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Poisson source localization on the plane: change-point case

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Abstract
We present a detection problem where several spatially distributed sensors observe Poisson signals emitted from a single radioactive source of unknown position. The measurements at each sensor are modeled by independent inhomogeneous Poisson processes. A method based on Bayesian change-point estimation is proposed to identify the location of the source's coordinates. The asymptotic behavior of the Bayesian estimator is studied. In particular, the consistency and the asymptotic efficiency of the estimator are analyzed. The limit distribution and the convergence of the moments are also described. The similar statistical model could be used in GPS localization problems.

Keywords Inhomogeneous Poisson process · Change-point problem · Bayesian estimator · Likelihood ratio process · Radioactive source · Sensors · GPS localization

1 Introduction

In this work, we study the properties of Bayesian estimators for the localization of a radioactive source emitting a signal that propagates over an area monitored by a set of sensors. This mathematical model could be used for the description of a radioactive emission, an explosion, a seismic activity or the detection of weak optical signals. Sensors are electronic devices that can measure changes in the environment around them; for instance, there are light sensors, proximity sensors, pressure sensors, heat sensors, radiation sensors, etc. The model under study could describe such data if the

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sequence of observed random events is of Poisson nature. Data obtained from a single sensor are often not fully reliable and incomplete due to single device's technical limitations. Using data from several sensors has advantages over data collected from a single sensor. If several identical sensors are employed, the observation process can be improved by combining individual information to generate a more complete picture of the environment monitoring. We refer the interested reader to Magee and Aggarwal (1985) or Chao et al. (1987) for the advantages of using multiple sensors. It has been shown that the probability of measurement error decreases with the size of the sensor network. However, it is worth mentioning the complexity of the monitoring system will increase with the number of sensors. Source tracking and localization are a problem of considerable importance that has attracted the scientific interest. Many examples of applications for such problems can be found in environmental monitoring, industrial sensing, infrastructure security, military tracking and diverse areas of security and defense, see, for instance, Zhao and Guibas (2004) and Chong and Kumar (2003). The present work focuses on the detection of explosion sources. Due to the recent events, security issues have become more and more concerning and the problem of detecting radioactive sources, more specifically the detection of illicit radioactive substances, stored or in transit, has received great deal of attention by the engineering community.

The detection of hidden nuclear material by means of sensors is an active area of research as part of defensive strategies. One can consult the work of Baidoo-Williams et al. (2015), Liu and Nehorai (2004) and Rao et al. (2008) for details and references on this topic. Nuclear radiations are a probabilistic physical process consisting of discrete emissions of particles that can be recorded by radiation sensors. Those emissions have been mathematically modeled with help of Poisson point processes which provide natural models describing their properties, see, for instance, Evans (1963) or Knoll (2010). Apart from radiation measurements, typical examples on the use of Poisson point processes include modeling streams of photo-electrons produced by light on photosensitive surfaces Mandel (1958), laser radar detection and ranging of objects Karr (1991), earthquake aftershocks Ogata (1994), electrical response of nerves to stimulus Snyder and Miller (1991) and others; for application to tracking and sensing, we refer to the book of Streit (2010). Special cases of the source localization problem have been studied in the past; for instance, Howse et al. (2011) described least squares estimation algorithms to estimate the location of a possibly moving source by a fixed number of sensors. For multiple sources, maximum likelihood estimation (MLE) was considered by Morelande et al. (2007). An iterative procedure for calculating MLEs of a single nuclear source from radiation measurements as well as corresponding Cramer–Rao bounds for localization accuracy was given by Baidoo-Williams et al. (2015). Concerning Bayesian statistics Liu and Nehorai (2004) presented a technique to locate a source according to Bayesian update methods. The results of Pahlajani et al. (2013) are also noteworthy: their paper studies the presence of a source using Likelihood ratio calculation and a Neyman–Pearson test. In what follows, we suppose that there is a single source generating a signal. Our goal is to describe the asymptotic behavior of the Bayesian estimator (BE) of its coordinates through the method developed by Ibragimov and Khasminskii (1981) for the study of such estimators. We show that the rate of convergence of the estimator is n and that the limit distribution is not

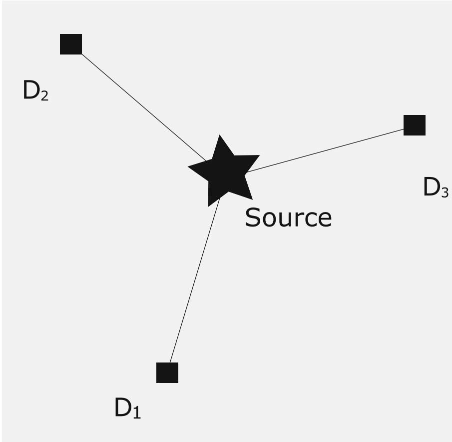

Fig. 1 Model of observations

Gaussian. A lower bound on the mean square risk is proposed, and the BE is proved to be asymptotically efficient.

Note that the same mathematical model can be used in the problem of GPS localization on the plane Luo (2013). Indeed, in this case the signals are emitted by k fixed emitters and an object receiving these signals has to define its own position. Here once more we have k signals with unknown moments of arriving, and using the estimators of these moments, the object can construct the estimator of its position.

2 Statement of the problem

We are interested in locating the source of an event with the help of several spatially distributed independent sensors monitoring an area over a fixed time interval. For example, if we have a radioactive source, then each sensor records ambient measurements, for instance, radiations due to natural isotopes in the environment. When the event occurs, then the sensors record the sum of ambient measurements and the measurements related to the event. The two signals are independently, and we consider that each sensor records a single inhomogeneous Poisson process whose intensity is the sum of the intensities due to both ambient and background event measurements.

Popular network topologies for source localization problems that were considered in other studies are grids of sensors Liu and Nehorai (2004) and triangular arrays Chin et al. (2010). In order to identify the source location, we use a configuration of sensors forming a triangle.

In our case, we have sequences of measurements from three sensors and collected within the same time window. The measurements from each sensor are sent to a central processing unit (fusion center) that combines the data and estimates the coordinates of the source (Fig. 1).

The source is located at an unknown position D_0 with coordinates $\vartheta_0 = (x_0, y_0)$ inside a convex set $\Theta \subset \mathcal{R}^2$. Three sensors are placed in the field at known positions at points D_1, D_2, D_3 with the coordinates $\vartheta_j = (x_j, y_j)$, $j = 1, 2, 3$. Each sensor records on the time interval $[0, T]$ a signal modeled by a Poisson point process $X_j = \left\{X_j(t),\ 0 \leq t \leq T\right\}$, $j = 1, 2, 3$ of intensity function $\lambda_j(\vartheta_0, t), 0 \leq t \leq T$. These intensity functions are supposed to be of the form

$$\lambda_j(\vartheta_0, t) = \lambda\left(t - \tau_j\right) + \lambda_0, \qquad 0 \leq t \leq T.$$

Here $\lambda_0 > 0$ is a known intensity of the background noise, $\lambda(t)$ is the known intensity function of the signal, and $\tau_j = \tau_j(\vartheta_0)$ is the arrival time of the signal to the j-th sensor (delay). This delay is calculated following the usual rule

$$\tau_j(\vartheta_0) = \frac{||\vartheta_j - \vartheta_0||}{\nu}, \tag{1}$$

where $||\cdot||$ is the Euclidean norm and ν is the known rate of propagation of the signal in the monitored area. We suppose that $\lambda(t) = 0$ for $t < 0$. At time $t = 0$ the emission of signals begins and τ_j is the arrival time of the signal to the j-th sensor. We are concerned by estimating the position ϑ_0 of the radioactive source. We are interested in the models of observations which allow the estimation with small errors such that $\mathbf{E}_{\vartheta_0}\left(\overline{\vartheta} - \vartheta_0\right)^2 = o(1)$. As usual such situations are considered in an asymptotic framework. The small errors can be obtained if the intensity of the signal takes large values or a periodical Poisson process could describe the data. Another possibility is to have many sensors. We take the model with large intensity functions $\lambda_j(\vartheta_0, t) = \lambda_{j,n}(\vartheta_0, t)$ which can be written as follows

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda\left(t - \tau_j\right) + n\lambda_0, \qquad 0 \leq t \leq T. \tag{2}$$

Here n is a "large parameter," and we study estimators as $n \to \infty$. For example, such a model could be obtained in the case of three clusters, where each cluster includes n detectors.

The likelihood ratio function $L(\vartheta, X^n)$ is

$$\ln L\left(\vartheta, X^n\right) = \sum_{j=1}^{3} \int_{\tau_j}^{T} \ln\left(1 + \frac{\lambda\left(t - \tau_j\right)}{\lambda_0}\right) \mathrm{d}X_j(t) - n\sum_{j=1}^{3}\int_{\tau_j}^{T} \lambda\left(t - \tau_j\right) \mathrm{d}t. \tag{3}$$

Here $\tau_j = \tau_j(\vartheta)$ and $X^n = \left(X_j(t), 0 \leq t \leq T, j = 1, 2, 3\right)$ are counting processes from three detectors. Based on this likelihood ratio formula, we define the maximum likelihood estimator (MLE) $\hat{\vartheta}_n$ and Bayesian estimator (BE) $\widetilde{\vartheta}_n$ by

$$L\left(\hat{\vartheta}_n, X^n\right) = \sup_{\vartheta \in \Theta} L\left(\vartheta, X^n\right), \tag{4}$$

and

$$\widetilde{\vartheta}_n = \frac{\int_{\Theta} \vartheta\, p(\vartheta)\, L(\vartheta, X^n)\, \mathrm{d}\vartheta}{\int_{\Theta} p(\vartheta)\, L(\vartheta, X^n)\, \mathrm{d}\vartheta}, \tag{5}$$

respectively.

Here $p(\vartheta), \vartheta \in \Theta$ is the prior density. As the limit properties of the BE do not depend on the prior density, we could consider a non-informative prior such as the uniform density. For any other positive continuous function $p(\cdot)$, the limit properties will remain the same.

Recall that in the case of a discontinuous intensity function $\lambda(\cdot)$ the definition of the MLE has to be modified since

$$\ln L\left(\vartheta, X^n\right) = \sum_{j=1}^{3} \sum_{i=1}^{N_j} \ln\left(1 + \frac{\lambda\left(t_{i,j} - \tau_j(\vartheta)\right)}{\lambda_0}\right) - n \sum_{j=1}^{3} \int_{\tau_j(\vartheta)}^{T} \lambda\left(t - \tau_j(\vartheta)\right) \mathrm{d}t.$$

Here $t_{i,j}, i = 1, \ldots, N_j$ are the registration times of the events in the j-th sensor, and N_j is the total number of events in this sensor. Of course, if $N_j = 0$, then we set

$$\sum_{i=1}^{N_j} \ln\left(1 + \frac{\lambda\left(t_{i,j} - \tau_j(\vartheta)\right)}{\lambda_0}\right) = 0.$$

We write formally

$$\max\left(L(\hat\vartheta_n -, X^n), L(\hat\vartheta_n +, X^n)\right) = \sup_{\vartheta\in\Theta} L(\vartheta, X^n)$$

which we understand as follows. The function

$$M\left(\tau_1(\vartheta), \tau_2(\vartheta), \tau_3(\vartheta), X^n\right) = L\left(\vartheta, X^n\right)$$

has jumps at points $\tau_j(\vartheta) = t_{i,j}$, and its supremum is in one of the jump points. It can be written

$$\sup_{\vartheta\in\Theta} L\left(\vartheta, X^n\right) = \max M\left(\tau_1(\hat\vartheta_n)\pm, \tau_2(\hat\vartheta_n)\pm, \tau_3(\hat\vartheta_n)\pm, X^n\right).$$

Here $M(\tau_1(\vartheta)\pm, \tau_2(\vartheta)\pm, \tau_2(\vartheta)\pm, X^n)$ are left and right limits of the function $M(\tau_1(\vartheta), \tau_2(\vartheta), \tau_2(\vartheta), X^n)$ at the points $\tau_j(\vartheta)$.

There are several different types of problems associated with the identification of the location depending on the regularity of the function $\lambda(t)$. In particular, the rate of convergence of the mean square error of the estimators $\overline{\vartheta}_n$ is

$$\mathbf{E}_{\vartheta_0}\left(\overline{\vartheta}_n - \vartheta_0\right)^2 = \frac{C}{n^\gamma}\left(1 + o(1)\right),$$

where the parameter $\gamma > 0$ depends on the regularity of the function $\lambda(\cdot)$.

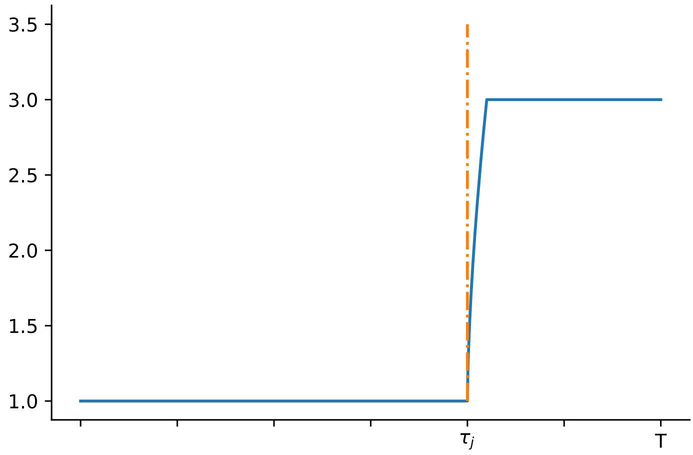

Fig. 2 Intensity (6) with $\kappa = \frac{5}{8}$, $\delta = 0, 1$

Let us present three of them. All the cases are illustrated using the following model

$$\lambda(\vartheta, t) = 2\left|\frac{t - \tau_j(\vartheta)}{\delta}\right|^{\kappa} \mathbb{1}_{\{0 \le t - \tau_j(\vartheta) < \delta\}} + 2\mathbb{1}_{\{t - \tau_j(\vartheta) \ge \delta\}} + 1. \tag{6}$$

Statistical problems related to different types of regularity could be obtained according to the values of parameter κ.

• **Smooth case.** Suppose that the function $\lambda(\cdot)$ in (2) is sufficiently smooth, for example, continuously differentiable, then the problem of parameter estimation is regular.

The MLE $\hat{\vartheta}_n$ and BE $\widetilde{\vartheta}_n$ (under regularity conditions) are consistent, asymptotically normal

$$\sqrt{n}\left(\hat{\vartheta}_n - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}(\vartheta_0)^{-1}\right), \quad \sqrt{n}\left(\widetilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}(\vartheta_0)^{-1}\right),$$

the moments converge, and both estimators are asymptotically efficient. Here $\mathbb{I}(\vartheta_0)$ represents the Fisher information matrix. For the mean square error, the following relation holds true:

$$\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_n - \vartheta_0\right\|^2 = \frac{C}{n}(1 + o(1)),$$

i.e., $\gamma = 1$.

This kind of regularity corresponds to the intensity function (6) with $\kappa > \frac{1}{2}$. An example of such an intensity function is given in Fig. 2.

It is worth mentioning that the derivative of this function is a discontinuous function; however, it is continuous in $L_2(0, T)$ and the MLE has all the aforementioned properties.

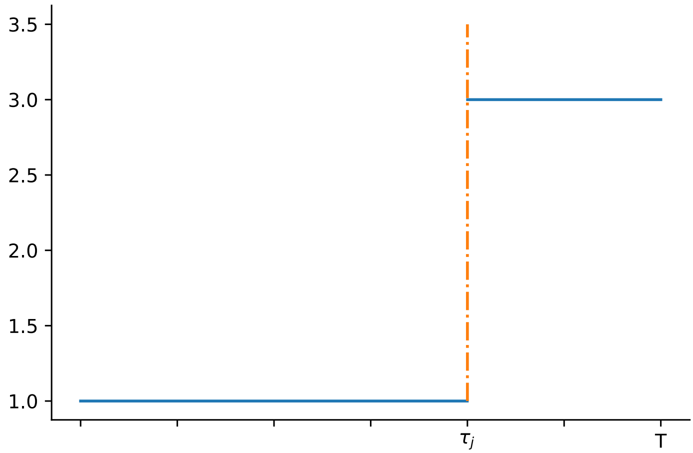

Fig. 3 Intensity (6) with $\kappa = 0$, $\delta = 0$

We describe these properties of estimators in the problem of radioactive source localization in the forthcoming work Chernoyarov and Kutoyants (2018).

• **Change-point case.** Suppose that the intensity function in (2) has the following form

$$\lambda_{j,n}(\vartheta, t) = n\lambda_1\left(t - \tau_j\right)\mathbb{1}_{\{t\geq\tau_j\}} + n\lambda_0, \qquad 0 \leq t \leq T.$$

Here $\lambda_1(t) > 0$, $t \geq 0$ and $\lambda_0 > 0$ are known.

This type of statistical problems corresponds to the intensity function (6) with $\kappa = 0$ and $\delta = 0$ (see Fig. 3).

In this situation, the intensities of the observed Poisson processes have positive jumps equal to $n\lambda_1(0)$ at the points $t = \tau_j = \tau_j(\vartheta_0)$. This is a non-regular parameter estimation problem, where the MLE and BE have the normalization n and different limit distributions

$$n\left(\hat{\vartheta}_n - \vartheta_0\right) \Longrightarrow \hat{\zeta}, \qquad n\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \tilde{\zeta}.$$

The moments of these estimators converge, but only the BE is asymptotically efficient. The random vectors $\hat{\zeta}$ and $\tilde{\zeta}$ are exponential functionals of some Poisson processes. The mean square error decreases as follows

$$\mathbf{E}_{\vartheta_0}\left\|\tilde{\vartheta}_n - \vartheta_0\right\|^2 = \frac{C}{n^2}\left(1 + o(1)\right),$$

i.e., $\gamma = 2$. Similar results in the case of an one-dimensional parameter ϑ could be found in Kutoyants (1998).

Here we focus on the study of the BE for this model of observations.

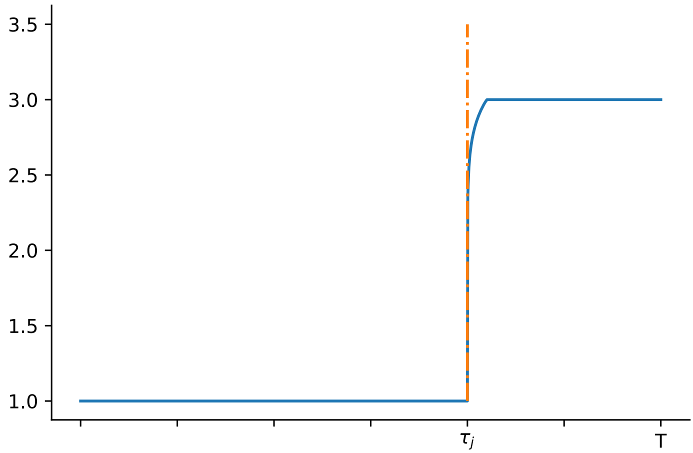

Fig. 4 Intensity (6) with $\kappa = 0, 1$, $\delta = 0, 1$

• **Cusp-type case.** This case is in some sense intermediate between the smooth and change-point cases. Suppose that the intensity function has the following form

$$\lambda_{j,n}(\vartheta, t) = n\,\lambda_1\left(t - \tau_j\right)\left|\frac{t - \tau_j}{\delta}\right|^{\kappa} \mathbb{1}_{\{0 \le t - \tau_j \le \delta\}} + n\lambda_1\left(t - \tau_j\right)\mathbb{1}_{\{t - \tau_j > \delta\}} + n\lambda_0.$$

The parameter $\kappa \in (0, \frac{1}{2})$, the parameter $\delta > 0$ takes small values, and the function $\lambda_1(t) = \lambda_1(0) + O(t) > 0$, where $\lambda_1(0) > 0$.

An example of such a function is given in Fig. 4.

In the statistical literature, change-point models are well studied, but in some real cases the intensity function could not have pure discontinuity since due to the physical laws the electrical current could not have jumps and the cusp-type model fits much better to the real data with strongly increased intensities. The intensity of the signal increases from zero to $\lambda(\tau + \delta)$ in the small interval $[\tau, \tau + \delta]$. Note that for these values of κ the Fisher information does not exist which leads to a singular estimation problem. The MLE and BE for this model of observations are consistent and have different limit distributions

$$n^{\frac{1}{2\kappa+1}}\left(\hat{\vartheta}_n - \vartheta_0\right) \Longrightarrow \hat{\xi}, \qquad n^{\frac{1}{2\kappa+1}}\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \tilde{\zeta},$$

the moments converge, and only the BE is asymptotically efficient. The random vectors $\hat{\xi}$ and $\tilde{\xi}$ are exponential functionals of the fractional Brownian motions.

$$\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_n - \vartheta_0\right\|^2 = \frac{C}{n^{\frac{2}{2\kappa+1}}}\left(1 + o(1)\right),$$

i.e., $\gamma = \frac{2}{2\kappa+1}$ and $1 < \gamma < 2$. These cases will be studied in the forthcoming work Dachian et al. (2018a). For the one-dimensional parameter case, see Dachian (2003).

The properties of the MLE and BE of the one-dimensional parameter in such three types of regularity problems for the signals observed in the white Gaussian noise are discussed in Dachian et al. (2018b).

3 Main results

There are three sensors with coordinates $\vartheta_j = \left(x_j, y_j\right), j = 1, 2, 3$ which measure the particles emitted by some source at the point $\vartheta_0 = (x_0, y_0)$. The observations are modeled by three independent inhomogeneous Poisson processes $X^n = \left(X_j(t), 0 \le t \le T, j = 1, 2, 3\right)$ with respective intensity functions

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda\left(t - \tau_j(\vartheta)\right)\mathbb{1}_{\{t \ge \tau_j(\vartheta)\}} + n\lambda_0, \qquad 0 \le t \le T$$

where $\lambda(t) > 0$ and $\lambda_0 > 0$. The arrival times of the signals in the j-th sensor according to (1) are $\tau_j = \tau_j(\vartheta_0)$, and the position of the source $\vartheta_0 = (x_0, y_0) \in \Theta \subset \mathcal{R}^2$ will be estimated. We suppose that the set Θ is non-empty, open and convex subset of $(\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$ and such that for all $\vartheta \in \Theta$ the corresponding $\tau(\vartheta) \in (0, T)$.

Note that if the model of observations with the constant intensities of the signal and noise is considered, i.e.,

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda_1\mathbb{1}_{\{t \ge \tau_j(\vartheta)\}} + n\lambda_0, \qquad 0 \le t \le T,$$

where $\lambda_1 = \lambda(0) > 0$, then the asymptotic properties of the estimators will be the same.

Let us recall the notations: λ_0 is known positive constant (intensity of the noise), $\lambda(\cdot)$ is known smooth positive function (intensity function of the signal), $\lambda_1 = \lambda(0)$ known constant, $\nu > 0$ is known rate of propagation of the signals, $\vartheta_j, j = 1, \ldots, k$ are known positions of the detectors, $\vartheta_0 = (x_0, y_0)$ unknown two-dimensional parameter (position of the source).

We study the asymptotic ($n \to \infty$) behavior of the Bayesian estimator of the unknown parameter $\vartheta_0 = (x_0, y_0)$. It is worth noticing that in such non-regular estimation problems the asymptotic results could be applied even for moderate values of n since we have faster convergence of estimators (rate n and not $\sqrt{n}$ as in the regular case).

Let us introduce the quantities

$$\underline{\tau} = \min_{j=1,2,3}\inf_{\vartheta \in \Theta} \tau_j(\vartheta), \qquad \overline{\tau} = \max_{j=1,2,3}\sup_{\vartheta \in \Theta} \tau_j(\vartheta), \qquad \mathbb{T} = [0, T - \overline{\tau}].$$

We suppose that $T > \overline{\tau}$. At this point we have to suppose some conditions providing the identifiability of the position of the source.

Conditions $\mathcal{I}$:

$\mathcal{I}_1$. *The location of the source is different from the sensor location. Consequently, we suppose that there exists a constant* $\varepsilon > 0$ *such that for every possible position of the source* $\vartheta_0 \in \Theta$ *and* $j = 1, 2, 3$

$$\rho_j = ||\vartheta_j - \vartheta_0|| \geq \varepsilon.$$

$\mathcal{I}_2$. *The function* $\lambda(s)$, $s \in \mathbb{T}$ *has two continuous derivatives*
$\mathcal{I}_3$. *The sensors are not aligned, therefore*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0.$$

By condition $\mathcal{I}_1$ the case $\tau_j = 0$ is excluded. Due to condition $\mathcal{I}_1$, we restrain the parameter space to

$$\Theta = \left[(\alpha_1, \alpha_2) \times (\beta_1, \beta_2)\right] \Big\backslash \left[\bigcup_{j=1}^{3} B(\vartheta_j, \varepsilon)\right],$$

where $B(\vartheta_j, \varepsilon) = \{z \in \mathbb{R}^2 : ||\vartheta_j - z|| \leq \varepsilon\}$. If the position of the source coincides with the position of one of the sensors, then for this sensor $\tau_j = 0$ and the properties of the estimators will be different. For example, the limit likelihood ratio $Z(u)$ can be defined for the positive values of one component of u only. This situation corresponds to the case, where the true value of the unknown parameter is on the border of a parametric set [see, e.g., Kutoyants (1998), where such situation was described]. Remark that if the condition $\mathcal{I}_2$ is not fulfilled and the sensors are in the same line, then the consistent estimation of the position ϑ_0 is not feasible. Of course, such conclusion depends on the set Θ too. Suppose that the detectors are on a line on the seashore and the source can be only be located on one side, then two detectors are sufficient for the consistent estimation of the position of the radioactive source.

The likelihood $L(\vartheta, X^n)$ according to (3) is given by [see, for example, Kutoyants (1998)].

$$\begin{aligned} \ln L\left(\vartheta, X^{(n)}\right) &= \sum_{j=1}^{3} \int_0^T \ln \frac{\lambda_{j,n}(\vartheta, t)}{n\lambda_0} \mathrm{d}X_j(t) - \sum_{j=1}^{3} \int_0^T \left(\lambda_{j,n}(\vartheta, t) - n\lambda_0\right) \mathrm{d}t \\ &= \sum_{j=1}^{3} \int_{\tau_j}^T \ln\left(1 + \frac{\lambda\left(t - \tau_j\right)}{\lambda_0}\right) \mathrm{d}X_j(t) - n \sum_{j=1}^{3} \int_{\tau_j}^T \lambda\left(t - \tau_j\right) \mathrm{d}t. \end{aligned}$$

Recall that here $\tau_j = \tau_j(\vartheta)$.

If the intensity function of the signal is constant $\lambda(t) \equiv \lambda_1 > 0$, then the likelihood ratio is simplified

$$\ln L\left(\vartheta, X^{(n)}\right) = \ln\left(1 + \frac{\lambda_1}{\lambda_0}\right) \sum_{j=1}^{3} \left[X_j(T) - X_j\left(\tau_j\right)\right] - n\lambda_1 \sum_{j=1}^{3} \left[T - \tau_j\right].$$

The Bayesian estimator $\widetilde{\vartheta}_n = (\widetilde{x}_n, \widetilde{y}_n)$ of the parameter $\vartheta_0 = (x_0, y_0)$ with respect to the quadratic loss function is defined by a conditional expectation which can be written as follows

$$\widetilde{\vartheta}_n = \mathbf{E}\left(\vartheta / X^{(n)}\right) = \int_\Theta \vartheta\, p(\vartheta) L\left(\vartheta, X^{(n)}\right) \,\mathrm{d}\vartheta \left(\int_\Theta p(\vartheta) L\left(\vartheta, X^{(n)}\right) \,\mathrm{d}\vartheta\right)^{-1}.$$

Even if the vector ϑ is not random with a given prior density we can use this formula to calculate $\widetilde{\vartheta}_n$ which is no more a conditional expectation, but just some way to construct the estimator. In this case it can be called *generalized Bayesian estimator* Ibragimov and Khasminskii (1981). Therefore, we can take any positive continuous function $p(\vartheta)$, $\vartheta \in \Theta$. For example, as the set Θ is bounded, we can put $p(\vartheta) = 1$.

Note that if the intensity of the signal is constant $\lambda(t) \equiv \lambda_1$, then the estimator can be calculated as follows

$$\widetilde{\vartheta}_n = \frac{\int_\Theta \vartheta \prod_{j=1}^3 \left(1 + \frac{\lambda_1}{\lambda_0}\right)^{-X_j(\tau_j(\vartheta))} e^{n\lambda_1 \sum_{j=1}^3 \tau_j(\vartheta)} \mathrm{d}\vartheta}{\int_\Theta \prod_{j=1}^3 \left(1 + \frac{\lambda_1}{\lambda_0}\right)^{-X_j(\tau_j(\vartheta))} e^{n\lambda_1 \sum_{j=1}^3 \tau_j(\vartheta)} \mathrm{d}\vartheta},$$

where $\tau_j(\vartheta) = \nu^{-1} \left\|\vartheta_j - \vartheta\right\|$.

In order to describe the properties of the Bayesian estimator, we need some additional notations. First let us introduce the unit vectors m_j, for $j = 1, \ldots, 3$

$$m_j = \left(\frac{x_j - x_0}{\rho_j}, \frac{y_j - y_0}{\rho_j}\right), \qquad \rho_j = \left\|\vartheta_j - \vartheta_0\right\|, \quad \left\|m_j\right\| = 1$$

and the sets

$$\mathbb{B}_j = \left\{u : \quad \langle m_j, u\rangle \geq 0\right\}, \quad \mathbb{B}_j^c = \left\{u : \quad \langle m_j, u\rangle < 0\right\}.$$

Here $\langle m_j, u\rangle$ denotes the Euclidean scalar product of the vectors m_j and $u = (u_1, u_2)$. The limit likelihood ratio $Z(u)$, $u \in \mathbb{R}^2$ we denote as follows

$$\begin{aligned}\ln Z(u) = \ell \sum_{j=1}^3 &\left[\Pi_{j,+}(u)\, \mathbb{1}_{\{u \in \mathbb{B}_j\}} - \Pi_{j,-}(u)\, \mathbb{1}_{\left\{u \in \mathbb{B}_j^c\right\}}\right] \\ &- \frac{\lambda_1}{\nu} \langle m_1 + m_2 + m_3, u\rangle,\end{aligned}$$

where $\ell = \ln\left(1 + \frac{\lambda_1}{\lambda_0}\right)$, $\Pi_{j,+}(u)$, $u \in \mathbb{B}_j$ and $\Pi_{j,-}(u)$, $u \in \mathbb{B}_j^c$ are independent Poisson random fields such that

$$\mathbf{E}_{\vartheta_0} \Pi_{j,+}(u) = \frac{\lambda_0}{\nu} \langle m_j, u\rangle, \qquad \mathbf{E}_{\vartheta_0} \Pi_{j,-}(u) = -\frac{\lambda_1 + \lambda_0}{\nu} \langle m_j, u\rangle.$$

Remark that such fields can be described better using 6 one-dimensional Poisson processes as follows. Let us put

$$\Pi_{j,+}(u) = \widetilde{\Pi}_{j,+}(s_+(u)), \qquad \Pi_{j,-}(u) = \widetilde{\Pi}_{j,-}(s_-(u)), \quad j = 1, 2, 3.$$

Here

$$s_+(u) = \frac{\lambda_0}{\nu}\langle m_j, u\rangle \in [0,\infty), \qquad s_-(u) = -\frac{\lambda_1+\lambda_0}{\nu}\langle m_j, u\rangle \in [0,\infty)$$

and $\widetilde{\Pi}_{j,+}(s_+)$, $s_+ \in [0,\infty)$, $\widetilde{\Pi}_{j,-}(s_-)$, $s_- \in [0,\infty)$ are independent Poisson processes on the half-line $[0,\infty)$.

For example, if $u, u_* \in \mathbb{B}_j$ and $u - u_* \perp m_j$, then $\Pi_{j,+}(u) = \Pi_{j,+}(u_*)$.

Second, we define the random vector $\widetilde{\zeta} = (\widetilde{\zeta}_1, \widetilde{\zeta}_2)$ with the components

$$\widetilde{\zeta}_1 = \int_{\mathcal{R}^2} u_1 Z(u_1,u_2)\, \mathrm{d}u_1\mathrm{d}u_2\left(\int\int_{\mathcal{R}^2} Z(u_1,u_2)\, \mathrm{d}u_1\mathrm{d}u_2\right)^{-1}$$

and

$$\widetilde{\zeta}_2 = \int_{\mathcal{R}^2} u_2 Z(u_1,u_2)\, \mathrm{d}u_1\mathrm{d}u_2\left(\int\int_{\mathcal{R}^2} Z(u_1,u_2)\, \mathrm{d}u_1\mathrm{d}u_2\right)^{-1}.$$

The main results of this work are the following two theorems. We first introduce the lower bound on the risk of all estimators.

Theorem 1 *Let the conditions $\mathcal{I}$ be fulfilled. Then for all $\vartheta_0 \in \Theta$ and a quadratic loss function,*

$$\lim_{\delta\to 0}\varliminf_{n\to\infty}\inf_{\overline{\vartheta}_n}\sup_{||\vartheta-\vartheta_0||<\delta} n^2\mathbf{E}_\vartheta\left\|\overline{\vartheta}_n - \vartheta\right\|^2 \geq \mathbf{E}\|\widetilde{\zeta}\|^2. \tag{7}$$

Here the inf is taken over all possible estimators $\overline{\vartheta}_n$ of the parameter ϑ. The inequality (7) allows us to give the following definition of efficient estimator.

Definition 1 Let the conditions $\mathcal{I}$ be satisfied. The estimator ϑ_n^* is asymptotically efficient, if for all $\vartheta_0 \in \Theta$ we have

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\sup_{||\vartheta-\vartheta_0||<\delta} n^2\mathbf{E}_\vartheta\left\|\vartheta_n^* - \vartheta\right\|^2 = \mathbf{E}\|\widetilde{\zeta}\|^2. \tag{8}$$

The second theorem describes the asymptotic behavior of the estimator $\widetilde{\vartheta}_n = (\widetilde{x}_n, \widetilde{y}_n)$.

Theorem 2 *Let the conditions $\mathcal{I}$ be fulfilled. Then the Bayesian estimator $\widetilde{\vartheta}_n$ is uniformly on compacts $\mathbb{K} \subset \Theta$ consistent: for any $\gamma > 0$*

$$\sup_{\vartheta_0\in\mathbb{K}} \mathbf{P}_{\vartheta_0}\left(\|\widetilde{\vartheta}_n - \vartheta_0\| > \gamma\right) \longrightarrow 0,$$

we have convergence in distribution

$$n\left(\widetilde{\vartheta}_n-\vartheta_0\right)\quad\Longrightarrow\quad\widetilde{\zeta},$$

and convergence of moments: for any $p>0$

$$\lim_{n\to\infty} n^p\mathbf{E}_{\vartheta_0}\|\widetilde{\vartheta}_n-\vartheta_0\|^p=\mathbf{E}_{\vartheta_0}\|\ \widetilde{\zeta}\ \|^p,$$

and $\widetilde{\vartheta}_n$ *is asymptotically efficient.*

The proofs of these theorems are given in the next section. They are based on the general results of Ibragimov and Khasminskii (1981) for the problem of parameter estimation in the case of i.i.d. observations with a discontinuous density function and the application of their results to the study of Bayesian estimators for inhomogeneous Poisson processes see Kutoyants (1998), Chapter 5.

Let us remind the main steps of these proofs. Introduce the normalized likelihood ratio random field

$$Z_n\left(u\right)=\frac{L\left(\vartheta_0+\frac{u}{n},X^n\right)}{L\left(\vartheta_0,X^n\right)},\qquad u\in\mathbb{U}_n,$$

where

$$\mathbb{U}_n=\left\{u:\ \vartheta_0+\frac{u}{n}\in\Theta\right\}.$$

Moreover, we extend the set $\mathbb{U}_n$ to cover the balls around the sensors

$$\mathbb{U}_n=\Big(n\left(\alpha_1-x_0\right),n\left(\alpha_2-x_0\right)\Big)\times\Big(n\left(\beta_1-y_0\right),n\left(\beta_2-y_0\right)\Big)\nearrow\mathcal{R}^2,$$

as $n\to\infty$, i.e., we extended the process $Z_n\left(u\right)$ on the values u belonging to the balls $\vartheta_0+\frac{u}{n}\in B\left(\vartheta_j,\varepsilon\right)$. This requires certain modifications of the general method developed in Ibragimov and Khasminskii (1981), which can be done without difficulties. What is important is to respect the condition $\vartheta_0\notin B\left(\vartheta_j,\varepsilon\right)$.

Suppose that we have already proved the convergence of finite-dimensional distributions $Z_n\left(\cdot\right)\Longrightarrow Z\left(\cdot\right)$. Below we change the variables $\vartheta=\vartheta_0+\frac{u}{n}$. We have

$$\begin{aligned}\widetilde{\vartheta}_n&=\int_\Theta\vartheta\frac{L\left(\vartheta,X^n\right)}{L\left(\vartheta_0,X^n\right)}\,\mathrm{d}\vartheta\left(\int_\Theta\frac{L\left(\vartheta,X^n\right)}{L\left(\vartheta_0,X^n\right)}\,\mathrm{d}\vartheta\right)^{-1}\\&=\vartheta_0+\frac{1}{n}\int_{\mathbb{U}_n}uZ_n\left(u\right)\,\mathrm{d}u\left(\int_{\mathbb{U}_n}Z_n\left(u\right)\,\mathrm{d}u\right)^{-1}\end{aligned}$$

and

$$n\left(\widetilde{\vartheta}_n-\vartheta_0\right)=\int_{\mathbb{U}_n} uZ_n\left(u\right)\,\mathrm{d}u\left(\int_{\mathbb{U}_n} Z_n\left(u\right)\,\mathrm{d}u\right)^{-1}.$$

If we prove the convergence

$$\left(\int_{\mathbb{U}_n} u_1Z_n\left(u\right)\mathrm{d}u,\int_{\mathbb{U}_n} u_2Z_n\left(u\right)\mathrm{d}u,\int_{\mathbb{U}_n} Z_n\left(u\right)\,\mathrm{d}u\right)$$
$$\Longrightarrow\left(\int_{\mathcal{R}^2} u_1Z\left(u\right)\mathrm{d}u,\int_{\mathcal{R}^2} u_2Z\left(u\right)\mathrm{d}u,\int_{\mathcal{R}^2} Z\left(u\right)\,\mathrm{d}u\right),$$

then we obtain the limit

$$n\left(\widetilde{\vartheta}_n-\vartheta_0\right)\Longrightarrow\widetilde{\zeta}.$$

To obtain the convergence of moments, we have to check the uniform integrability of the random variables $\left\|n\left(\widetilde{\vartheta}_n-\vartheta_0\right)\right\|^p$ for any $p>0$.

This work was realized in Ibragimov and Khasminskii (1981) in a sufficiently general framework (see Theorem 1.10.2 there). In the next section we verify the conditions of this theorem.

Suppose that we already proved Theorem 2, then the proof of Theorem 1 could be done as follows. Let us fix some small $\delta>0$, then

$$\begin{aligned}\sup_{||\vartheta-\vartheta_0||<\delta} n^2\mathbf{E}_\vartheta\|\overline{\vartheta}_n-\vartheta\|^2 &\geq n^2\int_{B(\vartheta_0,\delta)}\mathbf{E}_\vartheta\|\overline{\vartheta}_n-\vartheta\|^2q\left(\vartheta\right)\mathrm{d}\vartheta\\ &\geq n^2\int_{B(\vartheta_0,\delta)}\mathbf{E}_\vartheta\|\widetilde{\vartheta}_{q,n}-\vartheta\|^2q\left(\vartheta\right)\mathrm{d}\vartheta,\end{aligned}$$

where $q\left(\vartheta\right),\ \vartheta\in B\left(\vartheta_0,\delta\right)$ is some positive continuous density on $B\left(\vartheta_0,\delta\right)$ and $\widetilde{\vartheta}_{q,n}$ is a BE, which corresponds to this prior density. From the convergence of second moments, we have

$$n^2\int_{B(\vartheta_0,\delta)}\mathbf{E}_\vartheta\|\widetilde{\vartheta}_{q,n}-\vartheta\|^2q\left(\vartheta\right)\mathrm{d}\vartheta\longrightarrow\int_{B(\vartheta_0,\delta)}\mathbf{E}_\vartheta\|\widetilde{\zeta}\|^2\,q\left(\vartheta\right)\,\mathrm{d}\vartheta.$$

The continuity of $\mathbf{E}_\vartheta\|\widetilde{\zeta}\|^2$ w.r.t. ϑ allows us to write the last limit

$$\int_{B(\vartheta_0,\delta)}\mathbf{E}_\vartheta\|\widetilde{\zeta}\|^2\,q\left(\vartheta\right)\,\mathrm{d}\vartheta\longrightarrow\mathbf{E}_{\vartheta_0}\|\widetilde{\zeta}\|^2$$

as $\delta\to0$. Note that the lower bound (7) is a particular case of more general result in Ibragimov and Khasminskii (1981).

4 Proofs

Introduce the normalized likelihood random field

$$Z_n(u) = \exp\Bigg\{\sum_{j=1}^{3}\int_0^T \ln\frac{\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)}{\lambda_{j,n}\left(\vartheta_0,t\right)}\mathrm{d}X_j(t) \\ -\sum_{j=1}^{3}\int_0^T\left(\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)-\lambda_{j,n}\left(\vartheta_0,t\right)\right)\mathrm{d}t\Bigg\},$$

where $u=(u_1,u_2)\in\mathbb{U}_n$.

Lemma 1 *Let the conditions $\mathcal{I}_1,\mathcal{I}_2$ be satisfied, then the finite-dimensional distributions of the process $Z_n(u)$, $u\in\mathbb{U}_n$ converge to the finite-dimensional distributions of the process $Z(u)$, $u\in\mathcal{R}^2$, and this convergence is uniform with respect to $\vartheta_0\in\mathbb{K}$.*

Proof The characteristic function of $\ln Z_n(u)$ is calculated as follows [see Kutoyants (1998)].

$$\begin{aligned}\Phi_n(\mu;u) &= \mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z_n(u)\right]\\ &=\exp\Bigg\{\sum_{j=1}^{3}\int_0^T\left[\exp\left(i\mu\ln\frac{\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)}{\lambda_{j,n}\left(\vartheta_0,t\right)}\right)-1\right]\lambda_{j,n}\left(\vartheta_0,t\right)\mathrm{d}t\\ &\quad -i\mu\sum_{j=1}^{3}\int_0^T\left(\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)-\lambda_{j,n}\left(\vartheta_0,t\right)\right)\mathrm{d}t\Bigg\}.\end{aligned}$$

Introduce the sets A_k^n for $k=1,\ldots,8$, and $u=(u_1,u_2)\in\mathbb{U}_n$

$$\begin{aligned}
A_1^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\geq 0, \langle u,m_2\rangle\leq 0,\quad \langle u,m_3\rangle\leq 0\},\\
A_2^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\geq 0, \langle u,m_2\rangle\geq 0,\quad \langle u,m_3\rangle\leq 0\},\\
A_3^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\geq 0, \langle u,m_2\rangle\geq 0,\quad \langle u,m_3\rangle\geq 0\},\\
A_4^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\leq 0, \langle u,m_2\rangle\geq 0,\quad \langle u,m_3\rangle\geq 0\},\\
A_5^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\leq 0, \langle u,m_2\rangle\leq 0,\quad \langle u,m_3\rangle\geq 0\},\\
A_6^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\leq 0, \langle u,m_2\rangle\leq 0,\quad \langle u,m_3\rangle\leq 0\}.\\
A_7^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle\geq 0, \langle u,m_2\rangle< 0,\quad \langle u,m_3\rangle\geq 0\}.\\
A_8^n &= \{u\in\mathbb{U}_n,\quad \langle u,m_1\rangle< 0, \langle u,m_2\rangle\geq 0,\quad \langle u,m_3\rangle< 0\}.
\end{aligned}$$

Define $\vartheta_u=\vartheta_0+\frac{u}{n}$, $\tau_j=\tau_j\left(\vartheta_0\right)$, $\rho_j=\nu\tau_j$ and

$$\tau_j(\vartheta_u)=\frac{1}{\nu}\sqrt{\left(x_j-x_0-\frac{u_1}{n}\right)^2+\left(y_j-y_0-\frac{u_2}{n}\right)^2}.$$

It follows from condition $\mathcal{I}_1$ that $\tau_j(\vartheta_u)$ is differentiable w.r.t. u on $\mathbb{U}_n$. Using the Taylor expansion we obtain

$$
\begin{aligned}
\tau_j(\vartheta_u) &= \tau_j - \frac{u_1(x_j - x_0) + u_2(y_j - y_0)}{\nu n \rho_j} + \varepsilon_n(u) \\
&= \tau_j - \frac{1}{\nu n}\langle u, m_j\rangle + \varepsilon_n(u),
\end{aligned}
$$

where $n\varepsilon_n(u) \to 0$ uniformly on compacts u as $n \to \infty$. Thus

$$
\tau_j(\vartheta_u) - \tau_j = -\frac{1}{\nu n}\langle u, m_j\rangle + \varepsilon_n(u).
$$

Therefore, for all $j = 1, 2, 3$, bounded sets of u and n sufficiently large we have

$$
\begin{cases}
\tau_j \geq \tau_j(\vartheta_u), & if \quad \langle u, m_j\rangle \geq 0, \\
\tau_j \leq \tau_j(\vartheta_u), & if \quad \langle u, m_j\rangle \leq 0.
\end{cases}
$$

We will use this fact to calculate the characteristic function $\Phi_n(\mu; u)$ for each set A_k^n, $k = 1, \ldots, 8$ and obtain its limit.

If $u \in A_1^n$, then $\tau_1 \geq \tau_1(\vartheta_u)$, $\tau_2 \leq \tau_2(\vartheta_u)$ and $\tau_3 \leq \tau_3(\vartheta_u)$. Therefore, we can write

$$
\begin{aligned}
&\int_0^T \left[\exp\left(i\mu \ln \frac{\lambda_{1,n}\left(\vartheta_0 + \frac{u}{n}, t\right)}{\lambda_{1,n}(\vartheta_0, t)}\right) - 1\right]\lambda_{1,n}(\vartheta_0, t)\,\mathrm{d}t \\
&\quad = n\lambda_0 \int_{\tau_1(\vartheta_u)}^{\tau_1} \left[\exp\left(i\mu \ln \frac{\lambda\,(t - \tau_1(\vartheta_u)) + \lambda_0}{\lambda_0}\right) - 1\right]\mathrm{d}t \\
&\qquad + n \int_{\tau_1}^{T} \left[\exp\left(i\mu \ln \frac{\lambda\,(t - \tau_1(\vartheta_u)) + \lambda_0}{\lambda\,(t - \tau_1) + \lambda_0}\right) - 1\right][\lambda\,(t - \tau_1) + \lambda_0]\,\mathrm{d}t.
\end{aligned}
$$

Using once again Taylor's expansions by the powers of $\frac{u}{n}$, we obtain the representation

$$
\begin{aligned}
&\int_0^T \left[\exp\left(i\mu \ln \frac{\lambda_{1,n}\left(\vartheta_0 + \frac{u}{n}, t\right)}{\lambda_{1,n}(\vartheta_0, t)}\right) - 1\right]\lambda_{1,n}(\vartheta_0, t)\,\mathrm{d}t \\
&\quad = \left[\exp\left\{i\mu \ln \frac{\lambda_1 + \lambda_0}{\lambda_0}\right\} - 1\right] \frac{\lambda_0}{\nu}\langle u, m_1\rangle + o\,(1)\,.
\end{aligned}
$$

The similar arguments give us the relations

$$
\begin{aligned}
&\int_0^T \left[\exp\left(i\mu \ln \frac{\lambda_{2,n}\left(\vartheta_0 + \frac{u}{n}, t\right)}{\lambda_{2,n}(\vartheta_0, t)}\right) - 1\right]\lambda_{2,n}(\vartheta_0, t)\,\mathrm{d}t \\
&\quad = -\left[\exp\left\{-i\mu \ln \frac{\lambda_1 + \lambda_0}{\lambda_0}\right\} - 1\right] \frac{\lambda_1 + \lambda_0}{\nu}\langle u, m_2\rangle + o\,(1)
\end{aligned}
$$

and

$$\int_0^T\left[\exp\left(i\mu\ln\frac{\lambda_{3,n}\left(\vartheta_0+\frac{u}{n},t\right)}{\lambda_{3,n}\left(\vartheta_0,t\right)}\right)-1\right]\lambda_{3,n}\left(\vartheta_0,t\right){\rm d}t$$
$$=-\left[\exp\left\{-i\mu\ln\frac{\lambda_1+\lambda_0}{\lambda_0}\right\}-1\right]\frac{\lambda_1+\lambda_0}{\nu}\langle u,m_3\rangle+o\left(1\right).$$

Therefore, for $u\in A_1^n$ we obtain the limit

$$\lim_{n\to\infty}\Phi_n(\mu;u)=\exp\left\{\left[\exp\left(i\mu\ell\right)-1\right]\frac{\lambda_0}{\nu}\langle u,m_1\rangle\right.$$
$$\left.-\left[\exp\left(-i\mu\ell\right)-1\right]\frac{\lambda_0+\lambda_1}{\nu}\langle u,m_2+m_3\rangle-i\mu r(u)\right\}.$$

If $u\in A_2^n$, then similar arguments allow us to verify that

$$\lim_{n\to\infty}\Phi_n(\mu;u)=\exp\left\{\left[\exp\left(i\mu\ell\right)-1\right]\frac{\lambda_0}{\nu}\langle u,m_1+m_2\rangle\right.$$
$$\left.-\left[\exp\left(-i\mu\ell\right)-1\right]\frac{\lambda_0+\lambda_1}{\nu}\langle u,m_3\rangle-i\mu r(u)\right\}.$$

For $u\in A_3^n$ we have

$$\lim_{n\to\infty}\Phi_n(\mu;u)=\exp\left\{\left[\exp\left(i\mu\ell\right)-1\right]\frac{\lambda_0}{\nu}\langle u,m_1+m_2+m_3\rangle-i\mu r(u)\right\}.$$

For other sets A_k^n, we have the corresponding limits. For all sets, these limits provide the convergence of characteristic functions

$$\mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z_n\left(u\right)\right]\quad\longrightarrow\quad\mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z\left(u\right)\right].$$

Therefore, we have the convergence of one-dimensional distributions.

Using the same arguments, it is possible to verify the convergence of the finite-dimensional distributions too, i.e., for any $u_1,\ldots,u_L$ and real $\mu_1,\ldots,\mu_L$ we have

$$\mathbf{E}_{\vartheta_0}\exp\left[i\sum_{l=1}^L\mu_l\ln Z_n\left(u_l\right)\right]\quad\longrightarrow\quad\mathbf{E}_{\vartheta_0}\exp\left[i\sum_{l=1}^L\mu_l\ln Z\left(u_l\right)\right].$$

Moreover, from the presented proofs it follows that the convergence of finite-dimensional distributions is uniform on the compacts $\mathbb{K}\subset\Theta$. In particular,

$$\lim_{n\to\infty}\sup_{\vartheta_0\in\mathbb{K}}\left|\mathbf{E}_{\vartheta_0}\exp\left[i\sum_{l=1}^L\mu_l\ln Z_n\left(u_l\right)\right]-\mathbf{E}_{\vartheta_0}\exp\left[i\sum_{l=1}^L\mu_l\ln Z\left(u_l\right)\right]\right|=0.$$

□

Further, we need the following result.

Lemma 2 *Let the condition $\mathcal{I}_1, \mathcal{I}_2$ be fulfilled, then for any $R > 0$ and $\|u\| + \|v\| \leq R,\ u, v \in \mathbb{U}_n$ we have*

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0} \left| Z_n^{\frac{1}{2}}(u) - Z_n^{\frac{1}{2}}(v) \right|^2 \leq C\,(1+R)\,\|u-v\|\,,$$

where $C > 0$.

Proof According to Lemma 1.5 in Kutoyants (1998), we have

$$\begin{aligned}
&\mathbf{E}_{\vartheta_0} \left| Z_n^{\frac{1}{2}}(u) - Z_n^{\frac{1}{2}}(v) \right|^2 \\
&\quad \leq \sum_{j=1}^{3} \int_0^T \left[\sqrt{\lambda_{j,n}\left(\vartheta_0 + \frac{u}{n}, t\right)} - \sqrt{\lambda_{j,n}\left(\vartheta_0 + \frac{v}{n}, t\right)} \right]^2 \mathrm{d}t \\
&\quad \leq n \sum_{j=1}^{3} \int_0^T \Bigg[\sqrt{\lambda\left(t - \tau_j\left(\vartheta_u\right)\right) \mathbb{1}_{\{t > \tau_j(\vartheta_u))\}} + \lambda_0} \\
&\qquad - \sqrt{\lambda\left(t - \tau_j\left(\vartheta_v\right)\right) \mathbb{1}_{\{t > \tau_j(\vartheta_v)\}} + \lambda_0} \Bigg]^2 \mathrm{d}t \\
&\quad \leq Cn \sum_{j=1}^{3} \int_0^T \left[\lambda\left(t - \tau_j\left(\vartheta_u\right)\right) \mathbb{1}_{\{t > \tau_j(\vartheta_u)\}} - \lambda\left(t - \tau_j\left(\vartheta_v\right)\right) \mathbb{1}_{\{t > \tau_j(\vartheta_v)\}} \right]^2 \mathrm{d}t.
\end{aligned}$$

Here we use the elementary relations

$$\left[\sqrt{a} - \sqrt{b}\right]^2 = \frac{[a-b]^2}{\left[\sqrt{a} + \sqrt{b}\right]^2} \leq C\ [a-b]^2\,, \qquad C = \frac{1}{4M},$$

where $a > 0$, $b > 0$ and $M \leq (a \wedge b)$.

Consider the values $|u| + |v| \leq R$ with some $R > 0$. Then using once again Taylor's expansions we obtain

$$\lambda\left(t - \tau_j\left(\vartheta_u\right)\right) - \lambda\left(t - \tau_j\left(\vartheta_v\right)\right) = \frac{1}{\nu n} \lambda'\left(t - \tau_j\right) \langle u - v, m_j \rangle + \varepsilon_n(u, v)$$

and for large n

$$\left|\tau_j\left(\vartheta_u\right) - \tau_j\left(\vartheta_v\right)\right| \leq \frac{2}{\nu n} \left|\langle u - v, m_j\rangle\right| \leq \frac{C}{n} \|u - v\|\,.$$

These two estimates allow us to write

$$n\sum_{j=1}^{3}\int_{0}^{T}\left[\lambda\left(t-\tau_j\left(\vartheta_u\right)\right)\mathbb{1}_{\{t>\tau_j(\vartheta_u)\}}-\lambda\left(t-\tau_j\left(\vartheta_v\right)\right)\mathbb{1}_{\{t>\tau_j(\vartheta_v)\}}\right]^2\mathrm{d}t$$
$$\leq C\left\|u-v\right\|+\frac{C}{n}\left\|u-v\right\|^2\leq C\left(1+R\right)\left\|u-v\right\|.$$

□

The last result is given in the next lemma.

Lemma 3 *Let conditions $\mathcal{I}$ be fulfilled, then for $u\in\mathbb{U}_n$*

$$\sup_{\vartheta_0\in\mathbb{K}}\mathbf{E}_{\vartheta_0}Z_n^{\frac{1}{2}}\left(u\right)\leq e^{-\kappa||u||},\tag{9}$$

where $\kappa>0$.

Proof According to Lemma 1.5 of Kutoyants (1998), we can write

$$\mathbf{E}_{\vartheta_0}\left[Z_n^{\frac{1}{2}}\left(u\right)\right]=\exp\left\{-\frac{1}{2}\sum_{j=1}^{3}\int_0^T\left[\sqrt{\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)}-\sqrt{\lambda_{j,n}\left(\vartheta_0,t\right)}\right]^2\mathrm{d}t\right\}.$$

Elementary calculations lead to

$$\left[\sqrt{\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)}-\sqrt{\lambda_{j,n}\left(\vartheta_0,t\right)}\right]^2$$
$$=\frac{n\left[\lambda\left(t-\tau_j\left(\vartheta_u\right)\right)\mathbb{1}_{\{t>\vartheta_j(\vartheta_u)\}}-\lambda\left(t-\tau_j\right)\mathbb{1}_{\{t>\tau_j\}}\right]^2}{\left[\sqrt{\lambda\left(t-\tau_j\left(\vartheta_u\right)\right)\mathbb{1}_{\{t>\tau_j(\vartheta_u)\}}+\lambda_0}+\sqrt{\lambda\left(t-\tau_j\right)\mathbb{1}_{\{t>\tau_j\}}+\lambda_0}\right]^2}$$
$$\geq nc\left[\lambda\left(t-\tau_j\left(\vartheta_u\right)\right)\mathbb{1}_{\{t>\vartheta_j(\vartheta_u)\}}-\lambda\left(t-\tau_j\right)\mathbb{1}_{\{t>\tau_j\}}\right]^2$$

where $c=\frac{1}{4\lambda_M}$ with the constant $\lambda_M\geq\lambda\left(t\right)+\lambda_0$.

Let us now consider ϑ such that $\left\|\vartheta-\vartheta_0\right\|\leq\delta$ with small $\delta>0$ and such that $\tau_j\left(\vartheta\right)>\tau_j$. Then for sufficiently small δ, we can write

$$\int_0^T\left[\lambda\left(t-\tau_j\left(\vartheta\right)\right)\mathbb{1}_{\{t>\tau_j(\vartheta)\}}-\lambda\left(t-\tau_j\right)\mathbb{1}_{\{t>\tau_j\}}\right]^2\mathrm{d}t$$
$$=\int_{\tau_j}^{\tau_j(\vartheta)}\lambda\left(t-\tau_j\right)^2\mathrm{d}t+\int_{\tau_j(\vartheta)}^{T}\left[\lambda\left(t-\tau_j\left(\vartheta\right)\right)-\lambda\left(t-\tau_j\right)\right]^2\mathrm{d}t$$
$$\geq k\left[\tau_j\left(\vartheta\right)-\tau_j\right]-c_j\left\|\vartheta-\vartheta_0\right\|^2\geq k_j\left[\tau_j\left(\vartheta\right)-\tau_j\right]$$

with $k = \min_t \lambda(t)^2 > 0$ and some positive constant c_j, k_j.

Using this last inequality, we obtain

$$\sum_{j=1}^{3}\int_0^T\left[\sqrt{\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)}-\sqrt{\lambda_{j,n}(\vartheta_0,t)}\right]^2\mathrm{d}t \geq n\gamma\sum_{j=1}^{3}\left|\tau_j(\vartheta)-\tau_j(\vartheta_0)\right|$$
$$\geq \gamma\sum_{j=1}^{3}\left|\langle m_j,u\rangle\right|+\varepsilon_n(\delta)\geq\gamma_1\sum_{j=1}^{3}\left|\left\langle m_j,\frac{u}{\|u\|}\right\rangle\right|\|u\|$$
$$\geq\gamma_1\inf_{\|e\|=1}\sum_{j=1}^{3}\left|\langle m_j,e\rangle\right|||u||\geq\kappa_1\|u\|, \tag{10}$$

where $\kappa_1 > 0$.

Next we consider the case $\|\vartheta-\vartheta_0\| = \left\|\frac{u}{n}\right\| > \delta$. Let us denote

$$g(\vartheta_0,\delta)=\inf_{\|\vartheta-\vartheta_0\|>\delta}\sum_{j=1}^{3}\int_0^T\left[\lambda\left(t-\tau_j(\vartheta)\right)\mathbb{1}_{\{t>\tau_j(\vartheta)\}}-\lambda\left(t-\tau_j\right)\mathbb{1}_{\{t>\tau_j\}}\right]^2\mathrm{d}t.$$

Remark that for any compact $\mathbb{K}\subset\Theta$

$$g_{\mathbb{K}}(\delta)=\inf_{\vartheta_0\in\mathbb{K}}g(\vartheta_0,\delta)>0.$$

Indeed, if $g_{\mathbb{K}}(\delta)=0$, then there exists $\vartheta_1\neq\vartheta_0$, such that

$$\sum_{j=1}^{3}\int_0^T\left[\lambda\left(t-\tau_j(\vartheta_1)\right)\mathbb{1}_{\{t>\tau_j(\vartheta_1)\}}-\lambda\left(t-\tau_j(\vartheta_0)\right)\mathbb{1}_{\{t>\tau_j(\vartheta_0)\}}\right]^2\mathrm{d}t=0.$$

Due to the indicator functions, this equality is possible iff $\tau_j(\vartheta_1)=\tau_j(\vartheta_0)$, $j = 1,2,3$ but from the geometrical consideration this is impossible. Therefore, $g_{\mathbb{K}}(\delta)>0$ and for $\|\vartheta-\vartheta_0\|\geq\delta$ we can write

$$n\sum_{j=1}^{3}\int_0^T\left[\lambda\left(t-\tau_j(\vartheta)\right)\mathbb{1}_{\{t>\tau_j(\vartheta)\}}-\lambda\left(t-\tau_j(\vartheta_0)\right)\mathbb{1}_{\{t>\tau_j(\vartheta_0)\}}\right]^2\mathrm{d}t$$
$$\geq ng_{\mathbb{K}}(\delta)\geq ng_{\mathbb{K}}(\delta)\frac{\|\vartheta-\vartheta_0\|}{D(\Theta)}\geq\kappa_2\|u\|. \tag{11}$$

Here

$$D(\Theta)=\sup_{\vartheta,\vartheta_0\in\Theta}\|\vartheta-\vartheta_0\|,\qquad\kappa_2=\frac{g_{\mathbb{K}}(\delta)}{D(\Theta)}.$$

From the estimates (10) and (11), it follows that there exists $\kappa > 0$ such that

$$\sum_{j=1}^{3}\int_0^T \left[\sqrt{\lambda_{j,n}\left(\vartheta_0+\frac{u}{n},t\right)}-\sqrt{\lambda_{j,n}\left(\vartheta_0,t\right)}\right]^2 \mathrm{d}t \geq 2\kappa \left\|u\right\|.$$

This last estimate proves (9). □

The properties of the normalized likelihood ratio $Z_n(u), u \in \mathbb{U}_n$ described in Lemmas 1–3 allow us to cite Theorem 1.10.2 in Ibragimov and Khasminskii (1981), and according to this theorem, the BE $\widetilde{\vartheta}_n$ has all the properties mentioned in Theorem 1.

5 Simulations

We illustrate the convergence of the estimators by means of numerical simulations. Consider the problem of localization of a radioactive source at the point $\vartheta_0 = (0, 0)$. We have three sensors ϑ_j ($j = 1, 2, 3$), respectively, located at coordinates $\vartheta_1 = (8.5, 0)$, $\vartheta_2 = (0, 8.5)$ and $\vartheta_3 = \left(8.5\cos\left(\frac{5\pi}{4}\right), 8.5\sin\left(\frac{5\pi}{4}\right)\right)$. We choose the values $\lambda_0 = 1, \lambda_1 = 2$ and for convenience $\nu = 1$. Each sensor located at position ϑ_j records in the fixed time interval [0, 10] measurements that are modeled by a Poisson point processes of intensity function

$$\lambda_j(\vartheta_0, t) = n + 2n\mathbb{1}_{\{t\geq\tau_j\}}.$$

The parameter space of the unknown coordinates of the source ϑ_0 was chosen as $\Theta = (-1, 1) \times (-1, 1)$, and the prior density of ϑ_0 is the uniform density in the unit square, i.e., $p(\vartheta) = \frac{1}{4}\mathbb{1}_{\{(x,y)\in[-1,1]^2\}}$. The BE $\widetilde{\vartheta}_n$ was calculated using simulations for n running in the range [1, 100]. Figure 5 displays the evolution of the Euclidean distance between the BE $\widetilde{\vartheta}_n = (\widetilde{x}_n, \widetilde{y}_n)$ and ϑ_0 with n.

As can be seen, the distance between ϑ_0 and the BE after initial fluctuations quickly decreases toward zero which illustrates the consistency of the BE.

We also made simulations for the MLE $\hat{\vartheta}_n$ of the same parameter ϑ_0.

In what follows, we present the graphs of the corresponding error obtained for the same simulation model with n running in the range [1, 100] (Fig. 6).

As can be seen, the Euclidean distance between the MLE and ϑ_0 quickly decreases toward zero after initial fluctuations:

We can see that the fluctuations of the MLE at the beginning are more important than those of the BE.

6 Discussions

Let us mention now some problems related to the study of the MLE. The main technical difficulty to apply the Ibragimov and Khasminskii approach in the study of the MLE in this change-point statement is in the checking of the tightness of the family of

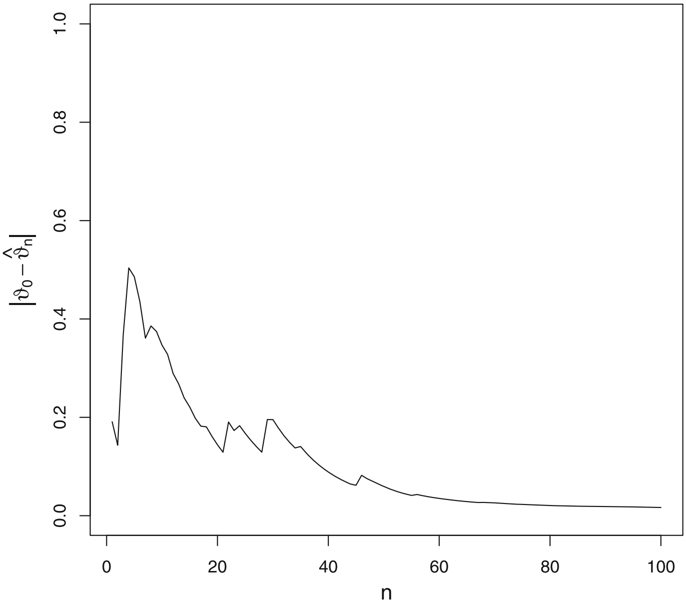

Fig. 5 Evolution of error $\|\widetilde{\vartheta}_n - \vartheta_0\|$

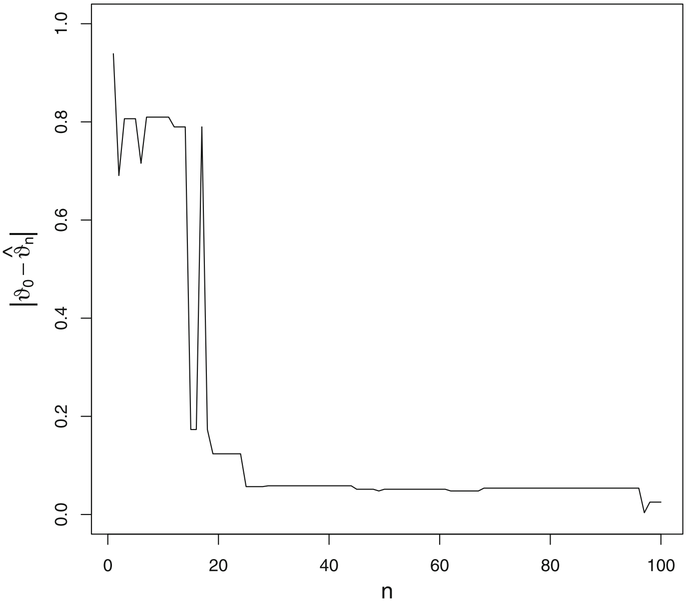

Fig. 6 Evolution of error $\|\hat{\vartheta}_n - \vartheta_0\|$

measures induced by the likelihood ratio random field $Z_n(u)$, $u \in \mathbb{U}_n$ in the space of its realizations. Recall that this is the space of surfaces with discontinuities along some curves.

Here we supposed that the signal and noise are of the same magnitude n, where $n \to \infty$. However, in some cases the signal can be much larger than the noise, say,

$$\lambda_{j,n}(\vartheta, t) = n\lambda\left(t - \tau_j(\vartheta)\right) \mathbb{1}_{\{t > \tau_j(\vartheta)\}} + \lambda_0, \qquad 0 \le t \le T, \qquad j = 1, 2, 3.$$

This case could be studied as well by means of the presented method, but the limit $Z(u), u \in \mathcal{R}^2$ of the normalized likelihood ratio function $Z_n(u), u \in \mathbb{U}_n$ will be different.

As mentioned in Introduction, there are several other statements related to the problem of radioactive source localization depending on the regularity of the signals. The cases of smooth signals and cusp-type signals are considered in the works Chernoyarov and Kutoyants (2018) and Dachian et al. (2018a), respectively. In particular, in Chernoyarov and Kutoyants (2018) the estimation of the parameter ϑ_0 by $k \ge 3$ sensors was made in two steps. First, we estimate the moments of the arrival times of the signals, say, $\overline{\tau}_{1,n}, \ldots, \overline{\tau}_{k,n}$; then given these estimators, the localization $\overline{\vartheta}_n$ of the source is found by solving the system of equations

$$\overline{\tau}_{1,n}^2 \nu^2 = \left\|\vartheta_1 - \overline{\vartheta}_n\right\|^2, \qquad \ldots, \qquad \overline{\tau}_{k,n}^2 \nu^2 = \left\|\vartheta_k - \overline{\vartheta}_n\right\|^2.$$

It is shown that the estimator $\overline{\vartheta}_n$ is consistent and asymptotically normal. It will be interesting to study the similar estimator in the change-point case.

Another question concerns the robustness of the estimators (MLE and BE) with respect to the knowledge of the model. Suppose that the signal $\lambda(t), t \ge 0$ is not exactly known and we use just a constant value $\lambda_1 > 0$. We can see what are the limits of the MLE and BE in such situations. It is known that in this case both estimators converge to the value $\hat{\vartheta}$ which minimizes the corresponding Kullback–Leibler distance. The one-dimensional case was studied in Dabye et al. (2003), where it was shown that for a wide range of values of λ_1 the BE is consistent even for the wrong model. We could suppose that the model considered in the present work has a similar property. Then the consistent estimation is possible in the case of misspecification as well.

Of course, a similar problem could be studied for the models of signals in white Gaussian noise. Indeed, suppose that we have the same positions of the source and the detectors (see Fig. 1), but the signals are Gaussian

$$\mathrm{d}X_{j,t} = S\left(t - \tau_j(\vartheta)\right) \mathbb{1}_{\{t \ge \tau_j(\vartheta)\}} \mathrm{d}t + \varepsilon \mathrm{d}W_{j,t}, \quad X_0 = 0, \quad 0 \le t \le T.$$

Here $j = 1, 2, 3$ and $W_{j,t}, 0 \le t \le T, j = 1, 2, 3$ are independent Wiener processes. Then we can describe the properties of the MLE and BE of the coordinates of the source in the asymptotics of small noise ($\varepsilon \to 0$) in the cases of different regularity of the signals [see, e.g., Dachian et al. (2018b)].

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