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Test for tail index constancy of GARCH innovations based on conditional volatility

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Abstract This study considers the problem of testing whether the tail index of the GARCH innovations undergoes a change according to the values of conditional volatilities. Special attention is paid to power-transformed and threshold generalized autoregressive conditional heteroscedasticity processes that can accommodate the GARCH family. We show that the proposed test asymptotically follows a functional of a standard Brownian motion under some regularity conditions. To evaluate our method, we carry out a simulation study and real data analysis using the return series of the Google stock price and DowJones index.

Keywords Constancy test for tail index · Heavy-tailed distribution · Conditional volatility · GARCH model · PTTGARCH model

1 Introduction

Financial asset returns are characterized by volatility clustering, heavy-tailedness violating normal assumption and mild skewness. In order to accommodate such properties, Bollerslev (1986) introduced the generalized autoregressive conditional heteroscedasticity (GARCH) models, and thereafter, several authors have proposed various GARCH-type models. In most cases, asymmetry and nonlinearity are the main objectives that give rise to various GARCH models. Among such models, we refer to the exponential GARCH (Nelson 1991), nonlinear GARCH (Engle and Ng 1993), GJR-GARCH (Glosten et al. 1993), threshold GARCH (TGARCH) (Zakoian

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1994), and power-transformed and threshold GARCH (PTTGARCH) (Pan et al. 2008) models; see also Higgins and Bera (1992), Li and Li (1996), Hwang and Kim (2004), and Hwang and Basawa (2004) for relevant references. To cope with heavy-tailedness and skewness, Bollerslev (1987) introduced the t-GARCH models, and later, Hansen (1994) proposed to use the skewed t-distribution for several GARCH-type models. Since then, the various extensions of the t-distribution have been explored by numerous researchers: see, for instance, Zhu and Galbraith (2010) and the articles cited therein. For the GARCH-type models with heavy-tailed innovations, we refer to Peng and Yao (2003), Berkes and Horváth (2004) and Pan et al. (2008).

Value-at-risk (VaR) has been playing an important role as a risk measurement in quantitative risk management. Its estimating method has been explored by numerous researchers: for a general review of quantitative risk management, see Embrechts et al. (2005). In practice, the risk of financial assets is highly influenced by the structure of time-varying volatility and the possibility of an extreme loss. Engle and Manganelli (2004) propose to use conditional autoregressive VaR (CAViaR) models and the quantile regression to obtain the conditional VaR (see also Lee and Noh 2013). Alternatively, the VaR can be estimated by assuming the models and their innovation distribution family and then by substituting suitable estimators into the closed form of given VaR formulas: see Lee and Lee (2011), Kim and Lee (2016a) and Kim and Lee (2016c), who study VaR forecasting using Gaussian mixture and asymmetrically skewed ARMA–GARCH models. Much attention should be paid in dealing with heavy-tailed financial time series, because the VaR estimation should be conducted for extremely small tail probabilities. For relevant references, see Drees (2003) who examine the VaR estimation in heavy-tailed distributions, and Chan et al. (2007) who consider the VaR estimation in the GARCH models with innovations having a regularly varying tail, wherein the tail heaviness is measured with the negative exponent called 'tail index,' and the extreme quantile of the innovation is used for VaR estimation. Owing to the time-varying nature of conditional volatility, this estimator can vary dynamically, and thus, it becomes more important to check the constancy of conditional volatility and the shape of innovation distribution. It is well known that ignoring a change could lead to a false conclusion and finally result in a critical loss.

To our knowledge, researchers have paid little attention to the interaction between the tail index of innovation distribution and the value of conditional volatility of GARCH processes, and few studies rigorously investigate whether the high and low values of conditional volatilities could affect the characteristics of innovation distributions. Motivated by this, we investigate this issue within the framework of a change point analysis on the tail index for GARCH models. In particular, we consider this problem on a larger family that accommodates GARCH models, namely PTTGARCH models, because the task of extension to the PTTGARCH family is not difficult to carry out. It is noteworthy that unlike the conventional methods, such as those of Quintos et al. (2001) and Kim and Lee (2011), the test is designed to check the constancy of tail index over the values of conditional volatilities rather than over time flow. The proposed test is shown to asymptotically follow the law of a functional of a standard Brownian motion: see Theorems 1 and 3 stated below. We demonstrate its validity through a simulation study and real data analysis.

The remainder of this paper is organized as follows. Section 2.1 introduces a constancy test for tail index and presents its asymptotic null distribution; Sect. 2.2 considers the PTTGARCH models that use heterogeneous innovations to formulate an alternative hypothesis and proposes a residual-based test; Sect. 3 conducts a simulation study and analyzes the return series of the Google stock price and DowJones index; Sect. 3.2 gives concluding remarks; Sect. 5 provides the proofs of the theorems stated in Sect. 2.

2 Main result

2.1 Change point test for tail index

Let $\{\boldsymbol{U}_i := (U_{i,1}, U_{i,2}) : i \in \mathbb{Z}\}$ and $\{\mathrm{s}_i : i \in \mathbb{Z}\}$ be sequences of random elements defined on probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that

$$\boldsymbol{U}_i,\, i \in \mathbb{Z}, \text{ are i.i.d. random vectors;} \tag{1}$$

$$\begin{aligned}&\mathrm{s}_i \text{ are uniformly distributed over } [0,1] \text{ and measurable with respect to } \sigma\{\boldsymbol{U}_j{:}j{<}i\}.\\ &\text{Moreover, } \{(\mathrm{s}_i, \boldsymbol{U}_i) : i \in \mathbb{Z}\} \text{ is ergodic and strictly stationary.}\end{aligned} \tag{2}$$

Note that $\boldsymbol{U}_i$ is independent of $\{\mathrm{s}_j : j \le i\}$ for each $i \in \mathbb{Z}$.

For any $\tau^\circ \in (0, 1]$, assume that

$$U_i := U_{i,1}\mathrm{I}(\mathrm{s}_i \le \tau^\circ) + U_{i,2}\mathrm{I}(\mathrm{s}_i > \tau^\circ) \quad \text{for } i \in \mathbb{Z}. \tag{3}$$

Later, the $\{U_i\}$ and $\{\mathrm{s}_i\}$ in (3), respectively, play the role of the innovations and the probability transforms of the logarithm of conditional volatilities of PTTGARCH models: see (13) in Sect. 2.2.

Suppose that one wishes to check whether the tail heaviness of U_i depends on s_i, that is, $0 < \tau^\circ < 1$. For this task, we assume that F_j, the distribution function of $U_{0,j}$ for $j = 1, 2$, has positive numbers $\alpha_1 \ne \alpha_2$ satisfying

$$1 - F_j(x) = x^{-\alpha_j}\ell_j(x) \quad \text{for } j = 1, 2, \tag{4}$$

where α_j is the tail index of F_j and each $\ell_j(x)$ is slowly varying as $x \to \infty$. Further, we set up the following hypotheses:

$$\mathcal{H}_0 : \tau^\circ = 1 \quad \text{vs.} \quad \mathcal{H}_1 : 0 < \tau^\circ < 1. \tag{5}$$

Assume that $\{(\mathrm{s}_i, U_i) : i = 1, \ldots, n\}$ are observed ($n \in \mathbb{N}$). Let $k = k_n < n$ be a positive integer satisfying $k \to \infty$ and $k = o(n)$ as $n \to \infty$. Then, for $0 \le \tau_1 < \tau_2 \le 1$ with either $\tau_1 = 0$ or $\tau_2 = 1$, we set

$$U_{(\tau_1,\tau_2)} := \begin{cases} \lfloor k(\tau_2 - \tau_1) + 1 \rfloor\text{-th largest order statistic of } \{U_i : \tau_1 < \mathrm{s}_i \le \tau_2\}, & \tau_1 = 0;\\ \lceil k(\tau_2 - \tau_1) + 1 \rceil\text{-th largest order statistic of } \{U_i : \tau_1 < \mathrm{s}_i \le \tau_2\}, & \tau_2 = 1,\end{cases}$$

and define the Hill estimator:

$$\mathrm{H}_n(\tau_1, \tau_2) := \frac{1}{k(\tau_2 - \tau_1)} \sum_{i=1}^{n} \left(\log U_i - \log U_{(\tau_1,\tau_2)}\right)_+ \mathrm{I}(\tau_1 < \mathrm{s}_i \le \tau_2),$$

which is consistent with the reciprocal of tail index (cf. Hall 1982; Hsing 1991). To test (5), we employ

$$\mathrm{Q}_n := k \sup_{\tau_0 \le \tau \le 1-\tau_0} \tau(1-\tau) \left\{\log \frac{\mathrm{H}_n(\tau, 1)}{\mathrm{H}_n(0, \tau)}\right\}^2, \quad 0 < \tau_0 < \frac{1}{2}. \tag{6}$$

Clearly, the above test statistic diverges to ∞ under $\mathcal{H}_1$.

Now, we investigate the asymptotic distribution of Q_n under $\mathcal{H}_0$. Under the null, we particularly have $U_i = U_{i,1}$ for every $i \in \mathbb{Z}$. By setting

$$\alpha := \alpha_1, \quad F := F_1, \quad \ell := \ell_1, \quad b(x) := \inf\left\{y : F(y) \ge 1 - x^{-1}\right\}, \quad x \ge 1, \tag{7}$$

we impose some regularity conditions as follows:

(A1) There exist $\mathrm{C} > 0$, $\gamma < 0$, and nonzero $\mathrm{D} \in \mathbb{R}$ such that

$$\ell(x) = \mathrm{C}\left(1 + \mathrm{D}x^{\gamma} + x^{\gamma}\Lambda(x)\right),$$

where $\Lambda(x)$ is differentiable in x,

$$\Lambda(x) \to 0 \quad \text{and} \quad \frac{d}{\mathrm{d}x}\Lambda(x) = o(x^{-1}) \quad \text{as } x \to \infty.$$

Furthermore, there exists $0 \le \mathrm{M} < \infty$ such that

$$\lim_{n\to\infty} \sqrt{k}(b(n/k))^{\gamma} = \mathrm{M}.$$

(A2) Let $\varrho(h) := \sup_{m\in\mathbb{N}} \mathrm{E}\sup\{|\mathrm{P}(B|\mathcal{A}_m) - \mathrm{P}(B)| : B \in \mathcal{A}^{m+h}\}$, where $\mathcal{A}_i := \sigma\{(\mathrm{s}_j, U_j) : j \le i\}$ and $\mathcal{A}^i := \sigma\{(\mathrm{s}_j, U_j) : j \ge i\}$ for $i \in \mathbb{N}$. Then,

$$\{(\mathrm{s}_i, U_i) : i \in \mathbb{Z}\} \text{ is absolutely regular, i.e., } \varrho(h) \to 0 \text{ as } h \to \infty.$$

(A3) There exist $l_n, s_n \in \mathbb{N}$ and $\Delta = \Delta_n > 0$ such that as $n \to \infty$,

$$l_n \to \infty; \quad l_n = o(n); \quad \frac{l_n k}{n} \to 0; \quad \sqrt{k}\Delta \to 0; \quad \frac{l_n^3}{k}\log_2^4 \frac{1}{\Delta} \to 0;$$
$$s_n = o(l_n); \quad \frac{n}{l_n}\varrho(s_n) \to 0.$$

Theorem 1 *Suppose that* **(A1)–(A3)** *hold.*
Then, under $\mathcal{H}_0$,

$$\mathrm{Q}_n \Rightarrow \sup_{\tau_0 \le \tau \le 1-\tau_0} \frac{\{\mathrm{B}(\tau) - \tau \mathrm{B}(1)\}^2}{\tau(1-\tau)}, \tag{8}$$

where B *stands for a standard Brownian motion.*

Remark 1 Although the result of Theorem 1 looks similar to Theorem 1 of Kim and Lee (2016b), the motivation and approach taken for the two are quite different in that Kim and Lee (2016b) deal with a change point test for α over time flow, whereas this is not the case here as seen in Sect. 2.2: the change is assumed to occur in terms of the values of s_i's. If $\{\mathrm{s}_i\}$ is independent of $\{\boldsymbol{U}_i\}$, (8) could be proven by following the lines similar to those in the proof of Theorem 1 in their paper, because $\{\mathrm{s}_i\}$ can play the same role as the time flow technically.

Remark 2 For p-value calculation, we use

$$\mathrm{P}\left(\sup_{\tau_0 \le \tau \le 1-\tau_0} \frac{\{\mathrm{B}(\tau) - \tau \mathrm{B}(1)\}^2}{\tau(1-\tau)} > x \right)$$
$$= \frac{\sqrt{x} e^{-x/2}}{\sqrt{2\pi}} \left\{ \left(1 - \frac{1}{x}\right) \log \frac{(1-\tau_0)^2}{\tau_0^2} + \frac{4}{x} + o(x^{-2}) \right\},$$

as $x \to \infty$ (cf. Csörgö and Horváth 1997, p. 25).

Remark 3 Since Q_n can depend on k, the optimal choice of k could be a concern for practitioners. Kim and Lee (2016b) consider to use the average of test statistics on a range of k in dealing with the problem of testing for the constancy of α over time. Such a method is not available at this stage and is beyond the scope of this paper. Since this issue deserves a further investigation, it is left as our future study. For a practical usage, though, we recommend to use k with $k/n \le 0.1$ at which $\mathrm{H}_n(0, 1)$ is stable, which is usually unstable for too small k's. In case $\mathcal{H}_0$ is rejected by Q_n, it is recommendable to confirm the tail index change again by a further analysis as done for the DowJones data in Sect. 3.2.

2.2 Change point test for PTTGARCH model

2.2.1 PTTGARCH(1,1) processes with heterogeneous innovations

In this subsection, we introduce PTTGARCH processes with heterogeneous innovations: see Pan et al. (2008) for the PTTGARCH model with i.i.d. innovations. For simplicity, here we focus on the PTTGARCH(1,1) model case: PTTGARCH models of higher order will be considered later in Sect. 2.2.2.

Let ω°, $\phi_{1,1}^\circ$, $\phi_{1,2}^\circ$, β_1°, δ°, and υ° be strictly positive constants, and $\{(U_{i,1}, U_{i,2}) : i \in \mathbb{Z}\}$ be a sequence of i.i.d. random vectors with $\mathrm{E}|U_{0,j}|^\nu < \infty$, $j = 1, 2$ for some

$\nu > 0$. Further, let $\{h_i^\circ\}$ be a Markov process satisfying

$$\begin{aligned} h_i^\circ = \omega^\circ + h_{i-1}^\circ \left(\phi_{1,1}^\circ (U_{i-1,1})_+^{2\delta^\circ} + \phi_{1,2}^\circ (U_{i-1,1})_-^{2\delta^\circ} + \beta_1^\circ\right) \mathrm{I}\left(h_{i-1}^\circ \leq \upsilon^\circ\right) \\ + h_{i-1}^\circ \left(\phi_{1,1}^\circ (U_{i-1,2})_+^{2\delta^\circ} + \phi_{1,2}^\circ (U_{i-1,2})_-^{2\delta^\circ} + \beta_1^\circ\right) \mathrm{I}\left(h_{i-1}^\circ > \upsilon^\circ\right). \end{aligned} \quad i \in \mathbb{N}. \tag{9}$$

Assume that $U_{0,1}$ and $U_{0,2}$ are absolutely continuous. If f_j denotes a density function of $\mathrm{sgn}(U_{0,j})|U_{0,j}|^{2\delta^\circ}$ for $j = 1, 2$, the transition density of (9) is given by

$$\begin{aligned} p(x, y) &:= \begin{cases} f_1^*(y|x)\mathrm{I}(x < \upsilon^\circ) + f_2^*(y|x)\mathrm{I}(x \geq \upsilon^\circ), & y > \omega^\circ + \beta_1^\circ x; \\ 0, & y \leq \omega^\circ + \beta_1^\circ x, \end{cases} \\ f_1^*(y|x) &:= \frac{1}{\phi_{1,1}^\circ x} f_1\left(\frac{y - \omega^\circ - \beta_1^\circ x}{\phi_{1,1}^\circ x}\right) + \frac{1}{\phi_{1,2}^\circ x} f_1\left(-\frac{y - \omega^\circ - \beta_1^\circ x}{\phi_{1,2}^\circ x}\right), \\ f_2^*(y|x) &:= \frac{1}{\phi_{1,1}^\circ x} f_2\left(\frac{y - \omega^\circ - \beta_1^\circ x}{\phi_{1,1}^\circ x}\right) + \frac{1}{\phi_{1,2}^\circ x} f_2\left(-\frac{y - \omega^\circ - \beta_1^\circ x}{\phi_{1,2}^\circ x}\right). \end{aligned} \tag{10}$$

Without loss of generality, we can assume that the state space of the Markov process in (9) is $[\omega^\circ/(1 - \beta_1^\circ), \infty)$ when $\beta_1^\circ < 1$. The following guarantees the stationarity and ergodicity of $\{h_i^\circ\}$ in (9) under regularity conditions.

Theorem 2 *Suppose that* $\{(U_{i,1}, U_{i,2}) : i \in \mathbb{Z}\}$ *are i.i.d. random vectors and* $U_{0,j}$, $j = 1, 2$, *are absolutely continuous with* $\mathrm{E}|U_{0,j}|^\nu < \infty$ *for some* $\nu > 0$. *Then, (9) has a strictly stationary and ergodic solution* $\{h_i^\circ\}$ *when*

$$\int_{-\infty}^{\infty} \log\left(\phi_{1,1}^\circ z_+ + \phi_{1,2}^\circ z_- + \beta_1^\circ\right) f_2(z)\mathrm{d}z < 0, \tag{11}$$

where f_2 *is a density of* $\mathrm{sgn}(U_{0,2})|U_{0,2}|^{2\delta^\circ}$, z_+ *and* z_- *are the positive and negative parts of* z*, respectively.*

Remark 4 Argument (11) corresponds to the negative top Lyapunov exponent condition in the i.i.d. innovation case, and further, implies that $\beta_1^\circ < 1$: see Bougerol and Picard (1992) for the top Lyapunov exponent. In (9), different $\phi_{1,2}^\circ$ and $\phi_{1,2}^\circ$ could be considered in the case of $h_{i-1}^\circ > \upsilon^\circ$. However, we used the same coefficients for simplicity. Moreover, their estimation can be an interesting issue in practice when $0 < \tau^\circ < 1$, which, however, is beyond the scope of this work and is carried over to our future project.

Theorem 2 implies that the invariance distribution of the Markov process (9) exists (cf. Tweedie 1975, Theorem 3.1). As an initial random variable, we take h_0° that follows the invariance distribution and is measurable with respect to $\sigma\{(U_{i,1}, U_{i,2}) : i \leq 0\}$, thus, is independent of $\{(U_{i,1}, U_{i,2}) : i \in \mathbb{N}\}$. Then using (9), we recursively define

$$U_i := U_{i,1}\mathrm{I}\left(h_i^\circ \leq \upsilon^\circ\right) + U_{i,2}\mathrm{I}\left(h_i^\circ > \upsilon^\circ\right), \quad \sigma_i := \left\{h_i^\circ\right\}^{1/(2\delta^\circ)}, X_i := \sigma_i U_i \quad \text{for } i \in \mathbb{N}. \tag{12}$$

It can be seen that σ_i is measurable with respect to $\sigma\{(U_{j,1}, U_{j,2}) : j < i\}$ for each $i \in \mathbb{N}$, and $\{X_i\}$ in (12) becomes an ergodic and strictly stationary process whose innovation distribution at time i depends on conditional volatility σ_i. Moreover, the following equation:

$$\begin{cases} X_i = \left\{\mathrm{h}_i^\circ\right\}^{1/(2\delta^\circ)} U_i, \\ \mathrm{h}_i^\circ = \omega^\circ + \left\{\phi_{1,1}^\circ (X_{i-1})_+^{2\delta^\circ} + \phi_{1,2}^\circ (X_{i-1})_-^{2\delta^\circ}\right\} + \beta_1^\circ \mathrm{h}_{i-1}^\circ \end{cases} \quad \text{for } i \in \mathbb{N} \qquad (13)$$

can be derived, and the $\{\mathrm{h}_i^\circ\}$ in (9) is ergodic and strictly stationary. If the distribution G of $\log \mathrm{h}_0^\circ$ is continuous, and

$$\mathrm{s}_i = \mathrm{G}(\log \mathrm{h}_i^\circ), \quad \tau^\circ = \begin{cases} \mathrm{G}(\log \upsilon^\circ), & \upsilon^\circ < \infty; \\ 1, & \upsilon^\circ = \infty \end{cases} \quad \text{for } i \in \mathbb{Z}^+ := \mathbb{N} \cup \{0\},$$

$\{(\mathrm{s}_i, U_{i,1}, U_{i,2}) : i \in \mathbb{N}\}$ satisfies (1)–(3).

2.2.2 Residual-based test

In this subsection, we consider the constancy test Q_n in (6) for the ergodic and strictly stationary PTTGARCH(p, q) process driven by i.i.d. innovations $\{U_i = U_{i,1}\}$ with $p, q \in \mathbb{N}$ as follows:

$$\begin{cases} X_i = \{\mathrm{h}_i^\circ\}^{1/(2\delta^\circ)} U_i, \\ \mathrm{h}_i^\circ = \omega^\circ + \sum_{j=1}^p \left\{\phi_{j,1}^\circ (X_{i-j})_+^{2\delta^\circ} + \phi_{j,2}^\circ (X_{i-j})_-^{2\delta^\circ}\right\} + \sum_{j=1}^q \beta_j^\circ \mathrm{h}_{i-j}^\circ, \\ \text{for } i \in \mathbb{Z}, \end{cases} \qquad (14)$$

(cf. (13)): we refer to Pan et al. (2008) for the existence of such processes. Let G be the distribution function of $\log \mathrm{h}_i^\circ$, where $\{\mathrm{h}_i^\circ\}$ is ergodic and strictly stationary, and let $\mathrm{s}_i = \mathrm{G}(\log \mathrm{h}_i^\circ)$ for each $i \in \mathbb{Z}$. Since s_i and U_i are unobservable, we consider the residual-based test Q_n in (6). Recall that $\mathcal{H}_0$ in (5) is equivalent to $\upsilon^\circ = \infty$ in (12).

Suppose that $X_{1-\mathrm{m}_n}, \ldots, X_n$ are observed from model (14), where $\mathrm{m}_n \in \mathbb{N}$ satisfies

$$\mathrm{m}_n := \lfloor n^\nu \rfloor \quad \text{for some } \nu > 0. \qquad (15)$$

Let $\boldsymbol{\theta}^\circ := (\omega^\circ, \phi_{1,1}^\circ, \ldots, \phi_{p,1}^\circ, \phi_{1,2}^\circ, \ldots, \phi_{p,2}^\circ, \beta_1^\circ, \ldots, \beta_q^\circ, \delta^\circ)$ be the true parameter vector and $\boldsymbol{\theta} = (\omega, \phi_{1,1}, \ldots, \phi_{p,1}, \phi_{1,2}, \ldots, \phi_{p,2}, \beta_1, \ldots, \beta_q, \delta)$ denote a generic one. For given $\boldsymbol{\theta}$, we recursively define

$$\hat{\mathrm{h}}_{n,i}(\boldsymbol{\theta}) = \omega + \sum_{j=1}^{p} \left\{\phi_{j,1}(\tilde{X}_{i-j})_+^{2\delta} + \phi_{j,2}(\tilde{X}_{i-j})_-^{2\delta}\right\} + \sum_{j=1}^{q} \beta_j \hat{\mathrm{h}}_{n,i-j}(\boldsymbol{\theta}) \quad \text{for } i = 1 - \mathrm{m}_n, \ldots, n \qquad (16)$$

by setting $\tilde{X}_i = X_i$ for $i \geq 1 - \mathrm{m}_n$ and $\tilde{X}_{-\mathrm{m}_n} = \cdots = \tilde{X}_{1-\mathrm{m}_n-p} = \hat{\mathrm{h}}_{n,-\mathrm{m}_n}(\boldsymbol{\theta}) = \cdots = \hat{\mathrm{h}}_{n,1-\mathrm{m}_n-q}(\boldsymbol{\theta}) = 0$. In recursion system (16), $i = 1 - \mathrm{m}_n, \ldots, 0$ is a burn-in period for obtaining stable estimates of $\mathrm{h}_1^\circ, \ldots, \mathrm{h}_n^\circ$.

Let $\hat{\boldsymbol{\theta}}_n = (\hat{\omega}, \hat{\phi}_{1,1}, \ldots, \hat{\phi}_{p,1}, \hat{\phi}_{1,2}, \ldots, \hat{\phi}_{p,2}, \hat{\beta}_1, \ldots, \hat{\beta}_q, \hat{\delta})$ be an estimator of $\boldsymbol{\theta}^\circ$. Setting

$$\hat{\mathrm{h}}_{n,i} := \hat{\mathrm{h}}_{n,i}(\hat{\boldsymbol{\theta}}_n), \quad \hat{\mathrm{s}}_{n,i} := \hat{\mathrm{G}}_n(\log \hat{\mathrm{h}}_{n,i}), \hat{U}_{n,i} := X_i/\{\hat{\mathrm{h}}_{n,i}\}^{1/(2\hat{\delta})} \quad \text{for } i = 1, \ldots, n, \tag{17}$$

where

$$\hat{\mathrm{G}}_n(x) := \frac{1}{n} \sum_{j=1}^{n} \mathrm{I}(\log \hat{\mathrm{h}}_{n,j} \leq x), \quad x \in \mathbb{R}, \tag{18}$$

we define $\widehat{\mathrm{Q}}_n$ in the same fashion as Q_n with U_i and s_i replaced by $\hat{U}_{n,i}$ and $\hat{\mathrm{s}}_{n,i}$, respectively (cf. (6)).

We assume that

- **(B1)** $\{\mathrm{h}_i^\circ\}$ is strongly mixing and satisfies (1.8) in Deo (1973);
- **(B2)** $\mathrm{E}\{(\mathrm{h}_0^\circ)^\nu\} < \infty$ for some $\nu > 0$;
- **(B3)** G is continuous and I_{G} constitutes a nondegenerated interval, where $I_{\mathrm{G}} := \{x \in \mathbb{R} : \mathrm{G}(x + \epsilon) - \mathrm{G}(x - \epsilon) > 0 \text{ for every } \epsilon > 0\}$;
- **(B4)** G is Lipschitz continuous on every compact interval contained in the interior of I_{G};
- **(B5)** (15) holds and $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}^\circ) = O_{\mathrm{P}}(1)$ as $n \to \infty$.

Then, we have the following.

Theorem 3 *Suppose that the distribution F of U_0 (cf. (7)) satisfies (4),* **(A1)–(A3)** *and* **(B1)–(B5)** *hold.*

Then, under $\mathcal{H}_0$, we have (8) with Q_n replaced by $\widehat{\mathrm{Q}}_n$.

Remark 5 The sufficient conditions for an ergodic and strictly stationary solution of (14) and **(B2)** to exist can be found in the Appendix of Pan et al. (2008). Moreover, several GARCH variants are known to be absolutely regular with an exponentially decaying rate under mild conditions (cf. Carrasco and Chen 2002). Such variants also satisfy **(A2)**, **(A3)** and **(B1)**.

Condition **(B5)** is fulfilled by the quasi-maximum likelihood estimator (QMLE) and least absolute deviations estimator (LADE) under some moment conditions (cf. Berkes and Horváth 2004 and Pan et al. 2008).

3 Simulation study and real data analysis

3.1 Simulation study

In this subsection, we carry out a simulation study to examine the validity of the proposed test $\widehat{\mathrm{Q}}_n$. To examine its size, we assume that under $\mathcal{H}_0$, $\{X_i : i = 1, \ldots, n\}$

is a PTTGARCH(1,1) process with $(\delta^\circ, \omega^\circ, \phi^\circ_{1,1}, \phi^\circ_{1,2}, \beta^\circ_1) = (.8, .2, .2, .1, .4)$ (cf. (14)). The innovations are assumed to follow a t-distribution with degree of freedom $\alpha > 1$. The LADE is used for a fit, and $\{\hat{\mathrm{h}}_{n,i} : i = 1, \ldots, n\}$ are constructed through the recursion formula in (16) with $p = q = 1$ (cf. Pan et al. 2008). Tables 1 and 2 present the sizes of $\widehat{\mathrm{Q}}_n$ at the level of 5% when $n = 1000, 2000, \alpha = 2.5, 3.5, 5.0, 7.0$, $k = 50, \ldots, 200$ (increased by 10), and $\tau_0 = 0.1$ is fixed. The sizes are calculated through 1000 repetitions. Tables 1 and 2 reveal that $\widehat{\mathrm{Q}}_n$ has no severe size distortions.

Next, we examine the power of $\widehat{\mathrm{Q}}_n$. We assume that $\{X_i : i = 1, \ldots, n\}$ are generated from the model specified by (9) and (12), wherein the parameters are the same as above. The $\{U_{i,1}\}$ are assumed to follow a scaled t-distribution with the degree of freedom $\alpha_1 = 2.5$, while $\{U_{i,2}\}$ are assumed to follow the same with the degree of freedom $\alpha_2 = 7$. This particularly renders extreme innovation with a higher probability, when the returns pass through a low volatility period than a high volatility period. Here, the common distributions must be suitably scaled so that the change in tail index does not change the PTTGARCH parameters (cf. Pan et al. 2008, (A4)). Tables 3, 4 and 5 present the powers of $\widehat{\mathrm{Q}}_n$ at the nominal level of 5% when $n = 1000, 2000, 3000$ and $\upsilon^\circ = 0.42, 0.52, 0.75$. It can be seen that $\widehat{\mathrm{Q}}_n$ generates higher powers as n gets larger. It is noteworthy that the power depends on k, which indicates the importance of the role of k.

Finally, we compare our test with the change point test as in Kim and Lee (2016b), designed to detect a change over time flow, by checking the performance of the latter in testing (5). This issue can be of much interest to practitioners since the two tests commonly share the same null hypothesis. More specifically, we consider $\widehat{\mathrm{Q}}^*_n$, defined in the same style as of Q_n, based on

$$\mathrm{H}^*_n(\tau_1, \tau_2) := \frac{1}{k(\tau_2 - \tau_1)} \sum_{i=\lfloor n\tau_1 \rfloor + 1}^{\lfloor n\tau_2 \rfloor} \left(\log \hat{U}_i - \log \hat{U}^*_{(\tau_1, \tau_2)} \right)_+, \quad 0 \le \tau_1 < \tau_2 \le 1,$$

instead of $\mathrm{H}_n(\tau_1, \tau_2)$, with

$$\hat{U}^*_{(\tau_1,\tau_2)} := \begin{cases} \lfloor k(\tau_2 - \tau_1) + 1 \rfloor\text{-th largest order statistic of } \{U_i : \tau_1 < i/n \le \tau_2\}, \ \tau_1 = 0; \\ \lceil k(\tau_2 - \tau_1) + 1 \rceil\text{-th largest order statistic of } \{U_i : \tau_1 < i/n \le \tau_2\}, \ \tau_2 = 1. \end{cases}$$

From Theorem 2 of Kim and Lee (2016b), we can derive that $\widehat{\mathrm{Q}}^*_n$ converges weakly to the same limiting distribution as Q_n under $\mathcal{H}_0$. Table 6 exhibits the power of $\widehat{\mathrm{Q}}^*_n$ in the same setting used for the evaluation of $\widehat{\mathrm{Q}}_n$ and shows that the test produces much lower powers. This result demonstrates that our test is much more suitable to testing (5) than $\widehat{\mathrm{Q}}^*_n$.

3.2 Real data analysis

3.2.1 Google Stock

We apply $\widehat{\mathrm{Q}}_n$ to the negative returns (unit:%) of Google's stock price (cf. Fig. 1). The data period is from August 20, 2004 to July 1, 2016, with $n = 2988$. Since the series

Table 1 The size of $\widehat{Q}_n$ at nominal level 5%, $n = 1000$

α	k															
	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
2.5	.065	.058	.059	.052	.052	.044	.039	.039	.041	.039	.035	.034	.034	.035	.038	.038
3.5	.063	.044	.042	.029	.030	.029	.031	.030	.028	.028	.029	.029	.031	.029	.029	.034
5	.054	.042	.037	.043	.052	.039	.040	.030	.033	.035	.033	.038	.040	.043	.044	.041
7	.064	.056	.039	.039	.038	.028	.018	.020	.023	.021	.022	.027	.021	.021	.026	.030

Table 2 The size of $\widehat{Q}_n$ at nominal level 5%, $n = 2000$

α	k															
	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
2.5	.075	.065	.061	.058	.061	.052	.045	.043	.037	.032	.038	.035	.041	.041	.040	.038
3.5	.071	.065	.048	.046	.043	.046	.039	.046	.040	.038	.036	.033	.037	.034	.028	.027
5	.078	.065	.067	.065	.055	.047	.036	.031	.030	.036	.033	.034	.032	.032	.032	.032
7	.082	.073	.065	.078	.054	.049	.050	.039	.042	.036	.033	.026	.030	.032	.030	.037

Table 3 The power of $\widehat{Q}_n$ at nominal level 5%, $n = 1000$

	k															
υ°	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.42	.130	.124	.142	.148	.126	.140	.152	.144	.130	.138	.132	.146	.148	.120	.144	.164
0.53	.200	.204	.210	.210	.212	.218	.220	.218	.210	.194	.204	.222	.218	.220	.198	.212
0.75	.202	.186	.184	.174	.192	.168	.156	.172	.170	.170	.176	.178	.170	.176	.156	.148

Table 4 The power of $\widehat{Q}_n$ at nominal level 5%, $n = 2000$

υ°	k															
	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.42	.228	.240	.240	.236	.222	.244	.272	.298	.306	.294	.288	.300	.282	.284	.302	.298
0.52	.330	.342	.376	.388	.408	.434	.414	.412	.404	.422	.442	.452	.462	.460	.468	.482
0.75	.274	.312	.308	.336	.350	.380	.404	.414	.418	.414	.396	.400	.404	.396	.404	.418

Table 5 The power of $\widehat{Q}_n$ at nominal level 5%, $n = 3000$

υ°	k															
	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.42	.314	.324	.344	.362	.398	.402	.414	.414	.420	.432	.422	.436	.446	.448	.464	.454
0.52	.416	.454	.484	.520	.536	.550	.576	.620	.614	.650	.660	.678	.690	.684	.690	.710
0.75	.346	.352	.386	.428	.424	.416	.474	.504	.510	.532	.542	.544	.538	.554	.572	.578

Table 6 The power of $\widehat{Q}_n^*$ at nominal level 5%, $n = 3000$

υ°	k															
	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.42	.108	.094	.080	.082	.072	.073	.074	.070	.060	.062	.060	.046	.044	.030	.034	.034
0.53	.074	.072	.072	.080	.078	.068	.056	.052	.054	.044	.046	.048	.042	.046	.030	.036
0.75	.064	.062	.054	.056	.054	.052	.048	.036	.046	.046	.044	.032	.030	.028	.030	.032

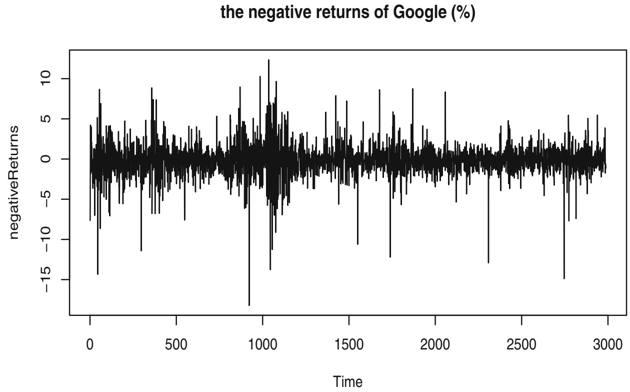

Fig. 1 The negative return series of Google stock price from August 20, 2004 to July 1, 2016

reveal the conditional heteroscedasticity as shown in Fig. 1, we fit a GARCH(1,1) model ($p = q = 1$, $\delta^\circ = 1$, $\phi^\circ_{1,1} = \phi^\circ_{1,2} =: \phi^\circ_1$) to the data, and obtain the QMLE based on the two-sided exponential density function (cf. Berkes and Horváth 2004), that is, $(\hat{\omega}, \hat{\phi}_1, \hat{\beta}_1) = (.109, .051, .834)$. Figures 2 and 3 list estimated conditional volatilities and residuals. A standard diagnostic test ensures the validity of the GARCH fitting. The objective here is to check whether or not the tail index of the residuals is varying according to the conditional volatility. The upper and lower tails are estimated for the residuals corresponding to the three different intervals of conditional volatility, that is, (0,1.09), (1.09,1.32) and (1.32, ∞), where 1.09 and 1.32 are 33 and 66% quantiles, respectively: the lines in Figs. 2 and 3 indicate these values. Figures 4 and 5 also present the Hill plots of the upper and lower tails, respectively (cf. Drees et al. 2000). The tail index in the upper tail part is seemingly varying, whereas the one in the low tail part is relatively stable. In order to check the significance of its variation, we apply $\widehat{\mathrm{Q}}_n$ with $\tau_0 = 0.1$ to the upper tail part. Figure 6 indicates that the tail index variation is not so significant. This suggests that the tail thickness is independent of the conditional volatility in the upper (and lower) tail part.

3.2.2 DowJones

Next, we apply the test to the negative returns of DowJones index from November 1, 2004 to November 4, 2016, with $n = 3026$ (cf. Fig. 7). A GARCH(1,1) model is fitted to the series. The resulting estimates appear to be $(\hat{\omega}, \hat{\phi}_1, \hat{\beta}_1) = (.012, .072, .851)$, and the residuals are displayed in Figs. 8 and 9. Figures 10 and 11 show the result on the tail index estimation for the upper and lower tail parts corresponding to the three different intervals of conditional volatility: (0, 0.52), (0.52, 0.72) and (0.72, ∞), where 0.52 and 0.72 denote 33 and 66% quantiles, respectively. Since the tail index in the upper tail part appears to be varying, we apply $\widehat{\mathrm{Q}}_n$ with $\tau_0 = 0.1$. Figure 12 shows that the result depends on the choice of k. More precisely, the test rejects the null at the level of 10% for $k = 80, 90, 100, 110, 120, 140, 150$, and further, at the level of 5% for $k = 80, 90, 110$. For $k = 90$, the value of $\widehat{\mathrm{Q}}_n$ is taken at $\tau = 0.38$, corresponding

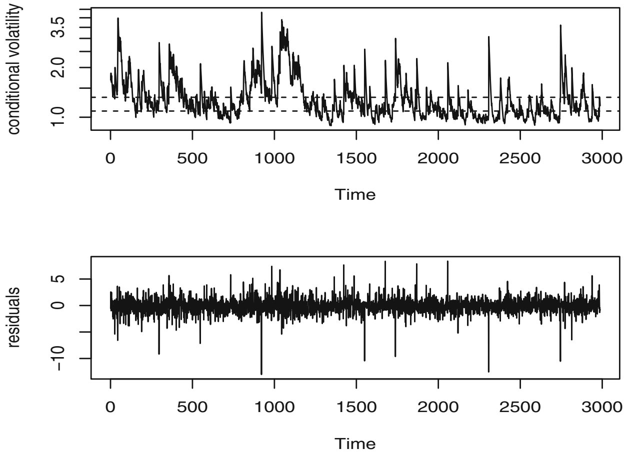

Fig. 2 The series of conditional volatilities and residuals with the dashed lines = 1.09 and 1.32

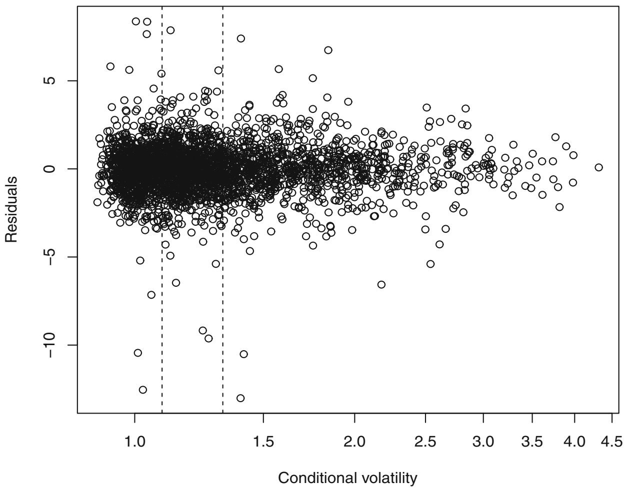

Fig. 3 The plot of conditional volatilities versus residuals with the dashed lines = 1.09 and 1.32

to the conditional volatility 0.53. Figure 13 shows the tail index estimates for the two intervals (0, 0.53) and (0.53, ∞), where the gray-colored lines stand for the 95%-level upper/lower limits for the lower/upper intervals, respectively. It is revealed that the tail indices are significantly distinct for a range of k's, wherein the estimates are stable. Our findings suggest that the thickness of the upper tail part depends on the conditional volatility.

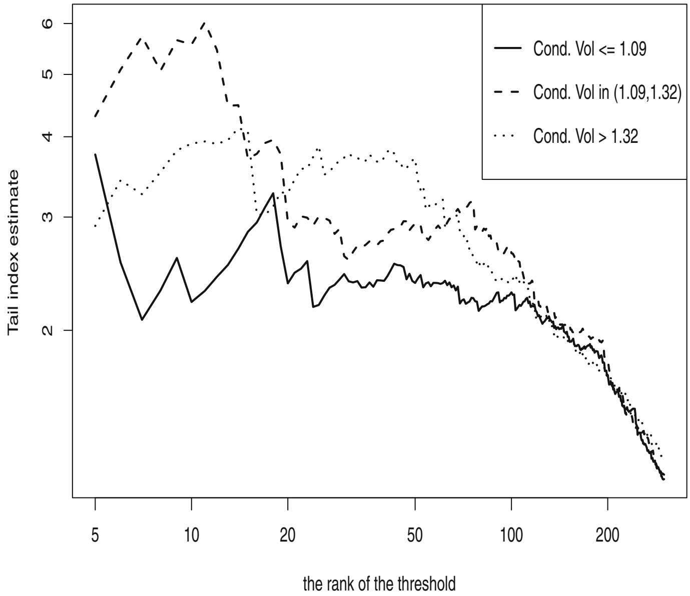

Fig. 4 The Hill plots of the upper tail for the three different intervals of conditional volatility

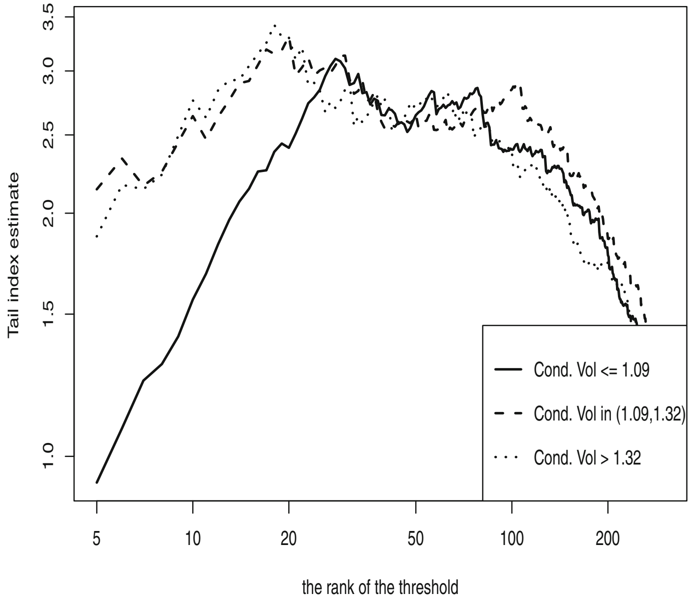

Fig. 5 The Hill plots of the lower tail for the three different intervals of conditional volatility

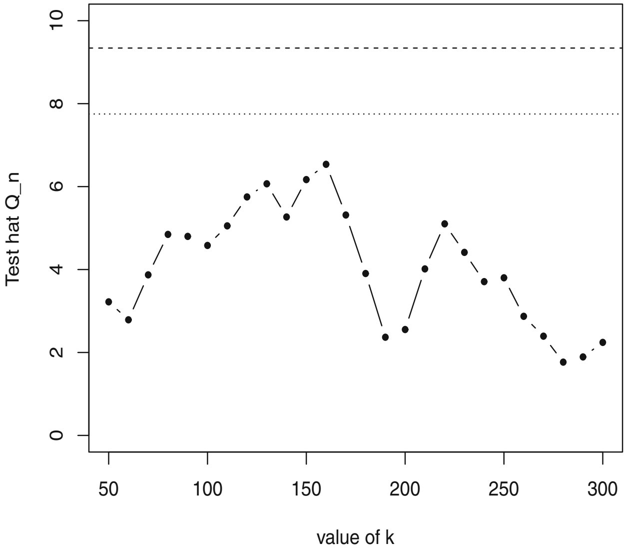

Fig. 6 The values of $\widehat{Q}_n$ against k with the critical values at 10 and 5% in line

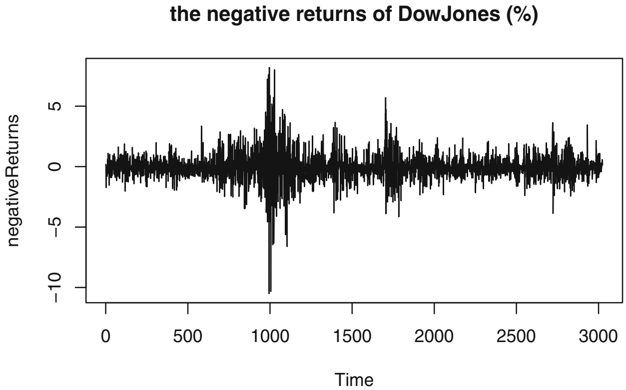

Fig. 7 The negative return series of DowJones from 1 Nov 2004 to 4 Nov 2016

4 Concluding remarks

In this study, we considered the problem of testing whether the tail index of the innovations of PTTGARCH models experiences a change at some threshold values of conditional volatility. We showed that under certain conditions, the proposed test asymptotically behaves like the functional of a standard Brownian motion. We evaluated the performance of the test through an empirical study and demonstrated that the tail index of underlying distributions can be influenced by conditional volatility. The result indicates that for practical applications, the underlying innovation distribution should be modeled in a more refined manner. For example, in VaR estimation, one can model the innovation distribution by employing an MLE method using the

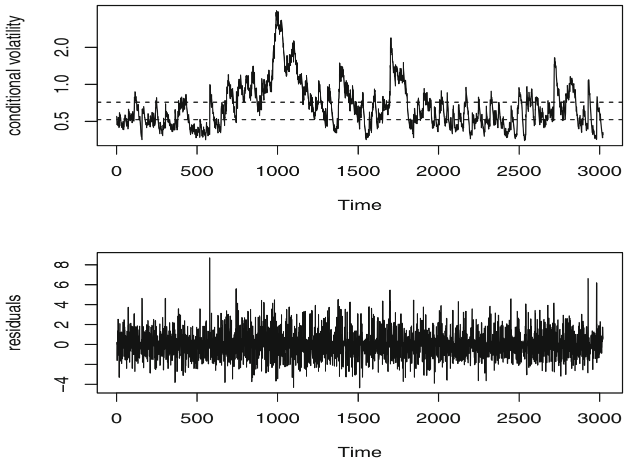

Fig. 8 The series of conditional volatilities and residuals with the dashed lines = 0.52 and 0.72

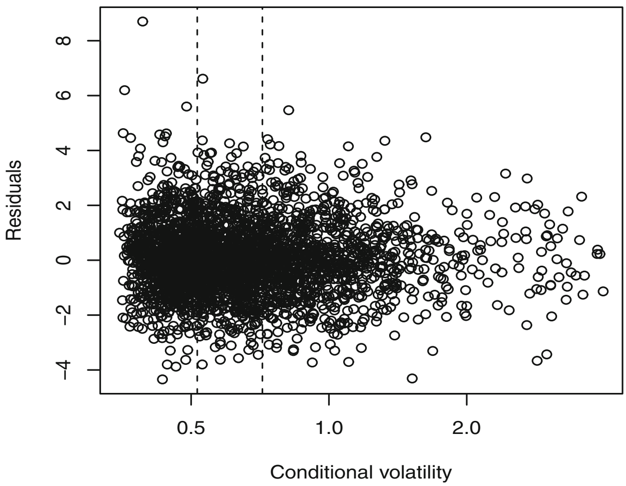

Fig. 9 The plot of conditional volatilities vs. residuals with the dashed lines = 0.52 and 0.72

asymmetric Student's *t* distribution (cf. Kim and Lee 2016c). Our findings warn that the direct usage of their method without considering the possibility of the underlying distributional change might lead to a false conclusion. In this work, we focused only on the PTTGARCH models, and thus, there is a demand to extend the current work to a more general class of location-scale models with heteroscedasticity. Moreover, the task of estimating coefficients and VaR forecasting is worth further investigation. All these issues are left as our future project.

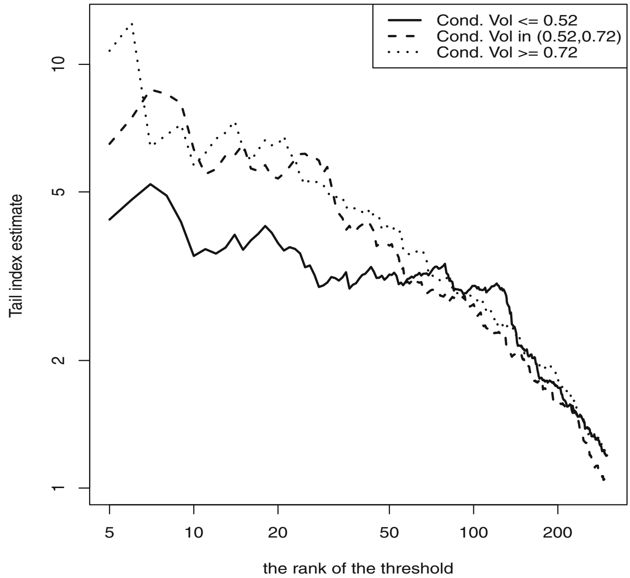

Fig. 10 The Hill plots of the upper tail for the three different intervals of conditional volatility

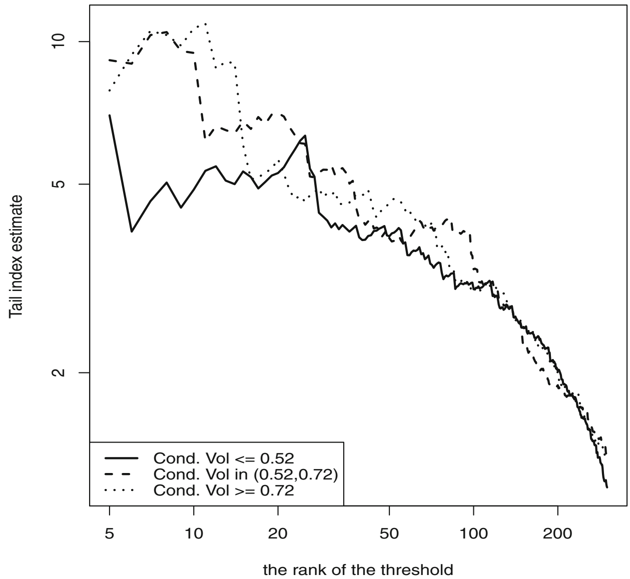

Fig. 11 The Hill plots of the lower tail for the three different intervals of conditional volatility

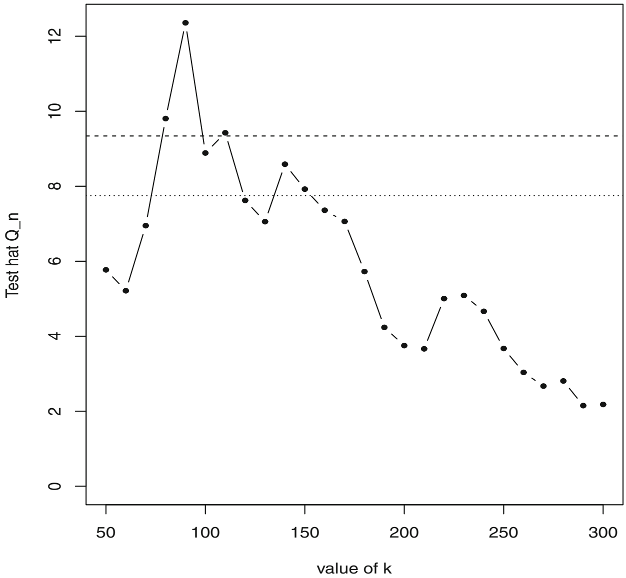

Fig. 12 The values of $\widehat{Q}_n$ against k with the critical values at 10 and 5%

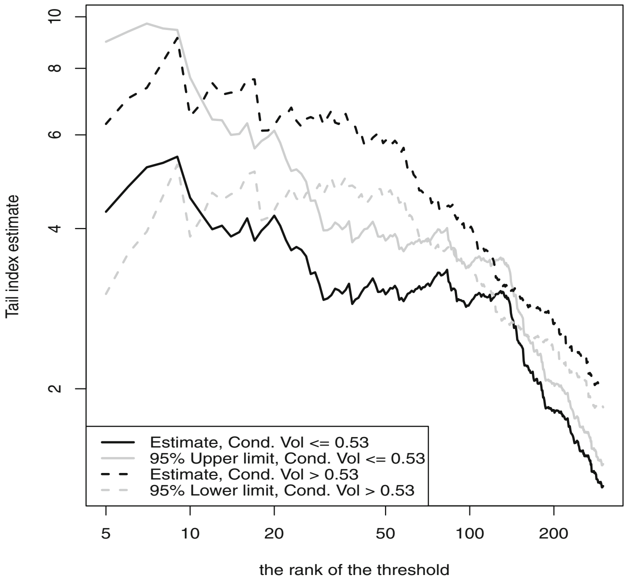

Fig. 13 Hill plots for the two intervals of conditional volatility

5 Proofs

5.1 Proof of Theorem 1

In this subsection, we prove Theorem 1. In what follows, $K > 0$ denotes a generic constant and $D[\,a, b\,]$ denotes the complete and separable metric space of all càdlàg functions defined over a compact interval $[\,a, b\,]$ $(0 \le a < b \le 1)$, equipped with the Skorohod metric (cf. Billingsley 1968).

We define the stochastic processes:

$$M_n(\tau, \zeta) := \sum_{i=1}^{n} Z_{ni}(\tau, \zeta), \quad L_n(\tau, \zeta) := \sum_{i=1}^{n} Y_{ni}(\tau, \zeta), \quad 0 \le \tau \le 1 \text{ and } \zeta \in \mathbb{R},$$

where

$$\begin{aligned} Z_{ni}(\tau, \zeta) &:= \frac{1}{\sqrt{k}} \left\{ \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n, \mathrm{s}_i \le \tau\right) - \mathrm{P}\left(U_i > e^{-\zeta/\sqrt{k}} b_n, \mathrm{s}_i \le \tau\right) \right\}, \\ A_{ni}(\tau, \zeta) &:= \left(\log U_i - \log b_n + \frac{\zeta}{\sqrt{k}} \right)_+ \mathrm{I}\,(\mathrm{s}_i \le \tau)\,, \\ Y_{ni}(\tau, \zeta) &:= \frac{1}{\sqrt{k}} \left\{ A_{ni}(\tau, \zeta) - \mathrm{E} A_{ni}(\tau, \zeta) \right\} \end{aligned} \tag{19}$$

with $b_n = b(n/k)$.

The proof comprises of three propositions and four lemmas. Among them, Proposition 3 plays a key role to prove the theorem. All the Lemmas and Proposition 2 are used to assert Proposition 3. We start with a lemma useful to prove Proposition 1, which is later used to prove Proposition 2. All the conditions in Theorem 1 are implicitly assumed without statement.

Lemma 1 *Let* $I_{ni} = \mathrm{I}\,(U_i > b_n, \tau_1 < \mathrm{s}_i \le \tau_2)$ *and* $J_{ni} = (\log U_i - \log b_n)_+ \mathrm{I}\,(\tau_1 < \mathrm{s}_i \le \tau_2)$ *for* $0 \le \tau_1 < \tau_2 \le 1$*. If* $l_n k = O(n)$ *as* $n \to \infty$*, then there exists* $K > 0$ *such that for* $0 \le \tau_1 < \tau_2 \le 1$ *and* $n \in \mathbb{N}$*,*

$$\frac{n}{l_n k} \mathrm{Var} \left\{ \sum_{i=1}^{l_n} I_{ni} \right\} \le K(\tau_2 - \tau_1), \quad \frac{n}{l_n k} \mathrm{Var} \left\{ \sum_{i=1}^{l_n} J_{ni} \right\} \le K(\tau_2 - \tau_1).$$

Moreover, if $l_n k = o(n)$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} \frac{n}{l_n k} \mathrm{Var} \left\{ \sum_{i=1}^{l_n} I_{ni} \right\} = (\tau_2 - \tau_1), \quad \lim_{n\to\infty} \frac{n}{l_n k} \mathrm{Var} \left\{ \sum_{i=1}^{l_n} J_{ni} \right\} = \frac{2}{\alpha^2}(\tau_2 - \tau_1).$$

Proof We only deal with I_{ni}, since J_{ni} can be handled similarly. Note that

$$\frac{n}{l_n k}\mathrm{Var}\left\{\sum_{i=1}^{l_n} I_{ni}\right\} = \frac{n}{k}\mathrm{Var}(I_{n,0}) + 2\sum_{h=1}^{l_n}\left(1-\frac{h}{l_n}\right)\frac{n}{k}\mathrm{Cov}(I_{n,0}, I_{n,h}).$$

Let $A_i = \{U_i > b_n\}$ and $B_i = \{\tau_1 < \mathrm{s}_i \le \tau_2\}$ for $0 \le \tau_1 < \tau_2 \le 1$ and $n \in \mathbb{N}$. Then, we get

$$\frac{n}{k}\mathrm{E}I_{n,0} = \frac{n}{k}\mathrm{P}A_0\,\mathrm{P}B_0 = (\tau_2 - \tau_1).$$

Moreover, we have that for $h = 1, \ldots, l_n$,

$$\begin{aligned}\left|\mathrm{Cov}(I_{n,0}, I_{n,h})\right| &= |\mathrm{P}A_h\mathrm{P}\left(B_h A_0 B_0\right) - \mathrm{P}A_h\mathrm{P}B_h\mathrm{P}A_0\mathrm{P}B_0| \\ &= |\mathrm{P}A_h\left\{\mathrm{P}B_h B_0 A_0 - \mathrm{P}B_h\mathrm{P}B_0\mathrm{P}A_0\right\}| \\ &\le \mathrm{P}A_h\mathrm{P}B_0\mathrm{P}A_0 = \left(\frac{k}{n}\right)^2(\tau_2-\tau_1),\end{aligned}$$

since $\mathrm{P}\left(B_h A_0 B_0\right) \vee \mathrm{P}B_h\mathrm{P}B_0\mathrm{P}A_0 \le \mathrm{P}B_0\mathrm{P}A_0$. Hence, the proof is completed. □

Proposition 1 *$\{M_n(\cdot, 0)\}$ and $\{L_n(\cdot, 0)\}$ are tight.*

Proof First, we deal with $M_n(\cdot, 0)$. Take $l_n \in \mathbb{N}$ and $\Delta = \Delta_n > 0$ so that $1/\Delta \in \mathbb{Z}$,

$$l_n = o(n), \quad \frac{n}{l_n}\varrho(l_n) \to 0, \quad \sqrt{k}\Delta \to 0, \quad \frac{l_n^2}{k}\log_2^4\frac{1}{\Delta} \to 0, \frac{l_n k}{n} \to 0 \quad \text{as } n \to \infty. \tag{20}$$

Let $m_n = \lfloor n/(2l_n)\rfloor$, $\mathcal{I}_{ni} = \{2(i-1)l_n + 1, \ldots, 2il_n - l_n\}$, $\mathcal{J}_{ni} = \{2il_n - l_n + 1, \ldots, 2il_n\}$,

$$B_{ni}^{(1)}(\tau) = \sum_{j\in\mathcal{I}_{ni}} Z_{nj}(\tau, 0), \quad B_{ni}^{(2)}(\tau) = \sum_{j\in\mathcal{J}_{ni}} Z_{nj}(\tau, 0), \quad B_n(\tau) = \sum_{j=2m_n l_n+1}^{n} Z_{nj}(\tau, 0), \tag{21}$$

$$M_n^{(r)}(\tau) = \sum_{i=1}^{m_n} B_{ni}^{(r)}(\tau), \quad r = 1, 2. \tag{22}$$

Due to (21) and (22), we can express $M_n(\tau, 0) = M_n^{(1)}(\tau) + M_n^{(2)}(\tau) + B_n(\tau)$. Thus, since $M_n(0, 0) = 0$ a.s., to prove the tightness of $\{M_n(\cdot, 0)\}$, it suffices to verify that $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically uniformly equicontinuous, because $\sup_{0\le\tau\le1}|B_n(\tau)| = o_{\mathrm{P}}(1)$ due to $\lim_{n\to\infty} l_n/\sqrt{k} = 0$. Moreover, if we set M_n^* to be the sum of i.i.d. m_n copies of $B_{n,1}^{(1)}$, the total variation between M_n^* and $M_n^{(1)}$ should go to 0 as $n \to \infty$, since $\lim_{n\to\infty} m_n \varrho(l_n) \to 0$ (cf. Eberlein 1984). Thus, it suffices to show that M_n^* is asymptotically uniformly equicontinuous.

Take $\tau_u = \min\{\Delta u, 1\}$ for $u \in \mathbb{Z}^+ := \mathbb{N} \cup \{0\}$ to form a partition of [0, 1]. Define $\xi_u^{(n)} := M_n^*(\tau_u) - M_n^*(\tau_{u-1})$. Then, for $u_0, u_1 \in \mathbb{Z}^+$ with $u_0 \le u_1$, we get

$$M_n^*(\tau_{u_1}) - M_n^*(\tau_{u_0}) = \sum_{u=u_0+1}^{u_1} \xi_u^{(n)},$$

and letting $I_{ni} = \mathrm{I}\left(U_i > b_n, \tau_{u_0} < \mathrm{s}_i \le \tau_{u_1}\right) - \mathrm{P}\left(U_i > b_n, \tau_{u_0} < \mathrm{s}_i \le \tau_{u_1}\right)$, we have

$$\mathrm{E}\left\{\sum_{u=u_0+1}^{u_1} \xi_u^{(n)}\right\}^4 = m_n \mathrm{E}\left\{\frac{1}{\sqrt{k}}\sum_{i=1}^{l_n} I_{ni}\right\}^4 + 3m_n(m_n-1)\mathrm{E}^2\left\{\frac{1}{\sqrt{k}}\sum_{i=1}^{l_n} I_{ni}\right\}^2.$$

Moreover, owing to the boundedness of the summands and Lemma 1, there exists $K > 0$ such that for $u_0, u_1 \in \mathbb{Z}^+$ with $u_0 < u_1$ and $n \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{k^2}\mathrm{E}\left\{\sum_{i=1}^{l_n} I_{ni}\right\}^4 &\le \frac{l_n^2}{k^2}\mathrm{E}\left\{\sum_{i=1}^{l_n} I_{ni}\right\}^2 \le K\frac{l_n^3}{nk}(u_1-u_0)\Delta, \\ \frac{1}{k^2}\mathrm{E}^2\left\{\sum_{i=1}^{l_n} I_{ni}\right\}^2 &\le K\frac{l_n^2}{n^2}(\tau_{u_1}-\tau_{u_0})^2 \le K\frac{l_n^2}{n^2}(u_1-u_0)^2\Delta^2. \end{aligned} \tag{23}$$

Thus, there exists $K_0 > 0$ such that for each $u_0, u_1 \in \mathbb{Z}^+$ with $u_0 < u_1$ and $n \in \mathbb{N}$,

$$\mathrm{E}\left\{\sum_{u=u_0+1}^{u_1} \xi_u^{(n)}\right\}^4 \le K_0(u_1-u_0)\Delta\left\{\frac{l_n^2}{k} + (u_1-u_0)\Delta\right\}.$$

Then, it follows from Theorem 1 in Móricz (1983) that for $u_0 \in \mathbb{Z}^+$, $\delta > 0$, and $n \in \mathbb{N}$ with $\delta > \Delta$,

$$\begin{aligned} &\mathrm{E}\max\left\{\left|M_n^*(\tau_{u_1}) - M_n^*(\tau_{u_0})\right|^4 : u_1 = u_0+1, \ldots, u_0 + \lfloor \delta/\Delta \rfloor\right\} \\ &= \mathrm{E}\max\left\{\left|\sum_{u=u_0+1}^{u_1} \xi_u^{(n)}\right|^4 : u_1 = u_0+1, \ldots, u_0 + \left\lfloor \frac{\delta}{\Delta} \right\rfloor\right\} \\ &\le \frac{5K_0}{2}\left\lfloor \frac{\delta}{\Delta} \right\rfloor \Delta \left[\sum_{i=0}^{\left\lfloor \log_2 \left\lfloor \frac{\delta}{\Delta} \right\rfloor \right\rfloor} \left\{\frac{l_n^2}{k} + \frac{\left\lfloor \frac{\delta}{\Delta} \right\rfloor}{2^i}\Delta\right\}^{\frac{1}{4}}\right]^4 \end{aligned}$$

$$\leq \frac{5K_0\delta}{2}\left[\left(\frac{l_n^2}{k}\right)^{\frac{1}{4}}\left(\log_2\left\lfloor\frac{\delta}{\Delta}\right\rfloor+1\right)+\frac{2^{1/4}}{2^{1/4}-1}\delta^{\frac{1}{4}}\right]^4 \leq \frac{80K_0\delta}{(2^{1/4}-1)^4}$$
$$\left[\frac{l_n^2}{k}\log_2^4\frac{2\delta}{\Delta}+\delta\right]. \tag{24}$$

Let $\epsilon, \eta > 0$. Due to (24), there exists $\delta_0 \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that for each $u_0 \in \mathbb{Z}^+$ and $n \geq n_0$,

$$\delta_0 > \Delta \quad \text{and} \quad \mathrm{P}\left(\max\left\{\left|M_n^*(\tau_{u_1}) - M_n^*(\tau_{u_0})\right| : u_1 = u_0+1, \ldots, u_0 + \lfloor\delta_0/\Delta\rfloor\right\} > \epsilon\right) \leq \eta\delta_0.$$

On the other hand, take $n_1 \in \mathbb{N}$ such that $6/\delta_0 < 1/\Delta$ for $n \geq n_1$. Then, for $\tau' \in [0, 1)$ and $n \geq n_1$,

$$\max\left\{|M_n^*(\tau) - M_n^*(\tau')| : \tau' \leq \tau \leq (\tau' + \delta_0/2) \wedge 1\right\}$$
$$\leq 4\max\left\{\left|M_n^*(\tau_{u_1}) - M_n^*(\tau_{u_0})\right| : u_1 = u_0+1, \ldots, u_0 + \lfloor\delta_0/\Delta\rfloor\right\} + 2\sqrt{k}\Delta,$$

where $u_0 \in \mathbb{Z}^+$ satisfies $u_0\Delta < \tau' \leq (u_0+1)\Delta$. Moreover, since $\Delta\sqrt{k} \to 0$ as $n \to \infty$, we can take $n_2 \in \mathbb{N}$ such that $2\sqrt{k}\Delta < \epsilon$ for $n \geq n_2$. Therefore, for $\tau' \in [0, 1]$ and $n \geq \max\{n_0, n_1, n_2\}$,

$$\mathrm{P}\left(\max\left\{\left|M_n^*(\tau) - M_n^*(\tau')\right| : \tau' \leq \tau \leq (\tau' + \delta_0/2) \wedge 1\right\} > 5\epsilon\right) < \eta\delta_0.$$

Then, applying Theorem 8.3 in Billingsley (1968), we can check that M_n^* is asymptotically uniformly equicontinuous.

We can verify the tightness of $L_n(\cdot, 0)$ in a similar way. As above, we take l_n and Δ satisfying (20) together with $\frac{l_n^3}{k}\log_2^4\frac{1}{\Delta} \to 0$ as $n \to \infty$. Then, instead of (23), we use the fact that

$$\frac{1}{k^2}\mathrm{E}\left\{\sum_{i=1}^{l_n}(I_{ni} - \mathrm{E}I_{ni})\right\}^4 \leq K\frac{l_n^4}{nk}(u_1 - u_0)\Delta,$$

where $I_{ni} = (\log U_i - \log b_n)_+ \mathrm{I}(\tau_{u_0} < \mathrm{s}_i \leq \tau_{u_1})$. Since the remaining steps of the proof are essentially the same as in those for M_n, we complete the proof without detailing algebras. □

Proposition 2

$$L_n(\cdot, 0) \Rightarrow 2\alpha^{-1}\mathrm{B}, \quad M_n(\cdot, 0) \Rightarrow \mathrm{B}, \tag{25}$$
$$L_n(\cdot, 0) - \alpha^{-1}M_n(\cdot, 0) \Rightarrow \alpha^{-1}\mathrm{B}, \tag{26}$$

in $D[0, 1]$, *where* B *stands for a standard Brownian motion defined over* $[0, 1]$.

Proof We only prove (26) since (25) can be shown easily. Recall that $L_n(\cdot, 0) - \alpha^{-1} M_n(\cdot, 0)$ has been proved to be tight by Proposition 1. Thus, it suffices to check the convergence of its finite-dimensional distributions. Take $l_n, s_n \in \mathbb{N}$ and $m_n = \lfloor n/(l_n + s_n) \rfloor$ such that

$$l_n \to \infty, \quad s_n = o(l_n), \quad \frac{n}{l_n}\varrho(s_n) \to 0, \quad \frac{l_n^3}{k} \to 0, \quad \frac{l_n k}{n} \to 0 \quad \text{as } n \to \infty. \tag{27}$$

Let $\mathcal{I}_{ni} = \{(i-1)(l_n + s_n) + 1, \ldots, il_n + (i-1)s_n\}$, $\mathcal{J}_{ni} = \{il_n + (i-1)s_n + 1, \ldots, i(l_n + s_n)\}$,

$$\begin{aligned} S_{ni}(\tau) &= k^{-1/2}\left\{(\log U_i - \log b_n)_+ - \alpha^{-1} I(U_i > b_n)\right\} I\left(\sigma_i^2 \le \tau\right), \\ B_{ni}^{(1)}(\tau) &= \sum_{j \in \mathcal{I}_{ni}} \{S_{nj}(\tau) - \mathrm{E}S_{nj}(\tau)\}, \quad B_{ni}^{(2)}(\tau) = \sum_{j \in \mathcal{J}_{ni}} \{S_{nj}(\tau) - \mathrm{E}S_{nj}(\tau)\}, \\ B_n(\tau) &= \sum_{j=m_n(l_n+s_n)+1}^{n} \{S_{nj}(\tau) - \mathrm{E}S_{nj}(\tau)\}. \end{aligned}$$

Then,

$$L_n(\tau, 0) - \frac{1}{\alpha} M_n(\tau, 0) = \sum_{i=1}^{m_n} B_{ni}^{(1)}(\tau) + \sum_{i=1}^{m_n} B_{ni}^{(2)}(\tau) + B_n(\tau), \tag{28}$$

where $B_n(\tau)$ is negligible for each τ because $\lim_n l_n^2/k = 0$. Moreover, as in the proof of Proposition 1, we can regard the first and second terms in the right-hand side of the equality as sums of i.i.d. summands due to the mixing condition given in (27). Thus, below, we assume that they are sums of i.i.d. random variables.

We deal with the first term. Let $T_n(\tau) = \sum_{i=1}^{m_n} B_{ni}^{(1)}(\tau)$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_h = 1$, $h \in \mathbb{N}$. Then, applying the Lyapunov condition and using the fact that $\lim_n l_n^3/k = 0$ in (27), we get

$$\begin{aligned} &(T_n(\tau_1) - T_n(\tau_0), \ldots, T_n(\tau_h) - T_n(\tau_{h-1})) \\ &\quad \Rightarrow \mathrm{N}\left(\mathbf{0}, \alpha^{-2}\mathrm{diag}\{\tau_1 - \tau_0, \ldots, \tau_h - \tau_{h-1}\}\right), \end{aligned}$$

since it follows from $\lim_n l_n k/n = 0$ in (27) that

$$\begin{aligned} &\lim_{n \to \infty} m_n \mathrm{Cov}\left\{B_{n,1}^{(1)}(\tau_u) - B_{n,1}^{(1)}(\tau_{u-1}), B_{n,1}^{(1)}(\tau_v) - B_{n,1}^{(1)}(\tau_{v-1})\right\} \\ &\quad = \begin{cases} \alpha^{-2}(\tau_u - \tau_{u-1}), & u = v; \\ 0, & u \ne v. \end{cases} \end{aligned}$$

For the second term, using the Chebyshev inequality, we see that the right-hand side of (28) is negligible, owing to $\lim_n s_n/l_n = 0$ in (27). This completes the proof. □

Lemma 2 *Let* $K > 0$. *Then,*

$$\sup_{|\zeta|<K}\sup_{0\le\tau\le1}|M_n(\tau,\zeta)-M_n(\tau,0)|=o_{\mathrm{P}}(1), \tag{29}$$

$$\sup_{|\zeta|<K}\sup_{0\le\tau\le1}|L_n(\tau,\zeta)-L_n(\tau,0)|=o_{\mathrm{P}}(1). \tag{30}$$

Proof We only verify (29) since (30) can be proven similarly. First, we show that for $\tau\in[0,1]$ and $\zeta\in\mathbb{R}$,

$$|M_n(\tau,\zeta)-M_n(\tau,0)|=o_{\mathrm{P}}(1). \tag{31}$$

Similarly to (21), we set

$$B_{ni}^{(1)}(\tau,\zeta)=\sum_{j\in\mathcal{I}_{ni}}\{Z_{nj}(\tau,\zeta)-Z_{nj}(\tau,0)\},$$
$$B_{ni}^{(2)}(\tau,\zeta)=\sum_{j\in\mathcal{J}_{ni}}\{Z_{nj}(\tau,\zeta)-Z_{nj}(\tau,0)\},$$
$$B_n(\tau,\zeta)=\sum_{j=2m_nl_n+1}^{n}\{Z_{nj}(\tau,\zeta)-Z_{nj}(\tau,0)\},$$

where $m_n,\mathcal{I}_{ni}$ and $\mathcal{J}_{ni}, i=1,\ldots,m_n$, are those in (21). Note that $B_n(\tau,\zeta)\xrightarrow{\mathrm{P}}0$ as $n\to\infty$, and there exists $K>0$ such that

$$\mathrm{Var}\{B_{ni}^{(r)}(\tau,\zeta)\}\le Kl_n^2\{n\sqrt{k}\}^{-1}\quad\text{for } r=1,2 \text{ and } n\in\mathbb{N}.$$

Thus, since $m_n\varrho(l_n)\to0$ and $l_n^2/k\to0$ as $n\to\infty$, (31) holds.

Following the lines to prove (4.10) in Kim and Lee (2009) together with (31), we can see that for $\zeta\in\mathbb{R}$,

$$\sup_{0\le\tau\le1}|M_n(\tau,\zeta)-M_n(\tau,0)|=o_{\mathrm{P}}(1), \tag{32}$$

and for any $\epsilon>0$ and $K>0$, as $\rho\to0$,

$$\limsup_{n\to\infty}\mathrm{P}\left(\sup\left\{\frac{1}{\sqrt{k}}\sum_{i=1}^{n}|I_{ni}(1,\zeta_1)-I_{ni}(1,\zeta_2)|:|\zeta_1-\zeta_2|<\rho,|\zeta_1|\vee|\zeta_2|<K\right\}>\epsilon\right)\to0, \tag{33}$$

$$\limsup_{n\to\infty}\sup\left\{\frac{1}{\sqrt{k}}\sum_{i=1}^{n}|\mathrm{E}I_{ni}(1,\zeta_1)-\mathrm{E}I_{ni}(1,\zeta_2)|:|\zeta_1-\zeta_2|<\rho,|\zeta_1|\vee|\zeta_2|<K\right\}\to0, \tag{34}$$

where $I_{ni}(\tau, \zeta) = \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n, \mathrm{s}_i \le \tau\right)$ (see Lemma 4 in Kim and Lee 2009). Therefore, using the arguments in the proof of Lemma 5 of Kim and Lee (2009), we can obtain (29), due to (32), (33) and (34). This validates the lemma. □

Lemma 3 *Let* $\tau_0 \in (0, 1/2)$*. For* $\zeta \in \mathbb{R}$*, it holds that* $\alpha(\tau_2 - \tau_1)\sqrt{k}\left\{\log U_{(\tau_1,\tau_2)} - \log b_n\right\} > \zeta$ *if and only if*

$$M_n(\tau_2, 0) - M_n(\tau_1, 0) > \zeta + o_{\mathrm{P}}(1),$$

where the $o_{\mathrm{P}}(1)$ *denotes the term uniformly negligible in* $0 \le \tau_1 < \tau_2 \le 1$ *with either* $\tau_1 = 0$ *or* $\tau_2 = 1$*, and* $\tau_2 - \tau_1 \ge \tau_0$*.*

Proof Let $\zeta \in \mathbb{R}$ and $\delta = \tau_2 - \tau_1$. First, consider the case of $\tau_1 = 0$. Then, by definition, we can see that $\alpha\delta\sqrt{k}\left\{\log U_{(\tau_1,\tau_2)} - \log b_n\right\} > \zeta$ if and only if $\sum_{i=1}^n \mathrm{I}(U_i > e^{\zeta/\alpha\delta\sqrt{k}} b_n,\ \tau_1 < \mathrm{s}_i \le \tau_2) > \lfloor k(\tau_2 - \tau_1)\rfloor$, which can be rewritten as

$$\begin{aligned}
&\frac{1}{\sqrt{k}}\sum_{i=1}^{n}\left\{\mathrm{I}\left(U_i > e^{\zeta/\alpha\delta\sqrt{k}} b_n,\ \tau_1 < \mathrm{s}_i \le \tau_2\right) - \mathrm{P}\left(U_i > e^{\zeta/\alpha\delta\sqrt{k}} b_n,\ \tau_1 < \mathrm{s}_i \le \tau_2\right)\right\} \\
&\quad > \sqrt{k}\left\{\frac{\lfloor k(\tau_2 - \tau_1)\rfloor}{k} - \frac{n}{k}\mathrm{P}\left(U_i > e^{\zeta/\alpha\delta\sqrt{k}} b_n,\ \tau_1 < \mathrm{s}_i \le \tau_2\right)\right\} \\
&\quad = \sqrt{k}\left\{\delta - \left(1 - \frac{\zeta}{\delta\sqrt{k}} + o\left(\frac{1}{\sqrt{k}}\right)\right)\delta + O\left(\frac{1}{k}\right)\right\} = \zeta + o(1),
\end{aligned}$$

uniformly in $0 < \tau_2 \le 1$ and $\delta \ge \tau_0$. Next, the case of $\tau_2 = 1$ can be handled via following the same lines with $\lfloor k(\tau_2 - \tau_1)\rfloor$ replaced by $\lceil k(\tau_2 - \tau_1)\rceil$. Hence, the proof is completed by Lemma 2. □

Lemma 4 *Let* $\tau_0 \in (0, 1/2)$ *and* $W_n(\tau_1, \tau_2) := \alpha(\tau_2 - \tau_1)\sqrt{k}\left\{\log U_{(\tau_1,\tau_2)} - \log b_n\right\}$ *for* $0 \le \tau_1 < \tau_2 \le 1$ *with either* $\tau_1 = 0$ *or* $\tau_2 = 1$*. Then,* $\{\tau \mapsto W_n(0, \tau)\}$ *and* $\{\tau \mapsto W_n(\tau, 1)\}$*,* $\tau_0 \le \tau \le 1 - \tau_0$*, are asymptotically uniformly equicontinuous.*

Proof Since the proof is essentially the same as that to verify (4.8) of Kim and Lee (2011), owing to Proposition 1 and Lemma 3, it is omitted for brevity. □

Proposition 3 *Let* $\tau_0 \in (0, 1/2)$*. Then,*

$$\begin{aligned}
&\left(\tau \mapsto \sqrt{k}\{\alpha \mathrm{H}_n(0, \tau) - 1\}, \tau \mapsto \sqrt{k}\{\alpha \mathrm{H}_n(\tau, 1) - 1\}\right) \\
&\quad \Rightarrow (\tau \mapsto \mathrm{B}(\tau)/\tau + \Upsilon, \tau \mapsto \{\mathrm{B}(1) - \mathrm{B}(\tau)\}/(1 - \tau) + \Upsilon) \quad \text{in } D[\tau_0, 1 - \tau_0] \\
&\quad \times D[\tau_0, 1 - \tau_0],
\end{aligned}$$

where $\Upsilon = \gamma \mathrm{DM}/(\alpha - \gamma)$*.*

Proof Since the proof is essentially the same as that of Theorem 1 of Kim and Lee (2016b), owing to Proposition 2 and Lemmas 2–4, it is omitted for brevity. □

Proof of Theorem 1 By Proposition 3, we have

$$\alpha\sqrt{k\tau(1-\tau)}\{\mathrm{H}_n(0,\tau)-\mathrm{H}_n(\tau,1)\} \Rightarrow \frac{\mathrm{B}(\tau)-\tau\mathrm{B}(1)}{\sqrt{\tau(1-\tau)}} \quad \text{in } D[\tau_0, 1-\tau_0].$$

Moreover, since $x-1=\log(x)+O((x-1)^2)$ as $x\to 1$, and $\sup_{\tau_0\le\tau\le 1-\tau_0}|\alpha\mathrm{H}_n(\tau,1)-1| = o_{\mathrm{P}}(1)$, we get

$$\sqrt{k\tau(1-\tau)}\log\frac{\mathrm{H}_n(0,\tau)}{\mathrm{H}_n(\tau,1)} \Rightarrow \frac{\mathrm{B}(\tau)-\tau\mathrm{B}(1)}{\sqrt{\tau(1-\tau)}} \quad \text{in } D[\tau_0, 1-\tau_0].$$

Then, because the mapping: $f\mapsto \sup_{\tau_0\le\tau\le 1-\tau_0}\{f(\tau)\}^2$, $f\in D[\tau_0, 1-\tau_0]$, is continuous at every continuous function f with respect to the Skorohod metric, the theorem is established by a continuous mapping theorem. □

5.2 Proofs of Theorems 2 and 3

Proof of Theorem 2 We first show that there exists $x_* > \upsilon^\circ$ satisfying the following condition: for any fixed $x_0 > x_*$, there exists nonnegative function g such that

$$\int_{\omega^\circ/(1-\beta_1^\circ)}^{\infty} g(y)\mathrm{p}(x,y)\mathrm{d}y \le g(x)-1 \quad \text{for } \; x\in[\omega^\circ/(1-\beta_1^\circ),\infty)\backslash\mathrm{K}, \tag{35}$$

$$\sup_{x\in\mathrm{K}}\int_{\omega^\circ/(1-\beta_1^\circ)}^{\infty} g(y)\mathrm{p}(x,y)\mathrm{d}y < \infty, \tag{36}$$

where $\mathrm{K}=[\omega^\circ/(1-\beta_1^\circ), x_0]$ and $\mathrm{p}(x,y)$ is given in (10). Note that (35) and (36) imply the ergodicity of $\{\mathrm{h}_i^\circ\}$ (cf. Tweedie 1975, Theorem 3.1).

Let f_j be a density function of $\mathrm{sgn}(U_{0,j})|U_{0,j}|^{2\delta^\circ}$ for $j=1,2$. Take $x_* > \upsilon^\circ$, $g(x)=c\log(x/\omega^\circ)$ $(c>0)$ and $\eta>1$, such that

$$(\eta-1)\frac{\beta_1^\circ x_*}{\omega^\circ} > 1, \quad \int_{-\infty}^{\infty} c\log\left(\phi_{1,1}^\circ z_+ + \phi_{1,2}^\circ z_- + \beta_1^\circ\right)\mathrm{f}_2(z)\mathrm{d}z + c\log\eta < -1.$$

Set $\mathrm{K}=[\omega^\circ/(1-\beta_1^\circ), x_0]$ for $x_0 > x_*$. Then, for $x\in[\omega^\circ/(1-\beta_1^\circ),\infty)\backslash\mathrm{K}$,

$$\begin{aligned}
&\int_{\omega^\circ+\beta_1^\circ x}^{\infty} g(y)\mathrm{p}(x,y)\mathrm{d}y \\
&\quad = \sum_{j=1,2}\int_{\omega^\circ+\beta_1^\circ x}^{\infty} c\log(y/\omega^\circ)\frac{1}{\phi_{1,j}^\circ x}\mathrm{f}_2\left((-1)^{j+1}\frac{y-\omega^\circ-\beta_1^\circ x}{\phi_{1,j}^\circ x}\right)\mathrm{d}y \\
&\quad = \int_0^{\infty} c\log\left\{\left(\phi_{1,1}^\circ z+\beta_1^\circ\right)\frac{x}{\omega^\circ}+1\right\}\mathrm{f}_2(z)\,\mathrm{d}z \\
&\qquad + \int_{-\infty}^{0} c\log\left\{\left(-\phi_{1,2}^\circ z+\beta_1^\circ\right)\frac{x}{\omega^\circ}+1\right\}\mathrm{f}_2(z)\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} c \log \left\{ \left(\phi_{1,1}^{\circ} z_{+} + \phi_{1,2}^{\circ} z_{-} + \beta_1^{\circ}\right) \frac{x}{\omega^{\circ}} + 1 \right\} \mathrm{f}_2(z) \mathrm{d}z \\
&\le \int_{-\infty}^{\infty} c \log \left\{ \eta \left(\phi_{1,1}^{\circ} z_{+} + \phi_{1,2}^{\circ} z_{-} + \beta_1^{\circ}\right) \frac{x}{\omega^{\circ}} \right\} \mathrm{f}_2(z) \mathrm{d}z \\
&= \int_{-\infty}^{\infty} c \log \left(\phi_{1,1}^{\circ} z_{+} + \phi_{1,2}^{\circ} z_{-} + \beta_1^{\circ}\right) \mathrm{f}_2(z) \mathrm{d}z + c \log \eta + c \log(x/\omega^{\circ}) \\
&\le g(x) - 1.
\end{aligned}
$$

This asserts (35). To prove (36), note that due to $\max_{j=1,2} \mathrm{E}|U_{0,j}|^{\nu} < \infty$ for some $\nu > 0$,

$$
\max_{j=1,2} \int_{-\infty}^{\infty} c \log \left\{ \left(\phi_{1,1}^{\circ} z_{+} + \phi_{1,2}^{\circ} z_{-} + \beta_1^{\circ}\right) \frac{x_0}{\omega^{\circ}} + 1 \right\} \mathrm{f}_j(z) \mathrm{d}z < \infty.
$$

This establishes the theorem. □

Finally, we prove Theorem 3 using Lemma 5 and Proposition 4 stated below: the conditions in Theorem 3 are implicitly assumed therein.

Proof of Theorem 2 It suffices to show that for any $K > 0$,

$$
\sup_{|\zeta| \le K} \sup_{\tau_0 \le \tau \le 1-\tau_0} \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{n} \left\{ \mathrm{I}\left(\hat{U}_{n,i} > e^{-\zeta/\sqrt{k}} b_n \,,\; \hat{\mathrm{s}}_{n,i} \le \tau\right) - \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n \,,\; \mathrm{s}_i \le \tau\right) \right\} \right|, \tag{37}
$$

$$
\sup_{|\zeta| \le K} \sup_{\tau_0 \le \tau \le 1-\tau_0} \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{n} \left\{ \hat{A}_{ni}(\tau, \zeta) - A_{ni}(\tau, \zeta) \right\} \right| \tag{38}
$$

are $o_{\mathrm{P}}(1)$ as $n \to \infty$, where $\hat{A}_{ni}(\tau, \zeta) := (\log \hat{U}_{n,i} - \log b_n + \zeta/\sqrt{k})_{+} \mathrm{I}\{\hat{\mathrm{s}}_{n,i} \le \tau\}$: see (17), (18), and (19). Here, we can prove (37) using Lemma 5 and Proposition 4 below. Since (38) can be handled similarly, the proof is omitted and the theorem is asserted. □

Lemma 5 *Let* $\Gamma_n := \{1/(2\hat{\delta}) \vee 1\} \max_{1 \le i \le n} |\log \hat{\mathrm{h}}_{n,i} - \log \mathrm{h}_i^{\circ}| + |1/(2\hat{\delta}) - 1/(2\delta^{\circ})| \max_{1 \le i \le n} \log \mathrm{h}_i^{\circ}$. *Then,* $\sqrt{k}\, \Gamma_n, = o_{\mathrm{P}}(1)$ *as* $n \to \infty$.

Proof of Theorem 3 We only sketch the proof because the details are found in Kim and Lee (2016b). Let $\{\mathrm{h}_i(\boldsymbol{\theta})\}$ be the solution of

$$
\mathrm{h}_i(\boldsymbol{\theta}) = \omega + \sum_{j=1}^{p} \left\{ \phi_{j,1} (X_{i-j})_{+}^{2\delta} + \phi_{j,2} (X_{i-j})_{-}^{2\delta} \right\} + \sum_{j=1}^{q} \beta_j \mathrm{h}_{i-j}(\boldsymbol{\theta}) \quad \text{for } i \in \mathbb{Z}
$$

(cf. (16)). Let $N_n(\eta) = \{\boldsymbol{\theta} : |\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| \leq \eta/\sqrt{n}\}$ for $\eta > 0$. Owing to Lemma 10 of Kim and Lee (2016b), we obtain that for some $r \in (0, 1)$,

$$\sup_{\boldsymbol{\theta}\in N_n(\eta)} \max_{1\leq i\leq n} |\log \hat{\mathrm{h}}_{n,i}(\boldsymbol{\theta}) - \log \mathrm{h}_i(\boldsymbol{\theta})| \leq r^{\mathrm{m}_n} V_n \quad \text{for all large } n,$$

where $V_n \geq 0$ satisfies $\sup_n \mathrm{E}V_n^\nu < \infty$ for some $\nu > 0$. Also, it follows from Lemma 12 of Kim and Lee (2016b) that for any $\nu > 0$, with a probability tending to 1 as $n \to \infty$,

$$\max_{1\leq i\leq n} |\log \mathrm{h}_i(\boldsymbol{\theta}) - \log \mathrm{h}_i^\circ| \leq n^\nu |\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| \quad \text{as far as } \boldsymbol{\theta} \text{ stays in } N_n(\eta).$$

Then, the lemma is validated by **(B2)**, **(B5)** and (15). □

Proposition 4 *Let $K = K_0$ be the one in (37). Then, (37) is $o_{\mathrm{P}}(1)$ as $n \to \infty$.*

Proof of Theorem 3 By the definition of Γ_n, **(B1)** and**(B3)** we have

$$\begin{aligned}
\hat{\mathrm{s}}_{n,i} &\geq \hat{\mathrm{G}}_n\left(\log \mathrm{h}_i^\circ - \Gamma_n\right) \geq \frac{1}{n}\sum_{j=1}^n \mathrm{I}\left\{\log \mathrm{h}_j^\circ \leq \log \mathrm{h}_i^\circ - 2\Gamma_n\right\} \\
&= \frac{1}{n}\sum_{j=1}^n \mathrm{I}\left\{\mathrm{s}_j \leq \mathrm{G}\left(\log \mathrm{h}_i^\circ - 2\Gamma_n\right)\right\} \\
&= \mathrm{G}\left(\log \mathrm{h}_i^\circ - 2\Gamma_n\right) + \Delta_{n,i}, \quad \max_{1\leq i\leq n} |\Delta_{n,i}| = O_{\mathrm{P}}(n^{-1/2}),
\end{aligned} \tag{39}$$

where the last equality is due to Theorem 1 in Deo (1973). Similarly,

$$\hat{\mathrm{s}}_{n,i} \leq \mathrm{G}(\log \mathrm{h}_i^\circ + 2\Gamma_n) + \Delta'_{n,i}, \quad \max_{1\leq i\leq n} |\Delta'_{n,i}| = O_{\mathrm{P}}(n^{-1/2}).$$

Thus, $\sum_{i=1}^n \{\mathrm{I}(\hat{U}_{n,i} > e^{-\zeta/\sqrt{k}} b_n, \hat{\mathrm{s}}_{n,i} \leq \tau) - \mathrm{I}(U_i > e^{-\zeta/\sqrt{k}} b_n, \mathrm{s}_i \leq \tau)\}$ is bounded by

$$\begin{aligned}
\sum_{i=1}^n &\Big\{\mathrm{I}\left(U_i > \exp\{-(\zeta + \sqrt{k}\Gamma_n)/\sqrt{k}\}\, b_n\,,\ \mathrm{G}(\log \mathrm{h}_i^\circ - 2\Gamma_n) + \Delta_{n,i} \leq \tau\right) \\
&-\mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n\,,\ \mathrm{s}_i \leq \tau\right)\Big\}
\end{aligned}$$

and is bounded below by

$$\begin{aligned}
\sum_{i=1}^n &\Big\{\mathrm{I}\left(U_i > \exp\{-(\zeta - \sqrt{k}\Gamma_n)/\sqrt{k}\}\, b_n\,,\ \mathrm{G}\left(\log \mathrm{h}_i^\circ + 2\Gamma_n\right) + \Delta'_{n,1} \leq \tau\right) \\
&-\mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n\,,\ \mathrm{s}_i \leq \tau\right)\Big\}.
\end{aligned}$$

Note that the upper bound can be rewritten as

$$\sum_{i=1}^{n} \left\{ \mathrm{I}\left(U_i > \exp\{-(\zeta + \sqrt{k}\Gamma_n)/\sqrt{k}\}\, b_n\right) - \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n\right)\right\}$$
$$\mathrm{I}\left(\mathrm{G}\left(\log \mathrm{h}_i^\circ - 2\Gamma_n\right) + \Delta_{n,i} \le \tau\right)$$
$$+ \sum_{i=1}^{n} \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n\right) \left\{\mathrm{I}\left(\mathrm{G}\left(\log \mathrm{h}_i^\circ - 2\Gamma_n\right) + \Delta_{n,i} \le \tau\right) - \mathrm{I}\left(\mathrm{s}_i \le \tau\right)\right\},$$

where the supremum of the first sum over $|\zeta| < K_0$ and $\tau \in [0, 1]$ is easily verified to be $o_{\mathrm{P}}(\sqrt{k})$ owing to Lemma 4 of Kim and Lee (2009) and Lemma 5, and the second sum can be expressed as follows:

$$I_1 + I_2 := \sum_{i=1}^{n} \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n\right) \left\{\mathrm{I}\left(\mathrm{G}\left(\log \mathrm{h}_i^\circ - 2\Gamma_n\right) + \Delta_{n,i} \le \tau\right) - \mathrm{I}\left(\mathrm{s}_i \le \tau\right)\right\}$$
$$\mathrm{I}\{\tau_0/2 \le \mathrm{s}_i \le 1 - \tau_0/2\}$$
$$+ \sum_{i=1}^{n} \mathrm{I}\left(U_i > e^{-\zeta/\sqrt{k}} b_n\right) \left\{\mathrm{I}\left(\mathrm{G}\left(\log \mathrm{h}_i^\circ - 2\Gamma_n\right) + \Delta_{n,i} \le \tau\right) - \mathrm{I}\left(\mathrm{s}_i \le \tau\right)\right\}$$
$$(1 - \mathrm{I}\{\tau_0/2 \le \mathrm{s}_i \le 1 - \tau_0/2\}).$$

Note that the supremum of I_2 over $\zeta \in \mathbb{R}$ and $\tau \in [\tau_0, 1 - \tau_0]$ is bounded by

$$\sum_{i=1}^{n} \sup_{\tau_0 \le \tau \le 1-\tau_0} \left\{\mathrm{I}\left(\mathrm{G}(\log \mathrm{h}_i^\circ - 2\Gamma_n) + \Delta_{n,i} \le \tau\right) - \mathrm{I}\left(\mathrm{s}_i \le \tau\right)\right\}$$
$$(1 - \mathrm{I}\{\tau_0/2 \le \mathrm{s}_i \le 1 - \tau_0/2\}),$$

which is $o_{\mathrm{P}}(1)$ because $\max\{\max_{1 \le i \le n} |\Delta_{n,i}|, \Gamma_n\} = o_{\mathrm{P}}(1)$ and $\min\{\mathrm{G}^{-1}(\tau_0) - \mathrm{G}^{-1}(\tau_0/2), \mathrm{G}^{-1}(1 - \tau_0/2) - \mathrm{G}^{-1}(1 - \tau_0)\} > 0$, $(\mathrm{G}^{-1}(\tau) := \inf\{x : \mathrm{G}(x) \ge \tau\}$ for $\tau \in (0, 1])$. On the other hand, due to **(B3)** and **(B4)**, there exists $K_{\mathrm{G}} > 0$ such that $|\mathrm{G}(x) - \mathrm{G}(y)| \le K_{\mathrm{G}}|x - y|$ for every $x, y \in [\tau_0/4, 1 - \tau_0/4]$, so that $I_1/\sqrt{k}$ is no more than

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{n} \mathrm{I}\left(U_i > e^{-K_0/\sqrt{k}} b_n\right) \left\{\mathrm{I}\left(\mathrm{s}_i \le \tau + 2K_{\mathrm{G}}\Gamma_n - \Delta_{n,i}\right) - \mathrm{I}\left(\mathrm{s}_i \le \tau\right)\right\}$$
$$= M_n(\tau + 2K_{\mathrm{G}}\Gamma_n - \Delta_{n,i}, K_0) - M_n(\tau, K_0) + \frac{n}{\sqrt{k}} \mathrm{P}\left(U_i > e^{-K_0/\sqrt{k}} b_n\right)$$
$$(2K_{\mathrm{G}}\Gamma_n - \Delta_{n,i}),$$

with a probability tending to 1. Since a similar result can be obtained on the lower bound as well, the proposition is asserted by Proposition 1, (39) and Lemmas 2 and 5. □

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