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Waiting time for consecutive repetitions of a pattern and related distributions

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Abstract Let k be a positive integer. Some exact distributions of the waiting time random variables for k consecutive repetitions of a pattern are derived in a sequence of independent identically distributed trials. It is proved that the number of equations of conditional probability generating functions for deriving the distribution can be reduced to less than or equal to the length of the basic pattern to be repeated consecutively. By using the result, various properties of the distributions of usual runs are extended to those of consecutive repetitions of a pattern. These results include some properties of the geometric distribution of order k and those of the waiting time distributions of the (k_1, k_2)-events. Further, the probability generating function of the number of non-overlapping occurrences of k consecutive repetitions of a pattern can be written in an explicit form with k as a parameter. Some recurrence relations, which are useful for evaluating the probability mass functions, are also given.

Keywords Geometric distribution of order k · Repetition of a pattern · Waiting time for a pattern · Conditional probability generating function · (k_1, k_2)-Event

1 Introduction

The geometric distribution of order k is the distribution of the number of trials until a run of "1" of length k is observed for the first time in independent $\{0, 1\}$-valued

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random variables $X_1, X_2, \ldots$, where $P(X_i = 1) = p = 1 - q$ for every positive integer i. The distribution was studied by de Moivre in the eighteenth century (see Todhunter 1865) and introduced rigorously by Feller (1968). Since about 1980s, it has been applied to various fields such as reliability of consecutive systems, start-up demonstration tests, sampling inspection plans. (see for example, Balakrishnan and Koutras 2002; Johnson et al. 2005). In discrete distribution theory, the study has been extended from runs to general patterns mainly by the contributions of Fu and Koutras (1994) and Fu (1996). Thus, the exact distributions of waiting times for any patterns with finite length can be evaluated in principle. It is well known that the probability generating function (p.g.f.) of the geometric distribution of order k is represented explicitly with k as $\frac{(pt)^k(1-pt)}{1-t+p^kqt^{k+1}}$. Distributions of waiting times for a pattern of finite length can be obtained by the forward and backward principle introduced by Fu (1996). However, p.g.f.'s of waiting time distributions with explicit expressions are scarcely known except for a few distributions such as the geometric distribution of order k just above or waiting time distribution for the (k_1, k_2)-events (Huang and Tsai 1991; Dafnis et al. 2010; Stefanov and Manca 2013).

Many probability generating functions in the problems are rational functions. From the explicit forms of the probability generating functions, we can obtain the probability functions by using various methods such as recurrence relations of rational generating functions (Stanley 1997), and the partial fraction expansion (Shmueli and Cohen 2000). Both methods need an explicit form of the generating function. The motivation of the study is to provide explicit forms of the probability generating functions of the distributions related to discrete patterns with arbitrary length. Above all, the method of the partial fraction expansion is effective for long-tailed distributions. In the last section of Huang and Tsai (1991), the authors introduced a $(k_1, k_2; k_3)$-events which are exactly k_3 consecutive (k_1, k_2)-events. They studied the double generating function of the number of occurrences of $(k_1, k_2; k_3)$-events in n independent trials. Extending the idea, we study the distributions of the waiting time for k-times consecutive repetitions of a pattern.

In this paper, we extend the concept of a run of length k to that of a pattern which is k-times consecutive repetitions of another pattern. By the extension, we can obtain many examples with explicit expressions of the p.g.f. with parameter k. For example, as we will show in Sect. 2, the distribution of the waiting time for the pattern $\overbrace{1010\ldots10}^{2k}$, which is k-times consecutive repetitions of the pattern "10", has the explicit and simple p.g.f. as

$$\frac{(pqt^2)^k(1-pqt^2)}{1-t+(pqt^2)^k(t-pqt^2)}.$$

A pattern $\mathcal{P}$ is called the *run of* a *pattern* α *with* k *repetitions* and denoted by $[\alpha]^k$ if it is k-times consecutive repetitions of the pattern α. The above pattern $\overbrace{1010\ldots10}^{2k}$ is written as $[10]^k$. When $\mathcal{P} = [\alpha]^k$, the pattern α is called a basic pattern of $\mathcal{P}$. If two or more representations are possible for a pattern such as "10101010"$= [10]^4 = [1010]^2$, we define the basic pattern to be the one which has the shortest length. The usual 1-run of length k is denoted by $[1]^k$. The $(k_1, k_2; k_3)$-event defined by Huang and Tsai (1991) is written as $[[0]^{k_1}[1]^{k_2}]^{k_3}$ with our symbol.

When we want to obtain the p.g.f. of a pattern $\mathcal{P}$ by solving a system of equations of the conditional p.g.f.'s, we can reduce the number of equations to be solved to less than or equal to the length of the pattern $\mathcal{P}$ using the forward and backward principle. In Sect. 2, if the pattern $\mathcal{P}$ is a run of a basic pattern α, we show that the number of equations to be solved can be reduced to less than or equal to the length of the basic pattern α. By using the result, we give all the p.g.f.'s of runs of a basic {0, 1}-pattern of length less than 4 with k repetitions in an explicit form with parameter k in Table 1. From the result, we can see that the forms of the p.g.f.'s are very simple. In particular, the p.g.f. of the waiting time for $[110]^k$ can be obtained from the p.g.f. of the geometric distribution of order k only by substituting pt by p^2qt^3. By generalizing this property, we represent the p.g.f. of the waiting time for the pattern $[[1]^m[0]^n]^k$ in a closed form for any positive integers m and n. Further, we write the p.g.f. of the waiting time for $[[1]^{m_1}[0]^n[1]^{m_2}]^k$ in an explicit form for any positive integers m_1, n, m_2 and k. In Sect. 3, the distributions of non-overlapping number of runs of a pattern in $X_1, X_2, \ldots, X_x$ are studied for any positive integer x. Recurrence relations of the p.g.f.'s, which are useful for evaluating the probability mass functions, are given. An explicit form of the p.g.f. of the distribution is also obtained. In Sect. 4, we consider the problems in independent multi-state trials. Theorem 4 provides a very simple result of the waiting time for runs of a basic pattern, when the basic pattern satisfies a condition. Theorem 5 extends the result on runs studied by Aki and Hirano (1995) to that of runs of a basic pattern. All the proofs of the results are given in Sect. 5.

2 Waiting time for a run of a pattern

Let $X_1, X_2, \ldots$ be independent identically distributed {0, 1}-valued random variables, with $P(X_i = 1) = p = 1 - q$ for $i = 1, 2 \ldots$. First, we briefly review how to obtain the distribution of the waiting time for a pattern. For example, we obtain the exact distribution of the waiting time W for the pattern "1101". We set $\phi(t) = E[t^W]$, $\phi_1(t) = E[t^{W-1}|X_1 = 1]$, $\phi_{11}(t) = E[t^{W-2}|X_1 = 1, X_2 = 1]$, and $\phi_{110}(t) = E[t^{W-3}|X_1 = 1, X_2 = 1, X_3 = 0]$. Using the forward and backward principle introduced by Fu (1996), these conditional p.g.f.'s satisfy the relations

$$\begin{cases} \phi = pt\phi_1 + qt\phi \\ \phi_1 = pt\phi_{11} + qt\phi \\ \phi_{11} = pt\phi_{11} + qt\phi_{110} \\ \phi_{110} = pt + qt\phi, \end{cases} \tag{1}$$

where $\phi(t)$, $\phi_1(t)$, $\phi_{11}(t)$ and $\phi_{110}(t)$ are abbreviated to ϕ, ϕ_1, ϕ_{11} and ϕ_{110}, respectively. We use abbreviations like these when the variable is clearly given. By solving the above equations, we obtain the p.g.f. ϕ of W. Since the system of equations is given by considering the next state from each state, it corresponds to the transition matrix of the finite Markov chain embedding method. Here, the number of equations in the

Table 1 p.g.f.'s of the waiting time for the generalized run of the kth repetition with basic pattern of length less than or equal to 3

Pattern	p.g.f.
$[1]^k$	$\frac{(pt)^k(1-pt)}{1-t+p^kqt^{k+1}}$
$[0]^k$	$\frac{(qt)^k(1-qt)}{1-t+q^kpt^{k+1}}$
$[11]^k$	$\frac{(pt)^{2k}(1-pt)}{1-t+p^{2k}qt^{2k+1}}$
$[00]^k$	$\frac{(qt)^{2k}(1-qt)}{1-t+q^{2k}pt^{2k+1}}$
$[10]^k$, $[01]^k$	$\frac{(pqt^2)^k(1-pqt^2)}{1-t+(pqt^2)^k(t-pqt^2)}$
$[111]^k$	$\frac{(pt)^{3k}(1-pt)}{1-t+p^{3k}qt^{3k+1}}$
$[000]^k$	$\frac{(qt)^{3k}(1-qt)}{1-t+q^{3k}pt^{3k+1}}$
$[110]^k$, $[011]^k$	$\frac{(p^2qt^3)^k(1-p^2qt^3)}{1-t+(p^2qt^3)^k(t-p^2qt^3)}$
$[100]^k$, $[001]^k$	$\frac{(q^2pt^3)^k(1-q^2pt^3)}{1-t+(q^2pt^3)^k(t-q^2pt^3)}$
$[101]^k$	$\frac{(p^2qt^3)^k(1-p^2qt^3)}{1-t+pqt^2(p^2qt^3)^{k-1}(1-t+pt^2-p^2qt^3+p^2q^2t^4)}$
$[010]^k$	$\frac{(q^2pt^3)^k(1-q^2pt^3)}{1-t+pqt^2(q^2pt^3)^{k-1}(1-t+qt^2-q^2pt^3+q^2p^2t^4)}$

system agrees with the length of the pattern. Therefore, it will be difficult to solve the system of equations for long patterns. In order to reduce the number of equations, we consider an alternative conditioning. We propose conditioning on the location of the first failure of observing the pattern. If the first observation is "0", we fail in observing the pattern at the first trial. If the first observation is "1" and the second observation is "0", we fail in observing the pattern at the second trial for the first time. If the first observation is "111", we fail in observing at the third trial for the first time. And, if the first observation is "1100", we fail at the fourth trial for the first time. Considering conditioning on the location of the first failure, we obtain the system of equations,

$$\begin{cases} \phi = qt\phi + pqt^2\phi + p^3t^3\phi_{11} + p^2q^2t^4\phi + p^3qt^4 \\ \phi_{11} = pt\phi_{11} + q^2t^2\phi + qpt^2. \end{cases} \tag{2}$$

From the both systems of Eqs. (1) and (2), we can obtain the same ϕ. We note that the system (2) has only two equations. By solving the system (2), we obtain the p.g.f, $\phi = \frac{p^3qt^4}{1-t+p^2qt^3-p^2q^2t^4}$. We can obtain the p.g.f. of the distribution of the waiting time of a pattern from the both systems of equations above. However, the latter system has smaller number of equations than the former system generally. In particular, if the pattern is a run of a pattern, as we prove later, the number of equations of the latter system is no more than the length of the basic pattern.

For example, we consider 1-run of length k as the pattern. The basic pattern is "1" and 1-run of length k is written as $[1]^k$. Since the length of the basic pattern is one, we can obtain the p.g.f. by solving an equation. In fact, by conditioning on the location of the first "0", we obtain a recurrence relation,

$$\phi = \sum_{j=0}^{k-1} p^{j-1} q t^j \phi \ + p^k t^k.$$

By the above conditioning, the number of equations becomes one for all k. By solving the above equation, we obtain immediately

$$\phi = \frac{p^k t^k}{1 - \sum_{j=1}^{k} p^{j-1} q t^j} = \frac{p^k t^k (1 - pt)}{1 - t + p^k q t^{k+1}}.$$

As an example for obtaining the p.g.f. of the waiting time for a run of a pattern, we consider the pattern $[10]^k$ which is the kth repetition of the basic pattern "10". Let $\phi = \phi(t)$ be the p.g.f. of the waiting time, and let ϕ_1 be the conditional p.g.f. of the waiting time given that the first "1" has just been observed. In this case, as the basic pattern "10" is very simple, the conditional p.g.f.'s to be defined in advance are sufficient if we study the waiting time by conditioning on the first place where we fail in observing the pattern. Hence, we obtain the recurrence relation:

$$\phi = \sum_{j=0}^{k-1} (pqt^2)^j (qt\phi + p^2 t^2 \phi_1) + (pqt^2)^k.$$

Using $\phi = pt\phi_1 + qt\phi$, we have $\phi = \frac{(pqt^2)^k (1 - pqt^2)}{1 - t + (pqt^2)^k (t - pqt^2)}$.

Here, we note that the distribution of the waiting time for $[01]^k$ agrees with that of the waiting time for $[10]^k$. The distribution for $[01]^k$ can be obtained by exchanging 1 and 0. Therefore, we have the p.g.f. only by exchanging p and q in the above p.g.f. for $[10]^k$. It is easy to see that the p.g.f. does not change by exchanging p and q. Alternatively, we see that the pattern $[01]^k$ is the reversed pattern of $[10]^k$. Then the result can also be obtained by Aki and Hirano (2002).

Remark 1 Throughout the paper we assume that the sequence is independent and identically distributed random variables. However, the assumption is not for deriving the p.g.f.'s but for the results to be simple. In the proofs we use only temporal homogeneity of the sequence. Similarly as the finite Markov chain embedding method (see for example, Fu and Koutras 1994; Balakrishnan and Koutras 2002), the method of conditional p.g.f.'s can be used for temporally homogeneous sequence like (temporally homogeneous) Markov chain. For example, let $X_1, X_2, \ldots$ be Markov chain with $P(X_1 = 1) = \pi_1,\ P(X_1 = 0) = \pi_0,\ P(X_{i+1} = 1 | X_i = 1) = p1 = 1 - q1$ and $P(X_{i+1} = 1 | X_i = 0) = p_0 = 1 - q_0$. Then the corresponding p.g.f. of the waiting time of $[10]^k$ can be obtained similarly as

$$\phi(t) = \frac{\pi_0 t (1 - p_0 q_1 t^2)(p_0 q_1 t^2)^k + \pi_1 t q_1 t (1 - q_0 t)(1 - p_0 q_1 t^2)(p_0 q_1 t^2)^{k-1}}{1 - (q_0 + p_1) t + (p_1 q_0 - p_0 q_1) t^2 + ((q_0 + p_1) t - p_1 q_0 t^2)(p_0 q_1 t^2)^k}.$$

Let m be a positive integer and let k be an integer greater than one. For $i = 1, 2, \ldots, m$, we set $e_i \in \{0, 1\}$. Let us study the distribution of the waiting time W for the pattern $[e_1 \ldots e_m]^k$. We denote by $P(e_i)$ the probability that e_i occurs in every trial. As we suppose that the trials are independent and identically distributed $\{0, 1\}$-valued random variables,

$$P(e_i) = \begin{cases} p & \text{if } e_i = 1 \\ q & \text{if } e_i = 0. \end{cases}$$

For every $i = 1, 2, \ldots, m$, we set $\overline{e_i} = 1 - e_i$. Let $\mathcal{F} = \{\emptyset, e_1, e_1e_2, \ldots, e_1e_2 \ldots e_{m-1}\}$ be the set of sequential subpatterns of the basic pattern $e_1e_2 \ldots e_m$. For $j = 0, 1, 2, \ldots, k-1$ and $i = 1, 2, \ldots, m$, we define the pattern

$$\alpha(j, i) := [e_1 \ldots e_m]^j e_1e_2 \ldots e_{i-1}\overline{e_i}.$$

These patterns are typical subsequences until the first failure in observing the pattern $[e_1 \ldots e_m]^k$. Let $x = (x_1, x_2, \ldots, x_\nu)$ be a finite $\{0, 1\}$-sequence. We denote by $(x)_L = (x_a, x_{a+1}, \ldots, x_\nu)$ the longest ending block of x in the set of sequential subpatterns of $[e_1 \ldots e_m]^k$. Let f be the mapping from the set of finite $\{0, 1\}$-sequences to the sequential subpatterns of $[e_1 \ldots e_m]^k$ defined by $f(x) = (x)_L$. For every element $\alpha \in \mathcal{F}$, we set $\phi_\alpha(t) = E[t^{(W-|\alpha|)}|(X_1, \ldots, X_{|\alpha|}) = \alpha]$, where $|\alpha|$ denotes the length of α. In particular, for $\alpha = \emptyset$, we define $\phi_\emptyset(t) = \phi(t) = E[t^W]$.

For example, let the pattern be $[110]^3$ and the sequence $x = (1, 1, 0, 1, 1, 1)$. Then, the longest ending block $(x)_L$ is $(1, 1)$ and $f(x) = (1, 1)$.

Theorem 1 *For every $j = 0, 1, 2, \ldots, k-1$ and for every $i = 1, 2, \ldots, m$, it holds that $f(\alpha(j, i)) = f([e_1 \ldots e_m]^j e_1e_2 \ldots e_{i-1}\overline{e_i}) \in \mathcal{F}$. The p.g.f. $\phi = \phi(t)$ of the waiting time for the pattern $[e_1 \ldots e_m]^k$ is the solution of the following system of equations:*

$$\begin{aligned}
\phi = {} & P(\overline{e_1})t\phi_{f(\overline{e_1})} + P(e_1)P(\overline{e_2})t^2\phi_{f(e_1\overline{e_2})} \\
& + \cdots + P(e_1)\ldots P(e_{m-1})P(\overline{e_m})t^m\phi_{f(e_1\ldots e_{m-1}\overline{e_m})} \\
& + \sum_{j=1}^{k-1}(P(e_1)\ldots P(e_m)t^m)^j \sum_{i=1}^{m-1} P(e_1)\ldots P(e_{i-1})P(\overline{e_i})t^i\phi_{f(\alpha(j,i))} \\
& + \left(P(e_1)\ldots P(e_m)t^m\right)^k,
\end{aligned}$$

and for $i = 1, 2, \ldots, m-1$,

$$\begin{aligned}
\phi_{e_1\ldots e_i} = {} & P(\overline{e_{i+1}})t\phi_{f(\alpha(0,i+1))} + P(e_{i+1})P(\overline{e_{i+2}})t^2\phi_{f(\alpha(0,i+2))} + \ldots \\
& + P(e_{i+1})\ldots P(e_{m-1})P(\overline{e_m})t^{m-i}\phi_{f(\alpha(0,m))} \\
& + P(e_{i+1})\ldots P(e_m)\sum_{j=1}^{k-2}(P(e_1)\ldots P(e_m)t^m)^j
\end{aligned}$$

$$\times \sum_{i=1}^{m-1} P(e_1)\dots P(e_{i-1})P(\overline{e_i})t^i \phi_{f(\alpha(j,i))}$$
$$+ (P(e_1)\dots P(e_m)t^m)^k.$$

Example 1 (Waiting time for $[110]^k$) Let k be an integer greater than one, and let ϕ be the p.g.f. of the waiting time for $[110]^k$. We define $\phi_{11}(t) = E[t^{W-2}|X_1 = 1, X_2 = 1]$. From Theorem 1, ϕ and ϕ_{11} satisfy the following equations:

$$\phi = \sum_{j=0}^{k-1}(p^2qt^3)^j \left(qt\phi + pqt^2\phi + p^3t^3\phi_{11}\right) + (p^2qt^3)^k,$$
$$\phi_{11} = pt\phi_{11} + qt\sum_{j=0}^{k-2}(p^2qt^3)^j \left(qt\phi + pqt^2\phi + p^3t^3\phi_{11}\right) + qt(p^2qt^3)^{k-1}.$$

Solving the equations, we obtain $\phi = \frac{(p^2qt^3)^k(1-p^2qt^3)}{1-t+(p^2qt^3)^k(t-p^2qt^3)}$.

Example 2 (Waiting time for $[101]^k$) Let ϕ be the p.g.f. of the waiting time for the pattern $[101]^k$, and define the conditional p.g.f.'s $\phi_1(t) = E[t^{W-1}|X_1 = 1]$ and $\phi_{10}(t) = E[t^{W-2}|X_1 = 1, X_2 = 0]$. From Theorem 1, we have

$$\phi = qt\phi + p^2t^2\phi_1 + pq^2t^3\phi + \sum_{j=1}^{k-1}(p^2qt^3)^j \left(qt\phi_{10} + p^2t^2\phi_1 + pq^2t^3\phi\right) + (p^2qt^3)^k,$$
$$\phi_1 = pt\phi_1 + q^2t^2\phi + qpt^2\sum_{j=0}^{k-2}(p^2qt^3)^j \left(qt\phi_{10} + p^2t^2\phi_1 + pqt^3\phi\right) + qpt^2(p^2qt^3)^{k-1},$$
$$\phi_{10} = qt\phi + pt\sum_{j=0}^{k-2}(p^2qt^3)^j \left(qt\phi_{10} + p^2t^2\phi_1 + pq^2t^3\phi\right) + pt(p^2qt^3)^{k-1}.$$

Solving the system of equations, we obtain

$$\phi = \frac{(p^2qt^3)^k(1 - p^2qt^3)}{1 - t + pqt^2(p^2qt^3)^{k-1}(1 - t + pt^2 - p^2qt^3 + p^2q^2t^4)}.$$

Since we have obtained all the p.g.f.'s of the kth repetition with basic patterns of length less than or equal to 3, we tabulate them in Table 1.

When the basic pattern is relatively simple, we can obtain the exact distribution of the kth consecutive repetition of the basic pattern even the length of the basic pattern is general. For example, we have the following results.

Theorem 2 *For any positive integers m and n, the p.g.f. of the exact distribution of the waiting time W for the pattern $[[1]^m[0]^n]^k$ can be written as*

$$\phi(t) = \frac{(1 - p^m q^n t^{m+n})(p^m q^n t^{m+n})^k}{1 - t + (p^m q^n t^{m+n})^k (t - p^m q^n t^{m+n})}.$$

Let r be a positive integer. Generally, for the p.g.f. $\phi(t)$ of the waiting time for an event in independent trials, the p.g.f. $\phi_r(t)$ of the waiting time for the non-overlapping rth occurrence of the event satisfies that $\phi_r(t) = (\phi(t))^r$. Therefore, we obtain the next result.

Corollary 1 *The p.g.f. of the rth non-overlapping occurrence of $[[1]^m[0]^n]^k$ is given by*

$$\phi_r(t) = \left(\frac{(1 - p^m q^n t^{m+n})(p^m q^n t^{m+n})^k}{1 - t + (p^m q^n t^{m+n})^k (t - p^m q^n t^{m+n})} \right)^r.$$

Remark 2 When $k = 1$ in Theorem 2, the p.g.f. coincides with that of the waiting time for the first (k_1, k_2)-event with $k_1 = n$ and $k_2 = m$. In fact, by setting $k = 1$, we obtain the p.g.f.

$$\phi(t) = \frac{(1 - \alpha_{m,n}(t))(\alpha_{m,n}(t))}{1 - t + (\alpha_{m,n}(t))(t - \alpha_{m,n}(t))}, \text{ where } \alpha_{m,n}(t) = p^m q^n t^{m+n}.$$

Here, noting that

$$1 - t + (\alpha_{m,n}(t))(t - \alpha_{m,n}(t)) = (1 - \alpha_{m,n}(t))(1 - t + \alpha_{m,n}(t)),$$

we can write

$$\phi(t) = \frac{\alpha_{m,n}(t)}{1 - t + \alpha_{m,n}(t)},$$

(cf. Corollary 2 of Huang and Tsai 1991, Theorem 4.1 of Dafnis et al. 2010 with $r = 1$, or Corollary 1 of Stefanov and Manca 2013 with $m = 1$). Therefore, we can say that our distribution is an order-k version of the waiting time distribution for the first occurrence of the (k_1, k_2)-event.

As another result, we shall give a general form corresponding to the waiting time for $[101]^k$ treated in Example 2.

Theorem 3 *Let k, m_1, n and m_2 be any positive integers and we set $n_1 = \min(m_1, m_2)$ and $n_2 = \max(m_1, m_2)$. Then, the p.g.f. of the exact distribution of the waiting time W for the pattern $[[1]^{m_1}[0]^n[1]^{m_2}]^k$ can be written as*

$$\phi(t) = \frac{(1 - P(t))P(t)^k}{1 - t + q^n p^{n_2} t^{n+n_2} P(t)^{k-1}(f(t) - P(t)g(t))},$$

where $P(t) = p^{m_1+m_2}q^n t^{m_1+m_2+n}$, *and*

$$f(t) = p^{n_1}t^{n_1+1} - p^{n_1-1}t^{n_1} - \sum_{\ell=1}^{n_1-1} p^{n_1-\ell-1}qt^{n_1-\ell} + 1$$
$$g(t) = -p^{n_1-1}qt^{n_1} - p^{n_1-2}qt^{n_1-1} - \cdots - qt + 1.$$

Remark 3 When $m_1 = 1$ in the above theorem, $f(t) = pt^2 - t + 1$ and $g(t) = -qt + 1$ hold. By setting $m_1 = 1$, $n = 1$, and $m_2 = 1$, we see that the result of Theorem 3 agrees with that of Example 2.

3 Number of occurrences of runs of a pattern

Let $X_1, X_2, \ldots$ be i.i.d. $\{0, 1\}$-valued random variables with $P(X_i = 1) = p = 1-q$ for $i = 1, 2, \ldots$. For a given positive integer x, the distribution of the number of occurrences of runs of a pattern in $X_1, X_2, \ldots, X_x$ is as important as that of waiting times treated in the previous section. Various problems on the number of occurrences of run of a pattern will be studied as various problems on usual runs have been studied (see e.g., Balakrishnan and Koutras 2002; Johnson et al. 2005).

Here, we study the distribution of the number of occurrences of $[[1]^m[0]^n]^k$ in $X_1, X_2, \ldots, X_x$, where m, n, and k are positive integers. We denote by N_x the number of non-overlapping occurrences of $[[1]^m[0]^n]^k$ in $X_1, X_2, \ldots, X_x$. In particular, we set $N_0 = 0$ for notational convenience. We give the next result for the p.g.f. $\phi_x(t)$ of N_x and the double generating function of N_x. A formula of the double generating function by using the p.g.f. of the waiting time for the (k_1, k_2)-event is given in Theorem 2 of Huang and Tsai (1991). However, the next proposition is very simple and we can derive the p.g.f. of N_x by using it.

Proposition 1 *The double generating function* $\Phi(t, z) = \sum_{x=0}^{\infty} \phi_x(t)z^x$ *can be written as*

$$\Phi(t, z) = \frac{1 - (p^m q^n z^{m+n})^k}{1 - z + (p^m q^n z^{m+n})^k (z - t - (1-t)p^m q^n z^{m+n})}. \tag{3}$$

Remark 4 We have derived the double generating function $\Phi(t, z)$ directly in Sect. 5. However, it can be derived also from the p.g.f. of the waiting time W for the first occurrence of $[[1]^m[0]^n]^k$. In fact, by using the formula (5.9) of Balakrishnan and Koutras (2002), we can show that

$$\Phi(t, z) = \frac{1}{1-z}\left\{1 - \phi(z)\frac{1-t}{1-t\phi(z)}\right\},$$

where $\phi(t)$ is the p.g.f. of the waiting time for $[[1]^m[0]^n]^k$ given in Theorem 2.

One of the merits for finding double generating functions is to obtain an explicit expression of the p.g.f. of the distribution of number of occurrences of a pattern in finitely given trials. In fact, by picking up the coefficient of z^x after expanding the double generating function (3), we give the next result for an explicit expression of $\phi_x(t)$.

Proposition 2 *The p.g.f. of N_x is given as*

$$\phi_x(t) = \sum\nolimits^{*} \frac{(x_1+x_2+x_3+x_4)!}{x_1!x_2!x_3!x_4!} t^{x_2}(1-t)^{x_4}(-1)^{x_3}(p^m q^n)^{k(x_2+x_3+x_4)} - \sum\nolimits^{**} \frac{(y_1+y_2+y_3+y_4)!}{y_1!y_2!y_3!y_4!} t^{y_2}(1-t)^{y_4}(-1)^{y_3}(p^m q^n)^{k(1+y_2+y_3+y_4)},$$

where the summation $\sum^{*}$ extends to all nonnegative integers x_1, x_2, x_3 and x_4 satisfying the condition $x_1 + (m+n)kx_2 + ((m+n)k+1)x_3 + (m+n)(k+1)x_4 = x$, and the summation $\sum^{**}$ extends to all nonnegative integers y_1, y_2, y_3 and y_4 satisfying the condition $y_1 + (m+n)ky_2 + ((m+n)k+1)y_3 + (m+n)(k+1)y_4 = x - (m+n)k$.

4 Multi-state trials

In the case of independent {0, 1}-valued trials, we have observed that the p.g.f. of the waiting time for $[[1]^m[0]^n]^k$ becomes similar to that of the geometric distribution of order k. In the case of independent multi-state trials, it is shown that similar formulae hold even if the form of the pattern is not necessarily repetitions of the same basic pattern. Let $A = \{1, 2, \ldots, r\}$ for a positive integer $r > 1$. Suppose that the independent A-valued random variables $\{X_i\}$ satisfy $P(X_i = j) = p_j$ for $j = 1, 2, \ldots, r$ with $p_1 + p_2 + \cdots p_r = 1$. Let k, m_1 and m_2 be integers satisfying $k \geq 1$, $m_1 \geq 1$, $m_2 \geq 0$, respectively. Let $w = (w_1, w_2, \ldots, w_{m_2}) \in \{A \backslash \{1\}\}^{m_2}$ be a finite sequence of length m_2. For $i = 1, 2, \ldots, m_2$, we set $w[i] := (w_1, w_2, \ldots, w_i)$ and define $p_w = p_{w_1} p_{w_2} \cdots p_{w_{m_2}}$ and $p_{w[i]} = p_{w_1} p_{w_2} \cdots p_{w_i}$. Here, we examine the distribution of the waiting time for the pattern $[[1]^{m_1}w]^k$ in the sequence of independent multi-state trials $\{X_i\}$. For integers m_1, m_2, and k, we assume $m_1 \geq 1$, $m_2 \geq 0$, $k \geq 1$. However, when $m_2 = 0$, the pattern $[[1]^{m_1}w]^k$ becomes a 1-run of length $m_1 k$ and hence the p.g.f. is written as

$$\frac{(p_1 t)^{m_1 k}(1-p_1 t)}{1-t+(p_1 t)^{m_1 k}(t-p_1 t)}. \tag{4}$$

Therefore, we assume $m_2 > 0$ and we set $\alpha(t) = p_1^{m_1} p_w t^{m_1+m_2}$. Then, we have the following result.

Theorem 4 *For any integers $m_1 \geq 1$ and $m_2 > 0$, the p.g.f. of the waiting time for the pattern $[[1]^{m_1}w]^k$ can be written as*

$$\phi(t) = \frac{(\alpha(t))^k(1-\alpha(t))}{1-t+(\alpha(t))^k(t-\alpha(t))}.$$

Remark 5 Setting $r = 2$ in Theorem 4, and regarding "2" as "0", we can see that Theorem 2 is a corollary of Theorem 4.

Example 3 (Waiting time for $[1123234]^k$) We shall derive the exact distribution of the waiting time for $[1123234]^k$. From Theorem 4, the p.g.f. is obtained by considering $m_1 = 2$ and $w =$"23234" in the theorem. Therefore, substituting $\alpha(t) = p_1^2 p_2^2 p_3^2 p_4 t^7$ in the result of Theorem 4, we can write the p.g.f. as

$$\frac{(\alpha(t))^k(1-\alpha(t))}{1-t+(\alpha(t))^k(t-\alpha(t))}.$$

Noting that the p.g.f. is determined by $\alpha(t)$, if $\alpha(t)$ is common, the distributions of the waiting times for various patterns agree with each other. For example, the waiting time distributions for "$[1122334]^k$" and "$[1143223]^k$" agree. Moreover, if $p_1 = p_2 = \cdots = p_r$ can be assumed like dice, the distribution does not depend on m_1. Since $m_1 \geq 1$ and $m_2 \geq 1$ are assumed, though $m_1 = 0$ is not allowed, the waiting time distributions for "$[1222334]^k$", "$[1112334]^k$", "$[1111134]^k$", and "$[1111114]^k$" agree with each other. However, the waiting time distribution for "$[1111111]^k$", whose p.g.f. has been given in (4), does not agree with the above distributions even if $p_1 = p_2 = \cdots = p_r$ can be assumed.

We give here a method for evaluation of the probability function of the distribution given in Theorems 2 and 4. We denote by W the waiting time for the theorems. Its p.g.f is given by

$$\phi(t) = \frac{(\alpha(t))^k(1-\alpha(t))}{1-t+(\alpha(t))^k(t-\alpha(t))},$$

where $\alpha(t)$ can be written as at^ℓ by using a constant $a > 0$ and the length ℓ of the basic pattern. For example, if $\alpha(t) = \alpha_{m,n}(t) = p^m q^n t^{m+n}$ in Remark 2, we set $a = p^m q^n$ and $\ell = m + n$. If $\alpha(t) = p_1^2 p_2^2 p_3^2 p_4 t^7$ in Example 3, we can set $a = p_1^2 p_2^2 p_3^2 p_4$ and $\ell = 7$. Let $P_n = P(W = n),\ n = 0, 1, 2, \ldots$ be the probability function of W.

Proposition 3 *The probability function* $\{P_n\}$ *of* W *above satisfies for* $0 \leq n \leq 2\ell k$,

$$P_n = \begin{cases} 0 & \text{if } 0 \leq n < \ell k \\ a^k & \text{if } \ell k \leq n < \ell(k+1) \\ a^k - a^{k+1} & \text{if } \ell(k+1) \leq n \leq 2\ell k. \end{cases}$$

For $n > 2\ell k$, *the recurrence relations hold*

$$P_n = P_{n-1} - a^k P_{n-\ell k-1} + a^{k+1} P_{n-\ell(k+1)}.$$

Remark 6 Setting $\ell = 1$ and $a = p$, we obtain the recurrence relations of the probability function of the geometric distribution of order k [cf. the formula (2.5) in Balakrishnan and Koutras 2002].

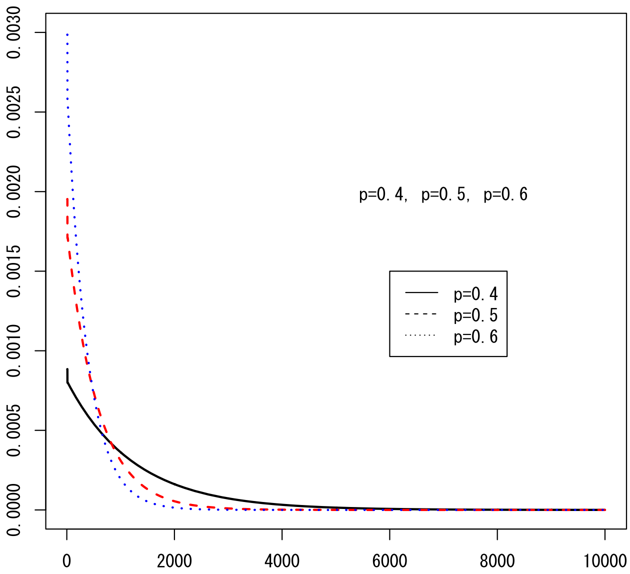

Fig. 1 Probability functions of the waiting time for $[[1]^2[0]^1]^3$ with $p = 0.4,\ p = 0.5$, and $p = 0.6$

The method for evaluation of the probability function given above is quite effective. Figure 1 illustrates the probability functions $P_n's$ of the distributions of the waiting time for $[[1]^2[0]^1]^3$ with $p = 0.4,\ p = 0.5$ and $p = 0.6$ for $n = 9, 10, \ldots, 10{,}000$.

Next, we investigate the distribution of the number of consecutive repetitions of $[[1]^{m_1}w]$ until the first occurrence of $[[1]^{m_1}w]^k$, where $w = (w_1, w_2, \ldots, w_{m_2}) \in (A\backslash\{1\})^{m_2}$. In the case of usual runs, the corresponding problem was studied by Aki and Hirano (1995). Let k and ℓ be positive integers with $1 \le \ell \le k-1$. Then, until the first occurrence of $[[1]^{m_1}w]^k$, $[[1]^{m_1}w]^\ell$ occurs necessarily. We introduce an overlapping counting scheme for $[[1]^{m_1}w]^\ell$. As the basic pattern of $[[1]^{m_1}w]^\ell$ is $[[1]^{m_1}w]$, we regard the basic pattern as one unit. In the overlapping counting scheme, when $[[1]^{m_1}w]$ occurs just after an occurrence of $[[1]^{m_1}w]^\ell$, it is defined that another $[[1]^{m_1}w]^\ell$ has occurred, that is, the ending part $[[1]^{m_1}w]^{\ell-1}$ of the $[[1]^{m_1}w]^\ell$ is overlapping. For example, in the overlapping counting scheme, there are $(k-\ell+1)$ $[[1]^{m_1}w]^\ell$'s in one $[[1]^{m_1}w]^k$.

Let W be the waiting time for $[[1]^{m_1}w]^k$ and for $\ell = 1, 2, \ldots, k-1$ let M_ℓ be the overlapping number of $[[1]^{m_1}w]^\ell$ until W. We define the joint p.g.f. of $(M_1, M_2, \ldots, M_{k-1}, W)$ by

$$\phi(\boldsymbol{s}, t) = \phi(s_1, s_2, \ldots, s_{k-1}, t) = E\left[s_1^{M_1} s_2^{M_2} \ldots s_{k-1}^{M_{k-1}} t^W\right].$$

Then we have the next theorem.

Theorem 5 *For integers $m_1 \geq 1$ and $m_2 \geq 0$, the joint p.g.f. $\phi(\boldsymbol{s}, t)$ of $(M_1, M_2, \ldots, M_{k-1}, W)$ is written by*

$$\phi(\boldsymbol{s}, t) = \frac{(p_1^{m_1} p_w)^k s_1^k s_2^{k-1} \ldots s_{k-1}^2 t^{k(m_1+m_2)}}{1 - \sum_{j=0}^{k-1} (p_1^{m_1} p_w)^j s_1^j s_2^{(j-1)\vee 0} \ldots s_{k-1}^{(j-(k-2))\vee 0} t^{j(m_1+m_2)} (t - (p_1 t)^{m_1} p_w t^{m_2})},$$

where $a \vee b$ means $\max(a, b)$.

Considering the marginal distributions, we obtain the next result.

Corollary 2 *Under the assumption of Theorem 5, the marginal distribution of M_ℓ is the geometric distribution of order $(k - \ell)$ with parameter $\pi = p_1^{m_1} p_w$ whose support is shifted so as to begin with $k - \ell + 1$.*

5 Proofs

5.1 Proof of Theorem 1

Proof From properties of the conditional expectation, it is obvious to see that the equations in Theorem 1 hold. Hence, it suffices to show

$$f(\alpha(j, i)) = f([e_1 \ldots e_m]^j e_1 e_2 \ldots e_{i-1} \overline{e_i}) \in \mathcal{F}$$

for every $j = 0, 1, 2, \ldots, k - 1$ and $i = 1, 2, \ldots, m$. When we fail in observing the pattern, we show that a coincidence between the ending subsequence with length more than m and a sequential subpattern of $[e_1 \ldots e_m]^k$ leads a contradiction. Assume that for some $j \geq 1$ and $1 \leq i \leq m$, there exist $1 \leq j_0 \leq j$ and $i_0 \neq i$ such that

$$f\left([e_1 \ldots e_m]^j e_1 \ldots \overline{e_i}\right) = [e_1 \ldots e_m]^{j_0} e_1 \ldots e_{i_0}$$

holds. If $i_0 < i$, then "$e_{i-i_0+1} \ldots e_m e_1 \ldots \overline{e_i}$" agrees with "$e_1 \ldots e_m e_1 \ldots e_{i_0}$", since they are the ending blocks of the same sequence. In particular, $\overline{e_i} = e_{i_0}$ holds, since they are the ending elements. Further, by comparing the i_0th elements, we see that $e_i = e_{i_0}$. Then, $\overline{e_i} = e_i$ holds. This is a contradiction. If $i_0 > i$, then '$e_{i_0-i+1} \ldots e_m e_1 \ldots e_{i_0}$" agrees with "$e_1 \ldots e_m e_1 \ldots \overline{e_i}$". Similarly in the above case, $\overline{e_i} = e_{i_0} = e_i$ holds, which leads a contradiction. □

5.2 Proof of Theorem 2

Proof Let ϕ be the p.g.f. of W and define the conditional p.g.f.'s $\phi_1(t) = E[t^{(W-1)} | X_1 = 1]$ and $\phi_m(t) = E[t^{(W-m)} | X_1 = 1, \ldots, X_m = 1]$. By using Theorem 1, we have

$$\phi = \sum_{j=0}^{k-1} (p^m q^n t^{m+n})^j \left(\sum_{\ell=1}^{m} p^{\ell-1} q t^\ell \phi + (pt)^{m+1} \phi_m + p^m q t^{m+1} \sum_{\ell=1}^{n-1} q^{\ell-1} p t^\ell \phi_1 \right) + (p^m q^n t^{m+n})^k. \tag{5}$$

Let us find a relation among ϕ, ϕ_1 and ϕ_m. Suppose that we have observed the first trial. Then, we see that $\phi = pt\phi_1 + qt\phi$ and hence we have $\phi_1 = \frac{1-qt}{pt}\phi$. Similarly, suppose we have observed the mth trial. Then we have $\phi = \sum_{\ell=1}^{m} p^{\ell-1}qt^{\ell}\phi + p^m t^m \phi_m$. and hence we observe $\phi_m = \frac{1-t+p^m qt^{m+1}}{(pt)^m(1-pt)}\phi$. Rewriting the summations in the right-hand side of (5) and using these relations, we obtain

$$\phi = (t - p^m q^n t^{m+n}) \frac{1-(p^m q^n t^{m+n})^k}{1-p^m q^n t^{m+n}} \phi + (p^m q^n t^{m+n})^k.$$

□

5.3 Proof of Theorem 3

Proof Generally, the exact distribution of the waiting time for a pattern agrees with that of the waiting time for the reversed pattern (see Corollary 1 of Aki and Hirano 2002). Therefore, since the reversed pattern of $[[1]^{m_1}[0]^n[1]^{m_2}]^k$ is $[[1]^{m_2}[0]^n[1]^{m_1}]^k$, the distributions agree with each other. Thus, it suffices to find the p.g.f. of the waiting time for $[[1]^{n_1}[0]^n[1]^{n_2}]^k$ and hence, without loss of generality, we assume that $0 < m_1 \le m_2$, $n_1 = m_1$ and $n_2 = m_2$.

We define $\phi_1 = \phi_1(t) = E[t^{W-1}|X_1 = 1]$,

$$\phi_{m_1} = \phi_{m_1}(t) = E\left[t^{W-m_1}|X_1 = 1, \ldots, X_{m_1} = 1\right],$$
$$\phi_{10} = \phi_{10}(t) = E\left[t^{W-m_1-1}|X_1 = 1, \ldots, X_{m_1} = 1, X_{m_1+1} = 0\right].$$

Then from Theorem 1, we obtain the following equation:

$$\begin{aligned}
\phi = & \sum_{\ell=1}^{m_1} p^{\ell-1}qt^{\ell}\phi + p^{m_1+1}t^{m_1+1}\phi_{m_1} + p^{m_1}qt^{m_1+1}\sum_{\ell=1}^{n-1} q^{\ell-1}pt^{\ell}\phi_1 \\
& + p^{m_1}q^n t^{m_1+n}\sum_{\ell=1}^{m_1} p^{\ell-1}qt^{\ell}\phi + p^{m_1}q^n t^{m_1+n}\sum_{\ell=m_1+1}^{m_2} p^{\ell-1}qt^{\ell}\phi_{10} \\
& + \sum_{j=1}^{k-1}\left(p^{m_1}q^n p^{m_2}t^{m_1+n+m_2}\right)^j \\
& \times \left\{\sum_{\ell=1}^{m_1} p^{\ell-1}qt^{\ell}\phi_{10} + p^{m_1+1}t^{m_1+1}\phi_{m_1} + p^{m_1}qt^{m_1+1}\sum_{\ell=1}^{n-1} q^{\ell-1}pt^{\ell}\phi_1\right. \\
& \left. + p^{m_1}q^n t^{m_1+n}\sum_{\ell=1}^{m_1} p^{\ell-1}qt^{\ell}\phi + p^{m_1}q^n t^{m_1+n}\sum_{\ell=m_1+1}^{m_2} p^{\ell-1}qt^{\ell}\phi_{10}\right\} \\
& + (p^{m_1}q^n p^{m_2}t^{m_1+n+m_2})^k. \qquad (6)
\end{aligned}$$

Noting that $\phi = pt\phi_1 + qt\phi$, we have $\phi_1 = \frac{1}{pt}(1-qt)\phi$. Similarly, from the equations

$$\phi = \sum_{\ell=1}^{m_1} p^{\ell-1}qt^{\ell}\phi + p^{m_1}t^{m_1}\phi_{m_1}, \text{ and } \phi_{m_1} = pt\phi_{m_1} + qt\phi_{10},$$

we have

$$\phi_{m_1} = \frac{1}{p^{m_1}t^{m_1}}\left(\frac{1-t+p^{m_1}qt^{m_1+1}}{1-pt}\right)\phi, \text{ and } \phi_{10} = \frac{1}{p^{m_1}qt^{m_1+1}}(1-t+p^{m_1}qt^{m_1+1})\phi.$$

Substituting these equations in the Eq. (6) and solving it, we obtain after some algebraic manipulations

$$\phi(t) = \frac{(1-pt)(1-P(t))P(t)^k}{(1-t)(1-pt) + q^n p^{m_2} t^{n+m_2} P(t)^{k-1} A(t)},$$

where $A(t) = 1 - t + p^{m_1}t^{m_1}(-1 + 2t - pt^2 - p^{m_2}q^n t^{m_2+n}(1 - t + p^{m_1}qt^{m_1+1}))$. Here, we show that $A(t)$ is divisible by $(1 - pt)$. We set

$$\begin{aligned}
F(t) &= -p^{m_1+1}t^{m_1+2} + 2p^{m_1}t^{m_1+1} - p^{m_1}t^{m_1} - t + 1, \\
G(t) &= p^{m_1}qt^{m_1+1} - t + 1.
\end{aligned}$$

Then, we see that $A(t) = F(t) - P(t)G(t)$. Since it is easy to see that $F(\frac{1}{p}) = 0,\ F(t)$ is divisible by $(t - \frac{1}{p})$. In fact, we have $F(t) = (1 - pt)f(t)$. Similarly, we see that $G(t) = (1 - pt)g(t)$. Consequently, we obtain

$$\phi(t) = \frac{(1-P(t))P(t)^k}{1 - t + q^n p^{n_2} t^{n+n_2} P(t)^{k-1}(f(t) - P(t)g(t))}.$$

This completes the proof. □

5.4 Proof of Proposition 1

Proof For every nonnegative integer x, we define

$$\begin{aligned}
\phi_x(t) &:= E[t^{N_x}], \\
\phi_x^{(1)}(t) &:= E[t^{N_{x+1}} | X_1 = 1], \text{ and} \\
\phi_x^{(m)}(t) &:= E[t^{N_{x+m}} | X_1 = 1, X_2 = 1, \ldots, X_m = 1].
\end{aligned}$$

Noting that the sequence $X_1, X_2, \ldots$ is temporally homogeneous, by conditioning on the location where the outcome does not match the pattern $[[1]^m[0]^n]^k$ for the first time, we observe after some algebraic manipulations for $x \geq (m+n)k$,

$$\phi_x(t) = \sum_{j=0}^{k-1} (p^m q^n)^j \left\{ \phi_{x-(m+n)j-1}(t) - p^m q^n \phi_{x-(m+n)j-m-n}(t) \right\} + (p^m q^n)^k t \phi_{x-(m+n)k}(t). \tag{7}$$

Since $\phi_x(t) = 1$ for $x < (m+n)k$, we can effectively compute the p.g.f. of N_x by the recurrence relation (7).

When a positive integer x is given, the formula (7) suffices to evaluate the probability function of N_x. However, if an explicit formula for the p.g.f. of N_x is necessary for every positive integer x, it is convenient to find the double generating function $\Phi(t,z) := \sum_{x=0}^{\infty} \phi_x(t) z^x$. By using the formula (7), we derive the function $\Phi(t,z)$. Multiplying both sides of (7) by z^k, and adding them for $x = (m+n)k, (m+n)k+1, \ldots,$ we obtain

$$\begin{aligned}\Phi(t,z) - \sum_{x=0}^{(m+n)k-1} z^x &= \sum_{j=0}^{k-1} (p^m q^n)^j \left\{ z^{(m+n)j+1} \left(\Phi(t,z) - \sum_{x=0}^{(m+n)(k-j)-2} z^x \right) \right\} \\ &\quad - \sum_{j=0}^{k-2} (p^m q^n)^j z^{(m+n)(j+1)} \left(\Phi(t,z) - \sum_{x=0}^{(m+n)(k-j-1)-1} z^n \right) \\ &\quad - (p^m q^n)^k z^{(m+n)k} \Phi(t,z) + (p^m q^n z^{(m+n)})^k t \Phi(t,z).\end{aligned}$$

Then, we obtain the desired result by simplifying the above formula. □

5.5 Proof of Theorem 4

Proof We define the conditional p.g.f. for m_1

$$\phi_{m_1}(t) := E\left[t^{W-m_1} | X_1 = 1, X_2 = 1, \ldots, X_{m_1} = 1 \right].$$

Note that the sequence $\{X_n\}$ is temporally homogeneous. Thus, if $m \geq m_1$, for any integer ν, $E[t^{W-\nu} | X_{\nu-m} = 1, X_{\nu-m+1} = 1, \ldots, X_\nu = 1] = \phi_{m_1}(t)$ holds. With this in mind, conditioning on the first location where we fail in observing the sequential subpattern of the pattern, we obtain

$$\begin{aligned}\phi(t) = \sum_{j=0}^{k-1} (\alpha(t))^j &\left\{ \sum_{\ell=1}^{m_1} p_1^{\ell-1} (1-p_1) t^\ell \phi(t) + (p_1 t)^{m_1+1} \phi_{m_1}(t) \right. \\ &+ (p_1 t)^{m_1} (1 - p_1 - p_{w_1}) t \phi(t)\end{aligned}$$

$$+ (p_1 t)^{m_1} \sum_{\ell=1}^{m_2-1} p_{w[\ell]} t^{\ell} \left(p_1 t \phi_1(t) + (1 - p_1 - p_{w_{\ell+1}}) t \phi(t) \right) \Bigg\} + (\alpha(t))^k. \tag{8}$$

Let us examine relations between $\phi_1(t)$ and $\phi(t)$, and between $\phi_{m_1}(t)$ and $\phi(t)$. It is easy to see that

$$\phi_1(t) = \frac{1 - (1 - p_1)t}{p_1 t} \phi(t). \tag{9}$$

and

$$\phi_{m_1}(t) = \frac{1 - t + (p_1 t)^{m_1}(1 - p_1)t}{(p_1 t)^{m_1}(1 - p_1 t)} \phi(t). \tag{10}$$

Substituting (9) and (10) in (8), we obtain

$$\begin{aligned}
\phi(t) = \sum_{j=0}^{k-1} (\alpha(t))^j \Bigg\{ &\sum_{\ell=1}^{m_1} p_1^{\ell-1}(1 - p_1) t^{\ell} \phi(t) + \frac{(p_1 t)(1 - t + (p_1 t)^{m_1}(1 - p_1)t)}{1 - p_1 t} \phi(t) \\
&+ \frac{(1 - p_1 t)(p_1 t)^{m_1}(1 - p_1 - p_{w_1})t}{1 - p_1 t} \phi(t) \\
&+ (p_1 t)^{m_1} \sum_{\ell=1}^{m_2-1} p_{w[\ell]} t^{\ell} \left((1 - (1 - p_1)t)\phi(t) + (1 - p_1 - p_{w_{\ell+1}}) t \phi(t) \right) \Bigg\} + (\alpha(t))^k.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\frac{(p_1 t)(1 - t + (p_1 t)^{m_1}(1 - p_1)t)}{1 - p_1 t} + \frac{(1 - p_1 t)(p_1 t)^{m_1}(1 - p_1 - p_{w_1})t}{1 - p_1 t} \\
&\quad = t - (p_1 t)^{m_1} p_{w_1} t,
\end{aligned}$$

we have $\phi(t) = \frac{(\alpha(t))^k (1 - \alpha(t))}{1 - t + (\alpha(t))^k (t - \alpha(t))}$. This completes the proof. □

5.6 Proof of Proposition 3

Proof Since $\alpha(t) = a t^{\ell}$, from the equation

$$\frac{(a t^{\ell})^k (1 - a t^{\ell})}{1 - t + (a t^{\ell})^k (t - a t^{\ell})} = \sum_{\nu} P_{\nu} t^{\nu},$$

we have

$$a^k t^{\ell k} - a^{k+1} t^{\ell(k+1)} = \sum_n \left(P_n - P_{n-1} + a^k P_{n-\ell k-1} + a^{k+1} P_{n-\ell(k+1)} \right) t^n.$$

By comparing the coefficients of t^n in both sides of the above equation, we obtain the recurrence relations. □

5.7 Proof of Theorem 5

Proof Conditioning on the location of the first failure of observing the pattern, we have

$$\begin{aligned}\phi(\mathbf{s}, t) = &\sum_{j=0}^{k-1}(p_1^{m_1} p_w)^j s_1^j s_2^{(j-1)\vee 0} \dots s_{k-1}^{(j-(k-2))\vee 0} t^{j(m_1+m_2)}\\ &\times \Bigg\{\sum_{\ell=1}^{m_1} p_1^{\ell-1}(1-p_1)t^\ell \phi(\mathbf{s}, t)\\ &\quad + (p_1 t)^{m_1+1}\phi_{m_1}(\mathbf{s}, t) + (p_1 t)^{m_1}(1-p_1-p_{w_1})t\phi(\mathbf{s}, t)\\ &+ (p_1 t)^{m_1}\sum_{\ell=1}^{m_2-1} p_{w[\ell]} t^\ell (p_1 t\phi_1(\mathbf{s}, t) + (1-p_1-p_{w_{\ell+1}})t\phi(\mathbf{s}, t))\Bigg\}\\ &+ (p_1^{m_1} p_w)^k s_1^k s_2^{k-1} \dots s_{k-1}^2 t^{k(m_1+m_2)}.\end{aligned}$$

Further algebraic manipulations on the above equation yield

$$\begin{aligned}\phi(\mathbf{s}, t) = &\sum_{j=0}^{k-1}(p_1^{m_1} p_w)^j s_1^j s_2^{(j-1)\vee 0} \dots s_{k-1}^{(j-(k-2))\vee 0} t^{j(m_1+m_2)}(t-(p_1 t)^{m_1} p_w t^{m_2})\phi(\mathbf{s}, t)\\ &+ (p_1^{m_1} p_w)^k s_1^k s_2^{k-1} \dots s_{k-1}^2 t^{k(m_1+m_2)}.\end{aligned}$$

Solving the equation with respect to $\phi(\mathbf{s}, t)$, we have the desired result. □

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