i}{G_0(Z_i)}\left[\rho_\tau\left(\varepsilon_i - \left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right) - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \right. \\ &\left.\rho_\tau\left(\varepsilon_i - \left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right)\right)\right] + \\ &\sum_{i=1}^n \frac{\delta_i\left(G_0(Z_i) - \widehat{G}(Z_i)\right)}{\widehat{G}(Z_i) G_0(Z_i)}\left[\rho_\tau\left(\varepsilon_i - \left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right) - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \right. \\ &\left.\rho_\tau\left(\varepsilon_i - \left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right)\right)\right] \\ &= R_{1n}\left(\gamma_{\beta_\tau}, G_0\right) + R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right). \end{aligned}$$

where $\widetilde{Z}_i^* = Z_i - X_{1i}^T\beta_\tau - \widetilde{\theta}(X_{2i})$. For the independent censoring case, proceeding as in the proof of Theorem 3 and using (1.1), we have that

$$R_{1n}(\gamma_{\beta_\tau}, G_0) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \frac{\delta_i}{G_0(Z_i)} X_{1i}\rho'_\tau(\varepsilon_i) + \frac{\gamma_{\beta_\tau}^T}{n} \sum_{i=1}^n f_{\varepsilon|X}(0|X_i) X_{1i}^{\otimes 2}\gamma_{\beta_\tau} -$$

$$\frac{\gamma_{\beta_\tau}^T}{n^{3/2}} \sum_{i=1}^n \sum_{j=1}^n f_{\varepsilon|X}(0|X_i) X_{1i}\eta(X_j) L_2(X_{2j} - X_{2i}) + o_p(1),$$

$$= \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \left\{ \frac{\delta_i}{G_0(Z_i)} X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1\eta(X_i) | X_2 = X_{2i}\right] \right\} + \gamma_{\beta_\tau}^T \Sigma_2 \gamma_{\beta_\tau} + o_p(1)$$

and

$$R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \frac{\left(G_0(Z_i) - \widehat{G}(Z_i)\right)}{\widehat{G}(Z_i)} \left[\frac{\delta_i}{G_0(Z_i)} X_{1i}\rho'_\tau(\varepsilon_i) -\right.$$

$$\left. E\left[f_{\varepsilon|X}(0|X) X_1\eta(X_i) | X_2 = X_{2i}\right]\right] + o_p(1) =$$

$$\frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \int_0^L \frac{E\left[\left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1\eta(X_i) | X_2 = X_{2i}\right]\right) I(Z_i \geq u)\right]}{S(u)} dM_i(u) +$$

$$o_p(1).$$

Thus by CL and QAL, we have $\widehat{\gamma}_{\beta_\tau} = \Sigma_2^{-1}\zeta'_{kn}$ where

$$\zeta'_{kn} = \left\{ \frac{1}{n^{1/2}} \sum_{i=1}^n \left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f(0|X) X_1\eta(X_i) | X_2 = X_{2i}\right]\right) - \right.$$

$$\frac{1}{n^{1/2}} \sum_{i=1}^n \int_0^L \left[X_{1i}\rho'_\tau(\varepsilon_i) - \frac{E\left[\left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1\eta(X_i) | X_2 = X_{2i}\right]\right) I(Z_i \geq u)\right]}{S(u)}\right] \times$$

$$\left. dM_i(u) \right\} + o_p(1),$$

and the conclusion follows by CLT and CMT, since

$$Var(\zeta'_{kn}) = E\left[\left(X_1\rho'_\tau(\varepsilon) - E\left[f_{\varepsilon|X}(0|X) X_1\eta(X) X_2\right]\right)^{\otimes 2}\right] + E\left[\int_0^L \left(X_1\rho'_\tau(\varepsilon) - \right.\right.$$

$$\left.\left. \frac{E\left[\left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1\eta(X_i) | X_2 = X_{2i}\right]\right) I(Z_i \geq u)\right]}{S(u)} I(Z > u)\right)^{\otimes 2} \frac{\lambda_0(u)}{G_0(u)} du\right].$$

The second and third conclusions follow by the same arguments as those used in the proofs of Theorems 4 and 5, with $E\left[f_{\varepsilon|X}(0|X) X_1\eta(X) | X_2\right]$ replacing $\varphi(X)$. ■

Proof of Theorem 14. As in the proof of Theorems 6 and 13, we have that, conditionally on

$\left(Z_i, \delta_i, X_i^T\right)_{i=1}^n$,

$$R_{\xi n}\left(\gamma_{\beta_\tau}, \widehat{G}_\xi\right) = \sum_{i=1}^n \frac{\delta_i \xi_i}{G_0\left(Z_i\right)}\left[\rho_\tau\left(\widetilde{Z}_i^* - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widetilde{Z}_i^*\right)\right] +$$

$$\sum_{i=1}^n \frac{\delta_i^L \xi_i \left(G_0\left(Z_i\right) - \widehat{G}_\xi\left(Z_i\right)\right)}{\widehat{G}_\xi\left(Z_i\right) G_0\left(Z_i\right)}\left[\rho_\tau\left(\widetilde{Z}_i^* - \frac{X_{1i}^T \gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widetilde{Z}_i^*\right)\right]$$

$$:= R_{\xi 1n}\left(\gamma_{\beta_\tau}, G_0\right) + R_{\xi 2n}\left(\gamma_{\beta_\tau}, \widehat{G}_\xi\right) + o_p(1),$$

where

$$R_{\xi 1n}\left(\gamma_{\beta_\tau}, G_0\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \frac{\delta_i \xi_i}{G_0\left(Z_i\right)} X_{1i} \rho_\tau'\left(\varepsilon_i\right) + \frac{\gamma_{\beta_\tau}^T}{n} \sum_{i=1}^n \xi_i f_{\varepsilon|X}\left(0|X_i\right) X_{1i}^{\otimes 2} \gamma_{\beta_\tau} + o_p(1),$$

$$R_{\xi 2n}\left(\gamma_\beta, \widehat{G}_\xi\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^n \int_0^\infty \frac{dM_{\xi i}(u)}{G_0(u)}\left[X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right] -$$

$$\left.\frac{E\left[\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right) I\left(Z_i \geq u\right)\right]}{S(u)}\right] + o_p(1).$$

Thus by CL and QAL, we have that $\widehat{\gamma}_{\xi\beta_\tau} = \Sigma_2^{-1}\zeta_{\xi kn}$, where

$$\zeta_{\xi kn} = \left\{\frac{1}{n^{1/2}} \sum_{i=1}^n \xi_i \left(X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right) -\right.$$

$$\frac{1}{n^{1/2}} \sum_{i=1}^n \int_0^L \xi_i \left[\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right) -\right.$$

$$\left.\left.\frac{E\left[\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right) I\left(Z_i \geq u\right)\right]}{S(u)}\right] dM_i(u)\right\} + o_p(1),$$

hence

$$n^{1/2}\left(\widetilde{\beta}_\tau^* - \widetilde{\beta}_\tau\right) = \Sigma_2^{-1} \frac{1}{n^{1/2}} \sum_{i=1}^n \left(\xi_i - 1\right)\left\{X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right] -\right.$$

$$\int_0^L \left[\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right) -\right.$$

$$\left.\left.\frac{E\left[\left(X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta\left(X_i\right)|X_2 = X_{2i}\right]\right) I\left(Z_i \geq u\right)\right]}{S(u)}\right] dM_i(u)\right\} + o_p(1),$$

and the conclusion follows by CMT and Lemma 2.9.5 of van der Vaart & Wellner (1996). For the dependent censoring case with the parametric specification, we have

$$\frac{1}{n} \sum_{i=1}^n \xi_i \psi\left(W_i, x\right) + r_{\xi n\psi}(x)$$

with $\sup_x |r_{\xi n\psi}(x)| = o_p(n^{-1/2})$ conditionally on $(Z_i, \delta_i, X_i^T)_{i=1}^n$. Then for

$$\zeta_{\xi\psi n}(W_i) = \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{\xi_i \delta_i}{G_0(Z_i|X_i)} [(X_{1i}\rho'_\tau(\varepsilon_i) -) -$$
$$\sum_{j=1}^n \frac{\xi_i \psi(W_j, X_i)}{G_0(Z_j|X_i)} \left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) | X_2 = X_{2i}\right]\right)\Bigg],$$

and

$$h_\xi(W_i, W_j) = \frac{\delta_i \xi_i}{G_0(Z_i|X_i)} \left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) X_2 = X_{2i}\right]\right) +$$
$$\frac{\delta_j \xi_j}{G_0(Z_j|X_j)} \left(X_{1j}\rho'_\tau(\varepsilon_j) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) | X_2 = X_{2j}\right]\right) -$$
$$\frac{\delta_i \xi_j}{G_0(Z_i|X_i)} \left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) | X_2 = X_{2i}\right]\right) \frac{\psi(W_j, X_i)}{G_0(Z_j|X_i)} -$$
$$\frac{\delta_j \xi_i}{G_0(Z_j|X_j)} \left(X_{1j}\rho'_\tau(\varepsilon_j) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_j) | X_2 = X_{2j}\right]\right) \frac{\psi(W_i, X_j)}{G_0(Z_i|X_j)},$$

we have that $\zeta_{\xi\psi n}(W_i) = n^2 2\zeta'_{\xi\psi n}(W_i)/n(n-1)$ is a U statistic with $E[h_\xi(W_i, W_j)] = 0$ and

$$\zeta'_{\xi\psi n}(W_i) = \frac{1}{n^2} \sum_{i<j=1}^n h_\xi(W_i, W_j) + r_{\xi hn}$$

with $r_{\xi hn} = n^{-3/2} \sum_{i=1}^n \xi_i \left(X_{1i}\rho'_\tau(\varepsilon_i) + E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) | X_2 = X_{2i}\right]\right)$. Clearly

$$h_\xi(W_1) := E[h_\xi(W_1, W_2) | W_1] = \frac{\delta_1 \xi_1}{G_0(Z_1|X_{11}, X_{21})} (X_1 \rho'_\tau(\varepsilon_1)$$
$$-E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_1) | X_2 = X_{21}\right]) + \xi_1 E[(X_{12}\rho'_\tau(\varepsilon_2) \times$$
$$-E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_{12}) | X_2 = X_{22}\right]) \frac{\psi(W_1, X_{12}, X_{22})}{G_0(Z_1|X_{12}, X_{22})} |W_1\Bigg],$$

hence

$$n^{1/2}\left(\widehat{\beta}^*_\tau - \widehat{\beta}_\tau\right) = \Sigma_2^{-1} \frac{1}{n^{1/2}} \sum_{i=1}^n (\xi_i - 1) h_1(W_i) + o_p(1)$$

and the conclusion follows by CMT and Lemma 2.9.5 of van der Vaart & Wellner (1996). Finally, for the nonparametric dependent censoring case, using the same arguments as those used in Theorem 5, we have by CL and QAL that $\widehat{\gamma}_{\xi\beta} = -\Sigma_2^{-1}\zeta_{\xi\psi npn}(W_i)$, where

$$\zeta_{\xi\psi npn}(W_i) = \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{\xi_i \delta_i}{G_0(Z_i|X_{2i})} \left(X_{1i}\rho'_\tau(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) | X_2 = X_{2i}\right]\right) -$$
$$\sum_{i=1}^n \xi_i f_{X_2}(X_{2i}) E\left[\frac{\psi(Z_i, \delta_i, Y_i, X_{2i}) \left(X_{1i}\rho(\varepsilon_i)' - E\left[f_{\varepsilon|X}(0|X) X_1 \eta(X_i) | X_2 = X_{2i}\right]\right)}{G_0(Z_i|X_{2i})} |X_{2i}\right],$$

hence

$$n^{1/2}\left(\widehat{\beta}_\tau^* - \widehat{\beta}_\tau\right) = \Sigma_2^{-1}\frac{1}{n^{1/2}}\sum_{i=1}^n (\xi_i - 1)\left\{\frac{\delta_i}{G_0(Z_i|X_{2i})}\left(X_{1i}\rho_\tau'(\varepsilon_i) - \right.\right.$$

$$\left. E\left[f_{\varepsilon|X}(0|X)X_1\eta(X_i)|X_2 = X_{2i}\right]\right) - \sum_{i=1}^n f_{X_2}(X_{2i}) \times$$

$$\left. E\left[\frac{\psi(Z_i,\delta_i,Y_i,X_{2i})\left(X_{1i}\rho(\varepsilon_i)' - E\left[f_{\varepsilon|X}(0|X)X_1\eta(X_i)|X_2 = X_{2i}\right]\right)}{G_0(Z_i|X_{2i})}|X_{2i}\right]\right\},$$

and the conclusion follows by CMT and Lemma 2.9.5 of van der Vaart & Wellner (1996). ■

Proof of Theorem 15. The proof is similar to that of Theorem 8, so we simply sketch it. Note that, for $\widetilde{Z}_i^* = Z_i - X_{1i}^T\beta_\tau - X_{3i}^T\widetilde{\theta}(X_{2i})$,

$$R_n\left(\gamma_{\beta_\tau},\widehat{G}\right) = \sum_{i=1}^n \frac{\delta_i}{\widehat{G}(\cdot)}\left[\rho_\tau\left(\widetilde{Z}_i^* - \frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right) - \rho_\tau\left(\widetilde{Z}_i^*\right)\right] =$$

$$\sum_{i=1}^n \frac{\delta_i}{G_0(Z_i)}\left[\rho_\tau\left(\varepsilon_i - X_{3i}^T\left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right) - \frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right) - \right.$$

$$\left.\rho_\tau\left(\varepsilon_i - X_{3i}^T\left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right)\right)\right] +$$

$$\sum_{i=1}^n \frac{\delta_i\left(G_0(Z_i) - \widehat{G}(Z_i)\right)}{\widehat{G}(Z_i)G_0(Z_i)}\left[\rho_\tau\left(\varepsilon_i - X_{3i}^T\left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right) - \frac{X_{1i}^T\gamma_{\beta_\tau}}{n^{1/2}}\right) - \right.$$

$$\left.\rho_\tau\left(\varepsilon_i - X_{3i}^T\left(\widetilde{\theta}(X_{2i}) - \theta_0(X_{2i})\right)\right)\right]$$

$$= R_{1n}(\gamma_{\beta_\tau}, G_0) + R_{2n}\left(\gamma_{\beta_\tau},\widehat{G}\right),$$

and

$$R_{1n}(\gamma_{\beta_\tau}, G_0) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\left\{\frac{\delta_i}{G_0(Z_i)}X_{1i}\rho_\tau'(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X)X_1X_3^T\eta(X_i)|X_2 = X_{2i}\right]\right\} + \gamma_{\beta_\tau}^T\Sigma_2\gamma_{\beta_\tau} +$$

$$o_p(1).$$

For the independent censoring case, we have that

$$R_{2n}\left(\gamma_{\beta_\tau},\widehat{G}\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\frac{\left(G_0(Z_i) - \widehat{G}(Z_i)\right)}{\widehat{G}(Z_i)}\times$$

$$\left[\frac{\delta_i}{G_0(Z_i)}X_{1i}\rho_\tau'(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X)X_1X_3^T\eta(X_i)|X_2 = X_{2i}\right]\right] + o_p(1) =$$

$$\frac{\gamma_{\beta_\tau}^T}{n^{1/2}}\sum_{i=1}^n\int_0^L\frac{E\left[\left(X_{1i}\rho_\tau'(\varepsilon_i) - E\left[f_{\varepsilon|X}(0|X)X_1X_3^T\eta(X_i)|X_2 = X_{2i}\right]\eta(X_i)\right)I(Z_i \geq u)\right]}{S(u)}dM_i(u) +$$

$$o_p(1),$$

hence the conclusion follows by CL, QAL, CLT and CMT as in the proof of Theorem 13. For the two dependent censoring cases, we have that

$$R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^{n} \frac{\delta_i}{G_0\left(Z_i|X_i\right)} \left[X_{1i}\rho_\tau'\left(\varepsilon_i\right) - E\left[f_{\varepsilon|X}\left(0|X\right) X_1 X_3^T \eta\left(X_i\right)|X_2 = X_{2i}\right]\right] + o_p\left(1\right),$$

$$R_{2n}\left(\gamma_{\beta_\tau}, \widehat{G}\right) = \frac{\gamma_{\beta_\tau}^T}{n^{1/2}} \sum_{i=1}^{n} f_{X_2}\left(X_{2i}\right) E\left[\frac{\psi\left(Z_i, \delta_i, Y_i, X_{2i}\right)}{G_0\left(Z_i|X_{2i}\right)}\left(X_{1i}\rho\left(\varepsilon_i\right)' - \frac{E\left[f_{\varepsilon|X}\left(0|X\right) X_1 X_3^T \eta\left(X_i\right)|X_2 = X_{2i}\right]\right)}{G_0\left(Z_i|X_{2i}\right)}|X_{2i}\right] + o_p\left(1\right)$$

hence the conclusion follows by CL, QAL, CLT and CMT as in the proof of Theorem 13. ■

Proof of Theorem 16. The proof is similar to that of Theorem 6, hence is omitted. ■

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