Ann Inst Stat Math (2018) 70:303–321
https://doi.org/10.1007/s10463-016-0594-5

The continuous-time triangular Pólya process

Chen Chen[1] · **Hosam Mahmoud**[1]

Received: 21 April 2016 / Revised: 24 October 2016 / Published online: 2 January 2017

Abstract We study poissonized triangular (reducible) urns on two colors, which we take to be white and blue. We analyze the number of white and blue balls after a certain period of time has elapsed. We show that for balanced processes in this class, a different scaling is needed for each color to produce nontrivial limits, contrary to the distributions in the usual irreducible urns which only require the same scaling for both colors. The limit distributions (of the scaled variables) underlying triangular urns are Gamma. The technique we use couples partial differential equations with the method of moments applied in a bootstrapped manner to produce exact and asymptotic moments. For the dominant color, we get exact moments, while relaxing the balance condition. The exact moments include alternating signs and Stirling numbers of the second kind.

Keywords Urn · Pólya urn · Pólya process · Branching process · Gamma distribution · Partial differential equation

1 Introduction

Owing to their conceptual simplicity and versatility, urn schemes have become a fundamental mathematical tool. They are widely used in modeling, simulation and testing. The applications are limitless, spanning a range of important modern areas such as algorithmics, genetics, epidemiology, physics, engineering, economics, networks and much more. For centuries, urns containing balls and similar constructs have been used as models (there is a mention of them in the Talmud; see Rabinovitch 1969),

✉ Chen Chen
cchen@gwu.edu

1 Department of Statistics, The George Washington University, Washington, DC 20052, USA

sometimes in a different terminology like particles at various energy levels or objects in boxes. Perhaps the earliest contributions in the flavor that in our time got to be commonly named Pólya urns are Eggenberger and Pólya (1923) and Ehrenfest and Ehrenfest (1907), where the authors used urns as models for contagion and for gas diffusion. Numerous Pólya urn models useful for applications appeared thereafter, too many (literally hundreds) to be cited individually. For a classic survey see Johnson and Kotz (1977), Kotz and Balakrishnan (1997), or Mahmoud (2008) for the basics and many applications including informatics and biosciences.

To keep our discussion simple, we treat the two-color Pólya urn process; generalization to multicolor may be possible. In the classic flavor, the time $n = 0, 1, 2, \ldots$ is discrete. At the beginning of time, the urn is nonempty and contains a certain number of white and blue balls. At every subsequent epoch of time, a ball is chosen at random from the urn, its color is observed, then the ball is placed back in the urn. Depending on its color, the returning ball comes back accompanied with an additional number of white and blue balls. If we have chosen a white ball, we put in the urn a additional white balls and b blue, but if we have chosen a blue ball, we put in the urn c white balls and d additional blue. These dynamics of the urn scheme are captured by a ball replacement matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

The rows of this matrix are indexed with the color of the ball picked at a stage (white and blue, respectively, from top to bottom), and the columns are indexed with the color of the balls added (white and blue, respectively, from left to right). Entry (i, j) is the number of balls of color $j \in \{\text{white, blue}\}$ added upon drawing a ball of color $i \in \{\text{white, blue}\}$. We concentrate on cases where all matrix entries are nonnegative. Thus, tenability is not an issue.

The interest is often in the long-term composition of the urn or in waiting times till certain events occur. Several balanced classes have been studied, where balance means that the total number of balls added (regardless of the color of the ball withdrawn) is constant. In a balanced urn, we have $a + b = c + d$. This sum, say θ, is often called the balance factor of the urn. Balance is introduced for mathematical convenience. Nevertheless, the balanced triangular case, with replacement matrix

$$\begin{pmatrix} a & d-a \\ 0 & d \end{pmatrix}, \tag{2}$$

with $d > a > 0$, was a challenge for some time. Only recently has there been an intensified foray on this case, and we now have a few decent characterizations; see the studies by Flajolet et al. (2006), Janson (2006), Kuba and Panholzer (2014), and Zhang et al. (2015).

Associated with a Pólya urn scheme is a process obtained by embedding in continuous time, lately called the Pólya process; see Sparks and Mahmoud (2013). To create a clear distinction, we adhere to the term "scheme," when we speak of an urn that grows in discrete time, and to the term "process," when we speak of an urn that grows in continuous time. The process was introduced by Athreya and Karlin (1968) as a mathematical transform (poissonization) to understand Pólya urn schemes. Athreya and Karlin (1968) reports that depoissonization of the Pólya urn process (to recover results on the discrete urn scheme, i.e., produce an inverse transform) is fraught with difficulty.

2 Scope

Is the continuous-time balanced triangular Pólya urn process as hard to analyze as the discrete-time Pólya urn scheme, which stood as a challenge for a long time? In this manuscript we take on the analysis of the balanced triangular urn process. In the following two subsections, we define them first and mention the tools we need. In Sect. 3, we state the main results with interpretation. In Sect. 4, we describe the method of bootstrapping the moments, and give the proofs for the theorems on asymptotics. In Sect. 5, we derive exact moments. Section 6 concludes with some remarks.

2.1 Embedding in real time

What distinguishes continuous-time Pólya processes from the more conventional discrete-time Pólya urn schemes is the timing of the ball draws. In the discrete-time Pólya urn schemes, the balls are picked at equispaced time intervals (and the spacing is taken to be the unit of time). In the continuous-time Pólya processes, each ball is endowed with a clock that rings in exponential time Exp(1) (an exponential random variable with mean 1). All the clocks are independent of each other and of any other random variables related to the past. When the clock of a ball rings (a renewal point in the Pólya process or an epoch), the ball is immediately picked from the urn, and the rules associated with its color are instantaneously executed. All new balls come endowed with their own independent clocks. The collective process enjoys memorylessness, as it is induced by independent clocks with exponential ringing time. If a clock at an epoch is at a certain proportion of its ringing time, the associated ball does not carry over that time to the next renewal. By memorylessness of exponential random variables, the time remaining on the clock to ring is distributed as Exp(1), as if the process is reset to start afresh after each renewal. Thus, the Pólya process is Markovian, with no memory of the past, but the rates change (the process speeds up) depending on how many balls are added. In general, the Markovian process may be an inhomogeneous jump process, with the jumps occurring at the epochs.

Let $W(t)$ and $B(t)$, be the number of white, respectively, blue, balls in the urn at time t in a balanced triangular Pólya process. Formally, the two-color Pólya process is the two-component vector $\binom{W(t)}{B(t)}$. In what follows, we refer to the total number of balls at time t as $\tau(t)$, i.e., we have

$$\tau(t) = W(t) + B(t).$$

So, $W_0 = W(0)$ and $B_0 = B(0)$ are the initial numbers of white and, respectively, blue balls, and $\tau_0 = \tau(0)$ is the initial number of all balls in the urn.

Asymptotic results for a large class of balanced Pólya processes can be obtained from Janson (2004) and earlier classic results of branching processes from Athreya and Ney (1972), Chapter 3. Theorem 3.1 by Janson (2004), coupled with a discussion by Athreya and Ney (1972), yields

$$\mathrm{e}^{-\theta t}\begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\text{a.s.}} \text{Gamma}\Big(\frac{\tau_0}{\theta}, \theta\Big)\begin{pmatrix} v \\ 1-v \end{pmatrix}.$$

Note that, in a balanced urn with balance factor θ, the value θ is the principal (larger) eigenvalue of the transpose of the replacement matrix, and $\left(\begin{smallmatrix} v \\ 1-v \end{smallmatrix}\right)$ is the corresponding (principal) eigenvector (normalized to be of length 1).

The class covered by this result is constrained to meet certain conditions, including balance, tenability and irreducibility, and excludes triangular urns. A color is called dominant, if when the urn starts with one ball of that color, every other color appears infinitely often. Irreducibility, as defined in Janson (2004), comes down to the condition that every color is dominant. Triangular urns are not irreducible. It is clear that in the two-color flavor of the triangular Pólya process, if we start with only one blue ball, white balls never appear.

2.2 Tools: partial differential equations and bootstrapped moments

We shall establish here the counterpart of the results in Janson (2004) for the excluded triangular case via an analytic method. Let

$$\phi(t, u, v) := \mathbb{E}\big[\mathrm{e}^{uW(t)+vB(t)}\big]$$

be the joint moment generating function of the vector $\left(\begin{smallmatrix} W(t) \\ B(t) \end{smallmatrix}\right)$. For the general tenable Pólya urn (not necessarily balanced or triangular) with replacement matrix (1), the work by Balaji and Mahmoud (2006) gives the partial differential equation (PDE):

$$\frac{\partial\phi(t,u,v)}{\partial t} + \big(1-\mathrm{e}^{au+bv}\big)\frac{\partial\phi(t,u,v)}{\partial u} + \big(1-\mathrm{e}^{cu+dv}\big)\frac{\partial\phi(t,u,v)}{\partial v} = 0. \qquad (3)$$

This functional equation has only been solved in a limited number of cases, as for instance for the case of forward and backward diagonal processes, see Balaji and Mahmoud (2006); for the Ehrenfest process, see Balaji et al. (2006); for those balanced processes with Bernoulli matrix entries, see Sparks and Mahmoud (2013); for Apollonian process, see Zhang and Mahmoud (2016).

While the PDE (3) is of the first order, with a known general solution, there is a latent difficulty in extracting the joint or marginal distributions. The general solution via the method of characteristics gives the solution as an integration along characteristic curves; see Levine (1997). Unfortunately, these characteristics are difficult to determine for the PDE (3) and integration along the characteristics is far from easy.

The simplest of urn cases involve integrating expressions having the Lambert W function, which itself has an implicit definition. In the cases solved, the authors appealed to indirect methods of solution, relying more on the probabilistic meaning of the PDE.

For the special balanced triangular urn with replacement matrix (2), the governing PDE (3) becomes

$$\frac{\partial \phi}{\partial t} + \left(1 - \mathrm{e}^{au+(d-a)v}\right)\frac{\partial \phi}{\partial u} + \left(1 - \mathrm{e}^{dv}\right)\frac{\partial \phi}{\partial v} = 0. \tag{4}$$

We shall pursue a strategy that extracts moments, in a bootstrapped way (i.e., following a particular order) from this PDE, and analyze exact and limit moments for the ball counts associated with the balanced triangular Pólya processes. More about the specific order will be discussed in Sect. 4.

3 Main results

In this section, we state the salient results. The number of white balls needs the scaling e^{at} to converge nontrivially (even in the unbalanced case). Lemma 3 gives the limit distribution of $\mathrm{e}^{-at}W(t)$, which is implicit in prior work as will be mentioned.

By contrast, the number of blue balls needs the scaling e^{dt} to converge nontrivially.

Theorem 1 *Let $B(t)$ be the number of blue balls in the urn at time t in a balanced triangular Pólya process with replacement matrix*

$$\begin{pmatrix} a & d-a \\ 0 & d \end{pmatrix},$$

with $d > a > 0$. Assume the process starts with W_0 white balls and B_0 blue balls (i.e., it starts with a total of $\tau_0 = W_0 + B_0 > 0$). As $t \to \infty$, we have

$$\frac{B(t)}{\mathrm{e}^{dt}} \xrightarrow{D} \mathrm{Gamma}\left(\frac{\tau_0}{d}, d\right).$$

The exact moments of $W(t)$ involve alternating signs, Stirling numbers of the second kind and rising factorials. These exact moments are derived under relaxed balance conditions.

Pochhammer's symbol for the rising factorial is

$$\langle x \rangle_s = x(x+1)\cdots(x+s-1),$$

for any $x \in \mathbb{R}$, and any integer $s \geq 0$, with the interpretation that $\langle x \rangle_0 = 1$.

The numbers $\left\{ {r \atop j} \right\}$ are Stirling numbers of the second kind. This is the number of ways to partition a set of r distinct objects into j nonempty parts. For properties of these combinatorial numbers, we refer the reader to textbooks such as David and Barton (1962) and Graham et al. (1994).

Theorem 2 *Let $W(t)$ be the number of white balls in the urn at time t in a triangular Pólya process with replacement matrix*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

with $a > 0$, $b > 0$ and $d > 0$. Assume the process starts with W_0 white balls. The moments of $W(t)$ are

$$\mathbb{E}\big[W^n(t)\big] = a^n \sum_{i=1}^{n} (-1)^{n-i} \left\{ {n \atop i} \right\} \left\langle \frac{W_0}{a} \right\rangle_i \mathrm{e}^{ait}, \quad n \ge 1. \tag{5}$$

The method of bootstrapped moments will furnish a bivariate distribution for the balanced triangular Pólya process under suitable scaling, with both marginal distributions being Gamma limits.

4 The method of bootstrapped moments

For positive integers, $i, j \ge 0$, consider $\mathbb{E}[W^i(t)B^j(t)]$, the mixed moment of $W(t)$ and $B(t)$ of order (i, j). From the PDE (3) we develop ordinary differential equations for the mixed moments, one for each such moment. These ordinary differential equations will involve full history recurrences—the right-hand side of the recurrence for the mixed moment of order (i, j) involves a term with the mixed moment of order (i, j), and the functions $\mathbb{E}[W^r(t)B^s(t)]$, for every $1 \le r + s \le i + j$ and $s < j$. The exact forms are rather complex, but the general form is amenable to the extraction of simple asymptotic mixed moments. The asymptotic mixed moments are of the Gamma type (have certain forms of the Gamma function). They are sufficient for the purpose of identifying an asymptotic joint Gamma distribution.

Take the partial derivative of both sides of (4), i times with respect to u and j times with respect to v, then evaluate at $u = v = 0$. This operation produces a first-order ordinary differential equation for the mixed moment of order (i, j) as follows. First, note that

$$\frac{\partial^{i+j}}{\partial u^i\, \partial v^j} \phi(t, u, v)\Big|_{u=v=0} = \mathbb{E}\big[W^i(t)B^j(t)\big].$$

Next apply the differential operator $\frac{\partial^{i+j}}{\partial u^i\, \partial v^j}$ to both sides of the PDE (4) and evaluate at $u = v = 0$. We get

$$\frac{\partial^{i+j}}{\partial u^i\, \partial v^j} \left(\frac{\partial}{\partial t} \phi(t, u, v) \right) \Big|_{u=v=0} + \frac{\partial^{i+j}}{\partial u^i\, \partial v^j} (1 - \mathrm{e}^{au+(d-a)v}) \frac{\partial \phi}{\partial u} \Big|_{u=v=0}$$
$$+ \frac{\partial^{i+j}}{\partial u^i\, \partial v^j} (1 - \mathrm{e}^{dv}) \frac{\partial \phi}{\partial v} \Big|_{u=v=0} = 0. \tag{6}$$

We go through such applications one part at a time. Consider the leftmost term in (6):

$$\frac{\partial^{i+j}}{\partial u^i\,\partial v^j}\left(\frac{\partial}{\partial t}\phi(t,u,v)\right)\bigg|_{u=v=0} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\left[\frac{\partial^{i+j}}{\partial u^i\,\partial v^j}\mathrm{e}^{W(t)u+B(t)v}\Big|_{u=v=0}\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\big[W^i(t)B^j(t)\big].$$

For the middle part in the left-hand side of (6), we obtain (via Leibniz rule for differentiation)

$$\frac{\partial^{i+j}}{\partial u^i\,\partial v^j}\big(1-\mathrm{e}^{au+(d-a)v}\big)\frac{\partial\phi}{\partial u}\bigg|_{u=v=0}$$
$$= \frac{\partial^i}{\partial u^i}\sum_{s=0}^{j}\binom{j}{s}\frac{\partial^{j-s}}{\partial v^{j-s}}\big(1-\mathrm{e}^{au+(d-a)v}\big)\frac{\partial^s}{\partial v^s}\left(\frac{\partial\phi}{\partial u}\right)\bigg|_{u=v=0}$$
$$= \frac{\partial^i}{\partial u^i}\left(\big(1-\mathrm{e}^{au+(d-a)v}\big)\frac{\partial^j}{\partial v^j}\left(\frac{\partial\phi}{\partial u}\right)\right)\bigg|_{u=v=0}$$
$$-\frac{\partial^i}{\partial u^i}\sum_{s=0}^{j-1}\binom{j}{s}(d-a)^{j-s}\mathrm{e}^{au+(d-a)v}\frac{\partial^s}{\partial v^s}\left(\frac{\partial\phi}{\partial u}\right)\bigg|_{u=v=0}$$
$$= \sum_{r=0}^{i}\binom{i}{r}\frac{\partial^{i-r}}{\partial u^{i-r}}\big(1-\mathrm{e}^{au+(d-a)v}\big)\left(\frac{\partial^r}{\partial u^r}\left(\frac{\partial^{j+1}\phi}{\partial u\,\partial v^j}\right)\right)\bigg|_{u=v=0}$$
$$-\sum_{s=0}^{j-1}\sum_{r=0}^{i}\binom{i}{r}\binom{j}{s}(d-a)^{j-s}\frac{\partial^{i-r}}{\partial u^{i-r}}\mathrm{e}^{au+(d-a)v}\left(\frac{\partial^r}{\partial u^r}\left(\frac{\partial^{s+1}\phi}{\partial u\,\partial v^s}\right)\right)\bigg|_{u=v=0}$$
$$= -\sum_{r=0}^{i-1}\binom{i}{r}a^{i-r}\mathbb{E}\big[W^{r+1}(t)B^j(t)\big]$$
$$-\sum_{s=0}^{j-1}\sum_{r=0}^{i}\binom{i}{r}\binom{j}{s}a^{i-r}(d-a)^{j-s}\mathbb{E}\big[W^{r+1}(t)B^s(t)\big].$$

Likewise, operating on the rightmost term in the left-hand side of (6), we get

$$\frac{\partial^{i+j}}{\partial u^i\,\partial v^j}\big(1-\mathrm{e}^{dv}\big)\frac{\partial\phi}{\partial v}\bigg|_{u=v=0} = \frac{\partial^i}{\partial u^i}\sum_{s=0}^{j}\binom{j}{s}\frac{\partial^{j-s}}{\partial v^{j-s}}\big(1-\mathrm{e}^{dv}\big)\frac{\partial^s}{\partial v^s}\left(\frac{\partial\phi}{\partial v}\right)\bigg|_{u=v=0}$$
$$= \frac{\partial^i}{\partial u^i}\left(\big(1-\mathrm{e}^{dv}\big)\frac{\partial^{j+1}\phi}{\partial v^{j+1}}\right)\bigg|_{u=v=0}$$

$$
- \frac{\partial^i}{\partial u^i} \sum_{s=0}^{j-1} \binom{j}{s} d^{j-s} \mathrm{e}^{dv} \frac{\partial^{s+1}\phi}{\partial v^{s+1}}\bigg|_{u=v=0}
$$

$$
= - \sum_{s=0}^{j-1} \binom{j}{s} d^{j-s} \mathbb{E}\big[W^i(t) B^{s+1}(t)\big].
$$

Putting all the parts together, we find the first-order differential equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\big[W^i(t) B^j(t)\big] &= \sum_{r=0}^{i-1} \binom{i}{r} a^{i-r} \mathbb{E}\big[W^{r+1}(t) B^j(t)\big] \\
&\quad + \sum_{r=0}^{i} \sum_{s=0}^{j-1} \binom{i}{r}\binom{j}{s} a^{i-r} (d-a)^{j-s} \mathbb{E}\big[W^{r+1}(t) B^s(t)\big] \\
&\quad + \sum_{s=0}^{j-1} \binom{j}{s} d^{j-s} \mathbb{E}\big[W^i(t) B^{s+1}(t)\big].
\end{aligned} \tag{7}
$$

Proposition 1 *Let $W(t)$ and $B(t)$ be the number of white and blue balls, respectively, in the urn at time t in a balanced triangular Pólya process with replacement matrix*

$$
\begin{pmatrix} a & d-a \\ 0 & d \end{pmatrix},
$$

with $d > a > 0$. Assume the urn starts with W_0 white balls and B_0 blue balls (and thus we have $\tau_0 = W_0 + B_0$).

We have

$$
\begin{aligned}
\mathbb{E}\big[W(t)\big] &= W_0 \mathrm{e}^{at}, \\
\mathbb{E}\big[B(t)\big] &= \big(W_0 + B_0\big)\mathrm{e}^{dt} - W_0 \mathrm{e}^{at}, \\
\mathbb{V}\mathrm{ar}\big[W(t)\big] &= a W_0 \big(\mathrm{e}^{2at} - \mathrm{e}^{at}\big), \\
\mathbb{C}\mathrm{ov}\big[W(t), B(t)\big] &= a W_0 \big(\mathrm{e}^{(a+d)t} - \mathrm{e}^{2at}\big), \\
\mathbb{V}\mathrm{ar}\big[B(t)\big] &= d\big(W_0 + B_0\big)\mathrm{e}^{2dt} - 2a W_0 \mathrm{e}^{(a+d)t} + a W_0 \mathrm{e}^{2at} \\
&\quad - d\big(W_0 + B_0\big)\mathrm{e}^{dt} + a W_0 \mathrm{e}^{at}.
\end{aligned}
$$

Proof In the differential equation (7), set $i = 1$ and $j = 0$ to get

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\big[W(t)\big] = a\, \mathbb{E}\big[W(t)\big].
$$

The solution for this differential equation is

$$
\mathbb{E}[W(t)] = \mathrm{e}^{\int_0^t a\, \mathrm{d}x}\, \mathbb{E}[W(0)] = W_0 \mathrm{e}^{at}.
$$

Similarly, using (7), we can obtain the following differential equations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\big[B(t)\big] &= (d-a)\,\mathbb{E}\big[W(t)\big] + d\,\mathbb{E}\big[B(t)\big],\\
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\big[W^2(t)\big] &= a^2\,\mathbb{E}\big[W(t)\big] + 2a\,\mathbb{E}\big[W^2(t)\big],\\
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\big[W(t)B(t)\big] &= a(d-a)\,\mathbb{E}\big[W(t)\big] + (a+d)\,\mathbb{E}\big[W(t)B(t)\big]\\
&\quad + (d-a)\,\mathbb{E}\big[W^2(t)\big],\\
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\big[B^2(t)\big] &= (d-a)^2\,\mathbb{E}\big[W(t)\big] + 2(d-a)\,\mathbb{E}\big[W(t)B(t)\big] + d^2\,\mathbb{E}\big[B(t)\big]\\
&\quad + 2d\,\mathbb{E}\big[B^2(t)\big].
\end{aligned}$$

The solutions to these equations are straightforward, yielding the statement of the proposition. □

Remark 1 Note that in Proposition 1 we listed the expression for $\mathbb{C}\mathrm{ov}[W(t), B(t)]$ before that for $\mathbb{V}\mathrm{ar}[B(t)]$ to reflect the order in which the corresponding differential equations were solved. The solution to the differential equation for $\mathbb{E}[W(t)]$ was bootstrapped into the differential equation for $\mathbb{E}[B(t)]$, the solution to which was then bootstrapped into the differential equation for $\mathbb{E}[W(t)B(t)]$, the solution to which was then bootstrapped into the differential equation for $\mathbb{E}[B^2(t)]$.

We say that $\mathbb{E}\big[W^i(t)B^j(t)\big]$ is a mixed moment of total order $i+j$. We can continue in the fashion of the proof of Proposition 1 to produce any desired number of moments of higher total order. The strategy is to first produce a few moments of low total order. Whenever we get to a higher total order, we bootstrap ingredients that are available, which are the solutions of mixed moments of lower total order, or of the same total order that have already been computed. For a fixed total order $i + j = k$, we follow an order of decreasing i, i.e., we solve for $i = k, k-1, \ldots, 0$.

The first few are obtained without difficulty. However, the computational complexity is ever increasing with the total order. As an illustration, we list all the mixed moments of total order 3 in the appendix. The reader will notice that moments of total order 1 and 2 are quite simple and are given by short formulas, and these formulas become lengthy for moments of total order 3.

Toward asymptotics, we simplify notation and focus on the functional equation aspect of the formulation. Set

$$m_{i,j}(t) = \mathbb{E}\big[W^i(t)B^j(t)\big].$$

We prove by a bootstrapped induction on $i+j$ that the mixed moments of order (i, j) of the balanced triangular Pólya process are a combination of exponential functions.

Lemma 1 *The mixed moments $\mathbb{E}\big[W^i(t)B^j(t)\big]$ have the form*

$$m_{i,j}(t) = \sum_{\substack{1\le r+s\le i+j\\ s\le j}} \xi_{r,s}^{(i,j)} \mathrm{e}^{(ar+ds)t}, \tag{8}$$

where $\xi_{r,s}^{(i,j)} = \xi_{r,s}^{(i,j)}(a, d, W_0, B_0) \in \mathbb{R}$ are coefficients.[1]

Proof The bootstrapping requires a particular order. When we deal with total order $k = i + j$, we first take the differential equation for the kth moment of the number of white balls, i.e., the mixed moment of order $(k, 0)$. This moment involves lower order moments of the number of white balls. We then proceed to compute the mixed moment of order $(k - 1, 1)$. The mixed moment of order $(k - 1, 1)$ depends on moments of total order $k - 1 = i + j - 1$, and the mixed moment of the order $(k, 0)$, which we have already computed. We continue by following a decreasing order for i (that is, $i = k, k - 1, \ldots, 0$). This bootstrapping technique was illustrated in the computation of moments of total order up to 3 (see Remark 1 and the appendix).

The forms of the means of $W(t)$ and $B(t)$ provide a basis for the induction, at $i + j = 1$. For some value (i, j), assuming the differential equation (8) holds for $m_{r,s}$ with $r + s \le i + j$ and $s < j$. Then, the differential form (7) yields

$$\begin{aligned} m'_{i,j}(t) &= (ai + dj)m_{i,j}(t) + \sum_{r=0}^{i-2} \binom{i}{r} a^{i-r} m_{r+1,j}(t) \\ &\quad + \sum_{r=0}^{i}\sum_{s=0}^{j-1} \binom{i}{r}\binom{j}{s} a^{i-r}(d-a)^{j-s} m_{r+1,s}(t) \\ &\quad + \sum_{s=0}^{j-2} \binom{j}{s} d^{j-s} m_{i,s+1}(t) \\ &= (ai + dj)m_{i,j}(t) + \sum_{r=0}^{i-2} \binom{i}{r} a^{i-r} \sum_{\substack{1\le k+\ell\le r+1+j\\ \ell\le j}} \xi_{k,\ell}^{(r+1,j)} \mathrm{e}^{(ak+d\ell)t} \\ &\quad + \sum_{r=0}^{i}\sum_{s=0}^{j-1} \binom{i}{r}\binom{j}{s} a^{i-s}(d-a)^{j-r} \sum_{\substack{1\le k+\ell\le r+1+s\\ \ell\le s}} \xi_{k,\ell}^{(r+1,s)} \mathrm{e}^{(ak+d\ell)t} \\ &\quad + \sum_{s=0}^{j-2} \binom{j}{s} d^{j-s} \sum_{\substack{1\le k+\ell\le i+s+1\\ \ell\le s+1}} \xi_{k,\ell}^{(i,s+1)} \mathrm{e}^{(ak+d\ell)t}. \end{aligned} \tag{9}$$

[1] Some of the coefficients may be 0. For simplicity we drop the parameters in the parentheses from the notation.

This equation boils down to a simple differential form:

$$m'_{i,j}(t) = (ai + dj)m_{i,j}(t) + h_{i,j}(t), \tag{10}$$

where $h_{i,j}(t)$ is a function that collects all the sums. For the purpose of asymptotic analysis, the exact values of the coefficients $\xi_{r,s}^{(i,j)}$ do not weigh in heavily as the dominant asymptotic equivalent does not involve them.

First-order differential equations have a standard theory; see Edwards and Penney (2007), for example. Suppose $g(x)$ and $h(x)$ are continuous functions on an open interval containing the point x_0. Then, the differential equation

$$f'(x) + g(x)f(x) = h(x)$$

has a unique solution:

$$f(x) = \frac{1}{\mathrm{e}^{\int_{x_0}^{x} g(t)\mathrm{d}t}} \left(f(x_0) + \int_{x_0}^{x} \mathrm{e}^{\int_{x_0}^{t} g(s)\,\mathrm{d}s}\, h(t)\,\mathrm{d}t \right).$$

Therefore, the general solution of (10) is

$$\begin{aligned} m_{i,j}(t) &= \frac{1}{\mathrm{e}^{\int_0^t -(ai+dj)\mathrm{d}x}} \left(m_{i,j}(0) + \int_0^t \mathrm{e}^{\int_0^x -(ai+dj)\,\mathrm{d}s}\, h_{i,j}(x)\,\mathrm{d}x \right) \\ &= \mathrm{e}^{(ai+dj)t} \left(m_{i,j}(0) + \int_0^t \mathrm{e}^{-(ai+dj)x}\, h_{i,j}(x)\,\mathrm{d}x \right). \end{aligned} \tag{11}$$

Since all the functions involved in the integral are exponential, the integration preserves exponential forms, which completes the induction. □

Note that the function $m_{i,j}(t)$ is comprised of exponential functions of the form $\mathrm{e}^{\alpha_{i,j}t}$, where $\alpha_{i,j} \le ai + dj$. Therefore, as $t \to \infty$, we have

$$\mathbb{E}\big[W^i(t)B^j(t)\big] = m_{i,j}(t) \sim K_{i,j}\mathrm{e}^{(ai+dj)t}, \tag{12}$$

where $K_{i,j} = K(i, j, a, d, W_0, B_0)$ does not depend on t. For simplicity, we write only the two subscripts i, j.

The variable $W(t)$ is easier to analyze asymptotically than $B(t)$. Also, $W(t)$ is amenable to exact analysis. So, we postpone the work on $W(t)$ for later and take up the more demanding asymptotic analysis of $B(t)$. The derivation of the asymptotic moments of $B(t)$ needs those of $\tau(t)$, which we discuss first in the following lemma. The source Balaji and Mahmoud (2006) gives the exact moment generating function of this total $\tau(t)$. Asymptotics are also known via an argument based on the theory of branching processes by Athreya and Ney (1972), Chapter 3.

Lemma 2 *Let $\tau(t)$ be the total number of balls in the urn at time t in a balanced triangular Pólya process with replacement matrix*

$$\begin{pmatrix} a & d-a \\ 0 & d \end{pmatrix},$$

with $d > a > 0$. Assume the process starts with a total of τ_0 balls. The moment generating function of $\tau(t)$ is

$$\phi_{\tau(t)}(u) = \left(\frac{\mathrm{e}^{-dt}\mathrm{e}^{du}}{1-(1-\mathrm{e}^{-dt})\mathrm{e}^{du}} \right)^{\frac{\tau_0}{d}}.$$

As $t \to \infty$, we have

$$\frac{\tau(t)}{\mathrm{e}^{dt}} \xrightarrow{\mathcal{D}} \text{Gamma}\left(\frac{\tau_0}{d}, d\right).$$

Remark 2 Note that, when τ_0 is a multiple of d, the moment generating function of $\tau(t)$ is a convolution of τ_0/d independent geometric random variables, each with probability of success e^{-dt}. Also note that Lemma 2 remains valid if the balanced Pólya urn is not triangular.

In view of uniform integrability, we have the following corollary.

Corollary 1

$$\mathbb{E}\big[\tau^r(t)\big] \sim g_r \mathrm{e}^{drt},$$

where g_r is the well-known rth moment of the Gamma distribution.

We now turn to the proofs of the main results stated in Sect. 3.

Proof of Theorem 1 We start with the binomial expansion

$$B^n(t) = \big(\tau(t) - W(t)\big)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \tau^r(t) W^{n-r}(t).$$

Now, take expectations of both sides to get

$$\begin{aligned}\mathbb{E}\big[B^n(t)\big] &= \mathbb{E}\left[\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \tau^r(t) W^{n-r}(t)\right] \\ &= \mathbb{E}\big[\tau^n(t)\big] + \sum_{r=0}^{n-1} \binom{n}{r} (-1)^{n-r} \mathbb{E}\big[\tau^r(t) W^{n-r}(t)\big].\end{aligned}$$

We isolated the nth moment of τ on the right-hand side because, as we shall see, it provides the asymptotically dominant term.

Let us bound the terms in the sum via Cauchy–Schwarz inequality:

$$\begin{aligned}\left|\sum_{r=0}^{n-1}\binom{n}{r}(-1)^{n-r}\mathbb{E}\big[\tau^r(t)W^{n-r}(t)\big]\right| &\le \sum_{r=0}^{n-1}\binom{n}{r}\mathbb{E}\big[\tau^r(t)W^{n-r}(t)\big]\\ &\le \sum_{r=0}^{n-1}\binom{n}{r}\sqrt{\mathbb{E}\big[\tau^{2r}(t)\big]\,\mathbb{E}\big[W^{2(n-r)}(t)\big]}.\end{aligned}$$

According to Corollary 1 and (12) applied with $i = 2(n-r)$ and $j = 0$, we write

$$\begin{aligned}\left|\sum_{r=0}^{n-1}\binom{n}{r}(-1)^{n-r}\mathbb{E}\big[\tau^r(t)W^{n-r}(t)\big]\right| &\le \sum_{r=0}^{n-1}\sqrt{O\big(\mathrm{e}^{2rdt}\big)\times O\big(\mathrm{e}^{2(n-r)at}\big)}\\ &\le \sum_{r=0}^{n-1}O\big(\mathrm{e}^{rdt+(n-r)at}\big);\end{aligned}$$

we subsumed the binomial coefficient into the O notation, and we emphasize that the constants hidden in O depend on n, r, a, d, W_0 and B_0. Note that $0 < rd+(n-r)a \le (n-1)d+a$, for $0 \le r \le n-1$. Therefore, we have

$$\left|\sum_{r=0}^{n-1}\binom{n}{r}(-1)^{n-r}\mathbb{E}\big[\tau^r(t)W^{n-r}(t)\big]\right| = O\big(\mathrm{e}^{(n-1)dt+at}\big).$$

Consequently, we have

$$\mathbb{E}\big[B^n(t)\big] = \mathbb{E}\big[\tau^n(t)\big] + O\big(\mathrm{e}^{(n-1)dt+at}\big).$$

Upon scaling (by e^{ndt}), we write equivalently

$$\mathbb{E}\left[\frac{B^n(t)}{\mathrm{e}^{ndt}}\right] = \mathbb{E}\left[\frac{\tau^n(t)}{\mathrm{e}^{ndt}}\right] + O\big(\mathrm{e}^{-(d-a)t}\big).$$

By an argument based on the theory of branching processes by Athreya and Ney (1972), Chapter 3, and Janson (2004), we know that $\mathrm{e}^{-dt}\tau(t)$ converges (also follows from Lemma 2):

$$\frac{\tau(t)}{\mathrm{e}^{dt}} \xrightarrow{D} \text{Gamma}\left(\frac{\tau_0}{d}, d\right).$$

We now see that the moments of $\mathrm{e}^{-dt}B(t)$ converge to those of a certain Gamma random variable, and the Gamma distribution is uniquely characterized by its moment.

We conclude that

$$\frac{B(t)}{e^{dt}} \xrightarrow{\mathcal{D}} \text{Gamma}\left(\frac{\tau_0}{d}, d\right).$$

□

By the nature of a triangular urn, white draws feed into the supply of white balls in the urn, and blue draws do not. So, we can view $W(t)$ as a branching process originated from an initial population of W_0 particles. The same argument by Athreya and Ney (1972), Chapter 3 and Janson (2004) can be used to find the appropriate scaling and the limit distribution for the scaled process. For the white balls, we have a formulation similar to that for the total number of balls (see Lemma 2).

Lemma 3 *Let $W(t)$ be the number of white balls in the urn at time t in a triangular Pólya process with replacement matrix*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \tag{13}$$

with $a > 0$, $b > 0$ and $d > 0$. Assume the process starts with $W_0 > 0$ white balls. The moment generating function of $W(t)$ is

$$\phi_{W(t)}(u) = \left(\frac{e^{-at}e^{au}}{1-(1-e^{-at})e^{au}}\right)^{\frac{W_0}{a}}.$$

As $t \to \infty$,

$$\frac{W(t)}{e^{at}} \xrightarrow{D} \text{Gamma}\left(\frac{W_0}{a}, a\right).$$

Remark 3 Note that Lemma 3 is stated for more general triangular urns, not necessarily balanced.

5 Exact moments

The bootstrapping method discussed can be used to find the exact moments of $W(t)$. These exact moments are presented in terms of Pochhammer's symbols for the rising factorials and Stirling numbers of the second kind.

Proof of Theorem 2 We develop a generating function for the (rising) factorial (which we name $R(u)$) of $W(t)$ from the connection

$$\begin{aligned} R_{W(t)}(u) &= \sum_{n=0}^{\infty} \mathbb{E}\big[\langle W(t)\rangle_n\big] \frac{u^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{w=0}^{\infty} \langle w\rangle_n \, \mathbb{P}(W(t)=w)\right) \frac{u^n}{n!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{w=0}^{\infty} \mathbb{P}(W(t) = w) \sum_{n=0}^{\infty} \langle w \rangle_n \frac{u^n}{n!} \\
&= \sum_{w=0}^{\infty} \frac{1}{(1-u)^w} \mathbb{P}\big(W(t) = w\big) \\
&= \phi_{W(t)} \left(\ln \left(\frac{1}{1-u} \right) \right).
\end{aligned}
$$

From Lemma 3, we then proceed with

$$
\begin{aligned}
R_{W(t)/a}(u) &= \phi_{W(t)} \left(\frac{1}{a} \ln \left(\frac{1}{1-u} \right) \right) \\
&= \left(\frac{\mathrm{e}^{-at} \mathrm{e}^{\ln(1/(1-u)}}{1 - (1 - \mathrm{e}^{-at}) \mathrm{e}^{\ln(1/(1-u)}} \right)^{W_0/a} \\
&= \big(1 - u\mathrm{e}^{at}\big)^{-W_0/a}.
\end{aligned}
$$

Reverting to the definition of the (rising) factorial moment generating function, and an expansion of the simple function on the right-hand side, we obtain

$$
\sum_{n=0}^{\infty} \mathbb{E}\left[\left\langle \frac{W(t)}{a} \right\rangle_n \right] \frac{u^n}{n!} = \sum_{n=0}^{\infty} \left\langle \frac{W_0}{a} \right\rangle_n \mathrm{e}^{ant} \frac{u^n}{n!}.
$$

Extracting powers of u, we arrive at

$$
\mathbb{E}\left[\left\langle \frac{W(t)}{a} \right\rangle_n \right] = \left\langle \frac{W_0}{a} \right\rangle_n \mathrm{e}^{ant}, \quad \text{for} \quad n \geq 1. \tag{14}
$$

Table 250 in Graham et al. (1994) provides the identity

$$
x^n = \sum_{i=0}^{n} \left\{ {n \atop i} \right\} (-1)^{n-i} \langle x \rangle_i .
$$

By (14), we reach the conclusion that

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{W(t)}{a} \right)^n \right] &= \sum_{i=0}^{n} \left\{ {n \atop i} \right\} (-1)^{n-i} \mathbb{E}\left[\left\langle \frac{W(t)}{a} \right\rangle_i \right] \\
&= \sum_{i=0}^{n} \left\{ {n \atop i} \right\} (-1)^{n-i} \left\langle \frac{W_0}{a} \right\rangle_i \mathrm{e}^{ait}, \quad \text{for} \quad n \geq 1.
\end{aligned}
$$

Multiplying both sides by a^n, the result follows. □

White balls feed into white balls only. Likewise, all the balls feed into all the colors. Thus, again, we can view the total number of balls as a branching process with both colors combined into one to obtain a result for the total $\tau(t)$ quite similar to that for the white balls:

$$\mathbb{E}\big[\tau^n(t)\big] = d^n \sum_{i=1}^{n} (-1)^{n-i} \left\{ {n \atop i} \right\} \left\langle \frac{\tau_0}{d} \right\rangle_i \mathrm{e}^{dit}, \quad n \ge 1.$$

6 Concluding remarks

As discussed in the body of the article, the counts of white and blue balls in a large class of irreducible balanced urns (with balance factor θ) are given by

$$\mathrm{e}^{-\theta t} \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\text{a.s.}} \text{Gamma}\left(\frac{\tau_0}{\theta}, \theta\right) \begin{pmatrix} v \\ 1-v \end{pmatrix}, \tag{15}$$

where $\binom{v}{1-v}$ is the principal eigenvector (normalized to be of length 1); see Janson (2004). To put matters in perspective, let us present our findings in a similar manner. The results we obtained assert that

$$\mathrm{e}^{-dt} \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\mathcal{D}} \text{Gamma}\left(\frac{\tau_0}{d}, d\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{16}$$

The class of balanced triangular urns is reducible as the blue color is not dominant. Yet, we see by comparing the general forms (15) and (16) that it is plausible that the type of result in Janson (2004) may extend to cover additional reducible classes. We note the 0 in (16)—it appears because scaling by e^{dt} is too much for the number of white balls. As discussed in the manuscript, an appropriate scaling is a function of the order e^{at}.

Some asymptotic results in this manuscript can be obtained from the theory of branching processes. We used a technique that espouses partial differential equations a method of bootstrapped moments. The merits of the latter method lie in the potential of producing exact distributions and exact moments.

The Gamma distribution we obtained as a limit for the (scaled) number of blue balls was derived under a condition of balance. However, several other results in the manuscript relax this condition. For instance, the Gamma distribution we obtained as a limit for the (scaled) number of white balls remains valid if the triangular Pólya urn is not balanced. The exact distribution of the number of white balls remains unchanged for unbalanced triangular Pólya urns. Furthermore, the exact moments of total order up to 3 in Proposition 1 and the appendix can be easily adapted for unbalanced urns. For instance, for a triangular urn with the replacement matrix (13), the means become

$$\begin{aligned} \mathbb{E}\big[W(t)\big] &= W_0 \mathrm{e}^{at}, \\ \mathbb{E}\big[B(t)\big] &= \left(\frac{b}{d-a} W_0 + B_0\right) \mathrm{e}^{dt} - \frac{b}{d-a} W_0 \, \mathrm{e}^{at}. \end{aligned}$$

Albeit essential differences between discrete-time triangular Pólya schemes and continuous-time triangular Pólya processes, there are similarities. To realize the similarity, we recall a result from Zhang et al. (2015). Let W_s be the number of white balls in the urn after s draws from triangular Pólya scheme with the replacement matrix (13). We then have

$$\mathbb{E}[W_s^n] = \frac{a^n}{\langle \frac{\tau_0}{d} \rangle_s} \sum_{i=1}^{n} (-1)^{n-i} \left\{ {n \atop i} \right\} \left\langle \frac{W_0}{a} \right\rangle_i \left\langle \frac{\tau_0 + ia}{d} \right\rangle_s .$$

We note that several elements, including alternating signs, powers, Stirling numbers, rising factorials are common among the two formulas.

Acknowledgements The authors wish to thank Dr. Panpan Zhang for many insightful discussions. They also sincerely acknowledge the contribution of two anonymous referees. One referee gave the paper a very thorough reading. We believe the exposition and readability have been improved, owing largely to a long list of critical points provided by that referee. Our original proof of Theorem 2 was by induction. A second reviewer has graciously provided an alternative direct proof, which is the one now used in the paper. We thank this referee for this valuable contribution.

Appendix

The mixed moments of total order 3 are given by the following differential equations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[W^3(t)\big] &= a^3\,\mathbb{E}\big[W(t)\big] + 3a^2\,\mathbb{E}\big[W^2(t)\big] + 3a\,\mathbb{E}\big[W^3(t)\big],\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[W^2(t)B(t)\big] &= a^2(d-a)\,\mathbb{E}\big[W(t)\big] + a^2\,\mathbb{E}\big[W(t)B(t)\big] + 2a(d-a)\,\mathbb{E}\big[W^2(t)\big]\\
&\quad + (2a+d)\,\mathbb{E}\big[W^2(t)B(t)\big] + (d-a)\,\mathbb{E}\big[W^3(t)\big],\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[W(t)B^2(t)\big] &= a(d-a)^2\,\mathbb{E}\big[W(t)\big] + \big(2a(d-a)+d^2\big)\,\mathbb{E}\big[W(t)B(t)\big]\\
&\quad + (d-a)^2\,\mathbb{E}\big[W^2(t)\big] + 2(d-a)\,\mathbb{E}\big[W^2(t)B(t)\big]\\
&\quad + (a+2d)\,\mathbb{E}\big[W(t)B^2(t)\big],\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[B^3(t)\big] &= (d-a)^3\,\mathbb{E}\big[W(t)\big] + 3(d-a)^2\,\mathbb{E}\big[W(t)B(t)\big]\\
&\quad + 3(d-a)\,\mathbb{E}\big[W(t)B^2(t)\big] + d^3\,\mathbb{E}\big[B(t)\big]\\
&\quad + 3d^2\,\mathbb{E}\big[B^2(t)\big] + 3d\,\mathbb{E}\big[B^3(t)\big].
\end{aligned}$$

Solving these differential equations we obtain the mixed moments of total order 3:

$$\begin{aligned}
\mathbb{E}\big[W^3(t)\big] &= W_0\big(W_0+a\big)\big(W_0+2a\big)\mathrm{e}^{3at} - 3aW_0\big(W_0+a\big)\mathrm{e}^{2at} + a^2W_0\mathrm{e}^{at},\\
\mathbb{E}\big[W^2(t)B(t)\big] &= W_0\big(W_0+a\big)\big(W_0+B_0+d\big)\mathrm{e}^{(2a+d)t}\\
&\quad - W_0\big(W_0+a\big)\big(W_0+2a\big)\mathrm{e}^{3at}\\
&\quad - aW_0\big(W_0+B_0+a\big)\mathrm{e}^{(a+d)t} + aW_0\big(W_0+a\big)\mathrm{e}^{2at},
\end{aligned}$$

$$\begin{aligned}\mathbb{E}\big[W(t)B^2(t)\big] &= W_0\big(W_0+B_0+a\big)\big(W_0+B_0+a+d\big)\mathrm{e}^{(a+2d)t}\\ &\quad -2W_0\big(W_0+a\big)\big(W_0+B_0+2a\big)\mathrm{e}^{(2a+d)t}\\ &\quad +W_0\big(W_0+a\big)\big(W_0+2a\big)\mathrm{e}^{3at}\\ &\quad -dW_0\big(W_0+B_0+a\big)\mathrm{e}^{(a+d)t}+aW_0\big(W_0+a\big)\mathrm{e}^{2at},\\ \mathbb{E}\big[B^3(t)\big] &= \big(W_0+B_0\big)\big(W_0+B_0+d\big)\big(W_0+B_0+2d\big)\mathrm{e}^{3dt}\\ &\quad -3W_0\big(W_0+B_0+a\big)\big(W_0+B_0+a+d\big)\mathrm{e}^{(a+2d)t}\\ &\quad +3W_0\big(W_0+a\big)\big(W_0+B_0+2a\big)\mathrm{e}^{(2a+d)t}\\ &\quad -W_0\big(W_0+a\big)\big(W_0+2a\big)\mathrm{e}^{3at}\\ &\quad -3d\big(W_0+B_0\big)\big(W_0+B_0+d\big)\mathrm{e}^{2dt}\\ &\quad +3(a+d)W_0\big(W_0+B_0+a\big)\mathrm{e}^{(a+d)t}\\ &\quad -3aW_0\big(W_0+a\big)\mathrm{e}^{2at}+d^2\big(W_0+B_0\big)\mathrm{e}^{dt}-a^2W_0\mathrm{e}^{at}.\end{aligned}$$

References

Athreya, K., & Karlin, S. (1968). Embedding of urn schemes into continuous time Markov branching process and related limit theorems. *The Annals of Mathematical Statistics*, *39*, 1801–1817.

Athreya, K., & Ney, P. (1972). *Branching process*. Berlin: Springer.

Balaji, S., & Mahmoud, H. (2006). Exact and limiting distributions in diagonal Pólya processes. *Annals of the Institute of Statistical Mathematics*, *58*, 171–185.

Balaji, S., Mahmoud, H., & Watanabe, O. (2006). Distributions in the Ehrenfest process. *Statistics and Probability Letters*, *76*, 666–674.

David, F., & Barton, E. (1962). *Combinatorial chance*. London: Charles Griffin.

Edwards, C., & Penney, D. (2007). *Elementary differential equations*. New York: Pearson.

Eggenberger, F., & Pólya, G. (1923). Über die statistik verketteter vorgänge. *Zeitschrift für Angewandte Mathematik und Mechanik*, *3*, 279–289.

Ehrenfest, P., & Ehrenfest, G. (1907). Über zwei bekannte einwände gegen das Boltzmansche H-theorem. *Physikalische Zeitschrift*, *8*, 311–314.

Flajolet, P., Dumas, P., & Puyhaubert, V. (2006). *Some exactly solvable models of urn process theory* (pp. 59–118). AG: Discrete Mathematics and Computer Science.

Graham, R., Knuth, D., Patashnik, O. (1994). Concrete mathematics: A foundation for computer science. Reading: Addison-Wesley.

Janson, S. (2004). Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Processes and their Applications*, *110*, 177–245.

Janson, S. (2006). Limit theorems for triangular urn schemes. *Probability Theory and Related Fields*, *134*, 417–452.

Johnson, N., & Kotz, S. (1977). *Urn models and their application: An approach to modern discrete probability theory*. New York: Wiley.

Kotz, S., Balakrishnan, N. (1997). Advances in urn models during the past two decades. *Advances in Combinatorial Methods and Applications to Probability and Statistics* (pp. 203–257). Boston: Birkhäuser.

Kuba, M., & Panholzer, A. (2014). On moment sequences and mixed Poisson distribution. *Probability Surveys*, *13*(2016), 89–155.

Levine, H. (1997). *Partial differential equations*. Rhode Island: American Mathematical Society.

Mahmoud, H. (2008). *Pólya urn models*. Boca Raton: Chapman-Hall.

Rabinovitch, N. (1969). Studies in the history of probability and statistics. XXII. Probability in the Talmud. *Biometrika*, *56*(2), 437–441.

Sparks, J., & Mahmoud, H. (2013). Phases in the two-color tenable zero-balanced Pólya processes. *Statistics and Probability Letters*, *83*, 265–271.
Zhang, P., & Mahmoud, H. (2016). Distributions in a class of poissonized urns with an application to Apollonian networks. *Statistics and Probability Letters*, *115*, 1–7.
Zhang, P., Chen, C., & Mahmoud, H. (2015). Characterization of the moments of balanced triangular urn schemes via an elementary approach. *Statistics and Probability Letters*, *96*, 149–153.