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Reliability analysis of a k-out-of-n:F system under a linear degradation model with calibrations

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Abstract A k-out-of-n:F system with both of soft and hard failures is considered such that its components degrade through internal and external factors. A linear model is considered for degradation path of each component. Reliability function of the system is derived and the effect of varying the parameters are studied on reliability function for some systems. Moreover, the effect of calibration on reliability and maximum working time of such a system is investigated. The optimal number of calibrations is also determined for some special cases.

Keywords Calibration · Soft failure · Hard failure · Internal degradation · External degradation · Reliability · Sensitivity analysis · Optimization

1 Introduction

For systems with high reliability, it is difficult to assess reliability based only on lifetime data, because failures don't occur during short time at normal conditions. In this case, degradation data contains more information than lifetime data about system reliability, which records the accumulation of damage over time. Many authors have been investigated such data. Lu and Meeker (1993) presented an application of

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degradation modeling to estimate the time to failure distribution for a broad class of degradation models. Zhao and Xie (1994) and Zhao et al. (1995) studied the storage reliability and discussed the failure and deterioration of items in a dormant state. Coit et al. (2005) used degradation modeling to predict reliability and developed a method to correlate field life with observed degradation for electronics modules. Haghighi and Bae (2015) proposed a modeling approach for jointly analyzing linear degradation data and failure times which are simultaneously recorded during the step-stress accelerated degradation testing. Rodrguez-Narciso and Christen (2016) proposed a methodology that sequentially selects the optimal observation times to measure the degradation, using a convenient rule that maximizes the inference precision and minimizes test costs. Xu et al. (2016) described a class of general path models to incorporate dynamic covariates for modeling degradation paths.

So far, few studies have been done about degradation of engineering systems. Song et al. (2012) investigated reliability function of a k-out-of-n system that its components exposure to degradation and shock loads. The main goal of this paper is to investigate the reliability of a k-out-of-n:F system based on degradation data. As known in the literature of reliability, a k-out-of-n:F system consists of n components which fails if and only if at least k of its components fail. Such systems have various applications in engineering discipline. For more details, we refer to Lawless (2003), Asadi and Bayramoglu (2006), Bairamov and Arnold (2008) and Tavangar and Bairamov (2015).

In spite of degradation analysis, the effect of calibration is studied on performance of a k-out-of-n:F system. Moreover, the optimal number of calibrations are determined considering both of reliability and total cost of experiment. Calibration is one of the primary processes used in many practical operating systems to maintain instrument accuracy. Because it can rejuvenate systems completely or partly, a system is offered to calibrate periodically in order to correct its accuracy. There are some differences between calibrations and maintenances in real world. The calibrations can usually correct some biases for the systems, but cannot reduce completely the underlying or essential faults in the systems; in addition can be carried out easily in practice. The maintenance may reduce the underlying faults and usually takes more time for actions. In theory we can treat the calibration as a special case of maintenances. Kong and Cui (2015) studied Bayesian inference of multistage reliability for degradation systems with calibration. Cui et al. (2016) considered two degradation models and developed degradation signals of a product with some Wiener diffusion processes under pre-specified periodical calibrations.

The rest of the paper is organized as follows. Section 2 focuses on model description. In this section, soft and hard failures as well as internal and external degradations are introduced. In Sect. 3, the reliability function of a k-out-of-n:F system is derived. The effect of calibration on degradation and reliability is studied in this section. A sensitivity analysis of reliability function is done in Sect. 4 and the effect of varying the parameters on reliability function is investigated. Maximum working time of a system and optimal number of calibrations are obtained in Sect. 5. Finally, some conclusions are stated in Sect. 6.

2 Model description and preliminaries

A system may consist of multiple degrading components. One can divide these degradations into two categories, internal and external degradations. Internal degradation is actually unexplained degradation, that is relevant to inherent of each component. External degradation is obtained by environmental factors such as stress, humidity and heat. Li et al. (2011) considered component degradations as a linear combination of internal and external normal degradations. Here, let us consider a k-out-of-n:F system with degradation paths $X_1(t), \ldots, X_n(t)$. Also, suppose that components are degraded subject to different internal factors and a common external factor. Let $Y_1(t), \ldots, Y_n(t)$ be independent internal degradation paths and denote degradation path of external factor by $Z(t)$. Since external degradation of each component is dependent to its location in the system, thus we suppose that the ith component exposures the external degradation $Z(t)$ with impact element α_i, for $i = 1, \ldots, n$. Therefore, the following model is considered for degradation path of each component

$$X_i(t) = Y_i(t) + \alpha_i Z(t); \quad i = 1, \ldots, n. \tag{1}$$

In general, there are two classes of degradation models: stochastic models and path models. In practice, the continuous degradation process is often modeled by a stochastic model. Wiener and Gamma processes are two classes of stochastic models that have been widely used in degradation modeling. A distinct feature of Wiener process is that its degradation measures are not necessarily monotone, which is not applicable in many cases. As an alternative, Gamma process is often used when monotonicity is required. For more details, see Noortwijk (2009) and Liu et al. (2013). Hence, to have monotone degradation paths, we assume that $Z(t)$ in model (1) obeys a Gamma process with shape parameter $a(t)$ and scale parameter b, denoted by $Z(t) \sim \Gamma(a(t), b)$. Further, for $i = 1, \ldots, n$, it is assumed that $Y_i(t) \sim \Gamma(a_i(t), b_i)$. Empirical studies show that the shape parameter of Gamma process at time t can often be described by a power law model. Hence, for positive real constants a, a_i and γ, we consider shape parameters as $a(t) = at^{\gamma}$ and $a_i(t) = a_i t^{\gamma} (i = 1, \ldots, n)$.

In degradation analysis, it is necessary to define the level of degradation or performance such that when degradation path exceeds the specified threshold, a failure is said to have occurred. Such a failure is known as soft failure. In addition, a hard failure is said to have occurred, when component stops working. The varying time of soft and hard failures depends on the degradation level of both internal and external factors. Haghighi and Nikulin (2010) described a parametric method to estimate the survival function of general degradation model with one soft failure and several hard failures. In our model, the soft failure occurs when the degradation path $X_i(t)$ reaches or exceeds a specified level of degradation, say d_i. So, the soft failure time of the ith component, denoted by T_i^0, is defined as the time when the degradation path reaches or exceeds the critical threshold d_i, i.e.,

$$T_i^0 = \inf\{t > 0; \quad X_i(t) \geq d_i\}; \quad i = 1, \ldots, n. \tag{2}$$

The survival and distribution functions of $T_i^0 (i = 1, \ldots, n)$ are given by

$$S_{T_i^0}(t) = P(X_i(t) < d_i) = \int_0^{\frac{d_i}{\alpha_i}} F_{Y_i(t)}(d_i - \alpha_i z) f_{Z(t)}(z)\mathrm{d}z \tag{3}$$

and

$$F_{T_i^0}(t) = \int_0^{\frac{d_i}{\alpha_i}} \bar{F}_{Y_i(t)}(d_i - \alpha_i z) f_{Z(t)}(z)\mathrm{d}z + \bar{F}_{Z(t)}\left(\frac{d_i}{\alpha_i}\right), \tag{4}$$

respectively, where $f_{Z(t)}(\cdot)$ and $F_{Z(t)}(\cdot)$ stand for the probability density function (pdf) and cumulative distribution function (cdf) of $Z(t)$, respectively; also, $\bar{F}_{Z(t)}(\cdot) = 1 - F_{Z(t)}(\cdot)$.

As previously mentioned, the other types of failures are due to the hard (traumatic) failures. Indeed, when a component stops working, a hard failure is assumed to have occurred. Denote the hard failure time for the ith component by T_i^1 and show its hazard rate function by $\lambda_i(t)$. Since the hard failure is influenced by both of external and internal degradations, the hazard rate function depends on $Y_i(t)$ and $Z(t)$. So, we assume that given $Z(t) = z$ and $Y_i(t) = y_i$, the conditional hazard rate of T_i^1 has a multiplicative form as in the Cox model. That is,

$$\lambda_i(t \mid Z(t) = z, Y_i(t) = y_i) = \lambda_0(t, \theta)\lambda(\alpha_i z, y_i); \quad i = 1, \ldots, n, \tag{5}$$

where $\lambda_0(\cdot)$ and $\lambda(\cdot)$ are the baseline hazard and intensity functions, respectively. From model (5), it is observed that the conditional hazard rate function depends on both types of external and internal degradations only through the intensity function. Similar to Nikulin and Wu (2016), we assume that the baseline hazard rate function is of the form $\lambda_0(t, \theta) = (1 + t)^\theta$, for $\theta > 0$. Moreover, for intensity function, we consider a multiplicative form as follows

$$\lambda(\alpha_i z, y_i) = (\alpha_i z)^\beta y_i^{\beta_i}; \quad \beta, \beta_i > 0, \quad i = 1, \ldots, n.$$

Since external degradation $Z(t)$ is common for all components, we have shown its effect on intensity function by a common parameter β. But, internal degradations are different for the components, so we use a distinct parameter β_i to show the effect of Y_i on intensity function. Summing up, the conditional hazard rate function of T_i^1, for $i = 1, \ldots, n$, is considered to be

$$\lambda_i(t \mid Z(t) = z, Y_i(t) = y_i) = (1 + t)^\theta (\alpha_i z)^\beta y_i^{\beta_i}; \quad \theta, \beta, \beta_i > 0. \tag{6}$$

Therefore, the conditional survival function of T_i^1 is

$$\begin{aligned} S_{T_i^1|\{z,y_i\}}(t) &= P(T_i^1 > t \mid Z(t) = z, Y_i(t) = y_i) \\ &= \exp\left\{-\int_0^t \lambda_i(s \mid Z(s) = z, Y_i(s) = y_i)\mathrm{d}s\right\} \end{aligned}$$

$$= \exp\left\{-(\alpha_i z)^{\beta} y_i^{\beta_i} \int_0^t (1+s)^{\theta} \mathrm{d}s\right\}$$
$$= \exp\left\{-\frac{(\alpha_i z)^{\beta} y_i^{\beta_i}}{\theta+1}\left((1+t)^{\theta+1}-1\right)\right\}; \quad t > 0. \tag{7}$$

Differentiating (7), the conditional probability density function (pdf) of T_i^1 is obtained as

$$f_{T_i^1|\{z,y_i\}}(t) = (\alpha_i z)^{\beta} y_i^{\beta_i} (1+t)^{\theta} \exp\left\{-\frac{(\alpha_i z)^{\beta} y_i^{\beta_i}}{\theta+1}\left((1+t)^{\theta+1}-1\right)\right\}; t > 0. \tag{8}$$

Now, assume that the ith component fails as soon as either soft failure or hard failure occurs first. That is, the failure time of the ith component is given by $T_i = \min\{T_i^0, T_i^1\}$. In this paper, we assume that all factors which cause the dependency between components, affect only the external degradation $Z(t)$ and internal degradation $Y_i(t)$. Considering the common external degradation $Z(t)$ affects all lifetimes, $T_1, \ldots, T_n$ are dependent. Since, we do not assume any other dependence sources, when $Z(t)$ becomes known, there is no dependence relationship between components. On the other words, given $Z(t) = z$, $T_1, \ldots, T_n$ are conditionally independent. Furthermore, according to the definition, T_i is dependent to both soft failure time T_i^0 and hard failure time T_i^1. On the other hand, T_i^0 and T_i^1 are exposed both external degradation $Z(t)$ and internal degradation $Y_i(t)$. So, T_i^0 and T_i^1 are dependent. Hence, similar to above scenario, given $Y_i(t) = y_i$ and $Z(t) = z$, T_i^0 and T_i^1 are conditionally independent.

Of course, there may be circumstances that more factors affect on dependency between components. In such situations, we need to know the joint distribution of components which may be considered in future researches. In the next section, we first obtain reliability function of a k-out-of-n:F system and then, we study the effect of calibration on the reliability.

3 Reliability and calibrations

Let $T_1, \ldots, T_n$ be the failure times of components of a k-out-of-n:F system, such that $T_i = \min\{T_i^0, T_i^1\}$. Denote the corresponding order statistics by $T_{1:n} < \cdots < T_{n:n}$. Then the failure time of the system is given by $T_{k:n}$. For more details about coherent systems see Lawless (2003). To determine the reliability function of such a system, let us first present the conditional reliability function of the ith component in the following lemma.

Lemma 1 *Suppose that T_i^0 and T_i^1 are the soft and hard failure times of the ith component, respectively. Given $Z(t) = z$, the conditional reliability function of the ith component at time point t is*

$$R_{T_i|z}(t) = \int_0^{d_i-\alpha_i z} \exp\left\{-\frac{(\alpha_i z)^\beta y_i^{\beta_i}}{\theta+1}\left((1+t)^{\theta+1}-1\right) - \frac{y_i}{b_i}\right\} \frac{1}{b_i^{a_i t^\gamma}\Gamma(a_i t^\gamma)} y_i^{a_i t^\gamma - 1} dy_i, \tag{9}$$

when $d_i > \alpha_i z$ *and* $R_{T_i|z}(t) = 0$ *for* $d_i < \alpha_i z$.

Proof Note that given $Z(t) = z$ and $Y_i(t) = y_i$, the random variables T_i^0 and T_i^1 are assumed to be conditionally independent. Now, using the fact that $T_i = \min\{T_i^0, T_i^1\}$, we have

$$\begin{aligned} R_{T_i|z}(t) &= P(T_i > t \mid Z(t) = z) \\ &= \int_0^\infty P(T_i^0 > t \mid Z(t) = z, Y_i(t) = y_i) \\ &\quad \times P(T_i^1 > t \mid Z(t) = z, Y_i(t) = y_i) f_{Y_i(t)}(y_i) dy_i, \end{aligned} \tag{10}$$

where $f_{Y_i(t)}(\cdot)$ stands for the pdf of $Y_i(t)$. From (2), the event $\{T_i^0 > t\}$ is equivalent to $\{X_i(t) < d_i\}$. Hence, using (1), we have

$$P(T_i^0 > t \mid Z(t) = z, Y_i(t) = y_i) = \begin{cases} 1, & y_i < d_i - \alpha_i z \\ 0, & y_i > d_i - \alpha_i z \end{cases}$$

So, using (7), for $d_i > \alpha_i z$ we get

$$R_{T_i|z}(t) = \int_0^{d_i-\alpha_i z} \exp\left\{-\frac{(\alpha_i z)^\beta y_i^{\beta_i}}{\theta+1}\left((1+t)^{\theta+1}-1\right)\right\} f_{Y_i(t)}(y_i) dy_i.$$

Also, for $d_i < \alpha_i z$, $R_{T_i|z}(t) = 0$. Since $Y_i(t)$ has $\Gamma(a_i t^\gamma, b_i)$ distribution, the proof is completed. □

Theorem 1 *Let* $T_i = \min\{T_i^0, T_i^1\}$ *be the failure time of the* i*th component* ($i = 1, \ldots, n$). *Then, the reliability function of a k-out-of-n:F system at time point t, is given by*

$$R_{k:n}(t) = \sum_{i=0}^{k-1} \sum_{S_i} \int_0^{\min\{\frac{d_1}{\alpha_1},\ldots,\frac{d_n}{\alpha_n}\}} \left\{ \prod_{\ell=1}^{i} (1 - R_{T_{j_\ell}|z}(t)) \prod_{\ell=i+1}^{n} R_{T_{j_\ell}|z}(t) \right\} f_{Z(t)}(z)\, dz, \tag{11}$$

where $R_{T_{j_\ell}|z}(t)$ *is as defined in* (9) *and summation index* S_i *extends over all permutations* $(j_1, \ldots, j_n)$ *of integers* $\{1, \ldots, n\}$ *such that* $j_1 < \cdots < j_i$ *and* $j_{i+1} < \cdots < j_n$.

Proof Note that reliability of the mentioned system at time t is equivalent to the probability of failing at most $(k-1)$ components up to time t. As previously mentioned, given the observed value of $Z(t)$, the random variables $T_1, \ldots, T_n$ are conditionally independent but they are not identically distributed. So, from David and Nagaraja (2003) p. 96, we get

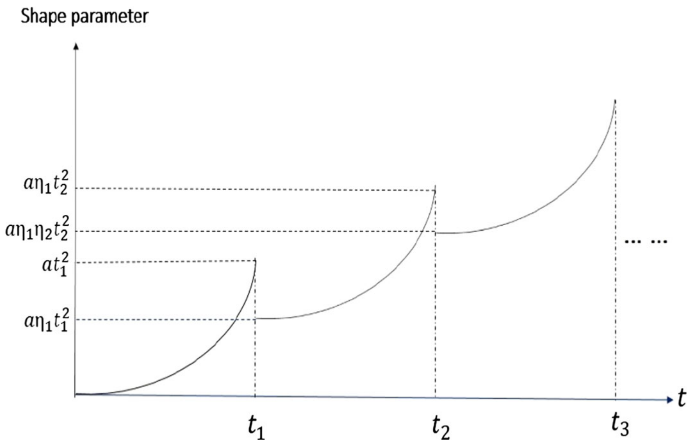

Fig. 1 Effect of calibration on shape parameter of external degradation

$$
\begin{aligned}
R_{k:n}(t) &= P(T_{k:n} > t) \\
&= \int_0^{\infty} P(T_{k:n} > t \mid Z(t) = z) f_{Z(t)}(z)\mathrm{d}z \\
&= \int_0^{\infty} \sum_{i=0}^{k-1} \sum_{S_i} \left\{ \prod_{\ell=1}^{i} (1 - R_{T_{j_\ell}|z}(t)) \prod_{\ell=i+1}^{n} R_{T_{j_\ell}|z}(t) \right\} f_{Z(t)}(z)\mathrm{d}z.
\end{aligned}
$$

From (9), it is easy to see that if $d_j - \alpha_j z < 0$, then $R_{T_j|z}(t) = 0$ for $j = 1, \ldots, n$. But if $d_j - \alpha_j z > 0$ for all $j = 1, \ldots, n$, then $R_{T_j|z}(t) > 0$; in this case $z < \min\{\frac{d_1}{\alpha_1}, \ldots, \frac{d_n}{\alpha_n}\}$. Therefore, the result is deduced. □

Using (11), the reliability of a k-out-of-n:F system may be determined at any given time point. It is obvious that degradation of a system increases over time and causes to decrease the system reliability. On the other hand, calibration is one way to increase reliability. Actually, calibration is an instrument for maintaining accuracy of system. An engineer can measure system characteristics and compare them with calibration standards to improve system with calibration degree η, if it is needed. So, degradation of system decreases and system works better.

In our model, the internal degradations are related to inherent of components and therefore they are not controllable. But the external degradation can be controlled via calibration. It is logical to perform calibration such that the expected value of external degradation decreases after each calibration. This goal may be attained by varying the shape parameter of Gamma process. Suppose that the inspections are done at pre-specified times t_m ($m \geq 1$) and the jth step of calibration is performed at an inspection time by calibration degree $\eta_j \in [0, 1]$, $j \geq 1$, such that when η_j decreases from 1 to 0, the calibration becomes more effective, i.e., degradation of the system is more decreased. In addition, assume that the effect of calibration at time t_m continues until time point t_{m+1}. Figure 1 shows the effect of calibration on shape parameter for times t_1 and t_2. In this figure, we consider the shape parameter of external degradation as a quadratic form $a(t) = at^2$. Therefore, the expected value of external degradation is $E(Z(t)) = abt^2$ on $[0, t_1]$. At time t_1, if a calibration is done with calibration degree η_1, then degradation may decreases on $[t_1, t_2]$ such that $E(Z(t)) = ab\eta_1 t^2$. Continuing this process, at the jth calibration we get $E(Z(t)) = ab(\prod_{i=1}^{j} \eta_i)t^2$ on the interval $[t_j, t_{j+1}]$. Therefore, at the jth calibration, the external degradation $Z(t)$ follows a $\Gamma(aL(j)t^2, b)$ process, where $L(j) = \prod_{i=1}^{j} \eta_i$.

Using the above scenario, the reliability function of k-out-of-n:F system after the jth calibration at time t, denoted by $R^j_{k:n}(t)$ may be computed using (11). In the next section, a sensitivity analysis performs to study the effects of varying the external and internal parameters on reliability function.

4 Sensitivity analysis

A sensitivity analysis is done here to illustrate the effect of varying parameters on reliability function of a four components system. For simplicity the interpretation, we consider shape parameters of Gamma processes $Z(t)$ and $Y_i(t)$ as a quadratic form, such that $a(t) = at^2$ and $a_i(t) = a_i t^2 (i = 1, \ldots, 4)$, respectively. In the first step, we investigate the effect of internal degradation parameters (a_i, b_i) of Gamma process, $Y_i(t)$. Toward this end, we fix all other parameters; the external degradation parameters of Gamma process $Z(t)$ are considered to be $(a, b) = (0.08, 1)$; the external parameter of intensity function $\lambda(\cdot, \cdot)$ and parameter of baseline hazard function $\lambda_0(\cdot, \cdot)$ for hard failure are considered to be $\beta = 2$ and $\theta = 1$, respectively. Also, we consider the impact elements $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha^* = 0.5$. Suppose that we would like to perform five steps for calibration and in each step, the external degradation only reduces five percent. That is, for $j = 1, \ldots, 4$, the calibration degree is $\eta_j = 0.95$. Moreover, we assume that the thresholds for degradation of the components are $d_1 = d_2 = d_3 = d_4 = d^* = 3$ and fix $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta^* = 1$. Now, three cases of parameters (a_i, b_i), for $i = 1, \ldots, 4$, are used as presented in Table 1. These parameters are chosen such that the expected internal degradation increases for each component from case I to case II to case III. Further, to study the effect of internal degradation parameter β^* on system reliability, we fix the parameters (a_i, b_i) as presented for case I of Table 1. The reliabilities are computed using (11) based on parameters of Table 1 and the results are shown in Fig. 2 for 3- and 4-out-of-4:F systems.

From Fig. 2, it is observed that

Table 1 Some choices of internal degradation parameters

	(a_1, b_1)	(a_2, b_2)	(a_3, b_3)	(a_4, b_4)
Case I	(0.03, 2)	(0.03, 1)	(0.02, 3)	(0.01, 1)
Case II	(0.05, 2)	(0.05, 1)	(0.04, 3)	(0.03, 1)
Case III	(0.05, 3.5)	(0.05, 2.5)	(0.04, 4.5)	(0.03, 2.5)

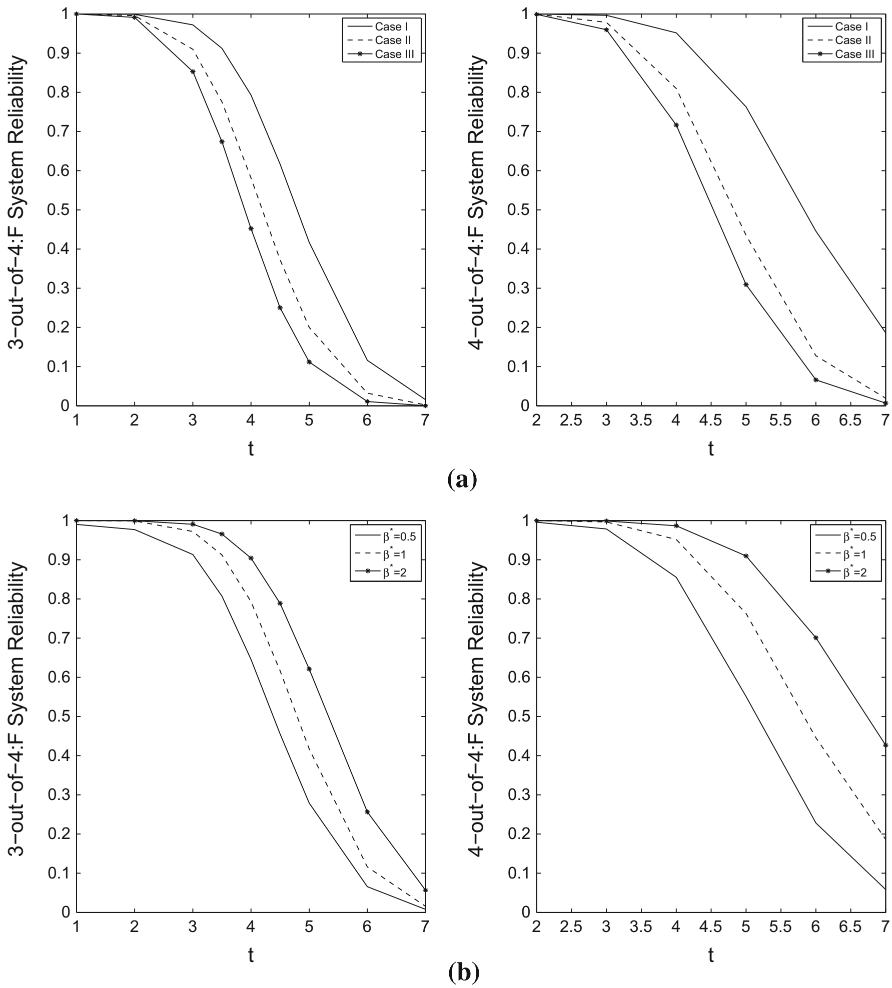

Fig. 2 The behavior of reliability function w.r.t. varying the internal parameters. **a** Reliabilities for some choices of (a_i, b_i), for $i = 1, \ldots, 4$. **b** Reliabilities for some choices of β^*

- The reliability functions of both systems are sensitive to varying the internal parameters (a_i, b_i). In fact, by increasing the expected internal degradations, the reliability functions of both systems decrease, when other parameters are fixed.
- By increasing parameter β^*, the reliabilities of both systems increase.

In the sequel, we investigate the effect of external parameters (a, b), β and θ on reliability function. For this purpose, internal degradation parameters (a_i, b_i) for $i = 1, \ldots, 4$, are considered as presented for case I of Table 1. Also, we put $d^* = 3$, $\alpha^* = 0.5$ and $\beta^* = 1$. Variations of reliabilities are shown in Fig. 3 with respect to varying each external parameter while the others are fixed. It is seen that

- By increasing the expected values of external degradation, the reliabilities of both systems decrease (Fig. 3a).

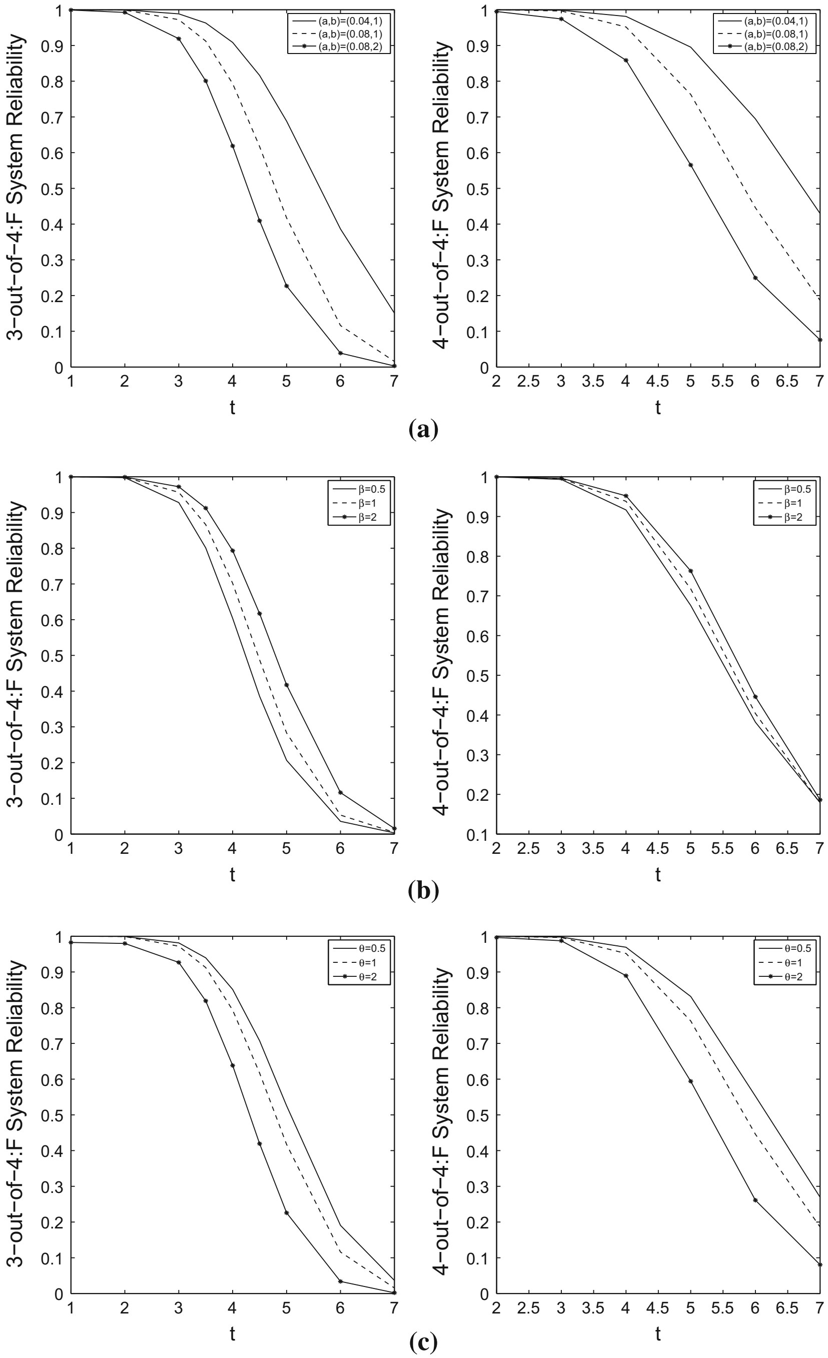

Fig. 3 The behavior of reliability function w.r.t. varying the external parameters. **a** Reliabilities for some choices of (a, b), when $\beta = 2$ and $\theta = 1$. **b** Reliabilities for some choices of β, when $(a, b) = (0.08, 1)$ and $\theta = 1$. **c** Reliabilities for some choices of θ, when $(a, b) = (0.08, 1)$ and $\beta = 2$

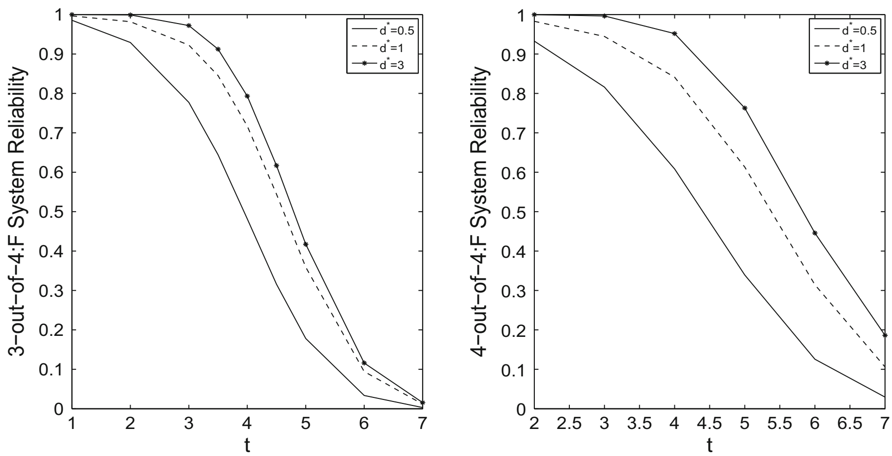

Fig. 4 The behavior of reliability function w.r.t. varying the threshold d^*

- When β increases, the reliabilities of both systems increase (Fig. 3b).
- The reliabilities of both systems are decreasing functions of θ. (Fig. 3c).

Now, to study the behavior of reliability with respect to the threshold d^*, we fix the other parameters such that (a_i, b_i), $1 \leq i \leq 4$, are as shown in case I of Table 1, $\beta^* = 1$, $(a, b) = (0.08, 1)$, $\beta = 2$, $\theta = 1$, $\alpha^* = 0.5$. The plots of reliability functions are shown in Fig. 4 for some choices of threshold d^*. As we expected, it is observed that by increasing the threshold d^*, the values of reliabilities increase.

Finally, we investigate the effect of impact element α^* on reliability function. As before, we fix the other parameters and vary the values of α^*. The behavior of reliabilities with respect to α^* are shown in Fig. 5 for 3- and 4-out-of-4:F systems. From Fig. 5, we conclude that by increasing the impact element α^*, the reliabilities of both systems decrease. Furthermore, from Figs. 2, 3, 4 and 5, it is seen that the reliability of 4-out-of-4:F system is more than the reliability of 3-out-of-4:F system. It is obvious; since, by increasing k, the reliability of a k-out-of-n : F system increases, for given n. In the next section, we investigate the effect of calibration on reliability of a k-out-of-n:F system. Also we find maximum working time of the system and optimal number of calibrations.

5 Optimal number of calibrations

Suppose that a k-out-of-n:F system have to function with reliability at least R^*. Some calibrations may also be needed to attain this purpose. On the other hand, the calibration is costly and so it is preferred to consider the total cost incurred for the system up to time t after the jth calibration, denoted by $c(t; j)$. Note that various sources such as the number of inspections, calibrations, soft and hard failures may affect the total cost. Denote the number of soft and hard failures up to time t after the jth calibration by $N_s(t; j) = \sum_{i=1}^{n} I(X_i(t) \geq d_i)$ and $N_h(t; j) = \sum_{i=1}^{n} I(T_i^1 \leq t)$, respectively, where $I(\cdot)$ stands for indicator function. Inspection times are also considered to be

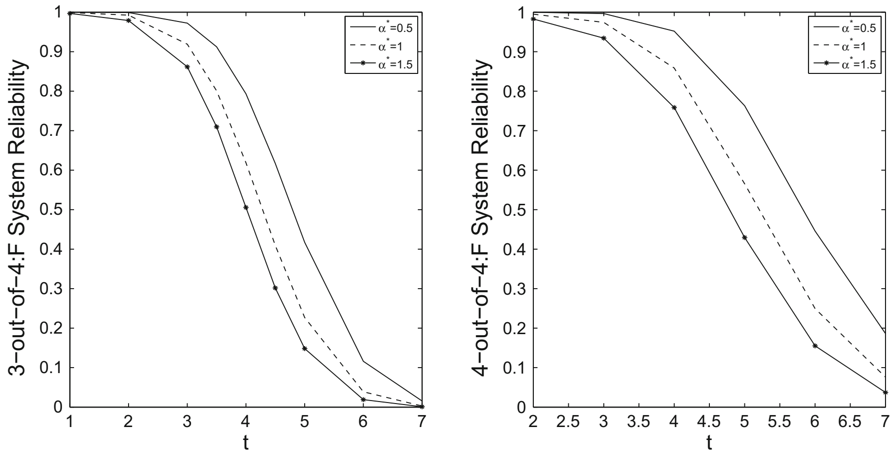

Fig. 5 The behavior of reliability function w.r.t. varying the impact element α^*

$t_m = 0.5m$ for $m \geq 1$. Then the expected total cost at the time of the mth inspection is defined as

$$E[c(t_m; j)] = m \times c_i + j \times c_c + E[N_s(t_m; j)] \times c_s + E[N_h(t_m; j)] \times c_h, \quad (12)$$

where m is the number of inspections up to time t_m and c_i, c_c, c_s and c_h are the costs of each inspection, calibration, soft failure and hard failure, respectively. Moreover, using (3) and (7), the expected number of soft and hard failures are derived as

$$E[N_s(t_m; j)] = \sum_{i=1}^{n} P(X_i(t_m) \geq d_i) = n - \sum_{i=1}^{n} S_{T_i^0}(t_m)$$

and

$$E[N_h(t_m; j)] = \sum_{i=1}^{n} P(T_i^1 \leq t)$$
$$= n - \sum_{i=1}^{n} \int_0^\infty \int_0^\infty S_{T_i^1|\{z,y_i\}}(t_m) f_{Z(t_m)}(z) f_{Y_i(t_m)}(y_i) dy_i \mathrm{d}z,$$

respectively.

Now, let us denote the maximum functioning time of the system and optimal number of calibrations by t^* and j^*, respectively. According to the above scenario, the following circumstances must be satisfied simultaneously to obtain these values:

1. The reliability of a k-out-of-n:F system after j calibrations is at least R^*, i.e., $R^j_{k:n}(t_m) > R^*$.
2. The expected total cost does not exceed threshold c^*, i.e., $E[c(t_m; j)] < c^*$.

Table 2 Values of $R^j_{3:4}(t_m)$ for some choices of t_m and $j = 0, \ldots, 10$

j	t_m								
	1	2	3	3.5	4	4.5	5	6	7
0	0.9996	0.9973	0.9729	0.9211	0.8137	0.6460	0.4457	0.1307	0.0194
1	0.9997	0.9975	0.9749	0.9265	0.8248	0.6634	0.4664	0.1440	0.0227
2	0.9997	0.9977	0.9767	0.9315	0.8354	0.6802	0.4869	0.1581	0.0265
3	0.9997	0.9979	0.9784	0.9361	0.8453	0.6964	0.5072	0.1730	0.0309
4	0.9997	0.9980	0.9799	0.9404	0.8546	0.7121	0.5272	0.1886	0.0358
5	0.9997	0.9982	0.9814	0.9444	0.8634	0.7270	0.5470	0.2050	0.0413
6	0.9997	0.9983	0.9827	0.9481	0.8717	0.7414	0.5663	0.2221	0.0476
7	0.9998	0.9984	0.9839	0.9515	0.8795	0.7551	0.5853	0.2399	0.0546
8	0.9998	0.9985	0.9850	0.9547	0.8868	0.7683	0.6038	0.2584	0.0625
9	0.9998	0.9986	0.9860	0.9577	0.8937	0.7808	0.6218	0.2774	0.0711
10	0.9998	0.9987	0.9869	0.9604	0.9002	0.7928	0.6393	0.2969	0.0808

Table 3 Values of $R^j_{4:4}(t_m)$ for some choices of t_m and $j = 0, \ldots, 10$

j	t_m								
	2.5	4	4.5	5	5.5	6	6.5	7.5	8.5
0	0.9952	0.9522	0.9017	0.8177	0.7034	0.5683	0.4307	0.2015	0.0687
1	0.9956	0.9566	0.9102	0.8316	0.7229	0.5923	0.4568	0.2244	0.0824
2	0.9959	0.9607	0.9179	0.8444	0.7412	0.6150	0.4819	0.2474	0.0970
3	0.9963	0.9642	0.9248	0.8561	0.7582	0.6364	0.5059	0.2703	0.1125
4	0.9966	0.9675	0.9311	0.8669	0.7739	0.6567	0.5289	0.2929	0.1286
5	0.9969	0.9704	0.9368	0.8767	0.7887	0.6757	0.5508	0.3151	0.1452
6	0.9971	0.9729	0.9420	0.8858	0.8024	0.6937	0.5718	0.3368	0.1621
7	0.9974	0.9753	0.9467	0.8942	0.8151	0.7107	0.5918	0.3579	0.1792
8	0.9976	0.9774	0.9510	0.9019	0.8270	0.7267	0.6109	0.3786	0.1963
9	0.9978	0.9793	0.9549	0.9089	0.8381	0.7418	0.6291	0.3986	0.2134
10	0.9979	0.9811	0.9584	0.9155	0.8484	0.7561	0.6466	0.4182	0.2303

In what follows, we illustrate the procedure for 3- and 4-out-of-4:F systems. Toward this end, we fix the parameters such that (a_i, b_i), $1 \leq i \leq 4$, are as shown in case I of Table 1, $\beta^* = 1$, $(a, b) = (0.08, 1)$, $\beta = 2$, $\theta = 1$, $\alpha^* = 0.5$. The thresholds for the degradation of components are also assumed to be $d_1 = 5$, $d_2 = 3$, $d_3 = 4$ and $d_4 = 2$. Suppose that we would like to perform at most 10 calibrations and the calibration degree is $\eta_j = 0.95$ and so $L(j) = (0.95)^j$, for $j = 0, 1, \ldots, 10$. Obviously, $j = 0$ is equivalent to no calibration. Using (11), values of $R^j_{3:4}(t_m)$ and $R^j_{4:4}(t_m)$ are derived for some choices of t_m and $j = 0, \ldots, 10$. The results are reported in Tables 2 and 3, respectively.

Table 4 Values of $E[c(t_m; j)]$ for 3- and 4-out-of-4:F systems

j	t_m								
	3-out-of-4:F system				4-out-of-4:F system				
	3.5	4	4.5	5	4	4.5	5	5.5	6
0	103.47	130.39	163.08	202.82	130.39	163.08	202.82	250.68	306.94
1	121.99	148.04	179.50	217.59	148.04	179.50	217.59	263.37	317.27
2	140.59	165.84	196.18	232.72	165.84	196.18	232.72	276.57	328.21
3	159.29	183.80	213.08	248.22	183.80	213.08	248.22	290.25	339.77
4	178.07	201.89	230.20	264.03	201.89	230.20	264.03	304.37	351.92
5	196.92	220.08	247.51	280.14	220.08	247.51	280.14	318.93	364.59
6	215.85	238.42	265.01	296.50	238.42	265.01	296.50	333.86	377.76
7	234.83	256.84	282.66	313.13	256.83	282.66	313.13	349.13	391.40
8	253.88	275.37	300.48	329.98	275.37	300.48	329.98	364.75	405.47
9	272.98	294.00	318.43	347.06	294.00	318.43	347.06	380.65	419.95
10	292.13	312.71	336.52	364.32	312.71	336.52	364.32	396.86	434.81

Table 5 Values of (t^*, j^*) for some choices of c^* and R^* for 3- and 4-out-of-4:F systems

R^*	c^*					
	3-out-of-4			4-out-of-4		
	250	300	350	260	310	360
0.95	(3.5, 7)	(3.5, 10)	(3.5, 10)	(4, 7)	(4.5, 8)	(4.5, 10)
0.85	(4, 6)	(4, 9)	(4, 10)	(5, 3)	(5, 6)	(5, 9)
0.7	(4.5, 5)	(4.5, 7)	(4.5, 10)	(5.5, 0)	(5.5, 4)	(5.5, 7)

From Tables 2 and 3, it is seen that the reliabilities of both systems increase when the number of calibrations increases, which is a trivial observation. However, it seems that no calibration is needed for initial times the system working. More precisely, a 3-out-of-4:F system works with probability at least 0.95 without any calibrations until time $t_6 = 3$; such time for 3-out-of-4:F system is $t_8 = 4$.

To obtain the optimal values t^* and j^*, we have to determine the values of $E[c(t_m; j)]$. Toward this end, we choose $c_i = 10\$, c_c = 20\$, c_s = 100\$$ and $c_h = 200\$$. The results are presented in Table 4 for some choices of t_m, when $j = 0, \ldots, 10$. Using Tables 2, 3 and 4, the optimal values of t^* and j^* may be obtained for given R^* and c^*. Table 5 shows the results for some choices of R^* and c^* for 3- and 4-out-of-4:F systems.

For interpretation the entries of Table 5, it may be stated for instance, when threshold of expected total cost is 250$, a 3-out-of-4:$F$ system attains the reliability at least 0.95 by choosing $t^* = 3.5$ with $j^* = 7$ calibrations. In the other words, considering the average cost of 250 dollars, a 3-out-of-4:F system can work up to time 3.5 with reliability at least 0.95 by 7 calibrations.

6 Conclusion

In this paper, degradation performance of a k-out-of-n:F system with both soft and hard failures was considered. It was assumed that the components degrade through internal and external degradations, such that external degradation is common among all components but internal degradations are different for the components. A linear model was considered for degradation path of each component. Reliability function of such system was derived and a sensitivity analysis of reliability with respect to varying all internal and external parameters was done for 3- and 4-out-of-4:F systems. It was seen that the reliability functions were sensitive to varying the parameters. Furthermore, the effect of calibration on degradation and reliability of system was investigated. It was shown that the reliability of the system increases by increasing the number of calibrations. Moreover, maximum functioning time of the system and optimal number of calibrations were derived based on maximum reliability and expected total cost criteria for 3- and 4-out-of-4:F systems. The results may be obtained for other values of k and n in a k-out-of-n:F system. Moreover, they may be extended to other coherent systems.

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