Supplement to "Maximum Likelihood Estimation of Autoregressive Models with a Near Unit Root and Cauchy Errors"

Jungjun Choi and In Choi

May, 2018

Abstract

This supplement contains simulation results.

1 Description of simulation results

This supplement reports simulation results that examine finite-sample properties of the estimators and test statistics of the previous sections.

Table 1 reports empirical biases and root mean square errors (RMSEs) of the MLE and LSE for the cases of $\gamma^o = 0$, 1 and 10. For Table 1, we drew $5,000$ samples of size $n = 50$, 100 and 250 from models (1), (3) and (5), with $Y_0 = 0$ and $\{\varepsilon_t\}$ being generated from a standard Cauchy distribution ($\sigma^o = 1$). Results in Table 1 are summarized as follows.

(i) The MLE is less biased than the LSE for models (3) and (5), and vice versa for model (1).

(ii) The MLE has smaller RMSEs than the LSE in the all the cases.

(iii) The closer ρ_n is to 1, the better the MLE performs. As ρ_n approaches 1, the

RMSEs of the MLE diminish although there a couple of exceptions.

(iv) Performance of the MLE and LSE deteriorates as more nonstochastic regressors are added to the AR(1) model.

To assess robustness of the results in Table 1, we consider errors generated by

$$\varepsilon_t = p\varepsilon_{1t} + (1-p)\varepsilon_{2t}, (p = 0,\ 0.1,\ 0.5,\ 0.8),$$

where $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are independent, standard normal and Cauchy errors, respectively, and report empirical biases and RMSEs of the MLE and LSE in Table 2. Note that the MLE is subject to misspecification under this experimental format. To save the space, we consider only the case $\gamma^o = 10$ in Table 2. When $p = 0$ (i.e., entirely normal errors), the LSE performs better than the MLE as expected. But when $p = 0.1,\ 0.5,\ 0.8$, the MLE is less biased and has smaller RMSEs than the LSE in all the cases. The results reported in Table 2 indicate that the MLE under the assumption of Cauchy errors may perform better than the LSE when the errors have infinite variance, although more experiments are needed to confirm this.

To evaluate finite-sample validity of the asymptotic distributions of the t-statistics studied in Theorem 4, empirical percentiles of the t-statistics are reported in Table 3. We drew $50,000$ samples of size $n = 100$, 250, 500 and $1,000$ from models (1), (3) and (5) with $Y_0 = 0$ and $\{\varepsilon_t\}$ being generated by a standard Cauchy distribution. We consider the cases $\gamma^o = 0$, 10, 50 and assume $a_0 = 0.5$, 1, 5 for model (3) and $a_0 = 0.5$, 5 and $b_0 = 0.5$, 5 for model (5). Results reported in Table 3 are summarized as:

(i) The empirical percentiles look more and more like those of the standard normal density as sample size increases and as ρ_n^o becomes closer to 1. However, when $\rho_n^o = 0.5$ with $n = 100$, the empirical percentiles look quite different from those of the standard normal density.

(ii) The empirical percentiles do not change much with the values of a_0 and b_0. This is well expected because the limiting distributions of the MLE are invariant to the values of of a_0 and b_0.

Table 4 reports size-unadjusted and size-adjusted empirical powers of the MLE-based t-test of this paper and the Dickey–Fuller t-test at the 5% nominal level. The Dickey–Fuller t-test uses its asymptotic critical values reported in Fuller (2009) for the size-unadjusted power simulation. The alternative hypotheses considered are $\rho_n^o = 0.999, 0.99, 0.97, 0.95, 0.9, 0.8$. The powers were computed for $5,000$ samples of size $n = 50,\ 100\ ,\ 250$. We consider the case $Y_0 = 0$, $\sigma^o = 1$ and assume $a_0 = 0$ for model (3) and $a_0 = b_0 = 0$ for model (5). The results in Table 4 are summarized as follows.

(i) The MLE-based t-test has empirical sizes reasonably close to the nominal size for model (1). But for models (3) and (5), size distortions are observed at $n = 50$.

(ii) The MLE-based t-test has much higher size-unadjusted and size-adjusted empirical powers than the Dickey–Fuller t-test for all the models.

(iii) The empirical power of the MLE-based t-test tends to decrease only slightly as higher order time polynomials are added to the model as regressors. This is due to the fact that the limiting distribution of the t-test is a standard normal for all the models. By contrast, the Dickey–Fuller test undergoes severe power reduction as higher order time polynomials are added to the AR(1) model as regressors because its null distributions shift leftward under such circumstance.

In summary, the simulation results of this section indicate that the MLE of the AR coefficient for a nearly non-stationary AR(1) model performs better than the LSE when the errors have a Cauchy distribution or a distribution which is a mixture of standard normal and Cauchy distributions. Furthermore, the finite-sample distribution of the MLE-based t-statistic becomes closer to a standard normal distribution as sample size increases. Last, the MLE-based t-test works reasonably well in finite samples and is more powerful than the Dickey–Fuller t-test.

2 Tables

Table 1. Empirical biases and RMSEs of MLE and LSE for ρ_n

Part A: model (1)

		Bias		RMSE	
γ^o	ρ_n^o	MLE	LSE	MLE	LSE
		$n = 50$			
10	0.80	-0.0026	-0.0186	0.0299	0.1421
1	0.98	-0.0017	-0.0232	0.0147	0.1052
0	1.00	-0.0025	-0.0211	0.0151	0.1713
		$n = 100$			
10	0.90	-0.0008	-0.0084	0.0092	0.1282
1	0.99	-0.0005	-0.0114	0.0049	0.0470
0	1.00	-0.0006	-0.0130	0.0046	0.1552
		$n = 250$			
10	0.96	-0.00003	-0.0045	0.0022	0.0212
1	0.996	-0.0001	-0.0070	0.0012	0.1742
0	1.00	-0.0001	-0.0044	0.0011	0.0118

Part B: model (3) with $a_0 = 1$

γ^o	ρ_n^o	Bias MLE	Bias LSE	RMSE MLE	RMSE LSE
			$n = 50$		
10	0.80	-0.0052	-0.0532	0.0312	0.4019
1	0.98	-0.0072	-0.0866	0.0231	0.2909
0	1.00	-0.0072	-0.0960	0.0241	0.1826
			$n = 100$		
10	0.90	-0.0013	-0.0326	0.0107	0.0579
1	0.99	-0.0016	-0.0463	0.0074	0.0641
0	1.00	-0.0015	-0.0509	0.0067	0.0849
			$n = 250$		
10	0.96	-0.0002	-0.0148	0.0026	0.0616
1	0.996	-0.0003	-0.0194	0.0017	0.0273
0	1.00	-0.0002	-0.0206	0.0016	0.0342

Part C: model (5) with $a_0 = 1$ and $b_0 = 1$

γ^o	ρ_n^o	Bias MLE	Bias LSE	RMSE MLE	RMSE LSE
			$n = 50$		
10	0.80	-0.0085	-0.1083	0.0366	0.1823
1	0.98	-0.0123	-0.1571	0.0360	0.3999
0	1.00	-0.0141	-0.1831	0.0377	0.3994
			$n = 100$		
10	0.90	-0.0024	-0.0598	0.0119	0.0848
1	0.99	-0.0032	-0.0854	0.0110	0.1107
0	1.00	-0.0034	-0.0910	0.0104	0.1219
			$n = 250$		
10	0.96	-0.0004	-0.0253	0.0027	0.0347
1	0.996	-0.0004	-0.0344	0.0022	0.0748
0	1.00	-0.0005	-0.0411	0.0023	0.1664

Table 2. Empirical biases and RMSEs of MLE and LSE for ρ_n
Note: p denotes the proportion of ε_{1t}(standard Cauchy distribution).

Part A: model (1) with $\gamma^o = 10$, $\sigma^o = 1$

p		0	0.1	0.5	0.8
		$n = 50$ ($\rho_n^o = 0.8$)			
Bias	MLE	-0.0319	-0.0185	-0.0035	-0.0028
	LSE	-0.0302	-0.0241	-0.0172	-0.0144
RMSE	MLE	0.1253	0.0924	0.0379	0.0293
	LSE	0.1005	0.1017	0.1053	0.1785
		$n = 100$ ($\rho_n^o = 0.9$)			
Bias	MLE	-0.0166	-0.0070	-0.0011	-0.0007
	LSE	-0.0170	-0.0126	-0.0115	-0.0100
RMSE	MLE	0.0655	0.0428	0.0140	0.0092
	LSE	0.0525	0.0469	0.0489	0.0521
		$n = 250$ ($\rho_n^o = 0.96$)			
Bias	MLE	-0.00660	-0.00225	-0.00027	-0.00013
	LSE	-0.00672	-0.00560	-0.00467	-0.00441
RMSE	MLE	0.0269	0.0142	0.0034	0.0023
	LSE	0.0217	0.0206	0.0328	0.0218

Part B: model (3) with $\gamma^o = 10$, $a_0 = 1$ and $\sigma^o = 1$

p		0	0.1	0.5	0.8
		$n = 50$ ($\rho_n^o = 0.8$)			
Bias	MLE	-0.0717	-0.0416	-0.0096	-0.0053
	LSE	-0.0734	-0.0653	-0.0550	-0.0563
RMSE	MLE	0.1509	0.1104	0.0452	0.0316
	LSE	0.1302	0.1591	0.3183	0.2371
		$n = 100$ ($\rho_n^o = 0.9$)			
Bias	MLE	-0.0396	-0.0181	-0.0026	-0.0015
	LSE	-0.0400	-0.0361	-0.0337	-0.0326
RMSE	MLE	0.0814	0.0516	0.0158	0.0106
	LSE	0.0694	0.0621	0.0617	0.0647
		$n = 250$ ($\rho_n^o = 0.96$)			
Bias	MLE	-0.0167	-0.0050	-0.0005	-0.0002
	LSE	-0.0166	-0.0150	-0.0159	-0.0141
RMSE	MLE	0.0339	0.0166	0.0038	0.0027
	LSE	0.0285	0.0264	0.1238	0.0251

Part C: model (5) with $\gamma^o = 10$, $a_0 = 1$, $b_0 = 1$ and $\sigma^o = 1$

p		0	0.1	0.5	0.8
		$n = 50$ ($\rho_n^o = 0.8$)			
Bias	MLE	-0.1176	-0.0650	-0.0153	-0.0096
	LSE	-0.1155	-0.1056	-0.1009	-0.1100
RMSE	MLE	0.1881	0.1327	0.0537	0.0397
	LSE	0.1632	0.1747	0.3525	0.3721
		$n = 100$ ($\rho_n^o = 0.9$)			
Bias	MLE	-0.0646	-0.0300	-0.0038	-0.0024
	LSE	-0.0656	-0.0609	-0.0599	-0.0582
RMSE	MLE	0.1022	0.0614	0.0178	0.0122
	LSE	0.0915	0.0849	0.0846	0.0881
		$n = 250$ ($\rho_n^o = 0.96$)			
Bias	MLE	-0.0272	-0.0084	-0.0006	-0.0004
	LSE	-0.0276	-0.0257	-0.0255	-0.0271
RMSE	MLE	0.0428	0.0204	0.0042	0.0029
	LSE	0.0385	0.0371	0.0368	0.1740

Table 3. Empirical percentiles of the MLE-based t-statistics

Part A: model (1) with $\sigma^o = 1$

Sample		Probability of a smaller value							
size (n)	ρ_n^o	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.932	-3.876	-2.725	-1.865	1.824	2.659	3.685	5.368
250	0.8	-3.141	-2.416	-1.926	-1.443	1.418	1.887	2.331	2.963
500	0.9	-2.581	-2.118	-1.742	-1.339	1.347	1.748	2.098	2.576
1000	0.95	-2.433	-2.036	-1.688	-1.309	1.295	1.684	2.024	2.411
					γ^o=10				
100	0.9	-2.777	-2.224	-1.812	-1.394	1.337	1.765	2.166	2.651
250	0.96	-2.454	-2.053	-1.715	-1.329	1.279	1.662	1.993	2.403
500	0.98	-2.396	-1.999	-1.663	-1.298	1.289	1.671	1.998	2.391
1000	0.99	-2.394	-2.017	-1.675	-1.314	1.269	1.626	1.947	2.340
					γ^o=0				
100	1	-2.545	-2.104	-1.772	-1.379	1.271	1.658	2.007	2.425
250	1	-2.433	-2.028	-1.712	-1.339	1.269	1.632	1.953	2.343
500	1	-2.391	-2.007	-1.688	-1.316	1.268	1.641	1.981	2.353
1000	1	-2.363	-1.996	-1.674	-1.305	1.263	1.634	1.939	2.313
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part B.1: model (3) with $a_0 = 0.5$ and $\sigma^o = 1$

Sample size (n)	ρ_n^o	Probability of a smaller value 0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.922	-4.012	-2.848	-1.923	1.780	2.579	3.609	5.339
250	0.8	-3.132	-2.459	-1.954	-1.472	1.413	1.871	2.326	2.990
500	0.9	-2.617	-2.140	-1.773	-1.372	1.330	1.735	2.090	2.565
1000	0.95	-2.440	-2.023	-1.708	-1.324	1.316	1.685	2.026	2.447
					γ^o=10				
100	0.9	-2.906	-2.353	-1.922	-1.484	1.285	1.711	2.127	2.663
250	0.96	-2.504	-2.106	-1.756	-1.381	1.258	1.642	1.985	2.381
500	0.98	-2.408	-2.004	-1.686	-1.324	1.252	1.624	1.931	2.317
1000	0.99	-2.386	-2.024	-1.691	-1.328	1.255	1.627	1.949	2.332
					γ^o=0				
100	1	-2.836	-2.367	-1.994	-1.587	1.142	1.551	1.924	2.389
250	1	-2.548	-2.173	-1.839	-1.452	1.170	1.563	1.897	2.287
500	1	-2.448	-2.074	-1.752	-1.391	1.188	1.573	1.898	2.282
1000	1	-2.406	-2.050	-1.731	-1.360	1.222	1.601	1.907	2.265
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part B.2: model (3) with $a_0 = 1$ and $\sigma^o = 1$

Sample size (n)	ρ_n^o	Probability of a smaller value 0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.957	-3.954	-2.795	-1.949	1.760	2.588	3.580	5.216
250	0.8	-3.133	-2.425	-1.948	-1.472	1.385	1.854	2.311	2.986
500	0.9	-2.565	-2.126	-1.745	-1.343	1.334	1.738	2.096	2.571
1000	0.95	-2.445	-2.052	-1.710	-1.332	1.286	1.667	1.994	2.397
					γ^o=10				
100	0.9	-2.918	-2.357	-1.953	-1.502	1.281	1.721	2.149	2.652
250	0.96	-2.480	-2.095	-1.749	-1.369	1.247	1.633	1.979	2.397
500	0.98	-2.408	-2.022	-1.706	-1.339	1.257	1.625	1.955	2.316
1000	0.99	-2.372	-2.007	-1.692	-1.323	1.251	1.615	1.943	2.310
					γ^o=0				
100	1	-2.853	-2.354	-1.983	-1.574	1.135	1.550	1.907	2.332
250	1	-2.520	-2.147	-1.824	-1.445	1.184	1.546	1.883	2.295
500	1	-2.451	-2.084	-1.766	-1.402	1.204	1.571	1.892	2.267
1000	1	-2.417	-2.050	-1.732	-1.363	1.220	1.584	1.888	2.287
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part B.3: model (3) with $a_0 = 5$ and $\sigma^o = 1$

Sample		Probability of a smaller value							
size (n)	ρ_n^o	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.674	-3.897	-2.800	-1.921	1.759	2.547	3.446	5.129
250	0.8	-3.063	-2.410	-1.938	-1.467	1.408	1.869	2.320	2.944
500	0.9	-2.566	-2.128	-1.745	-1.343	1.333	1.739	2.097	2.571
1000	0.95	-2.445	-2.053	-1.710	-1.332	1.286	1.666	1.993	2.398
					γ^o=10				
100	0.9	-2.940	-2.359	-1.923	-1.474	1.277	1.693	2.103	2.606
250	0.96	-2.515	-2.103	-1.773	-1.385	1.255	1.640	1.986	2.356
500	0.98	-2.444	-2.061	-1.726	-1.344	1.267	1.635	1.982	2.364
1000	0.99	-2.386	-2.003	-1.685	-1.326	1.259	1.626	1.950	2.325
					γ^o=0				
100	1	-2.843	-2.364	-2.000	-1.575	1.130	1.527	1.868	2.318
250	1	-2.568	-2.160	-1.838	-1.461	1.166	1.545	1.867	2.258
500	1	-2.429	-2.075	-1.755	-1.381	1.216	1.587	1.906	2.307
1000	1	-2.444	-2.064	-1.741	-1.365	1.228	1.603	1.913	2.319
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part C.1: model (5) with $a_0 = 0.5$, $b_0 = 0.5$ and $\sigma^o = 1$

Sample size (n)	ρ_n^o	Probability of a smaller value 0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.878	-4.020	-2.857	-1.984	1.742	2.566	3.513	5.140
250	0.8	-3.160	-2.460	-1.959	-1.488	1.370	1.836	2.298	2.981
500	0.9	-2.579	-2.143	-1.768	-1.361	1.330	1.733	2.098	2.576
1000	0.95	-2.454	-2.056	-1.709	-1.342	1.292	1.664	1.988	2.380
					γ^o=10				
100	0.9	-3.052	-2.470	-2.046	-1.586	1.227	1.657	2.058	2.550
250	0.96	-2.550	-2.126	-1.802	-1.409	1.217	1.598	1.940	2.337
500	0.98	-2.464	-2.070	-1.740	-1.365	1.236	1.611	1.917	2.290
1000	0.99	-2.402	-2.013	-1.699	-1.345	1.240	1.606	1.932	2.299
					γ^o=0				
100	1	-3.045	-2.543	-2.158	-1.725	1.062	1.476	1.848	2.354
250	1	-2.632	-2.233	-1.890	-1.499	1.122	1.506	1.843	2.247
500	1	-2.517	-2.133	-1.805	-1.436	1.159	1.532	1.853	2.218
1000	1	-2.420	-2.065	-1.747	-1.384	1.200	1.564	1.875	2.255
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part C.2: model (5) with $a_0 = 0.5$, $b_0 = 5$ and $\sigma^o = 1$

Sample		Probability of a smaller value							
size (n)	ρ_n^o	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.712	-3.940	-2.849	-1.972	1.752	2.562	3.455	5.062
250	0.8	-3.143	-2.470	-1.992	-1.513	1.378	1.872	2.344	2.972
500	0.9	-2.605	-2.155	-1.781	-1.383	1.308	1.713	2.087	2.586
1000	0.95	-2.435	-2.058	-1.725	-1.347	1.268	1.654	2.001	2.423
					γ^o=10				
100	0.9	-3.108	-2.494	-2.069	-1.602	1.234	1.671	2.064	2.578
250	0.96	-2.559	-2.159	-1.819	-1.435	1.227	1.613	1.966	2.399
500	0.98	-2.443	-2.064	-1.744	-1.372	1.225	1.605	1.942	2.317
1000	0.99	-2.416	-2.045	-1.710	-1.337	1.236	1.605	1.916	2.290
					γ^o=0				
100	1	-3.081	-2.572	-2.171	-1.723	1.071	1.477	1.860	2.302
250	1	-2.654	-2.258	-1.926	-1.540	1.132	1.508	1.841	2.252
500	1	-2.540	-2.156	-1.838	-1.456	1.158	1.529	1.866	2.263
1000	1	-2.466	-2.073	-1.764	-1.393	1.193	1.568	1.868	2.244
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part C.3: model (5) with $a_0 = 5$, $b_0 = 0.5$ and $\sigma^o = 1$

Sample size (n)	ρ_n^o	Probability of a smaller value 0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-5.916	-3.908	-2.835	-1.962	1.776	2.606	3.523	5.175
250	0.8	-3.186	-2.477	-1.986	-1.484	1.379	1.857	2.327	2.978
500	0.9	-2.626	-2.145	-1.777	-1.372	1.318	1.704	2.069	2.545
1000	0.95	-2.432	-2.049	-1.721	-1.342	1.275	1.646	1.996	2.413
					γ^o=10				
100	0.9	-3.016	-2.466	-2.036	-1.575	1.256	1.685	2.094	2.610
250	0.96	-2.584	-2.148	-1.819	-1.434	1.227	1.612	1.952	2.359
500	0.98	-2.436	-2.066	-1.743	-1.363	1.240	1.618	1.941	2.332
1000	0.99	-2.425	-2.052	-1.727	-1.350	1.238	1.615	1.924	2.287
					γ^o=0				
100	1	-3.050	-2.559	-2.160	-1.720	1.074	1.483	1.850	2.320
250	1	-2.662	-2.244	-1.905	-1.524	1.127	1.516	1.846	2.262
500	1	-2.536	-2.134	-1.811	-1.445	1.181	1.537	1.870	2.226
1000	1	-2.453	-2.079	-1.756	-1.396	1.190	1.564	1.882	2.263
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Part C.4: model (5) with $a_0 = 5$, $b_0 = 5$ and $\sigma^o = 1$

Sample size (n)	ρ_n^o	Probability of a smaller value 0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
					γ^o=50				
100	0.5	-6.056	-3.959	-2.889	-1.981	1.745	2.531	3.508	5.162
250	0.8	-3.151	-2.459	-1.976	-1.496	1.380	1.853	2.329	2.969
500	0.9	-2.635	-2.169	-1.820	-1.400	1.314	1.733	2.085	2.526
1000	0.95	-2.499	-2.094	-1.742	-1.354	1.269	1.647	1.989	2.400
					γ^o=10				
100	0.9	-3.070	-2.471	-2.028	-1.585	1.248	1.683	2.073	2.590
250	0.96	-2.613	-2.169	-1.829	-1.431	1.219	1.610	1.953	2.357
500	0.98	-2.483	-2.097	-1.770	-1.385	1.243	1.624	1.953	2.331
1000	0.99	-2.428	-2.055	-1.724	-1.344	1.233	1.606	1.928	2.334
					γ^o=0				
100	1	-3.060	-2.549	-2.157	-1.725	1.078	1.478	1.852	2.342
250	1	-2.660	-2.261	-1.917	-1.522	1.130	1.516	1.843	2.238
500	1	-2.525	-2.143	-1.813	-1.430	1.169	1.547	1.869	2.239
1000	1	-2.463	-2.097	-1.766	-1.394	1.180	1.539	1.860	2.234
Standard Normal		-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33

Table 4. Empirical powers of unit root tests
Note: Numbers in parentheses denote size-adjusted empirical powers.

Part A: model (1) with $\sigma^o = 1$

ρ_n^o	0.8	0.9	0.95	0.97	0.99	0.999	1
				$n = 50$			
MLE t-test	1.00(1.00)	0.99(0.98)	0.94(0.92)	0.86(0.83)	0.57(0.52)	0.16(0.13)	0.07(0.05)
DF t-test	0.85(0.95)	0.20(0.40)	0.08(0.15)	0.05(0.10)	0.04(0.07)	0.03(0.05)	0.03(0.05)
				$n = 100$			
MLE t-test	1.00(1.00)	1.00(1.00)	1.00(1.00)	0.98(0.98)	0.85(0.84)	0.28(0.26)	0.06(0.05)
DF t-test	0.99(1.00)	0.83(0.95)	0.20(0.40)	0.10(0.19)	0.04(0.08)	0.03(0.05)	0.03(0.05)
				$n = 250$			
MLE t-test	1.00(1.00)	1.00(1.00)	1.00(1.00)	1.00(1.00)	0.99(0.99)	0.61(0.59)	0.06(0.05)
DF t-test	1.00(1.00)	1.00(1.00)	0.94(0.98)	0.48(0.78)	0.08(0.14)	0.03(0.06)	0.03(0.05)

Part B: model (3) with $\sigma^o = 1$ and $a_0 = 0$

ρ_n^o	0.8	0.9	0.95	0.97	0.99	0.999	1
				$n = 50$			
MLE t-test	1.00(1.00)	0.99(0.96)	0.91(0.83)	081(0.68)	0.50(0.37)	0.18(0.10)	0.12(0.05)
DF t-test	0.23(0.07)	0.06(0.02)	0.04(0.02)	0.05(0.02)	0.07(0.04)	0.08(0.04)	0.09(0.05)
				$n = 100$			
MLE t-test	1.00(1.00)	1.00(1.00)	1.00(0.99)	0.97(0.95)	0.78(0.71)	0.25(0.18)	0.09(0.05)
DF t-test	0.94(0.75)	0.20(0.07)	0.07(0.03)	0.04(0.02)	0.06(0.03)	0.08(0.05)	0.08(0.05)
				$n = 250$			
MLE t-test	1.00(1.00)	1.00(1.00)	1.00(1.00)	1.00(1.00)	0.99(0.98)	0.49(0.44)	0.07(0.05)
DF t-test	1.00(1.00)	0.98(0.94)	0.34(0.14)	0.09(0.04)	0.04(0.02)	0.07(0.05)	0.07(0.05)

Part C: model (5) with $\sigma^o = 1$ and $a_0 = b_0 = 0$

ρ_n^o	0.8	0.9	0.95	0.97	0.99	0.999	1
				$n = 50$			
MLE t-test	1.00(0.99)	0.98(0.92)	0.90(0.75)	0.77(0.57)	0.47(0.27)	0.21(0.07)	0.16(0.05)
DF t-test	0.15(0.08)	0.08(0.05)	0.08(0.05)	0.07(0.05)	0.06(0.05)	0.06(0.05)	0.07(0.05)
				$n = 100$			
MLE t-test	1.00(1.00)	1.00(1.00)	1.00(0.98)	0.96(0.92)	0.73(0.60)	0.22(0.12)	0.12(0.05)
DF t-test	0.76(0.61)	0.13(0.09)	0.07(0.06)	0.07(0.06)	0.07(0.05)	0.07(0.06)	0.06(0.05)
				$n = 250$			
MLE t-test	1.00(1.00)	1.00(1.00)	1.00(1.00)	1.00(1.00)	0.99(0.98)	0.41(0.34)	0.09(0.05)
DF t-test	1.00(1.00)	0.92(0.85)	0.18(0.12)	0.09(0.06)	0.06(0.05)	0.06(0.05)	0.06(0.05)

References

Fuller, W. A. (2009). *Introduction to statistical time series* (Vol. 428). New Jersey: John Wiley & Sons.