Supplementary Material for the manuscript entitled, "Efficient and Robust Tests for Semiparametric Models" by Jingjing Wu and Rohana J. Karunamuni.

Here we present detailed proofs of Lemmas 1 to 5 used in the above paper.

Proof of Lemma 1. Note that

$$\begin{aligned}\int(\sqrt{\hat{f}(x)}-\sqrt{f(x)})^2dx &\leq 2\int(\sqrt{\hat{f}(x)}-\sqrt{\bar{f}(x)})^2dx+2\int(\sqrt{\bar{f}(x)}-\sqrt{f(x)})^2dx\\ &= 2\int_{B_n}(\sqrt{\hat{f}(x)}-\sqrt{\bar{f}(x)})^2dx+2\int_{B_n^c}(\sqrt{\hat{f}(x)}-\sqrt{\bar{f}(x)})^2dx\\ &\quad+2\int(\sqrt{\bar{f}(x)}-\sqrt{f(x)})^2dx,\end{aligned} \tag{53}$$

where $\bar{f}(x)=E[\hat{f}(x)]$. We have

$$\begin{aligned}nh\mathrm{Var}(\hat{f}(x)) = nhE(\hat{f}(x)-\bar{f}(x))^2 &\leq \int\frac{1}{h}K^2(\frac{x-y}{h})f(y)\,dy\\ &= \int f(x-hu)K^2(u)\,du \leq \|K\|_\infty\bar{f}(x)\end{aligned}$$

$$E\int_{B_n}\left(\sqrt{\hat{f}(x)}-\sqrt{\bar{f}(x)}\right)^2dx \leq E\int_{B_n}\frac{(\hat{f}(x)-\bar{f}(x))^2}{\bar{f}(x)}\,dx \leq 2\|K\|_\infty(nh)^{-1}c_n$$

$$\begin{aligned}E\int_{B_n^c}\left(\sqrt{\hat{f}(x)}-\sqrt{\bar{f}(x)}\right)^2dx &\leq E\int_{B_n^c}\left|\hat{f}(x)-\bar{f}(x)\right|\,dx\\ &\leq \int_{B_n^c}(\mathrm{Var}(\hat{f}(x))^{1/2}dx \leq (nh)^{-1/2}\|K\|_\infty^{1/2}\int_{B_n^c}(\bar{f}(x))^{1/2}dx.\end{aligned}$$

By a Taylor expansion, we obtain

$$\bar{f}(x)=f(x)+\frac{\mu_2}{2}h^2f^{(2)}(x)-\frac{1}{2}h^3\int\int_0^1(1-t)^2f^{(3)}(x-thu)u^3K(u)dtdu,$$

where $\mu_2=\int u^2K(u)du$. Then

$$\begin{aligned}\textstyle\int_{B_n^c}(\bar{f}(x))^{1/2}dx &\leq \textstyle\int_{B_n^c}(f(x))^{1/2}dx+(\frac{\mu_2}{2})^{1/2}h\int_{B_n^c}|f^{(2)}(x)|^{1/2}dx\\ &\quad\textstyle+(\frac{1}{2})^{1/2}h^{3/2}\int_{B_n^c}\left|\int\int_0^1(1-t)^2f^{(3)}(x-thu)u^3K(u)dudt\right|^{1/2}dx.\end{aligned}$$

It is easy to show that $0 \leq \overline{\lim}_n \int_{B_n^c} \left| \int \int_0^1 (1-t)^2 f^{(3)}(x - thu) u^3 K(u) du dt \right|^{1/2} dx \leq 0$. Therefore, we have

$$\int \left(\sqrt{\hat{f}(x)} - \sqrt{\bar{f}(x)}\right)^2 dx = O_P((nh)^{-1} c_n + (nh)^{-1/2} C_{n1} + n^{-1/2} h^{1/2} C_{n2}) + o_p(n^{-1/2} h). \quad (54)$$

Define $b(x,h) = \int f(x - hu) K(u)\, du$. Then the first two derivatives of $b(x,h)$ w.r.t. h are given by

$$\dot{b}(x,h) = -\int f^{(1)}(x - hu) u\, K(u) du \quad \text{and} \quad \ddot{b}(x,h) = \int f^{(2)}(x - hu) u^2 K(u)\, du.$$

Note that $b(x,0) = f(x)$ and $\dot{b}(x,0) = 0$. The first two derivatives of $s(x,h) = \sqrt{b(x,h)}$ w.r.t. h are

$$\dot{s}(x,h) = \frac{\dot{b}(x,h)}{2\sqrt{b(x,h)}} \quad \text{and} \quad \ddot{s}(x,h) = \frac{\ddot{b}(x,h)}{2\sqrt{b(x,h)}} - \frac{(\dot{b}(x,h))^2}{4b(x,h)^{3/2}}.$$

Thus, we can express

$$\sqrt{\bar{f}(x)} - \sqrt{f(x)} = s(x,h) - s(x,0) - h\dot{s}(x,0) = \int_0^1 (1-t) h^2 \ddot{s}(x, th)\, dt,$$

and by the Cauchy-Schwarz inequality and by Fubini's theorem then we obtain

$$\begin{aligned}\int (\sqrt{\bar{f}(x)} - \sqrt{f(x)})^2 dx &= \int (\int_0^1 (1-t) h^2 \ddot{s}(x, th)\, dt)^2 dx \\ &\leq h^4 \int_0^1 (1-t)^2 \int (\ddot{s}(x, th)\,)^2 dx dt.\end{aligned}$$

Applications of the Cauchy-Schwarz inequality yield

$$\begin{aligned}(\dot{b}(x,h))^2 &\leq \int \frac{(f^{(1)})^2}{f}(x - hu) u^2 K(u) du \int f(x - hu) K(u) du \\ &\leq (\int \frac{(f^{(1)})^4}{f^3}(x - hu) u^4 K(u) du)^{1/2} (\int f(x - hu) K(u) du)^{3/2}\end{aligned}$$

and

$$(\ddot{b}(x,h))^2 \leq \int \frac{(f^{(2)})^2}{f}(x - hu) u^4 K(u)\, du \int f(x - hu) K(u)\, du.$$

The above expressions show that

$$(\ddot{s}(x,h))^2 \leq 2\frac{(\ddot{b}(x,h))^2}{4b(x,h)} + \frac{(\dot{b}(x,h))^4}{16(b(x,h))^3} \leq \int \psi(x-hu)u^4K(u)du$$

with

$$\psi(x) = \frac{(f^{(2)}(x))^2}{2f(x)} + \frac{(f^{(1)}(x))^4}{8f^3(x)}.$$

Consequently, we have by Fubini's theorem

$$\begin{aligned}\int \left(\sqrt{\bar{f}(x)} - \sqrt{f(x)}\right)^2 dx &\leq h^4 \int_0^1 (1-t)^2 \int\int \psi(x-thu)u^4K(u)dudxdt \\ &\leq h^4 \int \psi(x)dx \int u^4K(u)du \int_0^1 (1-t)^2dt \\ &= O(h^4),\end{aligned} \tag{55}$$

since ψ is integrable by assumptions. The proof of (28) is now completed by combining (53), (54) and (55). The proof of (29) is similar. □

Proof of Lemma 2. Again writing $f = f_{\theta,\eta}$, observe that

$$\begin{aligned}& 2\int \dot{s}_{\theta,\eta}(x)\hat{f}^{1/2}(x)dx - \tfrac{1}{n}\textstyle\sum_{i=1}^n \frac{\dot{s}_{\theta,\eta}}{s_{\theta,\eta}}(X_i) \\ =\ & \tfrac{1}{2}\int f^{(1)}(x)\left[\frac{\hat{f}(x)}{f(x)} - \frac{(\hat{f}^{1/2}(x)-f^{1/2}(x))^2}{f(x)}\right]dx - \tfrac{1}{n}\textstyle\sum_{i=1}^n \frac{f^{(1)}}{2f}(X_i) \\ =\ & \tfrac{1}{2}\left[\int \frac{f^{(1)}}{f}(x)\hat{f}(x)dx - \tfrac{1}{n}\textstyle\sum_{i=1}^n \frac{f^{(1)}}{f}(X_i)\right] - \tfrac{1}{2}\int \frac{f^{(1)}}{f}(x)(\hat{f}^{1/2}(x) - f^{1/2}(x))^2dx \\ =\ & \tfrac{1}{2}I_3 - \tfrac{1}{2}I_4, \text{ say.}\end{aligned}$$

Using a Taylor expansion of order four with an integral form of the reminder term, by (30) and Fubini's theorem, we obtain

$$\begin{aligned}E\int \frac{f^{(1)}}{f}(x)\hat{f}(x)dx &= \int \frac{f^{(1)}}{f}(x)\big\{f(x) + \tfrac{h^2}{2}f^{(2)}(x)\int u^2K(u)du \\ &\qquad + \tfrac{h^4}{24}\int\int_0^1(1-t)^3f^{(4)}(x-thu)u^4K(u)dudt\big\}dx \\ &= \tfrac{h^4}{24}\int \frac{f^{(1)}}{f}(x)\left(\int\int_0^1(1-t)^3f^{(4)}(x-thu)u^4K(u)dudt\right)dx \\ &= O(h^4),\end{aligned}$$

where we used the fact that

$$
\begin{aligned}
&\lim_{n\to\infty}\int \frac{f^{(1)}}{f}(x)\Big(\int\int_0^1 (1-t)^3 f^{(4)}(x-thu)u^4K(u)dudt\Big)dx \\
&= \int \frac{f^{(1)}}{f}(x)f^{(4)}(x)dx\Big(\int\int_0^1 (1-t)^3 u^4K(u)dudt\Big).
\end{aligned}
$$

The last assertion can be verified using the Dominated Convergence theorem (DCT) and Fatou's lemma; see, e.g., the proof of Lemma 4 below. Furthermore, $E(\frac{1}{n}\sum_{i=1}^n \frac{f^{(1)}}{f}(X_i)) = 0$. Thus, using (31) we obtain

$$
|E(I_3)| = O(h^4). \tag{56}
$$

Again by direct calculation and using the Cauchy-Schwartz inequality and a Taylor expansion, we have

$$
\begin{aligned}
Var(I_3) &\leq \frac{1}{n}\int\Big[\int \frac{f^{(1)}}{f}(x)\frac{1}{h}K(\frac{x-y}{h})dx - \frac{f^{(1)}}{f}(y)\Big]^2 f(y)dy \\
&\leq \frac{1}{n}\int\int\Big(\frac{f^{(1)}}{f}(y-uh) - \frac{f^{(1)}}{f}(y)\Big)^2 K(u)f(y)dudy \\
&= O(n^{-1}h^2)
\end{aligned} \tag{57}
$$

Therefore, from (56) and (57) we obtain $I_3 = O_P(h^4 + n^{-1/2}h)$. Note that from (29), we have $I_4 = O_P(h^4 + (nh)^{-1}C_{n4} + (nh)^{-1/2}C_{n3} + n^{-1/2}h^{1/2})$. This completes the proof. $\square$

Proof of Lemma 3. Since $\hat{\eta}(x) = \frac{1}{2}(\hat{f}(x+\tilde{\theta}) + \hat{f}(-x+\tilde{\theta}))b_n^2(x)$ is a symmetric analogue of $\hat{f}(x+\tilde{\theta})b_n^2(x)$, it is enough to prove the lemma for $\hat{f}(x+\tilde{\theta})b_n^2(x)$. Thus, in what follows we assume that $\hat{\eta}(x) = \hat{f}(x+\tilde{\theta})b_n^2(x)$. Note that

$$
\begin{aligned}
\int (s_{t,\hat{\eta}}(x) - s_{t,\eta}(x))^2 dx &= \int (\hat{\eta}^{1/2}(x) - \eta^{1/2}(x))^2 dx \\
&= \int (\hat{\eta}^{1/2}(x-\tilde{\theta}) - \eta^{1/2}(x-\tilde{\theta}))^2 dx \\
&= \int (\hat{f}^{1/2}(x) b_n^2(x-\tilde{\theta}) - \eta^{1/2}(x-\tilde{\theta}))^2 dx \\
&\leq 4 \int \left(\hat{f}^{1/2}(x) b_n^2(x-\tilde{\theta}) - \eta^{1/2}(x-\theta) b_n^2(x-\tilde{\theta})\right)^2 dx \\
&+ 4 \int \left(\eta^{1/2}(x-\theta) b_n^2(x-\tilde{\theta}) - \eta^{1/2}(x-\theta)\right)^2 dx + \\
&2 \int \left(\eta^{1/2}(x-\theta) - \eta^{1/2}(x-\tilde{\theta})\right)^2 dx \\
&= 4I_7 + 4I_8 + 2I_9, \text{ say.}
\end{aligned}
$$

From Lemma 1, we have

$$
I_7 \leq O_P(h^4 + (nh)^{-1} c_n + (nh)^{-1/2} C_{n1} + n^{-1/2} h^{1/2} C_{n2} + n^{-1/2} h). \tag{58}
$$

By direct calculation,

$$
\begin{aligned}
I_8 &= \int (1 - b_n^2(x-\tilde{\theta}))^2 \eta(x-\theta) dx \\
&\leq 2 \int (1 - b_n^2(x-\theta))^2 \eta(x-\theta) dx + 2 \int (b_n^2(x-\theta) - b_n^2(x-\tilde{\theta}))^2 \eta(x-\theta) dx \\
&\leq 2 \int_{c_n \leq |x-\theta| \leq c_n+1} (1 - b_n^2(x-\theta))^2 \eta(x-\theta) dx + 4(\theta - \tilde{\theta})^2 \int (b_n^{(1)}(x-\theta^*))^2 \eta(x-\theta) dx \\
&\leq C_{n5} + O_P(n^{-1}),
\end{aligned}
\tag{59}
$$

last equality follows from the facts that $\tilde{\theta}$ is $\sqrt{n}$-consistent, $b_n^{(1)}$ is bounded and $\int_{c_n \leq |x-\theta| \leq c_n+1} (1 - b_n^2(x-\theta))^2 \eta(x-\theta) dx \leq \int_{c_n \leq |x-\theta|} \eta(x-\theta) dx$, where θ^* is a value between θ and $\tilde{\theta}$. Again using a Taylor expansion,

$$
\begin{aligned}
I_9 &= \frac{1}{2}(\theta-\tilde{\theta})^2\int(\eta^{(1)}(x-\theta^*))^2(\eta(x-\theta^*))^{-1}dx \\
&= \frac{1}{2}(\theta-\tilde{\theta})^2\int(\eta^{(1)}(x))^2(\eta(x))^{-1}dx \qquad (60)\\
&= O_P(n^{-1})
\end{aligned}
$$

by (32) and the $\sqrt{n}$-consistent property of $\tilde{\theta}$, where again $\tilde{\theta}^*$ is a value between $\tilde{\theta}$ and θ. Now (33) follows from (58), (59) and (60). This completes the proof. □

Proof of Lemma 4. As in the proof of Lemma 3, we assume that $\hat{\eta}(x) = \hat{f}(x+\tilde{\theta})b_n^2(x)$. Denote $\hat{g}_n(x) = \sqrt{\hat{\eta}(x-\tilde{\theta})}$, $g_n(x) = \sqrt{f(x)}b_n(x-\theta)$ and $g(x) = \sqrt{\eta(x)}$. Then by Minkowaski inequality we have

$$
\begin{aligned}
\int(\dot{s}_{t,\hat{\eta}}-\dot{s}_{t,\eta})^2dx &\le 2\int\left(\hat{g}_n^{(1)}(x)-g_n^{(1)}(x)\right)^2dx + 4\int\left(g_n^{(1)}(x)-g^{(1)}(x-\theta)\right)^2dx \\
&\quad +4\int\left(g^{(1)}(x-\theta)-g^{(1)}(x-\tilde{\theta})\right)^2dx \qquad (61)\\
&= 2I_{10}+4I_{11}+4I_{12}, \text{ say.}
\end{aligned}
$$

From (34) and a Taylor expansion, it follows that

$$
I_{12} = (\theta-\tilde{\theta})^2\int(g^{(2)}(x))^2dx = O_P(n^{-1}). \qquad (62)
$$

Again by Minkowaski inequality,

$$
\begin{aligned}
I_{11} &\le 4\int\left(\bar{s}^{(1)}(x)-s^{(1)}(x)\right)^2 b_n^2(x-\theta)dx + 4\int\left(s^{(1)}(x)\right)^2\left(1-b_n(x-\theta)\right)^2dx \\
&\quad +4\int\left(\bar{s}(x)-s(x)\right)^2\left(b_n^{(1)}(x-\theta)\right)^2dx + 4\int\left(s(x)\right)^2\left(b_n^{(1)}(x-\theta)\right)^2dx \qquad (63)\\
&= 4I_{13}+4I_{14}+4I_{15}+4I_{16}, \text{ say,}
\end{aligned}
$$

where $s(x) = \sqrt{f(x)}$ and $\bar{s}(x) = \sqrt{\bar{f}(x)}$. Clearly,

$$\begin{aligned} I_{16} &= \int \eta(x-\theta)(b_n^{(1)}(x-\theta))^2 dx \\ &= \int_{|x|\geq c_n+1} \eta(x)dx \\ &\leq C_{n5} \end{aligned} \tag{64}$$

and

$$I_{15} \leq \int \left(\sqrt{\bar{f}(x)} - \sqrt{f(x)}\right)^2 dx = O(h^4), \tag{65}$$

by (55). Similarly, by the definition of $b_n(x)$ we obtain

$$\begin{aligned} I_{14} &= \int_{c_n \leq |x-\theta| \leq c_n+1} (s^{(1)}(x))^2 (1 - b_n(x-\theta))^2 dx \\ &\leq \int_{c_n \leq |x-\theta|} (s^{(1)}(x))^2 dx \\ &= C_{n6}. \end{aligned} \tag{66}$$

To study I_{13}, write

$$\bar{s}^{(1)}(x) - s^{(1)}(x) = \frac{1}{s(x)}(\bar{f}^{(1)}(x) - f^{(1)}(x)) - \frac{\bar{f}^{(1)}(x)}{\bar{f}(x)}[\frac{1}{s(x)}(\bar{s}(x) - s(x))^2 + (\bar{s}(x) - s(x)),$$

then by Minkowaski inequality one obtains

$$\begin{aligned} I_{13} &\leq 2\int \frac{1}{f(x)}(\bar{f}^{(1)}(x) - f^{(1)}(x))^2 b_n^2(x-\theta)dx + 4\int (\frac{\bar{f}^{(1)}(x)}{\bar{f}(x)})^2 \frac{1}{f(x)}(\bar{s}(x) - s(x))^4 b_n^2(x-\theta)dx \\ &\quad +4\int (\frac{\bar{f}^{(1)}(x)}{\bar{f}(x)})^2 (\bar{s}(x) - s(x))^2 b_n^2(x-\theta)dx \\ &= 2I_{16} + 4I_{17} + 4I_{18}, \text{ say.} \end{aligned} \tag{67}$$

Using the proof of Lemma 1, we have

$$\int (\frac{\bar{f}^{(1)}(x)}{\bar{f}(x)})^2 (\bar{s}(x) - s(x))^2 dx \leq h^4 \int (\frac{\bar{f}^{(1)}(x)}{\bar{f}(x)})^2 \int\int_0^1 (1-t)^2 \psi(x - thu) u^4 K(u) du dt dx.$$

By the DCT, one obtains

$$\int\int_0^1 (1-t)^2\psi(x-thu)u^4K(u)dudt \to \psi(x)\int\int_0^1 (1-t)^2u^4K(u)dudt,$$

as $n\to\infty$, and hence, by Fatou's lemma, it follows that

$$\begin{aligned}&\int(\frac{f^{(1)}(x)}{f(x)})^2\psi(x)\int\int_0^1 (1-t)^2u^4K(u)dudtdx\\ \leq\;& \underline{\lim}_{n\to\infty}\int(\frac{\bar f^{(1)}(x)}{\bar f(x)})^2\int\int_0^1 (1-t)^2\psi(x-thu)u^4K(u)dudtdx,\end{aligned} \tag{68}$$

since $\bar f^{(1)}/\bar f \to f^{(1)}/f$ as $n\to\infty$ for each x. On the other hand,

$$\begin{aligned}&\overline{\lim}_{n\to\infty}\int(\frac{\bar f^{(1)}(x)}{\bar f(x)})^2\int\int_0^1 (1-t)^2\psi(x-thu)u^4K(u)dudtdx\\ \leq\;& \int(\frac{f^{(1)}(x)}{f(x)})^2\psi(x)\int\int_0^1 (1-t)^2u^4K(u)dudtdx.\end{aligned} \tag{69}$$

From the relations (68) and (69), we now obtain

$$\begin{aligned}&\lim_{n\to\infty}\int(\frac{\bar f^{(1)}(x)}{\bar f(x)})^2\int\int_0^1 (1-t)^2\psi(x-thu)u^4K(u)dudtdx\\ =\;& \int(\frac{f^{(1)}(x)}{f(x)})^2\psi(x)\int\int_0^1 (1-t)^2u^4K(u)dudtdx,\end{aligned} \tag{70}$$

and thus from (68) we have

$$\begin{aligned}I_{18} &\leq \int(\frac{\bar f^{(1)}(x)}{\bar f(x)})^2(\bar s(x)-s(x))^2dx\\ &= O(h^4)+o(h^4).\end{aligned} \tag{71}$$

Since $(\bar s(x)-s(x))^4 \leq \left(\sqrt{\bar f(x)}-\sqrt{f(x)}\right)^2$, a similar argument show that

$$I_{17} \leq O(h^4)+o(h^4), \tag{72}$$

provided $f^{(2)}$ is bounded and (36) hold. Similarly, we can show that

$$I_{16} \leq O(h^4)+o(h^4), \tag{73}$$

when $f^{(3)}$ is bounded and (37) hold. From Lemma 3 of Beran (1978), we also have

$$\begin{aligned} I_{10} &= \int (\hat{g}_n^{(1)}(x) - g_n^{(1)}(x))^2 dx \\ &= O_P((nh^3)^{-1}c_n. \end{aligned} \tag{74}$$

The proof of (38) is now completed by combining (61) to (74). □

Proof of Lemma 5. By (40) and (42), we obtain

$$\begin{aligned} &\int (\hat{\rho}_t(x) - \dot{s}_{t,\eta}(x))^2 dx \\ =& \int_{\gamma_n \le |x-t| \le \alpha_n} (\hat{\rho}_t(x) - \dot{s}_{t,\eta}(x))^2 dx + \int_{|x-t|<\gamma_n} \dot{s}^2_{t,\eta}(x) dx + \int_{|x-t|>\alpha_n} \dot{s}^2_{t,\eta}(x) dx \\ =& \frac{1}{4}\int_{\gamma_n}^{\alpha_n} (\frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)})^2 ds + \frac{1}{4}\int_0^{\gamma_n} \frac{[g^{(1)}(s)]^2}{g(s)} ds + \frac{1}{4}\int_{\alpha_n}^{+\infty} \frac{[g^{(1)}(s)]^2}{g(s)} ds. \end{aligned} \tag{75}$$

Let $\tilde{g}_n(s) = E\hat{g}_n(s)$. Obviously

$$\begin{aligned} &\int_{\gamma_n}^{\alpha_n} (\frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)})^2 ds \\ \le& \ 2\int_{\gamma_n}^{\alpha_n} (\frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)})^2 ds + 2\int_{\gamma_n}^{\alpha_n} (\frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)})^2 ds. \end{aligned} \tag{76}$$

By the arguments similar to those used in the proof of Lemma 3 of Beran (1978) and noting that $\inf_{s\in[\gamma_n,+\infty)}|\beta_1(s)\delta_0(s) - \beta_0(s)\delta_1(s)| \ge \gamma_0$, we deduce that

$$\int_{\gamma_n}^{\alpha_n} (\frac{\hat{g}_n^{(1)}(s)}{\hat{g}_n^{1/2}(s)} - \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)})^2 ds \ = \ O_p((nh^3)^{-1}\alpha_n). \tag{77}$$

By direct calculations,

$$\frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)} = \frac{\tilde{g}_n^{(1)}(s) - g^{(1)}(s)}{g^{1/2}(s)} - \frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n(s)}[\frac{1}{g^{1/2}(s)}(\tilde{g}_n^{1/2}(s) - g^{1/2}(s))^2 + (\tilde{g}_n^{1/2}(s) - g^{1/2}(s))].$$

Let $\beta(s) = \beta_1(s)\delta_0(s) - \beta_0(s)\delta_1(s)$. Then by a Taylor expansion, we have

$$\begin{aligned}\tilde{g}_n^{(1)}(s) &= \frac{1}{h^2\beta(s)}\int_0^{+\infty}[\beta_0(s)K'(\frac{s-y}{h}) - \delta_0(s)K(\frac{s-y}{h})]g(y)dy \\ &= \frac{1}{h\beta(s)}\int_{-\infty}^{s/h}[\beta_0(s)K'(u) - \delta_0(s)K(u)]g(s-hu)du \\ &= g^{(1)}(x) + \frac{h}{\beta(s)}\int_{-\infty}^{s/h}u^2[\beta_0(s)K'(u) - \delta_0(s)K(u)]\int_0^1(1-\tau)g^{(2)}(s-\tau hu)d\tau du.\end{aligned}$$

Then under the assumptions of Lemma 5, using a proof similar to that of (70) and noting that $0 \leq \beta_0(s) \leq \int_{-\infty}^{+\infty} K(s)ds = 1$ and $0 \leq \delta_0(s) < +\infty$, we obtain

$$\int_{\gamma_n}^{\alpha_n}\frac{[\tilde{g}_n^{(1)}(s) - g^{(1)}(s)]^2}{g(s)}ds = O(h^2). \quad (78)$$

Since

$$|\hat{g}_n^{(1)}(s)| \leq \frac{\beta_0(s)\sup_s|K'(s)/K(s)| + \delta_0(s)}{h\gamma_0}\frac{1}{nh}\sum_{i=1}^n K\big(\frac{s-Y_i}{h}\big)$$

and $\beta_0(s) \geq 1/2$, it follows that $\sup_{s\in[\gamma_n,+\infty)}|\frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n(s)}| \leq \frac{M_1}{h}$ for some positive constant M_1. Using a Taylor expansion, we have

$$\begin{aligned}\tilde{g}_n(s) &= \frac{1}{h\beta_0(s)}\int_0^{+\infty}K(\frac{s-y}{h})g(y)dy \\ &= g(s) - \frac{h\beta_1(s)}{\beta_0(s)}g^{(1)}(s) + \frac{h^2}{\beta_0(s)}\int_{-\infty}^{s/h}u^2K(u)\int_0^1(1-\tau)g^{(2)}(s-\tau hu)d\tau du.\end{aligned}$$

Therefore, under the assumptions of Lemma 5, we obtain that

$$\int_{\gamma_n}^{\alpha_n}(\frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n(s)})^2\frac{1}{g(s)}(\tilde{g}_n^{1/2}(s) - g^{1/2}(s))^4ds \;\leq\; \frac{M_2}{h^2}\int_{\gamma_n}^{\alpha_n}\frac{1}{g(s)}(\tilde{g}_n(s) - g(s))^2ds \;=\; O(h^2), \quad (79)$$

where M_2 is a positive constant. Similarly

$$\int_{\gamma_n}^{\alpha_n}(\tilde{g}_n^{1/2}(s) - g^{1/2}(s))^2ds \;\leq\; \int_{\gamma_n}^{\alpha_n}\frac{1}{g(s)}(\tilde{g}_n(s) - g(s))^2ds \;=\; O(h^4). \quad (80)$$

Combining (79) and (80), we conclude that

$$\int_{\gamma_n}^{\alpha_n} \left(\frac{\tilde{g}_n^{(1)}(s)}{\tilde{g}_n^{1/2}(s)} - \frac{g^{(1)}(s)}{g^{1/2}(s)}\right)^2 ds \;=\; O(h^2). \tag{81}$$

Now (43) follows from (74)-(76) and (81), and the proof of Lemma 5 is complete. □