Ann Inst Stat Math (2019) 71:599–626
https://doi.org/10.1007/s10463-018-0657-x

Testing for a δ-neighborhood of a generalized Pareto copula

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Received: 2 April 2014 / Revised: 5 February 2018 / Published online: 17 April 2018

Abstract A multivariate distribution function F is in the max-domain of attraction of an extreme value distribution if and only if this is true for the copula corresponding to F and its univariate margins. Aulbach et al. (*Bernoulli* 18(2), 455–475, 2012. https://doi.org/10.3150/10-BEJ343) have shown that a copula satisfies the extreme value condition if and only if the copula is tail equivalent to a generalized Pareto copula (GPC). In this paper, we propose a χ^2-goodness-of-fit test in arbitrary dimension for testing whether a copula is in a certain neighborhood of a GPC. The test can be applied to stochastic processes as well to check whether the corresponding copula process is close to a generalized Pareto process. Since the p value of the proposed test is highly sensitive to a proper selection of a certain threshold, we also present graphical tools that make the decision, whether or not to reject the hypothesis, more comfortable.

Keywords Multivariate max-domain of attraction · Multivariate extreme value distribution · Copula · D-norm · Generalized Pareto copula · χ^2-goodness-of-fit test · Max-stable processes · Functional max-domain of attraction

1 Introduction

Consider a random vector (rv) $\boldsymbol{U} = (U_1, \ldots, U_d)^\intercal$ whose distribution function (df) is a copula C, i.e., each U_i follows the uniform distribution on (0, 1). The copula C

Stefan Aulbach was supported by DFG Grant FA 262/4-2.

Electronic supplementary material The online version of this article (https://doi.org/10.1007/s10463-018-0657-x) contains supplementary material, which is available to authorized users.

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is said to be in the max-domain of attraction of an extreme value df (EVD) G on $\mathbb{R}^d$, denoted by $C \in \mathcal{D}(G)$, if

$$C^n\left(\mathbf{1}+\frac{1}{n}\boldsymbol{x}\right) \to_{n\to\infty} G(\boldsymbol{x}), \quad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d, \tag{1}$$

where $\mathbf{0} := (0,\dots,0)^\intercal \in \mathbb{R}^d$ and $\mathbf{1} := (1,\dots,1)^\intercal \in \mathbb{R}^d$. The characteristic property of the df G is its *max-stability*, precisely,

$$G^n\left(\frac{\boldsymbol{x}}{n}\right) = G(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \quad n \in \mathbb{N};$$

see, e.g., Falk et al. (2011, Section 4).

Let $\boldsymbol{U}^{(1)},\dots,\boldsymbol{U}^{(n)}$ be independent copies of $\boldsymbol{U}$. Equation (1) is equivalent with

$$P\left(n\left(\max_{1\le i\le n} \boldsymbol{U}^{(i)} - \mathbf{1}\right) \le \boldsymbol{x}\right) \to_{n\to\infty} G(\boldsymbol{x}), \quad \boldsymbol{x} \le \mathbf{0}.$$

All operations on vectors such as $\max_{1\le i\le n} \boldsymbol{U}^{(i)}$ are meant componentwise.

From Aulbach et al. (2012, Corollary 2.2), we know that $C \in \mathcal{D}(G)$ if and only if (iff) there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$ such that the copula C satisfies the expansion

$$C(\boldsymbol{u}) = 1 - \|\boldsymbol{u}-\mathbf{1}\| + o\left(\|\boldsymbol{u}-\mathbf{1}\|\right) \tag{2}$$

uniformly for $\boldsymbol{u} \in [0,1]^d$ as $\boldsymbol{u} \uparrow \mathbf{1}$, i.e.,

$$\lim_{t\downarrow 0} \sup_{\substack{\boldsymbol{u}\in[0,1]^d\setminus\{\mathbf{1}\} \\ \|\boldsymbol{u}-\mathbf{1}\|<t}} \frac{|C(\boldsymbol{u}) - (1-\|\boldsymbol{u}-\mathbf{1}\|)\,|}{\|\boldsymbol{u}-\mathbf{1}\|} = 0.$$

In this case, the norm $\|\cdot\|$ is called a *D-norm* and is commonly denoted by $\|\cdot\|_D$, where the additional character D means dependence. The corresponding EVD G is then given by

$$G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right), \quad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d; \tag{3}$$

we refer to Falk et al. (2011, Section 5.2) for further details. We have in particular independence of the margins of G iff the D-norm $\|\cdot\|_D$ is the usual L_1-norm $\|\cdot\|_1$, and we have complete dependence iff $\|\cdot\|_D$ is the maximum-norm $\|\cdot\|_\infty$.

If the copula C satisfies

$$C(\boldsymbol{u}) = 1 - \|\boldsymbol{u}-\mathbf{1}\|_D, \quad \boldsymbol{u}_0 \le \boldsymbol{u} \le \mathbf{1},$$

for some $\boldsymbol{u}_0 \in [0,1)^d$, then we refer to it as a *generalized Pareto copula* (GPC). The characteristic property of a GPC is its *excursion stability*: The rv $\boldsymbol{U} = (U_1,\dots,U_d)^\intercal$ follows a GPC iff there exists $\boldsymbol{u}_0 \in [0,1)^d$ such that

$$P(U_k - 1 > t(u_k-1),\ k \in K) = tP(U_k > u_k,\ k \in K), \quad t \in [0,1],$$

for all $\boldsymbol{u} \geq \boldsymbol{u}_0$ and each nonempty subset K of $\{1, \ldots, d\}$; see Falk et al. (2011, Proposition 5.3.4). Based on this characterization, Falk and Michel (2009) investigated a test whether $\boldsymbol{U}$ follows a GPC; see also Falk et al. (2011, Section 5.8).

If the remainder term in Eq. (2) satisfies

$$r(\boldsymbol{u}) := C(\boldsymbol{u}) - (1 - \|\boldsymbol{u} - \mathbf{1}\|_D) = O\left(\|\boldsymbol{u} - \mathbf{1}\|_D^{1+\delta}\right) \qquad (\delta - n)$$

as $\boldsymbol{u} \uparrow \mathbf{1}$ for some $\delta > 0$, then the copula C is said to be in the δ-*neighborhood* of a GPC. Note that (2) is already implied by ($\delta - n$) and that $O\left(\|\boldsymbol{u} - \mathbf{1}\|_D^{1+\delta}\right) = O\left(\|\boldsymbol{u} - \mathbf{1}\|^{1+\delta}\right)$ for an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^d$. The significance of such δ-neighborhoods is outlined in Sect. 2 where we also give some prominent examples. In Sect. 3.1, we will derive a χ^2-goodness-of-fit test based on $\boldsymbol{U}^{(1)}, \ldots, \boldsymbol{U}^{(n)}$ which checks whether the pertaining copula C satisfies condition ($\delta - n$).

Let $\boldsymbol{X} = (X_1, \ldots, X_d)^\intercal$ be a rv with arbitrary df F. It is well known (Deheuvels 1978, 1983; Galambos 1987) that F is in the max-domain of attraction of an EVD iff this is true for the univariate margins of F together with the condition that the copula C_F corresponding to F satisfies (1). While there are various tests which check for the univariate extreme value condition—see, e.g., Dietrich et al. (2002), Drees et al. (2006) as well as Reiss and Thomas (2007, Section 5.3)—much less has been done for the multivariate case. Utilizing the *empirical copula*, we can modify the test statistic from Sect. 3.1 and check, whether C_F satisfies condition ($\delta - n$), based on independent copies $\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(n)}$. This is the content of Sect. 3.2.

Sections 4 and 5 carry the results of Sect. 3 over to function space. The aim is to test whether the copula process of a given stochastic process in $C[0, 1]$ is in a δ-neighborhood of a generalized Pareto copula process (GPCP). While Sect. 4 deals with copula processes or general processes as a whole, respectively, Sect. 5 considers the case that the underlying processes are observed at a finite grid of points only.

It is by no means obvious to find a copula C, which does *not* satisfy $C \in \mathcal{D}(G)$ for any EVD G. An example is given in Kortschak and Albrecher (2009). A family of copulas, which are not in the max-domain of attraction of an EVD but come arbitrary close to a GPC, is given in supplementary material.

In order to demonstrate the performance of our test, supplementary material contains the results of a simulation study. Since the results from the main part of the work highly depend on a proper choice of some threshold, in supplementary material we also present graphical tools that make the decision, whether or not to reject the hypothesis, more comfortable.

2 δ-neigborhoods

The significance of the δ-neighborhood of a GPC can be seen as follows. Denote by $R := \{\boldsymbol{t} \in [0, 1]^d : \|\boldsymbol{t}\|_1 = \sum_{i=1}^d t_i = 1\}$ the unit sphere in $[0, \infty)^d$ with respect to the L_1-norm $\|\cdot\|_1$. Take an arbitrary copula C on $\mathbb{R}^d$ and put for $\boldsymbol{t} \in R$

$$C_{\boldsymbol{t}}(s) := C(\mathbf{1} + s\boldsymbol{t}), \quad s \leq 0.$$

Then $C_{\boldsymbol{t}}$ is a univariate df on $(-\infty, 0]$ and the copula C is obviously determined by the family

$$\mathcal{P}(C) := \{C_{\boldsymbol{t}} : \boldsymbol{t} \in R\}$$

of univariate *spectral df* $C_{\boldsymbol{t}}$. This family $\mathcal{P}(C)$ is the *spectral decomposition* of C; cf. Falk et al. (2011, Section 5.4). A copula C is, consequently, in $\mathcal{D}(G)$ with corresponding D-norm $\|\cdot\|_D$ iff its spectral decomposition satisfies

$$C_{\boldsymbol{t}}(s) = 1 + s\|\boldsymbol{t}\|_D + o(s), \quad \boldsymbol{t} \in R,$$

as $s \uparrow 0$. The copula C is in the δ-neighborhood of the GPC C_D with D-norm $\|\cdot\|_D$ iff

$$1 - C_{\boldsymbol{t}}(s) = \left(1 - C_{D,\boldsymbol{t}}(s)\right)\left(1 + O(|s|^{\delta})\right) \tag{4}$$

uniformly for $\boldsymbol{t} \in R$ as $s \uparrow 0$. In this case, we know from Falk et al. (2011, Theorem 5.5.5) that

$$\sup_{\boldsymbol{x}\in(-\infty,0]^d} \left| C^n\left(\boldsymbol{1} + \frac{1}{n}\boldsymbol{x}\right) - \exp\left(-\|\boldsymbol{x}\|_D\right) \right| = O(n^{-\delta}). \tag{5}$$

Under additional differentiability conditions on $C_{\boldsymbol{t}}(s)$ with respect to s, also the reverse implication (5) $\Longrightarrow$ (4) holds; cf. Falk et al. (2011, Theorem 5.5.5). Thus, the δ-neighborhood of a GPC, roughly, collects those copula with a polynomial rate of convergence of maxima.

Remark 1 A similar condition as ($\delta - n$) has also been considered in Einmahl et al. (2006), where a test for the bivariate extreme value condition is performed. Precisely, ($\delta - n$) corresponds to Einmahl et al. (2006, Equation 2.5) and assures that the underlying bivariate D-norm is estimated consistently. However, the cited authors rely on a certain representation of the bivariate spectral measure, which does not seem to extend to higher dimensions in an obvious manner.

Example 1 (Archimedean Copula) Let

$$C(\boldsymbol{u}) = \varphi^{[-1]}\left(\sum_{i=1}^{d}\varphi(u_i)\right), \quad \boldsymbol{u} = (u_1, \ldots, u_d)^{\mathsf{T}} \in [0,1]^d,$$

be an Archimedean copula with generator function $\varphi : [0,1] \to [0,\infty]$ and $\varphi^{[-1]}(t) := \inf\{u \in [0,1] : \varphi(u) \le t\}$, $0 \le t \le \infty$. The function φ is in particular strictly decreasing, continuous and satisfies $\varphi(1) = 0$; for a complete characterization of the function φ, we refer to McNeil and Nešlehová (2009).

Suppose that φ is differentiable on $[\varepsilon, 1]$ for some $\varepsilon < 1$ with derivative satisfying

$$\varphi'(1) < 0, \quad \varphi'(1-h) = \varphi'(1) + O(h^{\delta}) \tag{6}$$

for some $\delta > 0$ as $h \downarrow 0$. Then C is in the δ-neighborhood of a GPC with D-norm given by $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1$, $\boldsymbol{x} = (x_1, \ldots, x_d)^{\mathsf{T}} \in \mathbb{R}^d$.

The Clayton family with $\varphi_\vartheta(t) = \vartheta^{-1}\left(t^{-\vartheta} - 1\right)$, $\vartheta \in [-1, \infty)\backslash\{0\}$ satisfies condition (6) with $\delta = 1$ if $\vartheta > \frac{-1}{d-2}$, which becomes $\vartheta > -\infty$ for $d = 2$. In the case $d = 2$, the parameter $\vartheta = -1$ yields a GPC with D-norm $\|\cdot\|_1$.

The Gumbel–Hougaard family with $\varphi_\vartheta(t) = (-\log(t))^\vartheta$, $\vartheta \in [1, \infty)$, does not satisfy condition (6). But for $\vartheta \in [1, 2)$ it is in the δ-neighborhood with $\delta = 2 - \vartheta$ of a GPC having D-norm $\|\boldsymbol{x}\|_\vartheta = (\sum_{i=1}^d |x_i|^\vartheta)^{1/\vartheta}$. Moreover, for any $\vartheta \geq 1$, the corresponding Gumbel–Hougaard copula is in the 1-neighborhood of a GPC with D-norm $\|\cdot\|_\vartheta$. This follows from the easily proven fact that any copula $\tilde{C}$ of an EVD with D-norm $\|\cdot\|_D$—i.e., $\tilde{C}(\boldsymbol{u}) = \exp(-\|(\log(u_1), \ldots, \log(u_d))^\intercal\|_D)$ for $\boldsymbol{u} = (u_1, \ldots, u_d)^\intercal$ with $\boldsymbol{0} < \boldsymbol{u} \leq \boldsymbol{1}$, cf. (3)—is in the δ-neighborhood of a GPC with the same D-norm for $\delta = 1$.

For general results on the limiting distributions of Archimedean copulas, we refer to Charpentier and Segers (2009) and Larsson and Nešlehová (2011).

Example 2 (Normal Copula) Let C be a normal copula, i.e., C is the df of $\boldsymbol{U} = (\Phi(X_1), \ldots, \Phi(X_d))^\intercal$, where Φ denotes the standard normal df and $(X_1, \ldots, X_d)^\intercal$ follows a multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean vector $\boldsymbol{\mu} = \boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma} = (\rho_{ij})_{1\leq i,j\leq d}$, where $\rho_{ii} = 1$, $1 \leq i \leq d$. If $-1 < \rho_{ij} < 0$ for $1 \leq i \neq j \leq d$, then C is in the δ-neighborhood of a GPC C_D with $\|\cdot\|_D = \|\cdot\|_1$ and

$$\delta = \min_{1\leq i\neq j\leq d} \frac{\rho_{ij}^2}{1 - \rho_{ij}^2}.$$

A proof of this assertion is given in online supplementary material.

3 A test based on copula data

This section deals with deriving a test for condition ($\delta - n$) based on independent copies $\boldsymbol{U}^{(1)}, \ldots, \boldsymbol{U}^{(n)}$ of the rv $\boldsymbol{U} = (U_1, \ldots, U_d)^\intercal$ having df C. Put for $s < 0$

$$S_{\boldsymbol{U}}(s) := \sum_{i=1}^d 1_{(s,\infty)}(U_i - 1),$$

which is a discrete version of the sojourn time that $\boldsymbol{U}$ spends above the threshold $1+s$; see Falk and Hofmann (2012) for details as well as Sect. 4. If C satisfies condition ($\delta - n$), then we obtain with $\boldsymbol{s} := s\boldsymbol{1} \in (-\infty, 0)^d$

$$\begin{aligned} P\left(S_{\boldsymbol{U}}(s) = 0\right) &= P(\boldsymbol{U} \leq \boldsymbol{1} + \boldsymbol{s}) \\ &= C(\boldsymbol{1} + \boldsymbol{s}) \\ &= 1 - |s| m_D + O(|s|^{1+\delta}) \quad \text{as} \quad s \to 0. \end{aligned} \tag{7}$$

The constant $m_D := \|\boldsymbol{1}\|_D$, which is always between 1 and d, measures the tail dependence of the margins of C. It is the extremal coefficient (Smith 1990) and equal to one in case of complete dependence of the margins and equal to d in case of independence (Takahashi 1988); we refer to Falk et al. (2011, Section 4.4) for further details.

3.1 Observing copula data directly

In order to test for condition (7), we fit a grid in the upper tail of the copula C and observe the exceedances with respect to this grid: Choose $k \in \mathbb{N}$ and put for $0 < c < 1$

$$n_j(c) := \sum_{i=1}^{n} 1_{(0,\infty)}\left(S_{\boldsymbol{U}^{(i)}}\left(-\frac{c}{j}\right)\right), \quad 1 \le j \le k. \tag{8}$$

$n_j(c)$ is the number of those rv $\boldsymbol{U}^{(i)}$ among the independent copies $\boldsymbol{U}^{(1)}, \dots, \boldsymbol{U}^{(n)}$ of $\boldsymbol{U}$, whose sojourn times above $1 - \frac{c}{j}$ are positive, i.e., at least one component of $\boldsymbol{U}^{(i)}$ exceeds the threshold $1 - \frac{c}{j}$. On the other hand,

$$n - n_j(c) = \sum_{i=1}^{n} 1_{\{0\}}\left(S_{\boldsymbol{U}^{(i)}}\left(-\frac{c}{j}\right)\right) = \sum_{i=1}^{n} 1_{\left[\boldsymbol{0},\left(1-\frac{c}{j}\right)\boldsymbol{1}\right]}\left(\boldsymbol{U}^{(i)}\right)$$

is the number of those rv $\boldsymbol{U}^{(i)}$ whose realizations are below the vector with constant entry $1 - \frac{c}{j}$.

If C satisfies condition ($\delta - n$), then each $n_j(c)$ is binomial $B(n, p_j(c))$-distributed with

$$p_j(c) := 1 - P\left(S_{\boldsymbol{U}}\left(-\frac{c}{j}\right) = 0\right) = \frac{c}{j} m_D + O(c^{1+\delta}) \quad \text{as} \quad c \to 0.$$

Motivated by the usual χ^2-goodness-of-fit test, we consider in what follows the test statistic

$$T_n(c) := \frac{\sum_{j=1}^{k}\left(j\, n_j(c) - \frac{1}{k}\sum_{\ell=1}^{k} \ell\, n_\ell(c)\right)^2}{\frac{1}{k}\sum_{\ell=1}^{k} \ell\, n_\ell(c)} \tag{9}$$

which does not require the constant m_D to be known. By $\to_D$, we denote ordinary convergence in distribution as the sample size n tends to infinity.

Theorem 1 *Suppose that C satisfies condition ($\delta - n$) with $\delta > 0$. Let $c = c_n$ satisfy $c_n \to 0$, $nc_n \to \infty$ and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$. Then we obtain*

$$T_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i \xi_i^2,$$

where $\xi_1, \dots, \xi_{k-1}$ are independent and standard normal distributed rv and

$$\lambda_i = \frac{1}{4\sin^2\left(\frac{i}{k}\frac{\pi}{2}\right)}, \quad 1 \le i \le k-1.$$

Remark 2 We have $\lambda_1 = 1/2$ in case $k = 2$, and $\lambda_1 = 1$, $\lambda_2 = 1/3$ in case $k = 3$. If the copula C is a GPC, then the condition $nc_n^{1+2\delta} \to_{n\to\infty} 0$ in the preceding result can be dropped.

Proof (Theorem 1) Lindeberg's central limit theorem implies

$$\frac{1}{(nc_n)^{1/2}} \left(j\, n_j(c_n) - nc_n m_D \right) \to_D N\left(0, jm_D\right) \tag{10}$$

and, thus,

$$\frac{j\, n_j(c_n)}{nc_n} \to_{n\to\infty} m_D \quad \text{in probability}, \quad 1 \le j \le k, \tag{11}$$

yielding

$$\frac{1}{nc_n k} \sum_{j=1}^{k} j\, n_j(c_n) \to_{n\to\infty} m_D \quad \text{in probability.} \tag{12}$$

We, therefore, can substitute the denominator in the test statistic $T_n(c_n)$ by $nc_n m_D$, i.e., $T_n(c_n)$ is asymptotically equivalent with

$$\begin{aligned}
&\frac{1}{nc_n m_D} \sum_{j=1}^{k} \left(j\, n_j(c_n) - \frac{1}{k} \sum_{\ell=1}^{k} \ell\, n_\ell(c_n) \right)^2 \\
&= \frac{1}{nc_n m_D} \begin{pmatrix} 1 \cdot n_1(c_n) - nc_n m_D \\ \vdots \\ k \cdot n_k(c_n) - nc_n m_D \end{pmatrix}^{\mathsf{T}} \left(\boldsymbol{I}_k - \frac{1}{k} \boldsymbol{E}_k \right) \begin{pmatrix} 1 \cdot n_1(c_n) - nc_n m_D \\ \vdots \\ k \cdot n_k(c_n) - nc_n m_D \end{pmatrix} \\
&= \boldsymbol{Y}_n^{\mathsf{T}} \left(\boldsymbol{I}_k - \frac{1}{k} \boldsymbol{E}_k \right) \boldsymbol{Y}_n,
\end{aligned}$$

where $\boldsymbol{Y}_n = (Y_{n,1}, \ldots, Y_{n,k})^{\mathsf{T}}$ with

$$Y_{n,j} = \frac{1}{(nc_n m_D)^{1/2}} \left(j\, n_j(c_n) - nc_n m_D \right), \quad 1 \le j \le k,$$

$\boldsymbol{I}_k$ is the $k \times k$ unit matrix and $\boldsymbol{E}_k$ that $k \times k$-matrix with constant entry 1. Note that the matrix $\boldsymbol{P}_k := \boldsymbol{I}_k - k^{-1} \boldsymbol{E}_k$ is a *projection matrix*, i.e., $\boldsymbol{P}_k = \boldsymbol{P}_k^{\mathsf{T}} = \boldsymbol{P}_k^2$, and that $\boldsymbol{P}_k \boldsymbol{x} = \boldsymbol{0}$ for every vector $\boldsymbol{x} \in \mathbb{R}^k$ with constant entries.

The Cramér–Wold theorem and Lindeberg's central limit theorem imply $\boldsymbol{Y}_n \to_D N(\boldsymbol{0}, \boldsymbol{\Sigma})$, where the $k \times k$-covariance matrix $\boldsymbol{\Sigma} = (\sigma_{ij})$ is given by

$$\begin{aligned}
\sigma_{ij} &= \lim_{n\to\infty} \frac{1}{nc_n m_D} E\left(\left(i\, n_i(c_n) - nc_n m_D \right) \left(j\, n_j(c_n) - nc_n m_D \right) \right) \\
&= \lim_{n\to\infty} \frac{ij}{c_n m_D} E\left[1_{(0,\infty)} \left(S_U \left(-\frac{c_n}{i} \right) \right) 1_{(0,\infty)} \left(S_U \left(-\frac{c_n}{j} \right) \right) \right]
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{ij}{c_n m_D} P\left(S_U\left(-\frac{c_n}{\max(i,j)}\right) > 0\right)$$
$$= \frac{ij}{\max(i,j)} = \min(i,j)\,.$$

Note that $\boldsymbol{\Sigma} = \boldsymbol{M}_k \boldsymbol{M}_k^{\mathsf{T}}$ where the $k \times k$-matrix $\boldsymbol{M}_k$ is defined by

$$\boldsymbol{M}_k := \left(1_{[j,\infty)}(i)\right)_{1\le i,j\le k} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

Altogether we obtain

$$T_n(c_n) \to_D \boldsymbol{\xi}^{\mathsf{T}} \boldsymbol{M}_k^{\mathsf{T}} \left(\boldsymbol{I}_k - \frac{1}{k}\boldsymbol{E}_k\right) \boldsymbol{M}_k \boldsymbol{\xi}$$

with a k-dimensional standard normal rv $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k)^{\mathsf{T}}$. It is well known that the eigenvalues of

$$k\,\boldsymbol{M}_k^{\mathsf{T}} \left(\boldsymbol{I}_k - \frac{1}{k}\boldsymbol{E}_k\right) \boldsymbol{M}_k = \left(k\min(i-1, j-1) - (i-1)(j-1)\right)_{1\le i,j\le k}$$

are

$$k\lambda_j = \frac{k}{4\sin^2\left(\frac{j}{k}\frac{\pi}{2}\right)}, \quad j = 1, \dots, k-1, \quad \text{and} \quad \lambda_k = 0,$$

see, for example, Anderson and Stephens (1997) or Fortiana and Cuadras (1997), with corresponding orthonormal eigenvectors

$$\boldsymbol{r}_j = \sqrt{\frac{2}{k}} \left(\sin\left(\frac{(i-1)j\pi}{k}\right)\right)_{1\le i\le k}, \quad j = 1, \dots, k-1, \quad \text{and} \quad \boldsymbol{r}_k = (1, 0, \dots, 0)^{\mathsf{T}}.$$

This implies

$$T_n(c_n) \to_D \boldsymbol{\xi}^{\mathsf{T}} \operatorname{diag}(\lambda_1, \dots, \lambda_{k-1}, 0)\, \boldsymbol{\xi} = \sum_{i=1}^{k-1} \lambda_i \xi_i^2$$

as asserted. □

The proof of Theorem 1 shows that the distribution of $\sum_{i=1}^{k-1} \lambda_i \xi_i^2$ equals the one of $\sum_{i=1}^{k} (B(i) - \frac{1}{k}\sum_{j=1}^{k} B(j))^2$, where $(B(t))_{t\ge 0}$ is a standard Brownian motion on $[0, \infty)$. Computing the expected values, we obtain as a nice by-product the equation

$$\sum_{i=1}^{k-1} \frac{1}{4\sin^2\left(\frac{i}{k}\frac{\pi}{2}\right)} = \frac{(k-1)(k+1)}{6}, \quad k \ge 2.$$

Using characteristic functions, it is straightforward to prove that

$$\frac{6}{(k-1)(k+1)} \sum_{i=1}^{k-1} \lambda_i \xi_i^2 \to_D \frac{24}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \xi_i^2 \tag{13}$$

as $k \to \infty$. Taking expectations on both sides motivates the well known equality

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

Remark 3 The previous arguments suggest to replace k in Theorem 1 with a sequence $(k_n)_{n \in \mathbb{N}}$. If $k_n \to \infty$ as $n \to \infty$ at a proper rate of convergence, it should be possible to reproduce the limit distribution in (13). Although this might be of theoretical interest, we avoid doing so for several reasons: On the one hand, a data set will typically contain not too many exceedances above a high threshold. If c_n is sufficiently small to detect a δ-neighborhood, i.e., the threshold $(1 - c_n)\mathbf{1}$ is sufficiently large, there will be even less data that exceed $(1 - \frac{c_n}{k})\mathbf{1}$. This means that we would need a very large sample size in order to increase k *and* to assure that there are still sufficiently many exceedances in the outermost extremal region. On the other hand, it will be necessary to complement c_n and k with another parameter as soon as we consider more general data. While Theorem 1 allows to choose, e.g., $k = 2$ or $k = 3$ independently of the sample size, the problem of obtaining reasonable values for these parameters would become even more challenging if k depended on n as well. We also refer to Section 2.1 about parameter selection in supplementary material.

Remark 4 The test provided by Theorem 1 is based on $k + 1$ estimators of the value m_D in arbitrary dimension, cf. (11) and (12). In contrast, the test of Einmahl et al. (2006), cf. Remark 1, considers two different estimators of the D-norm on the whole set $(0, 1]^2$; recall that this approach is restricted to the bivariate case. However, the cited authors need to consider further technical details, such as the existence of certain continuous densities. Related tests for the copula of an EVD can be found in Ghoudi et al. (1998), Kojadinovic et al. (2011) and Berghaus et al. (2013), to name just a few. The latter tests could be exploited to test for the extreme value condition via a (componentwise) block maxima approach (even though it is not completely validated).

To evaluate the performance of the above test, we consider in what follows n independent copies $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ of the rv $\boldsymbol{U}$, whose df C satisfies for some $\delta > 0$ the expansion

$$C(\boldsymbol{u}) = 1 - \|\boldsymbol{u} - \mathbf{1}\|_D - J\left(\frac{\boldsymbol{u} - \mathbf{1}}{\|\boldsymbol{u} - \mathbf{1}\|_1}\right) \|\boldsymbol{u} - \mathbf{1}\|_D^{1+\delta} + o\left(\|\boldsymbol{u} - \mathbf{1}\|_D^{1+\delta}\right)$$

as $\boldsymbol{u} \uparrow 1$, uniformly for $\boldsymbol{u} \in [0, 1]^d$, where $J(\cdot)$ is an arbitrary function on the set $\{\boldsymbol{z} \leq \mathbf{0} \in \mathbb{R}^d : \|\boldsymbol{z}\|_1 = 1\}$ of directions in $(-\infty, 0]^d$. The above condition specifies the remainder term in the δ-neighborhood condition ($\delta - n$). We obtain for $c \in (0, 1)$ and $j \in \mathbb{N}$

$$p_j(c) = 1 - P\left(\boldsymbol{U} \leq 1 - \frac{c}{j}\right) = \frac{c}{j}m_D + K\left(\frac{c}{j}m_D\right)^{1+\delta} + o(c^{1+\delta}) \quad \text{as } c \to 0,$$

where $K := J(\mathbf{1}/d)$. For $n_j(c)$ as defined in (8), we obtain by elementary arguments

$$\frac{1}{(nc_n m_D)^{1/2}}(jn_j(c_n) - nc_n m_D) \to_D N\left(\frac{s^{1/2}m_D^{1/2+\delta}K}{j^\delta}, j\right)$$

if $c_n \to 0$, $nc_n \to \infty$ and $nc_n^{1+2\delta} \to s \geq 0$ as $n \to \infty$. Repeating the arguments in the proof of Theorem 1 then yields for the test statistic T_n defined in (9)

$$T_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i(\xi_i + \mu_i)^2,$$

where

$$\mu_i := K\sqrt{\frac{2s}{k}}m_D^{1/2+\delta}\sum_{j=1}^{k-1}\frac{1}{(j+1)^\delta}\sin\left(j\frac{i\pi}{k}\right), \quad 1 \leq i \leq k-1.$$

Note that each $\mu_i > 0$ if $Ks > 0$. With $K > 0$, the fact that C is not a GPC is, therefore, detected at an arbitrary level one error iff $nc_n^{1+2\delta} \to \infty$. Using the test statistic T_n, this way will even seperate a GPC from elements of its δ-neighborhood. That is why we use the rate $nc_n^{1+2\delta} \to 0$ for most part in this paper, as this relaxes the test not to reject elements of the δ-neighborhood.

3.2 The case of an arbitrary random vector without further assumptions

Consider a rv $\boldsymbol{X} = (X_1, \ldots, X_d)^\intercal$ whose df F is continuous and the copula $C_F(\boldsymbol{u}) = F\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right)$, $\boldsymbol{u} \in (0, 1)^d$, corresponding to F satisfies condition (2). Now we will modify the test statistic $T_n(c_n)$ from above to obtain a test which checks whether C_F satisfies condition ($\delta - n$) with F_i unknown, $i = 1, \ldots, d$. We denote in what follows by $H^{-1}(q) := \inf\{t \in \mathbb{R} : H(t) \geq q\}$, $q \in (0, 1)$, the generalized inverse of an arbitrary univariate df H.

Let $\boldsymbol{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_d^{(i)}\right)^\intercal$, $i = 1, \ldots, n$, be independent copies of $\boldsymbol{X}$ and fix $k \in \{2, 3, \ldots\}$. We now face the problem that we need to estimate the unknown margins $F_1, \ldots, F_d$ from the *same* sample as the test is based on. In order to guarantee that the approximations of the margins perform sufficiently well, the test statistic will be computed from the first $m_n < n$ observations, whereas the estimates of the margins consider the whole data set, cf. (16) and Lemma 1. Precisely, we put for $0 < c < 1$

$$n_{j,m_n}(c) := \sum_{i=1}^{m_n} 1_{(0,\infty)}\left(\sum_{r=1}^{d} 1_{\left(F_r^{-1}\left(1-\frac{c}{j}\right),\infty\right)}\left(X_r^{(i)}\right)\right)$$
$$= m_n - \sum_{i=1}^{m_n} 1_{(-\infty,\boldsymbol{\gamma}_j(c)]}\left(\boldsymbol{X}^{(i)}\right), \quad 1 \le j \le k, \tag{14}$$

which is the number of all rv $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(m_n)}$ exceeding the vector

$$\boldsymbol{\gamma}_j(c) := \left(F_1^{-1}\left(1-\frac{c}{j}\right), \dots, F_d^{-1}\left(1-\frac{c}{j}\right)\right)^{\mathsf{T}}$$

in at least one component. In what follows we show that we asymptotically may replace the quantiles in $\boldsymbol{\gamma}_j(c)$ with componentwise order statistics of $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$. If these quantiles were known, we would be in the same situation as in Sect. 3.1 and could simply choose $m_n = n$.

Since F is continuous, transforming each $X_r^{(i)}$ by its df F_r does not alter the value of $n_{j,m_n}(c)$ with probability one:

$$n_{j,m_n}(c) = m_n - \sum_{i=1}^{m_n} 1_{\left[\mathbf{0},\left(1-\frac{c}{j}\right)\mathbf{1}\right]}\left(\boldsymbol{U}^{(i)}\right) \tag{15}$$

where $\boldsymbol{U}^{(i)} = \left(U_1^{(i)}, \dots, U_d^{(i)}\right)^{\mathsf{T}}$, $U_r^{(i)} := F_r\left(X_r^{(i)}\right)$, and $\boldsymbol{U}^{(1)}, \dots, \boldsymbol{U}^{(n)}$ are iid with df C_F. By replacing $F_1, \dots, F_d$ with their empirical counterparts $\hat{F}_{n,r}(x) := n^{-1}\sum_{i=1}^{n} 1_{(-\infty,x]}(X_r)$, $x \in \mathbb{R}$, $1 \le r \le d$, we obtain analogously

$$\hat{n}_{j,m_n}(c) := \sum_{i=1}^{m_n} 1_{(0,\infty)}\left(\sum_{r=1}^{d} 1_{\left(\hat{F}_{n,r}^{-1}\left(1-\frac{c}{j}\right),\infty\right)}\left(X_r^{(i)}\right)\right)$$
$$= m_n - \sum_{i=1}^{m_n} 1_{\times_{r=1}^{d}[0,U_{\langle n(1-\frac{c}{j})\rangle:n,r}]}\left(\boldsymbol{U}^{(i)}\right) \tag{16}$$

with probability one. Note that $\hat{F}_{n,r}^{-1}(1-\frac{c}{j}) = X_{\langle n(1-\frac{c}{j})\rangle:n,r}$ where $\langle x\rangle := \min\{\ell \in \mathbb{N} : \ell \ge x\}$ and $X_{1:n,r} \le X_{2:n,r} \le \dots \le X_{n:n,r}$ denote the ordered values of $X_r^{(1)}, \dots, X_r^{(n)}$ for each $r = 1, \dots, d$. Thus, $U_{i:n,r} = F_r\left(X_{i:n,r}\right)$ and $X_{i:n,r} = F_r^{-1}\left(U_{i:n,r}\right)$, $1 \le i \le n$, almost surely. Two alternatives to this approach will be presented in Sect. 3.3.

Remark 5 In (14) and (16), we consider the (empirical) number of exceedances among $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(m_n)}$. We could also take any subset $M(n)$ of $\{1, \dots, n\}$ with m_n elements and use the observations $\boldsymbol{X}^{(i)}$, $i \in M(n)$, instead. However, since the test statistic in (9) requires the data to be iid, there seem to be no reasonable criteria for deriving a kind of "optimal" set $M(n)$. Thus, we chose $M(n) = \{1, \dots, m_n\}$ for convenience.

Since $\boldsymbol{U}^{(1)}, \ldots, \boldsymbol{U}^{(n)}$ are iid with df C_F, (16) shows that the distribution of $\left(\hat{n}_{1,m_n}(c), \ldots, \hat{n}_{k,m_n}(c)\right)^\intercal$ does not depend on the margins $F_1, \ldots, F_d$ but only on the copula C_F of the continuous df F. The following auxiliary result assures that we may actually consider $\hat{n}_{j,m_n}(c)$ instead of $n_{j,m_n}(c)$.

Lemma 1 *Suppose that the df F is continuous and that its copula C_F satisfies expansion ($\delta - n$) for some $\delta > 0$. Assume $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$, and let $c = c_n$ satisfy $c_n \to 0$, $m_n c_n \to \infty$, and $n c_n^{1+2\delta} \to 0$ as $n \to \infty$. Then we obtain for $j = 1, \ldots, k$*

$$(m_n c_n)^{-1/2} \left(n_{j,m_n}(c_n) - \hat{n}_{j,m_n}(c_n)\right) \to_{n\to\infty} 0 \quad \textit{in probability.}$$

Proof It suffices to show the assertion for $j = 1$. For $j > 1$, simply note that $n_{j,m_n}(c_n) = n_{1,m_n}(c_n/j)$ and $\hat{n}_{j,m_n}(c_n) = \hat{n}_{1,m_n}(c_n/j)$. Put

$$\delta_n := \max\left\{\frac{m_n}{n}, \frac{1}{m_n c_n}\right\} \quad \text{and} \quad \varepsilon_n := \left(\frac{c_n}{m_n}\delta_n\gamma_n\right)^{1/2}$$

with some sequence $\gamma_n > 0$ such that $\gamma_n \to \infty$ and $\delta_n\gamma_n \to 0$ as $n \to \infty$, e.g., $\gamma_n = \delta_n^{-1/2}$. This yields in particular $\varepsilon_n \to 0$, $\varepsilon_n/c_n \to 0$, $m_n\varepsilon_n^2/c_n \to 0$,

$$\frac{\varepsilon_n}{c_n^{1+\delta}} = \left(\gamma_n \max\left\{\frac{1}{n c_n^{1+2\delta}}, \frac{1}{(m_n c_n^{1+\delta})^2}\right\}\right)^{1/2} \geq \left(\frac{\gamma_n}{n c_n^{1+2\delta}}\right)^{1/2} \to \infty, \tag{17}$$

$$\frac{n\varepsilon_n^2}{c_n} = \gamma_n \max\left\{1, \frac{n}{m_n^2 c_n}\right\} \geq \gamma_n \to \infty, \tag{18}$$

$$m_n\varepsilon_n = \left(\gamma_n \max\left\{\frac{m_n^2 c_n}{n}, 1\right\}\right)^{1/2} \geq \gamma_n^{1/2} \to \infty$$

as $n \to \infty$.

With $\mu_n := E\left(U_{\langle n(1-c_n)\rangle:n,1}\right) = \langle n(1-c_n)\rangle/(n+1)$, we get for arbitrary $\eta > 0$ that

$$\begin{aligned}
&P\left(\left|\frac{n_{1,m_n}(c_n) - \hat{n}_{1,m_n}(c_n)}{(m_n c_n)^{1/2}}\right| > \eta\right)\\
&= P\left(\frac{n_{1,m_n}(c_n) - \hat{n}_{1,m_n}(c_n)}{(m_n c_n)^{1/2}} > \eta\right)\\
&\quad + P\left(\frac{n_{1,m_n}(c_n) - \hat{n}_{1,m_n}(c_n)}{(m_n c_n)^{1/2}} < -\eta\right)\\
&\leq P\left(\frac{n_{1,m_n}(c_n) - \hat{n}_{1,m_n}(c_n)}{(m_n c_n)^{1/2}} > \eta, \max_{1\leq r\leq d} |U_{\langle n(1-c_n)\rangle:n,r} - \mu_n| \leq \varepsilon_n\right)\\
&\quad + P\left(\frac{n_{1,m_n}(c_n) - \hat{n}_{1,m_n}(c_n)}{(m_n c_n)^{1/2}} < -\eta, \max_{1\leq r\leq d} |U_{\langle n(1-c_n)\rangle:n,r} - \mu_n| \leq \varepsilon_n\right)\\
&\quad + 2\,P\left(\max_{1\leq r\leq d} |U_{\langle n(1-c_n)\rangle:n,r} - \mu_n| > \varepsilon_n\right).
\end{aligned} \tag{19}$$

Furthermore, we deduce from Reiss (1989, Lemma 3.1.1) the exponential bound

$$P\left(\max_{1\le r\le d}|U_{\langle n(1-c_n)\rangle:n,r}-\mu_n|>\varepsilon_n\right)\le d\,P(|U_{\langle n(1-c_n)\rangle:n,1}-\mu_n|>\varepsilon_n)$$
$$\le d\exp\left(-\frac{\frac{n}{\sigma_n^2}\varepsilon_n^2}{3\left(1+\frac{\varepsilon_n}{\sigma_n^2}\right)}\right),$$

where $\sigma_n^2:=\mu_n(1-\mu_n)$. Note that $c_n\to 0$ and $nc_n\to\infty$ as $n\to\infty$ together with

$$\sigma_n^2\in\left(\frac{n^2(1-c_n)}{(n+1)^2}c_n,\left(1-\frac{nc_n}{n+1}\right)\frac{1+nc_n}{(n+1)c_n}c_n\right)$$

show $\sigma_n^2/c_n\to 1$ as $n\to\infty$. Thus, $\varepsilon_n/c_n\to 0$ and $n\varepsilon_n^2/c_n\to\infty$ as $n\to\infty$ prove that the above exponential bound has limit 0 as $n\to\infty$.

It remains to show that the both leading terms in (19) are asymptotically zero. Consider (15) and (16) and put $V_{n,i}:=1_{[\mathbf{0},(\mu_n+\varepsilon_n)\mathbf{1}]\setminus[\mathbf{0},(1-c_n)\mathbf{1}]}\big(\boldsymbol{U}^{(i)}\big)$ as well as $W_{n,i}:=1_{[\mathbf{0},(1-c_n)\mathbf{1}]\setminus[\mathbf{0},(\mu_n-\varepsilon_n)\mathbf{1}]}\big(\boldsymbol{U}^{(i)}\big)$ for $i=1,\dots,m_n$; note that $\varepsilon_n\to 0$, $\varepsilon_n/c_n\to 0$, and $n\varepsilon_n\to\infty$ as $n\to\infty$ yield

$$0\le\frac{n(1-c_n)}{n+1}-\varepsilon_n\le\mu_n-\varepsilon_n<1-c_n-\frac{1}{n}\left(n\varepsilon_n-\frac{nc_n}{n+1}\right)\le 1-c_n$$

as well as

$$1-c_n\le 1-c_n+\frac{1}{n}\left(n\varepsilon_n-\frac{n(1-c_n)}{n+1}\right)\le\mu_n+\varepsilon_n<1-c_n\left(\frac{n}{n+1}-\frac{\varepsilon_n}{c_n}\right)\le 1$$

whenever n is sufficiently large. Then we obtain

$$P\left(\frac{n_{1,m_n}(c_n)-\hat n_{1,m_n}(c_n)}{(m_nc_n)^{1/2}}>\eta,\ \max_{1\le r\le d}|U_{\langle n(1-c_n)\rangle:n,r}-\mu_n|\le\varepsilon_n\right)$$
$$\le P\left(\sum_{i=1}^{m_n}1_{[\mathbf{0},(\mu_n+\varepsilon_n)\mathbf{1}]}\big(\boldsymbol{U}^{(i)}\big)-\sum_{i=1}^{m_n}1_{[\mathbf{0},(1-c_n)\mathbf{1}]}\big(\boldsymbol{U}^{(i)}\big)>(m_nc_n)^{1/2}\eta\right)$$
$$=1-P\left(\sum_{i=1}^{m_n}V_{n,i}\le(m_nc_n)^{1/2}\eta\right)$$

and

$$P\left(\frac{n_{1,m_n}(c_n)-\hat n_{1,m_n}(c_n)}{(m_nc_n)^{1/2}}<-\eta,\ \max_{1\le r\le d}|U_{\langle n(1-c_n)\rangle:n,r}-\mu_n|\le\varepsilon_n\right)$$
$$\le 1-P\left(\sum_{i=1}^{m_n}W_{n,i}\le(m_nc_n)^{1/2}\eta\right)$$

for large n. Since C_F satisfies ($\delta - n$), we get

$$\begin{aligned} p_n &:= P(V_{n,1} = 1) = C_F((\mu_n + \varepsilon_n)\mathbf{1}) - C_F((1 - c_n)\mathbf{1}) \\ &= (\varepsilon_n + \mu_n - (1 - c_n))m_D + O\big(c_n^{1+\delta}\big) \\ &= \varepsilon_n\left[\left(1 + O\left(\frac{1}{n\varepsilon_n}\right)\right)m_D + O\left(\frac{c_n^{1+\delta}}{\varepsilon_n}\right)\right] \end{aligned} \tag{20}$$

and

$$q_n := P(W_{n,1} = 1) = \varepsilon_n\left[\left(1 + O\left(\frac{1}{n\varepsilon_n}\right)\right)m_D + O\left(\frac{c_n^{1+\delta}}{\varepsilon_n}\right)\right] \tag{21}$$

if n tends to infinity. Since $m_n\varepsilon_n \to \infty$ as $n \to \infty$, Lindeberg's central limit theorem and the continuity of Φ, the df of the standard normal distribution, show that for each $\tilde{\eta} > 0$ and sufficiently large n

$$\begin{aligned} P\left(\sum_{i=1}^{m_n} V_{n,i} \le (m_n c_n)^{1/2}\eta\right) &= P\left(\frac{\sum_{i=1}^{m_n}(V_{n,i} - p_n)}{(m_n p_n(1 - p_n))^{1/2}} \le \frac{(m_n c_n)^{1/2}\eta - m_n p_n}{(m_n p_n(1 - p_n))^{1/2}}\right) \\ &\ge \Phi\left(\frac{c_n^{1/2}}{p_n^{1/2}}\left(\eta - \frac{m_n^{1/2} p_n}{c_n^{1/2}}\right)\frac{1}{(1 - p_n)^{1/2}}\right) - \tilde{\eta} \\ &\to 1 - \tilde{\eta} \end{aligned} \tag{22}$$

since $p_n/\varepsilon_n \to m_D$, $\varepsilon_n/c_n \to 0$, and $m_n\varepsilon_n^2/c_n \to 0$ as $n \to \infty$. The same arguments also show

$$P\left(\sum_{i=1}^{m_n} W_{n,i} \le (m_n c_n)^{1/2}\eta\right) \to 1 \quad \text{as} \quad n \to \infty \tag{23}$$

and the proof is complete. □

Remark 6 The main idea of the preceding proof was the construction of a sequence ε_n such that all three terms in (19) have limit zero. For the last term, we applied Reiss (1989, Lemma 3.1.1) and needed $n\varepsilon_n^2/c_n \to \infty$ as $n \to \infty$. For the both leading terms, however, we exploited Lindeberg's central limit theorem and the convergence $m_n\varepsilon_n^2/c_n \to 0$ as $n \to \infty$. Thus, the condition $m_n/n \to 0$ as $n \to \infty$ in Lemma 1 cannot be dropped easily.

Another crucial condition on ε_n was

$$0 \le \mu_n - \varepsilon_n \le 1 - c_n \le \mu_n + \varepsilon_n \le 1 \quad \text{for large } n,$$

which led to suitable bounds of the difference $n_{1,m_n}(c_n) - \hat{n}_{1,m_n}(c_n)$ and thus allowed to apply Lindeberg's central limit theorem. While it is natural to assume $\mu_n - \varepsilon_n \le 1 - c_n \le \mu_n + \varepsilon_n$ since $\mu_n \sim 1 - c_n$ as $n \to \infty$, the bounds 0 and 1 implied the both representations (20) and (21). If we would have $\mu_n + \varepsilon_n > 1$ for large n, this would yield $p_n = c_n[m_D + O(c_n^\delta)]$ and (22) would fail since we would require $m_n c_n \to \infty$ and $m_n c_n^2/c_n \to 0$ as $n \to \infty$ at the same time. Similarly, $\mu_n - \varepsilon_n < 0$ for large n would mean (23) to fail.

Lemma 1 suggests a modification of our test statistic in (9)

$$\hat{T}_n(c) := \frac{\sum_{j=1}^{k}\left(j\,\hat{n}_{j,m_n}(c) - \frac{1}{k}\sum_{\ell=1}^{k}\ell\,\hat{n}_{\ell,m_n}(c)\right)^2}{\frac{1}{k}\sum_{\ell=1}^{k}\ell\,\hat{n}_{\ell,m_n}(c)}$$

which does not depend on the margins but only on the copula of the underlying df F. The following result is a consequence of Theorem 1 and Lemma 1.

Theorem 2 *Suppose that the df F is continuous and that its copula C_F satisfies expansion ($\delta - n$) for some $\delta > 0$. Assume $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$, and let $c = c_n$ satisfy $c_n \to 0$, $m_n c_n \to \infty$, and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$. Then we obtain*

$$\hat{T}_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i \xi_i^2$$

with ξ_i and λ_i as in Theorem 1.

The condition $nc_n^{1+2\delta} \to 0$ can again be dropped if the copula C_F is a GPC. Note that the assertion of Theorem 2 still holds if we replace the condition $nc_n^{1+2\delta} \to 0$ with, e.g., the two assumptions $m_n c_n^{1+2\delta} \to 0$ and $nc_n^{1+2\delta} \to s$ as $n \to \infty$ for some $s > 0$; cf. Theorem 1 and (17).

3.3 The case of an arbitrary random vector with further assumptions

The strategy in the previous section was to set aside a lot of data to estimate the marginal dfs of the random vector. Then a small part of the data is used to emulate iid observations of the copula. This emulation is good enough that the asymptotics in Theorems 1 and 2 are the same.

We will present two other ways to emulate iid observations of the copula without setting aside data. However, each approach comes with a cost.

The empirical copula way: Let $\boldsymbol{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_d^{(i)}\right)^{\mathsf{T}}$, $i = 1, \ldots, n$, be independent copies of a random vector $\boldsymbol{X}$. Then there are random ranks $R_i^j = |\{\ell | X_j^{(\ell)} \le X_j^{(i)}\}| = n \cdot \hat{F}_j(X_j^{(i)})$, where $\hat{F}_j$ is the empirical distribution function in the jth component. Scaling the ranks linearly to lie in the interval $[0, 1]$ produces what is known as the empirical copula. The question arises what happens if the test statistic T_n is applied to the empirical copula instead of the copula itself. The answer can be derived from Bücher et al. (2014). The positive message is that T_n applied to the empirical copula still converges in distribution to a random variable if condition ($\delta - n$) is fulfilled. The downside is that the distribution of this variable depends on the D-norm. So different underlying D-norms produce different critical values.

Theorem 3 *Let R_i^j be the ranks of n independent copies of a random vector whose copula fulfills ($\delta - n$) for some $\delta > 0$ and let $c = c_n$ satisfy $c_n \to 0$, $nc_n \to \infty$ and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$. Put*

$$n_j(c) := \sum_{i=1}^{n} 1_{(1-c/j,\infty)}\left(\max_{1\le j\le d}\frac{R_j^i - 1/2}{n}\right) \tag{24}$$

and $T_n(c)$ like in equation (9), then there exists a stochastic process $\mathbb{W}$ on $[0,\infty)^d$ such that the following convergences hold

$$\sqrt{nc_n}\left(\frac{j\,n_j(c_n) - m_D}{nc_n}\right)_{j=1}^{k} \to_D \left(j\cdot\mathbb{W}(\mathbf{1}/j)\right)_{j=1}^{k} \tag{25}$$

$$T_n(c_n) \to_D \frac{\sum_{j=1}^{k}\left(j\cdot\mathbb{W}(\mathbf{1}/j) - \frac{1}{k}\sum_{\ell=1}^{k}\ell\cdot\mathbb{W}(\mathbf{1}/\ell)\right)^2}{m_D} \tag{26}$$

$$T_n'(c_n) := nc_n\sum_{j=1}^{k}\left(\frac{jn_j(c_n)}{nc_n} - m_D\right)^2 \to_D \sum_{j=1}^{k}\left(j\cdot\mathbb{W}(\mathbf{1}/j)\right)^2 \tag{27}$$

as $n\to\infty$.

Proof Consult Section 5 of Bücher et al. (2014). Our paper's nc_n is their paper's k, our D-norm $\|\cdot\|_D$ is their stable tail dependence function L and our δ is their α. Thus, the $n_j(c_n)/(nc_n)$ above is their estimator $\hat{L}(\mathbf{1}/j)$. By homogeneity of L, we have $\frac{jn_j(c_n)}{nc_n} - m_d = j\cdot(\hat{L}-L)(\mathbf{1}/j)$. Because their paper's stochastic process convergence implies convergence in distribution on the compact set $\{\mathbf{1}/j,\ j=1,\dots,k\}$ equation (25) holds. For a precise definition of the limiting process $\mathbb{W}$, we refer to their paper. Equation (27) follows from the continuous mapping theorem. Further T_n can be rewritten as

$$T_n(c_n) = \frac{nc_n\sum_{j=1}^{k}\left(\left(\frac{jn_j(cn)}{nc_n} - m_D\right) - \frac{1}{k}\sum_{\ell=1}^{k}\left(\frac{\ell n_\ell(c_n)}{nc_n} - m_D\right)\right)^2}{\frac{1}{k}\sum_{\ell=1}^{k}\frac{\ell n_\ell(c_n)}{nc_n}}. \tag{28}$$

The denominator converges almost surely to m_D, and therefore, the limit in Eq. (26) holds as well. □

The effect of the D-norm on the right-hand side of Eq. (26) will not cancel. Therefore, the empirical copula approach of testing for a δ-neighborhood does not work without knowing the underlying D-norm in advance. But if you know the D-norm, you can instead evaluate the easier test statistic given by (27). The stochastic process $\mathbb{W}$ can be simulated. You therefore can get the critical values of both test statistics with a Monte Carlo procedure.

The parametric tail behavior way is another possibility to avoid splitting the data.

In this scenario, we have observations of an arbitrary random vector and assume to know nothing about the marginal distributions but their tails, which could, e.g., be given by generalized Pareto distributions.

Theorem 4 *Suppose $(X_1,\dots,X_d)^\mathsf{T}$ is a random vector with continuous univariate margins $F_j(y) = P(X_j \le y)$ and a copula C with realizations $(U_1,\dots,U_d)^\mathsf{T} = (F_1(X_1),\dots,F_d(X_d))^\mathsf{T}$, which fulfills ($\delta - n$) for some δ. Let G_j be dfs with*

$$G_j(F_j^{-1}(1-u)) = 1 - u + O(u^{1+\delta}) \tag{1D-δ-n}$$

for all j as $u \downarrow 0$. Let $c = c_n$ satisfy $c_n \to 0$, $nc_n \to \infty$ and $nc_n^{1+2\delta} \to 0$.

Then $(\tilde{U}_1, \ldots, \tilde{U}_d)^\top = (G_1(X_1), \ldots, G_d(X_d))^\top$ does not necessarily define a copula, but if we apply the test statistic $T_n(c_n)$ on these observations, we obtain the same limit distribution as in Theorem 1.

Note that this theorem is stated with deterministic G_j, $j = 1, \ldots, d$, which fulfill (1D-δ-n). In practice, only parametric estimates are available. In the light of Theorems 2 and 3, it is plausible that doing the tail estimation and testing on the same data leads to different limit distributions.

Proof It is sufficient to show that $P(T_n(c_n) = \tilde{T}_n(c_n)) \to 1$, where T_n is the test statistic applied on the true copula observations $\boldsymbol{U}^{(i)}$ and $\tilde{T}_n$ is the test statistic applied on $\tilde{\boldsymbol{U}}^{(i)}$.

Define $n_j(c)$ and $\tilde{n}_j(c)$ as in equation (8) with the only difference that $\tilde{n}_j$ uses $\tilde{\boldsymbol{U}}^{(i)}$. Because $\tilde{U}_j = G_j(F_j^{-1}(U_j))$ and U_j is uniformly distributed, we have the series of inequalities

$$\begin{aligned}
&P(n_j(c) \neq \tilde{n}_j(c))/n \\
&\quad \leq P(\exists \ell : U_\ell \leq 1 - \tfrac{c}{j}, \tilde{U}_\ell > 1 - \tfrac{c}{j} \text{ or } U_\ell > 1 - \tfrac{c}{j}, \tilde{U}_\ell \leq 1 - \tfrac{c}{j}) \\
&\quad \leq d \sup_\ell P(U_\ell \in (G_\ell(F_\ell^{-1}(1 - \tfrac{c}{j})), 1 - \tfrac{c}{j}] \text{ or } U_\ell \in (1 - \tfrac{c}{j}, G_\ell(F_\ell^{-1}(1 - \tfrac{c}{j}))]) \\
&\quad \leq d \sup_\ell 2 \cdot \left| G_\ell\left(F_\ell^{-1}\left(1 - \tfrac{c}{j}\right)\right) - \left(1 - \tfrac{c}{j}\right) \right| = n\, O\left(\left(\frac{c}{j}\right)^{1+\delta}\right)
\end{aligned}$$

and thus, we can assume $P(n_j(c) \neq \tilde{n}_j(c)) \leq n\, C(c/j)^{1+\delta}$ for some $C > 0$. Finally

$$P(T_n(c_n) \neq \tilde{T}_n(c_n)) \leq \sum_{j=1}^{k} P(n_j(c_n) \neq \tilde{n}_j(c_n)) \leq (nc_n^{1+\delta})\, C \sum_{j=1}^{k} (1/j)^{1+\delta} \to 0$$

as $n \to \infty$, and therefore, $\tilde{T}_n$ has the same limit distribution as T_n has in Theorem 1. □

Both the empirical copula way and the parametric tail behavior way require additional knowledge about the underlying distribution, but avoid splitting the data. This can also be seen in Fig. 1, which features a flowchart on how to test some rv for a δ-neighborhood with the methods presented in this paper.

4 Testing for δ-neighborhoods of a GPCP

In this section, we carry the results of Sect. 3 over to function space, namely the space $C[0, 1]$ of continuous functions on $[0, 1]$. A stochastic process $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$ with

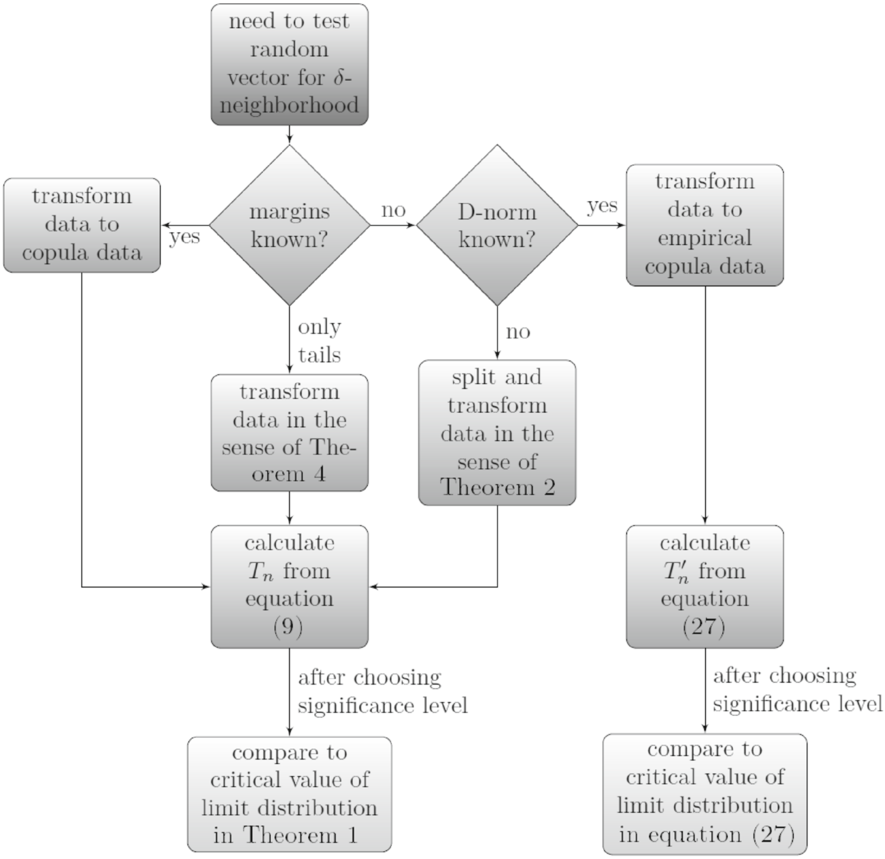

Fig. 1 Flowchart

sample paths in $C[0, 1]$ is called a *standard max-stable process* (SMSP), if $P(\eta_t \le x) = \exp(x)$, $x \le 0$, for each $t \in [0, 1]$, and if the distribution of $n \max_{1\le i\le n} \boldsymbol{\eta}^{(i)}$ equals that of $\boldsymbol{\eta}$ for each $n \in \mathbb{N}$, where $\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \ldots$ are independent copies of $\boldsymbol{\eta}$. All operations on functions such as max, + and / are meant pointwise. To improve the readability, we set stochastic processes such as $\boldsymbol{\eta}$ or $\boldsymbol{Z}$ in bold font and deterministic functions like f in default font.

From Giné et al. (1990)—see also Aulbach et al. (2013) as well as Hofmann (2013)—we know that a stochastic process $\boldsymbol{\eta} \in C[0, 1]$ is an SMSP iff there exists a *generator process* $\boldsymbol{Z} = (Z_t)_{t\in[0,1]} \in C[0, 1]$ with $0 \le Z_t \le q$, $t \in [0, 1]$, for some $q \ge 1$, and $E(Z_t) = 1$, such that

$$P(\boldsymbol{\eta} \le f) = \exp\left(-E\left(\sup_{t\in[0,1]} (|f(t)|Z_t)\right)\right), \quad f \in E^-[0, 1].$$

By $E[0, 1]$, we denote the set of those functions $f : [0, 1] \to \mathbb{R}$, which are bounded and have only a finite number of discontinuities; $E^-[0, 1]$ is the subset of those functions in $E[0,1]$ that attain only nonpositive values. Note that

$$\|f\|_D := E\left(\sup_{t\in[0,1]} (|f(t)|Z_t)\right), \quad f \in E^-[0,1],$$

defines a norm on $E[0,1]$.

Let $\boldsymbol{U} = (U_t)_{t\in[0,1]}$ be a *copula process*, i.e., each component U_t is uniformly distributed on $(0,1)$. A copula process $\boldsymbol{U} \in C[0,1]$ is said to be in the *functional max-domain attraction* of an SMSP $\boldsymbol{\eta}$, denoted by $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, if

$$P\left(n(\boldsymbol{U} - 1_{[0,1]}) \le f\right)^n \to_{n\to\infty} P(\boldsymbol{\eta} \le f) = \exp\left(-\|f\|_D\right), \quad f \in E[0,1], \quad (29)$$

where $1_{[0,1]}$ denotes the indicator function of the interval $[0,1]$. This is the functional version of (1). We refer to Aulbach et al. (2013) for details. A more restrictive definition of functional max-domain of attraction of stochastic processes in terms of usual weak convergence was introduced by de Haan and Lin (2001).

From Aulbach et al. (2013, Proposition 8), we know that condition (29) is equivalent with the expansion

$$P\left(\boldsymbol{U} \le 1_{[0,1]} + cf\right) = 1 - c\|f\|_D + o(c), \quad f \in E^-[0,1],$$

as $c \downarrow 0$. In particular, we obtain in this case

$$P\left(\boldsymbol{U} \le (1-c)1_{[0,1]}\right) = 1 - c\left(m_D + r(-c)\right), \quad (30)$$

where $m_D := E(\sup_{t\in[0,1]} Z_t) = \|1_{[0,1]}\|_D$ is the uniquely determined *generator constant* pertaining to $\boldsymbol{\eta}$, and the remainder term satisfies $r(-c) \to 0$ as $c \downarrow 0$.

A copula process $\boldsymbol{V} \in C[0,1]$ is a *generalized Pareto copula process* (GPCP), if there exists $\varepsilon_0 > 0$ such that

$$P\left(\boldsymbol{V} \le 1_{[0,1]} + f\right) = 1 - \|f\|_D, \quad f \in E^-[0,1],\ \|f\|_\infty \le \varepsilon_0.$$

A GPCP with a prescribed D-norm $\|\cdot\|_D$ can easily be constructed; cf. Aulbach et al. (2013, Example 5). Its characteristic property is its excursion stability; cf. de Haan and Ferreira (2006). A copula process $\boldsymbol{U} \in C[0,1]$, consequently, satisfies $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ iff there exists a GPCP $\boldsymbol{V}$ such that

$$P\left(\boldsymbol{U} \le 1_{[0,1]} + cf\right) = P\left(\boldsymbol{V} \le 1_{[0,1]} + cf\right) + o(c), \quad f \in E^-[0,1], \quad (31)$$

as $c \downarrow 0$. If the remainder term $o(c)$ in expansion (31) is in fact of order $O(c^{1+\delta})$ for some $\delta > 0$, then the copula process $\boldsymbol{U} \in C[0,1]$ is said to be in the δ*-neighborhood* of a GPCP; cf. ($\delta - n$).

4.1 Observing copula processes

The test statistic $T_n(c_n)$ investigated in Sect. 3.1 carries over to function space $C[0,1]$, which enables us to check whether a given copula process $\boldsymbol{U} = (U_t)_{t\in[0,1]}$ is in a δ-neighborhood of an SMSP $\boldsymbol{V}$. Put for $s < 0$

$$S_{\boldsymbol{U}}(s) := \int_0^1 1_{(s,\infty)}(U_t - 1)\, \mathrm{d}t \in [0,1],$$

which is the sojourn time that the process $\boldsymbol{U}$ spends above the threshold $1+s$. If $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, then we obtain from equation (30)

$$\begin{aligned} P\left(S_{\boldsymbol{U}}(s) > 0\right) &= 1 - P\left(S_{\boldsymbol{U}}(s) = 0\right) \\ &= 1 - P\left(\boldsymbol{U} \le (1+s)1_{[0,1]}\right) \\ &= |s|\left(m_D + r(s)\right). \end{aligned}$$

Choose again $k \in \mathbb{N}$, $k \ge 2$, and put for $j = 1,\dots,k$ and $c > 0$

$$n_j(c) := \sum_{i=1}^n 1_{(0,1]}\left(S_{\boldsymbol{U}^{(i)}}\left(-\frac{c}{j}\right)\right)$$

where $\boldsymbol{U}^{(1)},\dots,\boldsymbol{U}^{(n)}$ are independent copies of $\boldsymbol{U}$. Then $n_j(c)$ is the number of those processes among $\boldsymbol{U}^{(1)},\dots,\boldsymbol{U}^{(n)}$, which exceed the threshold $1-\frac{c}{j}$ in at least one point.

If $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, then each $n_j(c)$ is binomial $B(n, p_j(c))$ distributed with

$$p_j(c) = P\left(S_{\boldsymbol{U}}\left(-\frac{c}{j}\right) > 0\right) = \frac{c}{j}\left(m_D + r\left(-\frac{c}{j}\right)\right).$$

Put again

$$T_n(c) := \frac{\sum_{j=1}^k \left(j\, n_j(c) - \frac{1}{k}\sum_{\ell=1}^k \ell\, n_\ell(c)\right)^2}{\frac{1}{k}\sum_{\ell=1}^k \ell\, n_\ell(c)}. \tag{32}$$

Repeating the arguments in the proof of Theorem 1, one shows that its assertion carries over to the functional space as well.

Theorem 5 *Suppose that the copula process $\boldsymbol{U} \in C[0,1]$ is in the δ-neighborhood of a GPCP for some $\delta > 0$. In this case, the remainder term $r(s)$ in expansion (30) is of order $O\left(|s|^\delta\right)$ as $s \to 0$. Let $c = c_n$ satisfy $c_n \to 0$, $nc_n \to \infty$ and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$. Then we obtain*

$$T_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i \xi_i^2,$$

with ξ_i and λ_i as in Theorem 1.

4.2 The case of more general processes

In what follows we will extend Theorem 5 to the case when observing the underlying copula process is subject to a certain kind of nuisance. Let $\boldsymbol{X} = (X_t)_{t\in[0,1]} \in C[0,1]$ be a stochastic process with identical continuous univariate marginal df, i.e., $F(x) := P(X_0 \le x) = P(X_t \le x)$, $t \in [0,1]$, is a continuous function in $x \in \mathbb{R}$. $\boldsymbol{X}$ is said to be in the functional max-domain of attraction of a max-stable process $\boldsymbol{\xi} = (\xi_t)_{t\in[0,1]}$, if the copula process $\boldsymbol{U} = (F(X_t))_{t\in[0,1]}$ satisfies $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, where $\boldsymbol{\eta}$ is a SMSP, and the df F satisfies the univariate extreme value condition; for the univariate case, we refer to Falk et al. (2011, Section 2.1), among others.

Let $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$ be independent copies of the process $\boldsymbol{X}$ and denote the sample df pertaining to the univariate iid observations $X_0^{(1)}, \dots, X_0^{(n)}$ by $\hat{F}_n(x) := n^{-1}\sum_{i=1}^n 1_{(-\infty,x]}\big(X_0^{(i)}\big)$, $x \in \mathbb{R}$. As in Sect. 3.2, fix $k \in \{2,3,\dots\}$, choose some integer $m_n \le n$ and put for $c > 0$ and $j = 1,\dots,k$

$$
\begin{aligned}
n_{j,m_n}(c) &:= \sum_{i=1}^{m_n} 1_{(0,1]}\left(\int_0^1 1_{(\gamma(c),\infty)}\big(X_t^{(i)}\big)\,\mathrm{d}t\right)\\
&= \sum_{i=1}^{m_n} 1_{[0,1)}\left(\int_0^1 1_{(-\infty,\gamma(c)]}\big(X_t^{(i)}\big)\,\mathrm{d}t\right)\\
&= \sum_{i=1}^{m_n} 1_{[0,1)}\left(\int_0^1 1_{\left[0,1-\frac{c}{j}\right]}\big(U_t^{(i)}\big)\,\mathrm{d}t\right)\\
&= \sum_{i=1}^{m_n} 1_{\{\boldsymbol{U}^{(i)} \not\le (1-\frac{c}{j})1_{[0,1]}\}}
\end{aligned}
\tag{33}
$$

where $\gamma(c) := F^{-1}\big(1-\frac{c}{j}\big)$ and the next to last equation holds almost surely. Again, we replace the marginal df F with its empirical counterpart and obtain analogously with $\hat{\gamma}_n(c) := \hat{F}_n^{-1}\big(1-\frac{c}{j}\big)$

$$
\hat{n}_{j,m_n}(c) := \sum_{i=1}^{m_n} 1_{(0,1]}\left(\int_0^1 1_{(\hat{\gamma}_n(c),\infty)}\big(X_t^{(i)}\big)\,\mathrm{d}t\right)
$$

Thus, the rv $\hat{n}_{j,m_n}(c)$ is the total number of processes $\boldsymbol{X}^{(i)} = \big(X_t^{(i)}\big)_{t\in[0,1]}$ among $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(i)}$, which exceed the random threshold $\hat{F}_n^{-1}\big(1-\frac{c}{j}\big)$ for some $t \in [0,1]$. Note that the distribution of the rv $\big(\hat{n}_1(c), \dots, \hat{n}_k(c)\big)^{\mathsf{T}}$ does not depend on F but on the copula process $\boldsymbol{U}$ since

$$
\hat{n}_{j,m_n}(c) = \sum_{i=1}^{m_n} 1_{[0,1)}\left(\int_0^1 1_{(-\infty,\hat{\gamma}_n(c)]}\big(X_t^{(i)}\big)\,\mathrm{d}t\right)
$$

$$= \sum_{i=1}^{m_n} 1_{[0,1)}\left(\int_0^1 1_{[0,U_{\langle n(1-\frac{c}{j})\rangle:n}]}\big(U_t^{(i)}\big)\,\mathrm{d}t\right)$$
$$= \sum_{i=1}^{m_n} 1_{\left\{U^{(i)} \nleq U_{\langle n(1-\frac{c}{j})\rangle:n} 1_{[0,1]}\right\}} \tag{34}$$

with probability one, where $U_{1:n} \le \cdots \le U_{n:n}$ denote the ordered values of $U_0^{(1)}, \dots, U_0^{(n)}$. The following auxiliary result is the extension of Lemma 1 to function space. Its proof is similar to the one of Lemma 1 and is given in online supplementary material.

Lemma 2 *Suppose that the copula process* $\boldsymbol{U} \in C[0,1]$ *corresponding to* $\boldsymbol{X}$ *is in the* δ*-neighborhood of a GPCP for some* $\delta > 0$*. In this case, the remainder term* $r(s)$ *in expansion (30) is of order* $O(|s|^\delta)$ *as* $s \uparrow 0$*. Choose* $m_n \in \mathbb{N}$ *and* $c_n > 0$ *such that* $m_n \to \infty$*,* $m_n/n \to 0$*,* $c_n \to 0$*,* $m_n c_n \to \infty$*, and* $nc_n^{1+2\delta} \to 0$ *as* $n \to \infty$*. Then we obtain for* $j = 1, \dots, k$

$$(m_n c_n)^{-1/2}\left(n_{j,m_n}(c_n) - \hat{n}_{j,m_n}(c_n)\right) \to_{n\to\infty} 0 \quad \textit{in probability}.$$

Analogously to Sect. 3, we now choose $m_n \le n$ and replace $n_j(c)$ in (32) with $\hat{n}_{j,m_n}(c)$ and obtain

$$\hat{T}_n(c) := \frac{\sum_{j=1}^k \left(j\,\hat{n}_{j,m_n}(c) - \frac{1}{k}\sum_{\ell=1}^k \ell\,\hat{n}_{\ell,m_n}(c)\right)^2}{\frac{1}{k}\sum_{\ell=1}^k \ell\,\hat{n}_{\ell,m_n}(c)}.$$

By this statistic, we can in particular check whether the copula process $\boldsymbol{U} = (F(X_t))_{t\in[0,1]}$ pertaining to $\boldsymbol{X}$ is in a δ-neigborhood of some GPCP $\boldsymbol{V}$. Its distribution does not depend on the marginal df F of $\boldsymbol{X}$ but on the copula process $\boldsymbol{U}$. The next result follows from Lemma 2 and the arguments in the proof of Theorem 1.

Theorem 6 *We have under the conditions of Lemma 2*

$$\hat{T}_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i \xi_i^2,$$

with ξ_i *and* λ_i *as in Theorem 1.*

5 Testing via a grid of points

Observing a complete process on [0, 1] as in the preceding section might be a too restrictive assumption. Instead we will require in what follows that we observe stochastic processes with sample paths in $C[0,1]$ only through an increasing grid of points in [0, 1].

Let $\boldsymbol{V} \in C[0,1]$ be a GPCP with pertaining D-norm

$$\|f\|_D = E\left(\sup_{t\in[0,1]} (|f(t)|Z_t)\right), \quad f \in E[0,1].$$

Choose a grid of points $0 = t_1^{(d)} < t_2^{(d)} < \cdots < t_d^{(d)} = 1$. Then the rv

$$\boldsymbol{V}_d := \left(V_{t_1^{(d)}}, \ldots, V_{t_d^{(d)}}\right)^\intercal$$

follows a GPC, whose corresponding D-norm is given by

$$\|\boldsymbol{x}\|_{D,d} := E\left(\max_{1\le i\le d}\left(|x_i|Z_{t_i^{(d)}}\right)\right), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Let now $d = d_n$ depend on n. If we require that

$$\max_{1\le i\le d_n-1} |t_{i+1}^{(d_n)} - t_i^{(d_n)}| \to_{n\to\infty} 0,$$

then, by the continuity of $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$,

$$\max_{1\le i\le d_n-1} |Z_{t_{i+1}^{(d_n)}} - Z_{t_i^{(d_n)}}| \to_{n\to\infty} 0, \quad \max_{1\le i\le d_n} Z_{t_i^{(d_n)}} \to_{n\to\infty} \sup_{t\in[0,1]} Z_t \quad \text{a.s.},$$

and, thus, the sequence of generator constants converges:

$$m_{D,d_n} := E\left(\max_{1\le i\le d_n} Z_{t_i^{(d_n)}}\right) \to_{n\to\infty} E\left(\sup_{t\in[0,1]} Z_t\right) = m_D. \tag{35}$$

5.1 Observing copula data

Suppose we are given n independent copies of a copula process $\boldsymbol{U}$. The projection of each process onto the grid $0 = t_1^{(d_n)} < t_2^{(d_n)} < \cdots < t_{d_n}^{(d_n)} = 1$ yields n iid rv in $\mathbb{R}^{d_n}$, which follow a copula. Let $n_j(c)$ as defined in (8) be based on these rv. Note that $n_j(c)$ depends on d_n as well. But in order not to overload our notation, we suppress the dependence on the dimension.

Moreover, we require that

$$P\left(\boldsymbol{U} \le 1_{[0,1]} + cf\right) = 1 - c\|f\|_D + O(c^{1+\delta}) \quad \text{as} \quad c \downarrow 0 \tag{36}$$

holds uniformly for all $f \in E^-[0,1]$ satisfying $\|f\|_\infty \le 1$. Again a suitable version of the central limit theorem implies

$$\frac{1}{(nc_n)^{1/2}}\left(j\, n_j(c_n) - nc_n m_{D,d_n}\right) \to_D N(0, jm_D)$$

and thus,

$$\frac{j\,n_j(c_n)}{nc_n} \to_{n\to\infty} m_D \quad \text{in probability}, \quad 1 \le j \le k,$$

yielding

$$\frac{1}{nc_nk}\sum_{j=1}^{k} j\,n_j(c_n) \to_{n\to\infty} m_D \quad \text{in probability}.$$

Theorem 1 now carries over:

Theorem 7 *Let $\boldsymbol{U}$ be a copula process satisfying (36). Choose a grid of points $0 = t_1^{(d)} < t_2^{(d)} < \cdots < t_d^{(d)} = 1$ such that $d = d_n \to \infty$ as well as $\max_{1\le i\le d_n-1}|t_{i+1}^{(d_n)} - t_i^{(d_n)}| \to 0$ as $n \to \infty$. Let T_n as defined in (9) be based on the projections of n independent copies of $\boldsymbol{U}$ onto this increasing grid of points. If $c = c_n$ satisfies $c_n \to 0$, $nc_n \to \infty$ and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$, then we obtain*

$$T_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i \xi_i^2,$$

with ξ_i and λ_i as in Theorem 1.

5.2 The case of an arbitrary process

Now we will extend Theorem 7 to a general process $\boldsymbol{X} = (X_t)_{t\in[0,1]} \in C[0,1]$ with continuous marginal df F_t, $t \in [0,1]$. We want to test whether the copula process $\boldsymbol{U} := (F_t(X_t))_{t\in[0,1]} \in C[0,1]$ corresponding to $\boldsymbol{X}$ satisfies (36). As before this will be done by projecting the process $\boldsymbol{X}$ onto a grid of points $0 = t_1^{(d)} < \cdots < t_d^{(d)} = 1$ with $d = d_n \to_{n\to\infty} \infty$ and $\max_{1\le i\le d_n-1}\left|t_{i+1}^{(d_n)} - t_i^{(d_n)}\right| \to_{n\to\infty} 0$.

Let $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$ be independent copies of $\boldsymbol{X}$ and consider the n iid rv of projections $\boldsymbol{X}_{d_n}^{(i)} := \left(X_{t_1^{(d_n)}}^{(i)}, \dots, X_{t_{d_n}^{(d_n)}}^{(i)}\right)^{\mathsf{T}}$, $i = 1, \dots, n$. Fix $k \in \{2, 3, \dots\}$, choose an integer $m_n \le n$, and put for $0 < c < 1$

$$\begin{aligned} n_{j,m_n}(c) &:= \sum_{i=1}^{m_n} 1_{(0,\infty)}\left(\sum_{r=1}^{d_n} 1_{(\gamma_{j,r}(c),\infty)}\left(X_{t_r^{(d_n)}}^{(i)}\right)\right) \\ &= m_n - \sum_{i=1}^{m_n} 1_{(-\infty,\boldsymbol{\gamma}_j(c)]}\left(\boldsymbol{X}_{d_n}^{(i)}\right) \end{aligned}$$

which is the number of all rv $\boldsymbol{X}_{d_n}^{(i)}$, $i = 1, \dots, m_n$, exceeding the vector

$$\boldsymbol{\gamma}_j(c) := \left(\gamma_{j,1}(c), \dots, \gamma_{j,d_n}(c)\right)^{\mathsf{T}} := \left(F_{t_1^{(d_n)}}^{-1}\left(1 - \frac{c}{j}\right), \dots, F_{t_{d_n}^{(d_n)}}^{-1}\left(1 - \frac{c}{j}\right)\right)^{\mathsf{T}}$$

in at least one component. Clearly we have

$$n_{j,m_n}(c) = m_n - \sum_{i=1}^{m_n} 1_{\left[\mathbf{0},\left(1-\frac{c}{j}\right)\mathbf{1}\right]}\left(\boldsymbol{U}_{d_n}^{(i)}\right) \tag{37}$$

almost surely where $\boldsymbol{U}_{d_n}^{(i)} = \left(U_{t_1^{(d_n)}}^{(i)}, \dots, U_{t_{d_n}^{(d_n)}}^{(i)}\right)^{\mathsf{T}}$ is the copula process of $\boldsymbol{X}_{d_n}^{(i)}$.

Again we replace $\boldsymbol{\gamma}_j(c)$ with

$$\hat{\boldsymbol{\gamma}}_j(c) := \left(\hat{\gamma}_{j,1}(c), \dots, \hat{\gamma}_{j,d_n}(c)\right)^{\mathsf{T}}$$

where $\hat{\gamma}_{j,r}(c) := \hat{F}_{t_r^{(d_n)}}^{-1}\left(1-\frac{c}{j}\right)$ and $\hat{F}_{t_r^{(d_n)}}(x) := n^{-1}\sum_{i=1}^{n} 1_{(-\infty,x]}\left(X_{t_r^{(d_n)}}^{(i)}\right)$, $1 \le r \le d_n$, yielding an estimator of $n_j(c)$:

$$\begin{aligned}\hat{n}_{j,m_n}(c) &= \sum_{i=1}^{m_n} 1_{(0,\infty)}\left(\sum_{r=1}^{d} 1_{(\hat{\gamma}_{j,r}(c),\infty)}\left(X_{t_r^{(d_n)}}^{(i)}\right)\right)\\ &= m_n - \sum_{i=1}^{m_n} 1_{(-\infty,\hat{\boldsymbol{\gamma}}_j(c)]}\left(\boldsymbol{X}_{d_n}^{(i)}\right).\end{aligned}$$

We have

$$\hat{\gamma}_{j,r}(c) = X_{\langle n(1-\frac{c}{j})\rangle:n,r},$$

where $X_{1:n,r} \le X_{2:n,r} \le \dots \le X_{n:n,r}$ are the ordered values of $X_{t_r^{(d_n)}}^{(1)}, \dots, X_{t_r^{(d_n)}}^{(n)}$ for each $r = 1, \dots, d_n$, and $\langle x \rangle = \min\{k \in \mathbb{N} : k \ge x\}$ is the right integer neighbor of $x > 0$. Since transforming each $X_{t_r^{(d_n)}}^{(i)}$ by its df $F_{t_r^{(d_n)}}$ does not alter the value of $\hat{n}_{j,m_n}(c)$ with probability one, we obtain

$$\hat{n}_{j,m_n}(c) = m_n - \sum_{i=1}^{m_n} 1_{\times_{r=1}^{d_n}[0,U_{\langle n(1-\frac{c}{j})\rangle:n,r}]}\left(\boldsymbol{U}_{d_n}^{(i)}\right) \tag{38}$$

almost surely where $U_{1:n,r} \le U_{2:n,r} \le \dots \le U_{n:n,r}$ denote the order statistics of $U_{t_r^{(d_n)}}^{(1)}, \dots, U_{t_r^{(d_n)}}^{(n)}$. Since $\boldsymbol{U}_{d_n}^{(1)}, \dots, \boldsymbol{U}_{d_n}^{(n)}$ are independent copies of the rv

$$\boldsymbol{U}_{d_n} := \left(F_{t_1^{(d_n)}}\left(X_{t_1^{(d_n)}}\right), \dots, F_{t_{d_n}^{(d_n)}}\left(X_{t_{d_n}^{(d_n)}}\right)\right)^{\mathsf{T}},$$

the distribution of $\left(\hat{n}_{1,m_n}(c), \dots, \hat{n}_{k,m_n}(c)\right)^{\mathsf{T}}$ does not depend on the marginal df $F_{t_r^{(d)}}$. The following auxiliary result is crucial and its proof, which is similar to the one of Lemma 1, is given in online supplementary material.

Lemma 3 *Suppose that $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$. Let $c_n > 0$ satisfy $c_n \to 0$, $m_n c_n \to \infty$, and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$. Assume that*

$$d_n \to_{n\to\infty} \infty \quad \text{and} \quad \limsup_{n\to\infty} \left(\max \left\{ \frac{m_n^{1/2}}{n^{1/2}}, \frac{1}{(m_n c_n)^{1/2}} \right\} \log(d_n) \right) \le \frac{1}{4}. \tag{39}$$

If $\max_{1\le i\le d_n-1} |t_{i+1}^{(d_n)} - t_i^{(d_n)}| \to 0$ as $n \to \infty$ for a grid of points $0 = t_1^{(d_n)} < t_2^{(d_n)} < \cdots < t_{d_n}^{(d_n)} = 1$, then we obtain for $j = 1, \dots, k$

$$(m_n c_n)^{-1/2} \left(n_{j,m_n}(c_n) - \hat{n}_{j,m_n}(c_n)\right) \to_{n\to\infty} 0 \quad \textit{in probability}.$$

Now we consider the modified test statistic

$$\hat{T}_n(c) := \frac{\sum_{j=1}^k \left(j\,\hat{n}_{j,m_n}(c) - \frac{1}{k}\sum_{\ell=1}^k \ell\,\hat{n}_{\ell,m_n}(c)\right)^2}{\frac{1}{k}\sum_{\ell=1}^k \ell\,\hat{n}_{\ell,m_n}(c)}$$

which does not depend on the marginal df F_t, $t \in [0, 1]$, of the process $\boldsymbol{X}$. The following result is a consequence of the arguments in the proof of Theorem 1 and Lemma 3.

Theorem 8 *Suppose that the process $\boldsymbol{X} = (X_t)_{t\in[0,1]} \in C[0, 1]$ has continuous marginal df F_t, $t \in [0, 1]$ and that the pertaining copula process $\boldsymbol{U} = (F_t(X_t))_{t\in[0,1]}$ satisfies (36). Let $m_n \in \mathbb{N}$ and $c_n > 0$ be sequences such that $m_n \to \infty$, $m_n/n \to 0$, $c_n \to 0$, $m_n c_n \to \infty$, and $nc_n^{1+2\delta} \to 0$ as $n \to \infty$. Choose a grid of points $0 = t_1^{(d_n)} < t_2^{(d_n)} < \cdots < t_{d_n}^{(d_n)} = 1$ satisfying (39) and $\max_{1\le i\le d_n-1} |t_{i+1}^{(d_n)} - t_i^{(d_n)}| \to 0$ as $n \to \infty$. Then we obtain*

$$\hat{T}_n(c_n) \to_D \sum_{i=1}^{k-1} \lambda_i \xi_i^2$$

with ξ_i and λ_i as in Theorem 1.

Acknowledgements The authors are grateful to Kilani Ghoudi for his hint to compute the asymptotic distribution of the above test statistics using Imhof's (1961) method. The authors thank an anonymous associate editor and anonymous referee for their constructive notes and comments, from which this final version of the paper has benefited a lot.

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