Supplementary Material for the manuscript entitled, ”Efficient and Robust Tests for Semiparametric Models” by Jingjing Wu and Rohana J. Karunamuni.

Here we present detailed proofs of Lemmas 1 to 5 used in the above paper.

**Proof of Lemma 1.** Note that

\[
\int (\sqrt{\hat{f}(x)} - \sqrt{f(x)})^2 dx \leq 2 \int (\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)})^2 dx + 2 \int (\sqrt{\bar{f}(x)} - \sqrt{f(x)})^2 dx \\
= 2 \int_{B_n} (\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)})^2 dx + 2 \int_{B^n} (\sqrt{\bar{f}(x)} - \sqrt{f(x)})^2 dx \\
+ 2 \int (\sqrt{\bar{f}(x)} - \sqrt{f(x)})^2 dx,
\]

where \( \bar{f}(x) = E[\hat{f}(x)] \). We have

\[
nh \text{Var}(\hat{f}(x)) = nhE(\hat{f}(x) - \bar{f}(x))^2 \leq \int \frac{1}{h}K^2 \frac{(x - y)}{h} f(y) dy \leq \|K\|_\infty \bar{f}(x)
\]

\[
E \int_{B_n} (\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)})^2 dx \leq E \int_{B_n} \frac{(\hat{f}(x) - \bar{f}(x))^2}{\bar{f}(x)} dx \leq 2\|K\|_\infty (nh)^{-1} c_n
\]

\[
E \int_{B^n} (\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)})^2 dx \leq E \int_{B^n} \left| \hat{f}(x) - \bar{f}(x) \right| dx \\
\leq \int_{B^n} \left( \text{Var}(\hat{f}(x)) \right)^{1/2} dx \leq (nh)^{-1/2} \|K\|^{1/2}_\infty \int_{B^n} (\bar{f}(x))^{1/2} dx.
\]

By a Taylor expansion, we obtain

\[
\bar{f}(x) = f(x) + \frac{\mu_2}{2} h^2 f^{(2)}(x) - \frac{1}{2} h^3 \int_0^1 (1 - t)^2 f^{(3)}(x - thu) u^3 K(u) du,
\]

where \( \mu_2 = \int u^2 K(u) du \). Then

\[
\int_{B^n} (\bar{f}(x))^{1/2} dx \leq \int_{B^n} (f(x))^{1/2} dx + (\frac{\mu_2}{2})^{1/2} h^{1/2} \int_{B^n} \left| f^{(2)}(x) \right|^{1/2} dx \\
+ (\frac{1}{2})^{1/2} h^{3/2} \int_{B^n} \left| \int_0^1 (1 - t)^2 f^{(3)}(x - thu) u^3 K(u) du \right|^{1/2} dx.
\]
It is easy to show that $0 \leq \lim_n \int_{B_n^c} \left| \int_0^1 (1 - t)^2 f^{(3)}(x - thu)u^3 K(u)du \right|^{1/2} dx \leq 0$. Therefore, we have

$$\int \left( \sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)} \right)^2 dx = O_p((nh)^{-1}c_n + (nh)^{-1/2}C_{n1} + n^{-1/2}h^{1/2}C_{n2}) + o_p(n^{-1/2}h). \quad (54)$$

Define $b(x, h) = \int f(x - hu)K(u)du$. Then the first two derivatives of $b(x, h)$ w.r.t. $h$ are given by

$$\dot{b}(x, h) = -\int f^{(1)}(x - hu)uK(u)du \quad \text{and} \quad \ddot{b}(x, h) = \int f^{(2)}(x - hu)u^2 K(u)du.$$ 

Note that $b(x, 0) = f(x)$ and $\dot{b}(x, 0) = 0$. The first two derivatives of $s(x, h) = \sqrt{b(x, h)}$ w.r.t. $h$ are

$$\dot{s}(x, h) = \frac{\dot{b}(x, h)}{2\sqrt{b(x, h)}} \quad \text{and} \quad \ddot{s}(x, h) = \frac{\ddot{b}(x, h)}{2\sqrt{b(x, h)}} - \frac{\dot{b}(x, h)^2}{4b(x, h)^{3/2}}.$$ 

Thus, we can express

$$\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)} = s(x, h) - s(x, 0) - h\dot{s}(x, 0) = \int_0^1 (1 - t)h^2\ddot{s}(x, th)dt,$$

and by the Cauchy-Schwarz inequality and by Fubini’s theorem then we obtain

$$\int (\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)})^2 dx = \int \left( \int_0^1 (1 - t)h^2\ddot{s}(x, th)dt \right)^2 dx \leq h^4 \int_0^1 (1 - t)^2 \int (\ddot{s}(x, th))^2 dx dt.$$ 

Applications of the Cauchy-Schwarz inequality yield

$$(\dot{b}(x, h))^2 \leq \int \frac{(f^{(1)})^2}{f}(x - hu)u^2 K(u)du \int f(x - hu)K(u)du \leq \left( \int \frac{(f^{(1)})^4}{f^3}(x - hu)u^4 K(u)du \right)^{1/2} \left( \int f(x - hu)K(u)du \right)^{3/2}$$

and

$$(\ddot{b}(x, h))^2 \leq \int \frac{(f^{(2)})^2}{f}(x - hu)u^4 K(u) du \int f(x - hu)K(u) du.$$
The above expressions show that
\[(\hat{s}(x, h))^2 \leq 2\left(\frac{\hat{b}(x, h)}{4b(x, h)}\right)^2 + \frac{(\hat{b}(x, h))^4}{16(b(x, h))^3} \leq \int \psi(x - hu)u^4K(u)du\]

with
\[\psi(x) = \frac{(f^{(2)}(x))^2}{2f(x)} + \frac{(f^{(1)}(x))^4}{8f^3(x)}\]

Consequently, we have by Fubini’s theorem
\[\int (\sqrt{\hat{f}(x)} - \sqrt{f(x)})^2 dx \leq h^4 \int_0^1 (1 - t)^2 \int \psi(x - thu)u^4K(u)dudxdt \leq h^4 \int \psi(x)dx \int u^4K(u)du \int_0^1 (1 - t)^2dt = O(h^4)\]  

since \(\psi\) is integrable by assumptions. The proof of (28) is now completed by combining (53), (54) and (55). The proof of (29) is similar. \(\square\)

**Proof of Lemma 2.** Again writing \(f = f_{\theta, \eta}\), observe that

\[2 \int \hat{s}_{\theta, \eta}(x)\hat{f}^{1/2}(x)dx - \frac{1}{n} \sum_{i=1}^n \hat{s}_{\theta, \eta}(X_i) = \frac{1}{2} \int f^{(1)}(x)\left[\frac{f(x)}{f(\hat{f})} - \frac{(f^{(1/2)}(x)-f^{1/2}(x))^2}{f(x)}\right]dx - \frac{1}{n} \sum_{i=1}^n \frac{f^{(1)}(X_i)}{2f}(X_i) \]

\[= \frac{1}{2} \int f^{(1)}(x)\hat{f}(x)dx - \frac{1}{n} \sum_{i=1}^n \frac{f^{(1)}(X_i)}{f}(X_i) - \frac{1}{2} \int f^{(1)}(x)(\hat{f}^{1/2}(x) - f^{1/2}(x))^2dx \]

\[= \frac{1}{4}I_3 - \frac{1}{2}I_4, \text{ say.}\]

Using a Taylor expansion of order four with an integral form of the reminder term, by (30) and Fubini’s theorem, we obtain

\[E \int \frac{f^{(1)}}{f}(x)\hat{f}(x)dx = \int \frac{f^{(1)}}{f}(x)\left\{f(x) + \frac{h^2}{2}f^{(2)}(x) \int u^2K(u)du + \frac{h^4}{24} \int \int_0^1 (1 - t)^3 f^{(4)}(x - thu)u^4K(u)dudt\right\}dx \]

\[= \frac{h^4}{24} \int \frac{f^{(1)}}{f}(x)\left( \int \int_0^1 (1 - t)^3 f^{(4)}(x - thu)u^4K(u)dudt \right)dx \]

\[= O(h^4)\]

where we used the fact that
\[
\lim_{n \to \infty} \int f(x) \left( \int_0^1 (1-t)^3 f^{(4)}(x-thu)K(u)dudt \right) dx
= \int f(x) f^{(4)}(x) dx \left( \int_0^1 (1-t)^3 u^4 K(u) dudt \right).
\]

The last assertion can be verified using the Dominated Convergence theorem (DCT) and Fatou’s lemma; see, e.g., the proof of Lemma 4 below. Furthermore, \( E\left( \frac{1}{n} \sum_{i=1}^{n} \frac{f^{(1)}}{f}(X_i) \right) = 0 \). Thus, using (31) we obtain

\[
|E(I_3)| = O(h^4).
\]  

(56)

Again by direct calculation and using the Cauchy-Schwartz inequality and a Taylor expansion, we have

\[
\text{Var}(I_3) \leq \frac{1}{n} \int \left[ \int f(x) \left( \frac{1}{h} K(\frac{x-y}{h}) dx - \frac{f(y)}{f} \right)^2 f(y) dy \right] dx
\leq \frac{1}{n} \int \left( \frac{f(y)}{f} (y - uh) - \frac{f(y)}{f} \right)^2 K(u)f(y) dudy
= O(n^{-1}h^2).
\]  

(57)

Therefore, from (56) and (57) we obtain \( I_3 = O_P(h^4 + n^{-1/2}h) \). Note that from (29), we have \( I_4 = O_P(h^4 + (nh)^{-1}C_{n4} + (nh)^{-1/2}C_{n3} + n^{-1/2}h^{1/2}) \). This completes the proof. \( \square \)

**Proof of Lemma 3.** Since \( \hat{\eta}(x) = \frac{1}{2}(\hat{f}(x + \bar{\theta}) + \hat{f}(-x + \bar{\theta}))b_n^2(x) \) is a symmetric analogue of \( \hat{f}(x + \bar{\theta})b_n^2(x) \), it is enough to prove the lemma for \( \hat{f}(x + \bar{\theta})b_n^2(x) \). Thus, in what follows we assume that \( \hat{\eta}(x) = \hat{f}(x + \bar{\theta})b_n^2(x) \). Note that
\[ \int (s_{t, \tilde{\theta}}(x) - s_{t, \eta}(x))^2 dx = \int (\hat{\eta}^{1/2}(x) - \eta^{1/2}(x))^2 dx \]

\[ = \int (\hat{\eta}^{1/2}(x - \tilde{\theta}) - \eta^{1/2}(x - \tilde{\theta}))^2 dx \]

\[ = \int (\hat{f}^{1/2}(x)b_n^2(x - \tilde{\theta}) - \eta^{1/2}(x - \tilde{\theta}))^2 dx \]

\[ \leq 4 \int (\hat{f}^{1/2}(x)b_n^2(x - \tilde{\theta}) - \eta^{1/2}(x - \theta)b_n^2(x - \tilde{\theta}))^2 dx + 4 \int (\eta^{1/2}(x - \theta)b_n^2(x - \tilde{\theta}) - \eta^{1/2}(x - \tilde{\theta}))^2 dx \]

\[ + 2 \int (\eta^{1/2}(x - \theta) - \eta^{1/2}(x - \tilde{\theta}))^2 dx \]

\[ = 4I_7 + 4I_8 + 2I_9, \text{ say}. \]

From Lemma 1, we have

\[ I_7 \leq O_P(h^4 + (nh)^{-1}c_n + (nh)^{-1/2}C_n1 + n^{-1/2}h^{1/2}C_n2 + n^{-1/2}h). \] (58)

By direct calculation,

\[ I_8 = \int (1 - b_n^2(x - \tilde{\theta}))^2 \eta(x - \theta) dx \]

\[ \leq 2 \int (1 - b_n^2(x - \theta))^2 \eta(x - \theta) dx + 2 \int (b_n^2(x - \theta) - b_n^2(x - \tilde{\theta}))^2 \eta(x - \theta) dx \]

\[ \leq 2 \int_{c_n \leq |x - \theta| \leq c_n + 1} (1 - b_n^2(x - \theta))^2 \eta(x - \theta) dx + 4(\theta - \tilde{\theta})^2 \int (b_n^{(1)}(x - \theta^*))^2 \eta(x - \theta) dx \]

\[ \leq C_n5 + O_P(n^{-1}), \]

(59)

last equality follows from the facts that \( \tilde{\theta} \) is \( \sqrt{n} \)-consistent, \( b_n^{(1)} \) is bounded and \( \int_{c_n \leq |x - \theta| \leq c_n + 1} (1 - b_n^2(x - \theta))^2 \eta(x - \theta) dx \leq \int_{c_n \leq |x - \theta|} \eta(x - \theta) dx \), where \( \theta^* \) is a value between \( \theta \) and \( \tilde{\theta} \). Again using
a Taylor expansion,

\[ I_9 = \frac{1}{2}(\theta - \tilde{\theta})^2 \int (\eta^{(1)}(x - \theta^*))^2 (\eta(x - \theta^*))^{-1} dx \]

\[ = \frac{1}{2}(\theta - \tilde{\theta})^2 \int (\eta^{(1)}(x))^2 (\eta(x))^{-1} dx \]

\[ = O_P(n^{-1}) \quad (60) \]

by (32) and the \( \sqrt{n} \)-consistent property of \( \tilde{\theta} \), where again \( \tilde{\theta}^* \) is a value between \( \tilde{\theta} \) and \( \theta \). Now (33) follows from (58), (59) and (60). This completes the proof.

\[ \square \]

**Proof of Lemma 4.** As in the proof of Lemma 3, we assume that \( \hat{\eta}(x) = \hat{f}(x + \tilde{\theta}) b_n^2(x) \).

Denote \( \hat{g}_n(x) = \sqrt{\hat{\eta}(x - \tilde{\theta})} \), \( g_n(x) = \sqrt{f(x)} b_n(x - \theta) \) and \( g(x) = \sqrt{\eta(x)} \). Then by Minkowaski inequality we have

\[
\int (\hat{s}_{t,\tilde{\eta}} - \hat{s}_{t,n})^2 dx \leq 2 \int (\hat{g}_n^{(1)}(x) - g_n^{(1)}(x))^2 dx + 4 \int (g_n^{(1)}(x) - g^{(1)}(x - \theta))^2 dx \\
+ 4 \int (g^{(1)}(x - \theta) - g^{(1)}(x - \tilde{\theta}))^2 dx \\
= 2I_{10} + 4I_{11} + 4I_{12}, \text{ say.} \quad (61)
\]

From (34) and a Taylor expansion, it follows that

\[
I_{12} = (\theta - \tilde{\theta})^2 \int (g^{(2)}(x))^2 dx = O_P(n^{-1}). \quad (62)
\]

Again by Minkowaski inequality,

\[
I_{11} \leq 4 \int (\hat{s}^{(1)}(x) - s^{(1)}(x))^2 b_n^2(x - \theta) dx + 4 \int (s^{(1)}(x))^2 (1 - b_n(x - \theta))^2 dx \\
+ 4 \int (s(x) - s(x))^2 (b_n^{(1)}(x - \theta))^2 dx + 4 \int (s(x))^2 (b_n^{(1)}(x - \theta))^2 dx \\
= 4I_{13} + 4I_{14} + 4I_{15} + 4I_{16}, \text{ say,} \quad (63)
\]
where \( s(x) = \sqrt{f(x)} \) and \( \bar{s}(x) = \sqrt{\bar{f}(x)} \). Clearly,

\[
I_{16} = \int \eta(x - \theta) (b_n^{(1)}(x - \theta))^2 \, dx \\
= \int_{|x| \geq c_n + 1} \eta(x) \, dx \\
\leq C_{n5}
\]

and

\[
I_{15} \leq \int (\sqrt{f(x)} - \sqrt{\bar{f}(x)})^2 \, dx = O(h^4),
\]

by (55). Similarly, by the definition of \( b_n(x) \) we obtain

\[
I_{14} = \int_{c_n \leq |x - \theta| \leq c_n + 1} (s^{(1)}(x))^2 (1 - b_n(x - \theta))^2 \, dx \\
\leq \int_{c_n \leq |x - \theta|} (s^{(1)}(x))^2 \, dx \\
= C_{n6}.
\]

To study \( I_{13} \), write

\[
\bar{s}^{(1)}(x) - s^{(1)}(x) = \frac{1}{s(x)} (\bar{f}^{(1)}(x) - f^{(1)}(x)) - \frac{\bar{f}^{(1)}(x)}{f(x)} \left[ \frac{1}{s(x)} (\bar{s}(x) - s(x))^2 + (\bar{s}(x) - s(x)) \right],
\]

then by Minkowski inequality one obtains

\[
I_{13} \leq 2 \int \frac{1}{f(x)} (\bar{f}^{(1)}(x) - f^{(1)}(x))^2 b_n^2(x - \theta) \, dx + 4 \int (\frac{\bar{f}^{(1)}(x)}{f(x)})^2 \frac{1}{f(x)} (\bar{s}(x) - s(x))^4 b_n^2(x - \theta) \, dx \\
+ 4 \int (\frac{\bar{f}^{(1)}(x)}{f(x)})^2 (\bar{s}(x) - s(x))^2 b_n^2(x - \theta) \, dx \\
= 2I_{16} + 4I_{17} + 4I_{18}, \text{ say.}
\]

Using the proof of Lemma 1, we have

\[
\int (\frac{\bar{f}^{(1)}(x)}{f(x)})^2 (\bar{s}(x) - s(x))^2 \, dx \leq h^4 \int (\frac{\bar{f}^{(1)}(x)}{f(x)})^2 \int \int_0^1 (1 - t)^2 \psi(x - thu) u^4 K(u) \, du \, dt \, dx.
\]

By the DCT, one obtains
\[
\int_0^1 \int_0^1 (1-t)^2 \psi(x-thu)u^4 K(u)dudt \rightarrow \psi(x) \int_0^1 (1-t)^2 u^4 K(u)dudt,
\]
as \(n \rightarrow \infty\), and hence, by Fatou’s lemma, it follows that

\[
\int \int_0^1 (1-t)^2 \psi(x-thu)u^4 K(u)dudt dx \leq \lim_{n \rightarrow \infty} \int \int_0^1 (1-t)^2 \psi(x-thu)u^4 K(u)dudt dx.
\]

since \(\bar{f}^{(1)}/f \rightarrow f^{(1)}/f\) as \(n \rightarrow \infty\) for each \(x\). On the other hand,

\[
\lim_{n \rightarrow \infty} \int \int_0^1 (\bar{f}^{(1)}/f)^2 \psi(x-thu)u^4 K(u)dudt dx \leq \int \int_0^1 (f^{(1)}/f)^2 \psi(x-thu)u^4 K(u)dudt dx.
\]

From the relations (68) and (69), we now obtain

\[
\lim_{n \rightarrow \infty} \int \int_0^1 (\bar{f}^{(1)}/f)^2 \psi(x-thu)u^4 K(u)dudt dx = \int \int_0^1 (f^{(1)}/f)^2 \psi(x-thu)u^4 K(u)dudt dx,
\]

and thus from (68) we have

\[
I_{18} \leq \int (\bar{f}^{(1)}/f)^2 (\bar{s}(x) - s(x))^2 dx = O(h^4) + o(h^4).
\]

Since \((\bar{s}(x) - s(x))^2 \leq (\sqrt{\bar{f}(x)} - \sqrt{f(x)})^2\), a similar argument show that

\[
I_{17} \leq O(h^4) + o(h^4),
\]

provided \(f^{(2)}\) is bounded and (36) hold. Similarly, we can show that

\[
I_{16} \leq O(h^4) + o(h^4),
\]
when \( f^{(3)} \) is bounded and (37) hold. From Lemma 3 of Beran (1978), we also have
\[
I_{10} = \int (\hat{g}_n^{(1)}(x) - g_n^{(1)}(x))^2 dx = O_P((nh^3)^{-1}c_n). \tag{74}
\]

The proof of (38) is now completed by combining (61) to (74).

\[\square\]

**Proof of Lemma 5.** By (40) and (42), we obtain
\[
\int (\hat{p}_n(x) - \hat{s}_{t,\eta}(x))^2 dx
\]
\[
= \int_{\gamma_n \leq |x-t| \leq \alpha_n} (\hat{p}_n(x) - \hat{s}_{t,\eta}(x))^2 dx + \int_{|x-t| < \gamma_n} \hat{s}_{t,\eta}^2(x)dx + \int_{|x-t| > \alpha_n} \hat{s}_{t,\eta}^2(x)dx \tag{75}
\]
\[
= \frac{1}{4} \int_{\gamma_n}^{\alpha_n} \left( \frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)} \right)^2 ds + \frac{1}{4} \int_{0}^{\gamma_n} \left[ \frac{g^{(1)}(s)}{g(s)} \right]^2 ds + \frac{1}{4} \int_{\alpha_n}^{+\infty} \left[ \frac{g^{(1)}(s)}{g(s)} \right]^2 ds.
\]

Let \( \tilde{g}_n(s) = E\hat{g}_n(s) \). Obviously
\[
\int_{\gamma_n}^{\alpha_n} \left( \frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)} \right)^2 ds \leq 2 \int_{\gamma_n}^{\alpha_n} \left( \frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} \right)^2 ds + 2 \int_{\gamma_n}^{\alpha_n} \left( \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)} \right)^2 ds. \tag{76}
\]

By the arguments similar to those used in the proof of Lemma 3 of Beran (1978) and noting that \( \inf_{s \in [\gamma_n, +\infty)} |\beta_1(s)\delta_0(s) - \beta_0(s)\delta_1(s)| \geq \gamma_0 \), we deduce that
\[
\int_{\gamma_n}^{\alpha_n} \left( \frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} \right)^2 ds = O_p((nh^3)^{-1}\alpha_n). \tag{77}
\]

By direct calculations,
\[
\frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)} = \frac{\tilde{g}_n^{(1)}(s) - g^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} - \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} \left[ \frac{1}{g^{1/2}(s)}(\hat{g}_n^{1/2}(s) - g^{1/2}(s))^2 + (\hat{g}_n^{1/2}(s) - g^{1/2}(s))^2 \right].
\]
Let \( \beta(s) = \beta_1(s)\delta_0(s) - \beta_0(s)\delta_1(s) \). Then by a Taylor expansion, we have

\[
\tilde{g}_n^{(1)}(s) = \frac{1}{h^2\beta(s)} \int_0^{+\infty} \left[ \beta_0(s)K'(\frac{s-y}{h}) - \delta_0(s)K(\frac{s-y}{h}) \right] g(y) dy \\
= \frac{1}{h\beta(s)} \int_{-\infty}^{s/h} [\beta_0(s)K'(u) - \delta_0(s)K(u)] g(s-hu) du \\
= g^{(1)}(x) + \frac{h}{\beta(s)} \int_{-\infty}^{s/h} u^2 [\beta_0(s)K'(u) - \delta_0(s)K(u)] \int_0^1 (1 - \tau)g^{(2)}(s - \tau hu) d\tau du.
\]

Then under the assumptions of Lemma 5, using a proof similar to that of (70) and noting that
\( 0 \leq \beta_0(s) \leq \int_{-\infty}^{+\infty} K(s) ds = 1 \) and \( 0 \leq \delta_0(s) < +\infty \), we obtain

\[
\int_{\gamma_n} \frac{[\tilde{g}_n^{(1)}(s) - g^{(1)}(s)]^2}{g(s)} ds = O(h^2). \tag{78}
\]

Since
\[
|\tilde{g}_n^{(1)}(s)| \leq \frac{\beta_0(s) \sup_s |K'(s)/K(s)| + \delta_0(s)}{h\gamma_0} \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{s-Y_i}{h}\right)
\]
and \( \beta_0(s) \geq 1/2 \), it follows that \( \sup_{s \in [\gamma_n, +\infty)} \left| \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n(s)} \right| \leq \frac{M_1}{h} \) for some positive constant \( M_1 \). Using a Taylor expansion, we have

\[
\tilde{g}_n(s) = \frac{1}{h\beta_0(s)} \int_0^{+\infty} K(\frac{s-y}{h}) g(y) dy \\
= g(s) - \frac{h\beta_1(s)}{\beta_0(s)} g^{(1)}(s) + \frac{h^2}{\beta_0(s)} \int_{-\infty}^{s/h} u^2 K(u) \int_0^1 (1 - \tau)g^{(2)}(s - \tau hu) d\tau du.
\]

Therefore, under the assumptions of Lemma 5, we obtain that

\[
\int_{\gamma_n}^{\alpha_n} \left( \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n(s)} \right)^2 \frac{1}{g(s)} \left( \tilde{g}_n^{1/2}(s) - g^{1/2}(s) \right)^4 ds \leq \frac{M_2}{h^2} \int_{\gamma_n}^{\alpha_n} \frac{1}{g(s)} (\tilde{g}_n(s) - g(s))^2 ds = O(h^2), \tag{79}
\]

where \( M_2 \) is a positive constant. Similarly

\[
\int_{\gamma_n}^{\alpha_n} (\tilde{g}_n^{1/2}(s) - g^{1/2}(s))^2 ds \leq \int_{\gamma_n}^{\alpha_n} \frac{1}{g(s)} (\tilde{g}_n(s) - g(s))^2 ds = O(h^4). \tag{80}
\]
Combining (79) and (80), we conclude that

\[
\int_{\gamma_n}^{\alpha_n} \left( \frac{\tilde{g}_n^{(1)}(s)}{g_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)} \right)^2 ds = O(h^2). \tag{81}
\]

Now (43) follows from (74)-(76) and (81), and the proof of Lemma 5 is complete.