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On parameter estimation for cusp-type signals

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Abstract We consider the problem of parameter estimation by continuous time observations of a deterministic signal in white Gaussian noise. It is supposed that the signal has a cusp-type singularity. The properties of the maximum-likelihood and Bayesian estimators are described in the asymptotics of small noise. Special attention is paid to the problem of parameter estimation in the situation of misspecification in regularity, i.e., when the statistician supposes that the observed signal has this singularity, but the real signal is smooth. The rate and the asymptotic distribution of the maximum-likelihood estimator in this situation are described.

Keywords Parameter estimation · Cusp-type singularity · Small noise · Misspecification

1 Introduction

Consider the problem of parameter estimation by continuous time observations $X^T = (X_t,\ 0 \leq t \leq T)$ of a signal in a white Gaussian noise (WGN)

$$\mathrm{d}X_t = S(\vartheta, t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \leq t \leq T. \tag{1}$$

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Here, $S(\vartheta, t)$ is a known function (signal), $W_t, 0 \leq t \leq T$, is a Wiener process, and $\vartheta \in \Theta = (\alpha, \beta)$ is an unknown parameter.

We want to estimate the parameter ϑ by continuous time observations X^T and to describe the properties of the estimators in the asymptotics of *small noise*, i.e., $\varepsilon \in (0, 1]$ is known and the asymptotics corresponds to $\varepsilon \to 0$.

It is known that if the signal $S(\vartheta, \cdot)$ is a smooth function of ϑ, then the maximum-likelihood estimator (MLE) $\hat{\vartheta}_\varepsilon$ and the Bayesian estimators (BEs) $\tilde{\vartheta}_\varepsilon$ are consistent and asymptotically normal (with rate of convergence ε):

$$\varepsilon^{-1}\left(\hat{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}(\vartheta)^{-1}\right) \quad \text{and} \quad \varepsilon^{-1}\left(\tilde{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}(\vartheta)^{-1}\right),$$

where $\mathbb{I}(\vartheta)$ is the Fisher information

$$\mathbb{I}(\vartheta) = \int_0^T \dot{S}(\vartheta, t)^2 \, \mathrm{d}t \tag{2}$$

(here and in the sequel, "dot" means derivation w.r.t. ϑ). We also have the convergence of all polynomial moments and both estimators are asymptotically efficient (see Ibragimov and Has'minskii 1974, 1981).

If the signal $S(\vartheta, t) = S(t - \vartheta)$, where $S(t)$ is a discontinuous function of t having a jump at the point $t = 0$, then $\mathbb{I}(\vartheta) = \infty$. In this case, the MLE $\hat{\vartheta}_\varepsilon$ and the BEs $\tilde{\vartheta}_\varepsilon$ have the rate of convergence ε^2 with different limit distributions:

$$\varepsilon^{-2}\left(\hat{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \hat{u} \quad \text{and} \quad \varepsilon^{-2}\left(\tilde{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \tilde{u},$$

and the BEs are asymptotically efficient. Here, $\mathbf{E}(\hat{u}^2) > \mathbf{E}(\tilde{u}^2)$, i.e., the MLE is not asymptotically efficient. For the proofs, see Ibragimov and Has'minskii (1975).

We are interested in the properties of the MLE $\hat{\vartheta}_\varepsilon$ in the case of observations (1), where the signal $S(\vartheta, t)$ has a *cusp-type singularity*, i.e., at the vicinity of the point $t = \vartheta$, it has the representation $S(\vartheta, t) \approx a\,|t - \vartheta|^\kappa$, where $\kappa \in \left(0, \frac{1}{2}\right)$. Note that for these values of κ, we have $\mathbb{I}(\vartheta) = \infty$.

The problem of parameter estimation for cusp-type singular density functions by i.i.d. observations was considered in Prakasa (1968). It was shown that the rate of convergence of the MLE $\hat{\vartheta}_n$ is $n^{-\frac{1}{2\kappa+1}}$:

$$n^{\frac{1}{2\kappa+1}}\left(\hat{\vartheta}_n - \vartheta\right) \Longrightarrow \hat{\eta}.$$

An exhaustive study of singular estimation problems for i.i.d. observations, including cusp-type singularity, can be found in Ibragimov and Has'minskii (1981). For stochastic processes observed in continuous time, similar problems were considered in Dachian (2003) for inhomogeneous Poisson processes and in Dachian and Kutoyants (2003) and Fujii (2010) for ergodic diffusion processes. Note as well the works Döring (2015), Döring and Jensen (2015), and Prakasa (1985, 2004), where the problem of estimation of the cusp location was considered for regression models. Nonparametric regression model with cusp-type singularities was studied in Raimondo (1998).

This work is devoted to two problems. The first (auxiliary) one is to describe the asymptotics of the MLE $\hat{\vartheta}_\varepsilon$ and of the BEs $\tilde{\vartheta}_\varepsilon$ in the case of a signal with cusp-type singularity. This problem is studied in Sect. 2. The observed process is supposed to be

$$\mathrm{d}X_t = \left[a\,|t-\vartheta_0|^\kappa + h(\vartheta_0, t)\right]\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T.$$

Here, the parameters $a > 0$ and $\kappa \in \left(0, \frac{1}{2}\right)$, as well as the (smooth) function $h(\cdot)$ are known, while the location $\vartheta_0 \in (\alpha, \beta)$ of the cusp is unknown. We consider the problem of estimation of the true value ϑ_0 by continuous time observations $X^T = (X_t,\ 0 \le t \le T)$. The properties of the estimators are described in the asymptotics $\varepsilon \to 0$. It is shown that

$$\varepsilon^{-\frac{2}{2\kappa+1}}\left(\hat{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \hat{\xi} \quad \text{and} \quad \varepsilon^{-\frac{2}{2\kappa+1}}\left(\tilde{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \tilde{\xi},$$

where $\hat{\xi}$ and $\tilde{\xi}$ are two different r.v.'s, and the BEs are asymptotically efficient. Note that it is shown in Novikov (2014) that $\mathbf{E}(\hat{\xi}^2) > \mathbf{E}(\tilde{\xi}^2)$, i.e., the MLE is not asymptotically efficient.

The second (main) problem is to study the properties of the MLE, when the theoretical signal supposed by the statistician has cusp-type singularity, but the real signal is a smooth function. Let us consider the following example to illustrate this statement of the problem. Suppose that the model of observations chosen by the statistician (*theoretical model*) is

$$\mathrm{d}X_t = M(\vartheta, t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T, \tag{3}$$

where $\vartheta \in \Theta = (\alpha, \beta)$, $0 < \alpha < \beta < T$, and the signal is

$$M(\vartheta, t) = \operatorname{sgn}(t-\vartheta)\left[|t-\vartheta|^\kappa\,\mathbb{1}_{\{|t-\vartheta|\le 1\}} + \mathbb{1}_{\{|t-\vartheta|>1\}}\right]$$

with some known $a > 0$ and $\kappa \in \left(0, \frac{1}{2}\right)$. The observed process (*real model*) is

$$\mathrm{d}X_t = S(\vartheta_0, t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T, \tag{4}$$

where

$$S(\vartheta, t) = \operatorname{sgn}(t-\vartheta)\left[\frac{|t-\vartheta|}{\delta}\,\mathbb{1}_{\{|t-\vartheta|\le\delta\}} + \mathbb{1}_{\{|t-\vartheta|>\delta\}}\right]$$

with some $\delta > 0$. A plot of these two signals (with $\delta = \frac{1}{4}$) is given in Fig. 1.

We show that

$$\varepsilon^{-\frac{2}{3-2\kappa}}\left(\hat{\vartheta}_\varepsilon - \vartheta_0\right) \Longrightarrow \hat{\zeta},$$

where $\hat{\zeta}$ is some random variable described in Sect. 3.

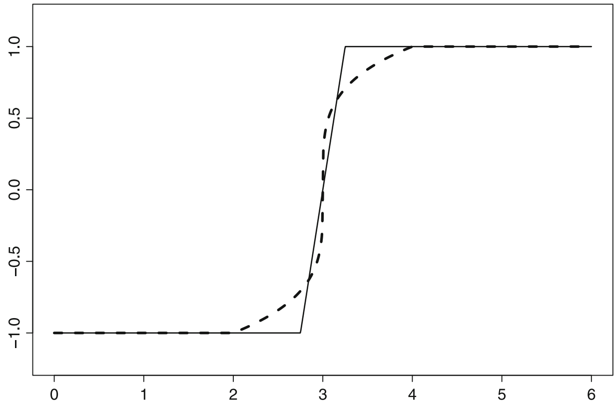

Fig. 1 Theoretical (*dashed line*) and real (*continuous line*) signals

This statement of the problem arose from our contacts with specialists in statistical radiophysics. The motivation is the following. The algorithms of detection of the change-point are based on the assumption that the signal has rectangular or cusp-type shape (see, e.g., (3)). However, the real electronic devices cannot produce signals with exactly perpendicular or cusp-type front, even if the smooth (real) signals can be close, in some sense, to the supposed shape. The properties of the estimators strongly depend on the analytic properties of the observed signal and the form of the supposed theoretical signal. Therefore, it is interesting to study the properties of the estimators of the location of a cusp or of a change-point in the situations of misspecification in regularity. We consider two different approximations. The first one is an approximation of a rectangular signal by a smooth or cusp-type real signal (considered in Chernoyarov et al. 2015). It is shown there that if the theoretical signal is of change-point type and the observed signal is smooth, then

$$\varepsilon^{-\frac{2}{3}}\left(\hat{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \hat{\eta}.$$

The second approximation is presented in this work. Here, the real signal is smooth (finite derivative), while the approximating cusp-type signal has infinite derivative. This problem is studied in Sect. 3, where we consider a slightly more general situation, where the MLE converges to the value $\hat{\vartheta}$ which minimizes the Kullback–Leibler distance between the measures corresponding to real and theoretical models.

The proofs in Sect. 2 are carried out following two general results: Theorems 1.10.1 and 1.10.2 from Ibragimov and Has'minskii (1981), i.e., we verify the conditions of these theorems for our model of observations. In Sect. 3, the direct application of the same method is not possible, since in the case of misspecification, the expectation of the pseudo-likelihood-ratio process $Z_\varepsilon(u)$ is no more equal to 1 and tends to zero. Recall that the property $\mathbf{E}_{\vartheta_0} Z_\varepsilon(u) = 1$ is widely used in the proofs of the above-mentioned theorems from Ibragimov and Has'minskii (1981). The idea is to introduce another normalized process $\hat{Z}_\varepsilon(u)$ and to describe the properties of the MLE with the help of this new process. We discuss as well the problem of the simultaneous estimation of the parameters ϑ, a, and κ.

2 Cusp location estimation

We consider the problem of parameter estimation by continuous time observations $X^T = (X_t,\ 0 \le t \le T)$ of a deterministic signal in the presence of a white Gaussian noise of small intensity:

$$\mathrm{d}X_t = S(\vartheta_0, t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T. \tag{5}$$

Here, the unknown parameter is $\vartheta_0 \in \Theta = (\alpha, \beta)$ and we are interested in the behavior of the estimators of this parameter in the asymptotics of *small noise*, i.e., as $\varepsilon \to 0$.

Suppose that the signal $S(\vartheta, t)$ has a *cusp-type singularity*:

$$S(\vartheta, t) = a\,|t - \vartheta|^{\kappa} + h(\vartheta, t),$$

where $0 < \alpha < \vartheta < \beta < T$. Here, $a > 0$ and $\kappa \in \left(0, \frac{1}{2}\right)$ are some known the constants, and $h(\vartheta, t)$ is continuously differentiable known function having bounded derivatives. Note that the Fisher information is not finite in this case, and so this parameter estimation problem is non-regular (singular).

The likelihood-ratio function is (see Liptser and Shiryayev 2001) is

$$V\left(\vartheta, X^T\right) = \exp\left\{\frac{1}{\varepsilon^2}\int_0^T S(\vartheta, t)\,\mathrm{d}X_t - \frac{1}{2\varepsilon^2}\int_0^T S(\vartheta, t)^2\,\mathrm{d}t\right\}, \quad \vartheta \in \Theta,$$

and the maximum-likelihood estimator (MLE) $\hat{\vartheta}_\varepsilon$ is defined by the equation

$$V\left(\hat{\vartheta}_\varepsilon, X^T\right) = \sup_{\vartheta \in \Theta} V\left(\vartheta, X^T\right). \tag{6}$$

Suppose that ϑ is a random variable with continuous positive density function $p(\theta), \alpha < \theta < \beta$. The Bayesian estimator (BE) $\tilde{\vartheta}_\varepsilon$ with prior density $p(\cdot)$ and quadratic loss function is

$$\tilde{\vartheta}_\varepsilon = \frac{\int_\alpha^\beta \theta\, p(\theta)\, V\left(\theta, X^T\right)\mathrm{d}\theta}{\int_\alpha^\beta p(\theta)\, V\left(\theta, X^T\right)\mathrm{d}\theta}. \tag{7}$$

We are interested in the properties of the estimators $\hat{\vartheta}_\varepsilon$ and $\tilde{\vartheta}_\varepsilon$ in the asymptotics $\varepsilon \to 0$.

Introduce the Hurst parameter $H = \kappa + \frac{1}{2}$ and the two-sided fractional Brownian motion (fBm) $W^H(u), u \in \mathbb{R}$. Recall, that $\mathbf{E}\,W^H(u) = 0$ and

$$\mathbf{E}\,W^H(u)\,W^H(v) = \frac{1}{2}\left[|u|^{2H} + |v|^{2H} - |u - v|^{2H}\right], \quad u, v \in \mathbb{R}. \tag{8}$$

In addition, introduce two random variables $\hat{\xi}$ and $\tilde{\xi}$ by the relations

$$Z\left(\hat{\xi}\right) = \sup_{u\in\mathbb{R}} Z(u) \quad \text{and} \quad \tilde{\xi} = \frac{\int_{\mathbb{R}} u\, Z(u)\,\mathrm{d}u}{\int_{\mathbb{R}} Z(u)\,\mathrm{d}u} ,$$

where the process

$$Z(u) = \exp\left\{\Gamma_a W^H(u) - \frac{\Gamma_a^2}{2}|u|^{2H}\right\}, \qquad u \in \mathbb{R}. \tag{9}$$

Here

$$\Gamma_a^2 = a^2 \int_{-\infty}^{\infty} \left[|v-1|^\kappa - |v|^\kappa\right]^2 \mathrm{d}v.$$

Introduce as well the process

$$Z^o(v) = \exp\left\{w^H(v) - \frac{1}{2}|v|^{2H}\right\}, \qquad v \in \mathbb{R},$$

where $w^H(v)$ is a fBm, and the corresponding random variables $\hat{\xi}_o$ and $\tilde{\xi}_o$ are defined by the relations

$$Z^o\left(\hat{\xi}_o\right) = \sup_{v\in\mathbb{R}} Z^o(v) \quad \text{and} \quad \tilde{\xi}_o = \frac{\int_{\mathbb{R}} v\, Z^o(v)\,\mathrm{d}v}{\int_{\mathbb{R}} Z^o(v)\,\mathrm{d}v} .$$

Note that

$$\hat{\xi} = \Gamma_a^{-1/H}\,\hat{\xi}_o \quad \text{and} \quad \tilde{\xi} = \Gamma_a^{-1/H}\,\tilde{\xi}_o. \tag{10}$$

The proof of (10) follows immediately by changing the variable $u = \Gamma_a^{-1/H} v$ in $Z(u)$ and putting $w^H(v) = \Gamma_a\, W^H\left(\Gamma_a^{-1/H} v\right)$.

Asymptotically efficient estimators are defined with the help of the following lower bound. For all $\vartheta_0 \in \Theta$ and all estimators $\bar{\vartheta}_\varepsilon$, we have the relation

$$\lim_{\delta\to 0} \underline{\lim}_{\varepsilon\to 0} \sup_{|\vartheta-\vartheta_0|<\delta} \varepsilon^{-2/H}\,\mathbf{E}_\vartheta \left|\bar{\vartheta}_\varepsilon - \vartheta\right|^2 \geq \mathbf{E}\left(\tilde{\xi}^2\right) = \Gamma_a^{-2/H}\,\mathbf{E}\left(\tilde{\xi_o}^{\,2}\right). \tag{11}$$

Therefore, we call an estimator ϑ_ε^* asymptotically efficient if for all $\vartheta_0 \in \Theta$, we have the equality

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \sup_{|\vartheta-\vartheta_0|<\delta} \varepsilon^{-2/H}\,\mathbf{E}_\vartheta \left|\vartheta_\varepsilon^* - \vartheta\right|^2 = \mathbf{E}\left(\tilde{\xi}^2\right). \tag{12}$$

The proof of the bound (11) follows from the general results presented in Section 1.9 (see (1.9.4)) of Ibragimov and Has'minskii (1981). We can recall here the sketch of the proof, supposing that the properties of the BEs for this model are already established (see Theorem 1 below). Introduce a continuous positive density function $q(\vartheta)$, $\vartheta_0-\delta<\vartheta<\vartheta_0+\delta$. Then, we can write

$$\sup_{|\vartheta-\vartheta_0|<\delta}\mathbf{E}_\vartheta\left|\bar\vartheta_\varepsilon-\vartheta\right|^2\geq\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\mathbf{E}_\vartheta\left|\bar\vartheta_\varepsilon-\vartheta\right|^2q(\vartheta)\,\mathrm{d}\vartheta$$
$$\geq\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\mathbf{E}_\vartheta\left|\tilde\vartheta_{q,\varepsilon}-\vartheta\right|^2q(\vartheta)\,\mathrm{d}\vartheta,$$

where we denoted $\tilde\vartheta_{q,\varepsilon}$ the BE for the prior density $q(\cdot)$. As we have the convergence of moments of the BEs, we obtain the limit

$$\varliminf_{\varepsilon\to0}\sup_{|\vartheta-\vartheta_0|<\delta}\varepsilon^{-2/H}\,\mathbf{E}_\vartheta\left|\vartheta_\varepsilon^*-\vartheta\right|^2\geq\lim_{\varepsilon\to0}\varepsilon^{-2/H}\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\mathbf{E}_\vartheta\left|\tilde\vartheta_{q,\varepsilon}-\vartheta\right|^2q(\vartheta)\,\mathrm{d}\vartheta$$
$$=\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\mathbf{E}\left|\tilde\xi\right|^2q(\vartheta)\,\mathrm{d}\vartheta=\mathbf{E}\big(\tilde\xi^2\big)$$

for all $\delta>0$. Remind that $\mathbf{E}\big(\tilde\xi^2\big)$ does not depend on ϑ. This proves the lower bound (11).

Theorem 1 *The MLE and the BEs are consistent, and have different limit distributions:*

$$\varepsilon^{-1/H}\big(\hat\vartheta_\varepsilon-\vartheta\big)\Longrightarrow\hat\xi\quad\text{and}\quad\varepsilon^{-1/H}\big(\tilde\vartheta_\varepsilon-\vartheta\big)\Longrightarrow\tilde\xi,$$

the polynomial moments converge:

$$\lim_{\varepsilon\to0}\mathbf{E}_\vartheta\left|\frac{\hat\vartheta_\varepsilon-\vartheta}{\varepsilon^{1/H}}\right|^p=\mathbf{E}_\vartheta\left|\hat\xi\right|^p\quad\text{and}\quad\lim_{\varepsilon\to0}\mathbf{E}_\vartheta\left|\frac{\tilde\vartheta_\varepsilon-\vartheta}{\varepsilon^{1/H}}\right|^p=\mathbf{E}_\vartheta\left|\tilde\xi\right|^p$$

for any $p>0$, and the BEs are asymptotically efficient.

Proof To prove this theorem, we put $\varphi_\varepsilon=\varepsilon^{1/H}$, introduce the normalized likelihood ratio

$$Z_\varepsilon(u)=\frac{V\big(\vartheta_0+\varphi_\varepsilon u,X^T\big)}{V\big(\vartheta_0,X^T\big)},\qquad u\in\mathbb{U}_\varepsilon=\big(\varepsilon^{-1/H}(\alpha-\vartheta_0),\varepsilon^{-1/H}(\beta-\vartheta_0)\big),$$

and check the conditions of general Theorems 1.10.1 and 1.10.2 from Ibragimov and Has'minskii (1981). The verification of these conditions is done with the help of several lemmas presented below.

Lemma 1 *We have the convergence of finite-dimensional distributions of $Z_\varepsilon(\cdot)$: for any any $k=1,2,\ldots$ and any $u_1,\ldots,u_k\in\mathbb{R}$, we have*

$$\left(Z_\varepsilon(u_1),\ldots,Z_\varepsilon(u_k)\right)\Longrightarrow\left(Z(u_1),\ldots,Z(u_k)\right). \tag{13}$$

Moreover, this convergence is uniform in ϑ on compacts $\mathbb{K}\subset\Theta$.

Proof We can write ($u>0$)

$$\begin{aligned}\ln Z_\varepsilon(u)&=\frac{1}{\varepsilon^2}\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]\mathrm{d}X_t\\&\quad-\frac{1}{2\varepsilon^2}\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)^2-S(\vartheta_0,t)^2\right]\mathrm{d}t\\&=\frac{1}{\varepsilon}\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]\mathrm{d}W_t\\&\quad-\frac{1}{2\varepsilon^2}\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]^2\mathrm{d}t.\end{aligned}$$

For the last integral, we have

$$\begin{aligned}&\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]^2\mathrm{d}t\\&\quad=\int_0^T\left[a\left|t-\vartheta_0-\varphi_\varepsilon u\right|^\kappa-a\left|t-\vartheta_0\right|^\kappa+h(\vartheta_0+\varphi_\varepsilon u,t)-h(\vartheta_0,t)\right]^2\mathrm{d}t\\&\quad=\int_{-\vartheta_0}^{T-\vartheta_0}\left[a\left|t-\varphi_\varepsilon u\right|^\kappa-a\left|t\right|^\kappa+\varphi_\varepsilon\,u\,\dot h\left(\tilde\vartheta,t-\vartheta_0\right)\right]^2\mathrm{d}t,\end{aligned}$$

where we changed the variable and used Taylor expansion for the function $h(\vartheta,t)$. The function $\dot h(\vartheta,t)$ is bounded, and hence, we have

$$\varepsilon^{-2}\int_0^T\left[\varphi_\varepsilon\,u\,\dot h\left(\tilde\vartheta,t\right)\right]^2\mathrm{d}t=\varepsilon^{\frac{2}{\kappa+\frac12}-2}\,u^2\int_0^T\dot h\left(\tilde\vartheta,t\right)^2\mathrm{d}t\le C\,u^2\,\varepsilon^{\frac{1-2\kappa}{\kappa+\frac12}}\longrightarrow 0.$$

Putting $t=\varphi_\varepsilon s$, we obtain

$$\begin{aligned}&\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]^2\mathrm{d}t\\&\quad=\varphi_\varepsilon^{2\kappa+1}\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\phi_\varepsilon}}\left[a\left|s-u\right|^\kappa-a\left|s\right|^\kappa+\varphi_\varepsilon^{1-\kappa}\,u\,\dot h\left(\tilde\vartheta,s\varphi_\varepsilon-\vartheta_0\right)\right]^2\mathrm{d}t\\&\quad=a^2\,\varphi_\varepsilon^{2\kappa+1}\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\phi_\varepsilon}}\left[\left|s-u\right|^\kappa-\left|s\right|^\kappa\right]^2\mathrm{d}t\,\left(1+o(1)\right).\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^T \left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]^2 \mathrm{d}t\\
&\quad=\frac{a^2\,\varphi_\varepsilon^{2\kappa+1}}{\varepsilon^2}\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}} \left[|s-u|^\kappa-|s|^\kappa\right]^2 \mathrm{d}s\,\left(1+o(1)\right)\\
&\quad=a^2\,|u|^{2\kappa+1}\int_{-\frac{\vartheta_0}{\varphi_\varepsilon u}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon u}} \left[|v-1|^\kappa-|v|^\kappa\right]^2 \mathrm{d}v\,\left(1+o(1)\right)\longrightarrow \Gamma_a^2\,|u|^{2\kappa+1},
\end{aligned}\tag{14}$$

where we put $s=v\,u$.

For the stochastic integral, we have

$$\begin{aligned}
\varepsilon^{-2}\,\mathbf{E}_{\vartheta_0}\left(\int_0^T\left[\varphi_\varepsilon\,u\,\dot h\left(\tilde\vartheta,t\right)\right]\mathrm{d}W_t\right)^2&=\varepsilon^{\frac{1-2\kappa}{\kappa+\frac12}}\,u^2\int_0^T \dot h\left(\tilde\vartheta,t\right)^2\mathrm{d}t\\
&\le C\,u^2\,\varepsilon^{\frac{1-2\kappa}{\kappa+\frac12}}\longrightarrow 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^T\left[S(\vartheta_0+\varphi_\varepsilon u,t)-S(\vartheta_0,t)\right]\mathrm{d}W_t\\
&\quad=a\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}}\left[|s-u|^\kappa-|s|^\kappa\right]\mathrm{d}\tilde W(s)\,\left(1+o(1)\right)\\
&\quad\Longrightarrow a\int_{-\infty}^{\infty}\left[|s-u|^\kappa-|s|^\kappa\right]\mathrm{d}W(s)\sim\mathcal{N}\left(0,|u|^{2H}\Gamma_a^2\right).
\end{aligned}$$

Here, $\tilde W(s)$, $s\in\left[\frac{-\vartheta_0}{\varphi_\varepsilon},\frac{T-\vartheta_0}{\varphi_\varepsilon}\right]$, and $W(v)$, $u\in\mathbb{R}$, are two-sided Wiener processes. For example, the process $W(\cdot)$ is defined as follows:

$$W(v)=\begin{cases}W_+(v), & \text{if } v>0,\\ W_-(-v), & \text{if } v\le 0,\end{cases}$$

where $W_+(\cdot)$ and $W_-(\cdot)$ are two independent Wiener processes.

Let us denote

$$W^H(u)=\Gamma_1^{-1}\int_{-\infty}^{\infty}\left[|s-u|^\kappa-|s|^\kappa\right]\mathrm{d}W(s)$$

and verify (8). Using the elementary equality $a\,b=\frac12\left[a^2+b^2-(a-b)^2\right]$, we obtain

$$\begin{aligned}\mathbf{E}\, W^H(u)\, W^H(v) \\ &= \frac{1}{2}\left[\mathbf{E}\big(W^H(u)\big)^2 + \mathbf{E}\big(W^H(v)\big)^2 - \mathbf{E}\big(W^H(u) - W^H(v)\big)^2\right] \\ &= \frac{1}{2}\left[|u|^{2H} + |v|^{2H} - |u-v|^{2H}\right]\end{aligned}$$

since

$$\begin{aligned}\mathbf{E}\big(W^H(u) - W^H(v)\big)^2 &= \Gamma_1^{-2}\int_{-\infty}^{\infty}\left[|s-u|^{\kappa} - |s-v|^{\kappa}\right]^2 \mathrm{d}s \\ &= \Gamma_1^{-2}\int_{-\infty}^{\infty}\left[\big|r-(u-v)\big|^{\kappa} - |r|^{\kappa}\right]^2 \mathrm{d}r = |u-v|^{2\kappa+1}.\end{aligned}$$

Hence, $W^H(u)$, $u \in \mathbb{R}$, is a two-sided fBm.

Therefore, we proved the convergence of one-dimensional distributions. The multidimensional case can be treated in a similar way: for arbitrary vectors $(\lambda_1, \ldots, \lambda_k)$ and $(u_1, \ldots, u_k)$, we can show the convergence

$$\sum_{j=1}^{k} \lambda_j \ln Z_\varepsilon(u_j) \Longrightarrow \sum_{j=1}^{k} \lambda_j \ln Z(u_j),$$

and so the lemma is proved. □

Let us now denote

$$\Phi(\vartheta, \vartheta_0) = \int_0^T \left[S(\vartheta, t) - S(\vartheta_0, t)\right]^2 \mathrm{d}t.$$

We have the following elementary estimate on $\Phi(\cdot)$.

Lemma 2 *There exists a constant $\mu > 0$, such that*

$$\Phi(\vartheta, \vartheta_0) \geq \mu\, |\vartheta - \vartheta_0|^{2H}. \tag{15}$$

Proof First, let us note that for any $\nu > 0$, we have

$$m(\nu) = \inf_{|\vartheta - \vartheta_0| > \nu} \Phi(\vartheta, \vartheta_0) > 0.$$

Indeed, if for some $\nu > 0$, we had $m(\nu) = 0$, then there would exist $\vartheta_1 \neq \vartheta_0$, such that for all $t \in [0, T]$, we would have

$$a\, |t - \vartheta_1|^{\kappa} + h(\vartheta_1, t) = a\, |t - \vartheta_0|^{\kappa} + h(\vartheta_0, t),$$

and the function

$$h(\vartheta_1, t) = a\, |t - \vartheta_0|^{\kappa} - a\, |t - \vartheta_1|^{\kappa} + h(\vartheta_0, t)$$

would have no continuous bounded derivative on t. Hence, for $|\vartheta - \vartheta_0| > \nu$, we get

$$\Phi(\vartheta, \vartheta_0) \geq m(\nu) \geq m(\nu)\, \frac{|\vartheta - \vartheta_0|^{2H}}{|\beta - \alpha|^{2H}}\,.$$

Further, for $|\vartheta - \vartheta_0| \leq \nu$ with sufficiently small ν, we have

$$\Phi(\vartheta, \vartheta_0) = |\vartheta - \vartheta_0|^{2H}\, \Gamma_a^2\, \big(1 + o(1)\big).$$

Therefore, for sufficiently small ν, we can write

$$\Phi(\vartheta, \vartheta_0) \geq \frac{1}{2}\, \Gamma_a^2\, |\vartheta - \vartheta_0|^{2H}.$$

Taking

$$\mu = \min\left(\frac{m(\nu)}{|\beta - \alpha|^{2H}}\,,\, \frac{\Gamma_a^2}{2}\right),$$

we obtain (15). □

This estimate allows us to verify the boundedness of all moments of the likelihood-ratio process.

Lemma 3 *There exists a constant* $c > 0$*, such that*

$$\mathbf{E}_{\vartheta_0}\, Z_\varepsilon^{1/2}(u) \leq \mathrm{e}^{-c\,|u|^{2H}}. \tag{16}$$

Proof We have

$$\mathbf{E}_{\vartheta_0}\, Z_\varepsilon(u)^{1/2} = \exp\left\{-\frac{1}{8\,\varepsilon^2}\, \Phi(\vartheta_0 + \varphi_\varepsilon u, \vartheta_0)\right\} \leq \exp\left\{-\frac{\mu}{8}\,|u|^{2H}\right\},$$

where we used (15). □

Lemma 4 *For any* $N > 0$, $|u_1| < N$*, and* $|u_2| < N$*, we have the estimate*

$$\mathbf{E}_{\vartheta_0}\big|Z_\varepsilon^{1/2}(u_2) - Z_\varepsilon^{1/2}(u_1)\big|^2 \leq C\,(1 + N)\,|u_2 - u_1|^{2H} \tag{17}$$

with some constant $C > 0$.

Proof We can write

$$\mathbf{E}_{\vartheta_0}\big|Z_\varepsilon^{1/2}(u_2) - Z_\varepsilon^{1/2}(u_1)\big|^2 = 2\left(1 - \mathbf{E}_{\vartheta_0 + \varphi_\varepsilon u_1}\left(\frac{Z_\varepsilon(u_2)}{Z_\varepsilon(u_1)}\right)^{\frac{1}{2}}\right)$$
$$= 2\left(1 - \exp\left\{-\frac{1}{8\varepsilon^2}\, \Phi(\vartheta_0 + \varphi_\varepsilon u_2, \vartheta_0 + \varphi_\varepsilon u_1)\right\}\right)$$

$$
\begin{aligned}
&\leq \frac{1}{4\varepsilon^2}\,\Phi(\vartheta_0+\varphi_\varepsilon u_2,\vartheta_0+\varphi_\varepsilon u_1)\\
&=\frac{1}{4\varepsilon^2}\int_0^T\Big[a\left|t-\vartheta_0-\varphi_\varepsilon u_2\right|^\kappa-a\left|t-\vartheta_0-\varphi_\varepsilon u_1\right|^\kappa\\
&\quad+h(\vartheta_0+\varphi_\varepsilon u_2,t)-h(\vartheta_0+\varphi_\varepsilon u_1,t)\Big]^2\,\mathrm{d}t\\
&\leq \frac{1}{2\varepsilon^2}\int_0^T\Big[a\left|t-\vartheta_0-\varphi_\varepsilon u_2\right|^\kappa-a\left|t-\vartheta_0-\varphi_\varepsilon u_1\right|^\kappa\Big]^2\,\mathrm{d}t\\
&\quad+\frac{1}{2\varepsilon^2}\int_0^T\Big[h(\vartheta_0+\varphi_\varepsilon u_2,t)-h(\vartheta_0+\varphi_\varepsilon u_1,t)\Big]^2\,\mathrm{d}t\\
&\leq \frac{\varphi_\varepsilon^{2\kappa+1}}{2\varepsilon^2}\,\Gamma_a^2\left|u_2-u_1\right|^{2\kappa+1}+\frac{\varphi_\varepsilon^2}{2\varepsilon^2}\int_0^T\dot h\big(\tilde\vartheta,t\big)^2\,\mathrm{d}t\;(u_2-u_1)^2\\
&\leq C\left(1+\left|u_2-u_1\right|^{1-2\kappa}\right)\left|u_2-u_1\right|^{2\kappa+1}\leq C\,(1+N)\left|u_2-u_1\right|^{2H}
\end{aligned}
$$

(note that $2\kappa < 1$ and $2H > 1$). □

The properties (13), (16), and (17) of the likelihood ratio correspond to the conditions of Theorems 1.10.1 and 1.10.2 from Ibragimov and Has'minskii (1981), and hence, the MLE $\hat\vartheta_\varepsilon$ and the BEs $\tilde\vartheta_\varepsilon$ have all the properties stated in Theorem 1.

Remark 1 More detailed analysis shows that if the signal has several points of cusp, say

$$
S(\vartheta,t)=\sum_{\ell=1}^{L}a_\ell\left|t-\vartheta\right|^{\kappa_\ell},
$$

where $\kappa_l\in\left(0,\frac{1}{2}\right)$, the result of Theorem 1 holds with

$$
\kappa=\min_{1\leq\ell\leq L}\kappa_\ell .
$$

The proof is similar to that of Theorem 1.

Remark 2 It is possible to study the properties of the estimators $\hat\vartheta_\varepsilon$ and $\tilde\vartheta_\varepsilon$ in the case of multiple different singularities. For example, suppose that

$$
S(\vartheta,t)=\sum_{\ell=1}^{L}a_\ell\left|t-\vartheta_l\right|^{\kappa_\ell},
$$

where $\vartheta=(\vartheta_1,\ldots,\vartheta_L)\in\Theta$. Here, $\Theta=(\alpha_1,\beta_1)\times\cdots\times(\alpha_L,\beta_L)$, $0<\alpha_\ell<\beta_\ell<T$ for $\ell=1,\ldots,L$, and $\beta_\ell<\alpha_{\ell+1}$ for $\ell=1,\ldots,L-1$.

Then, the limit of the normalized likelihood ratio

$$
Z_\varepsilon(u_1,\ldots,u_L)=\frac{V\left(\vartheta_1+\varepsilon^{1/H_1}u_1,\ldots,\vartheta_L+\varepsilon^{1/H_L}u_L,X^T\right)}{V\left(\vartheta_1,\ldots,\vartheta_L,X^T\right)}
$$

is the process

$$Z(u_1,\ldots,u_L)=\prod_{\ell=1}^{L}Z_\ell(u_\ell),\qquad u_\ell\in\mathbb{R},$$

with

$$Z_\ell(u_\ell)=\exp\left\{\gamma_\ell\,W_\ell^{H_\ell}(u_l)-\frac{\gamma_\ell^2}{2}|u_\ell|^{2H_\ell}\right\},\qquad u_\ell\in\mathbb{R},$$

where

$$\gamma_\ell^2=a_\ell^2\int_{-\infty}^{\infty}\left[|v-1|^{\kappa_\ell}-|v|^{\kappa_\ell}\right]^2\mathrm{d}v$$

and $W_1^{H_1}(\cdot),\ldots,W_L^{H_L}(\cdot)$ are independent fBm processes. The MLE $\hat\vartheta_\varepsilon=\left(\hat\vartheta_{1,\varepsilon},\ldots,\hat\vartheta_{L,\varepsilon}\right)$ and the BEs $\tilde\vartheta_\varepsilon=\left(\tilde\vartheta_{1,\varepsilon},\ldots,\tilde\vartheta_{L,\varepsilon}\right)$ are defined as before by the relations (6) and (7). They have different limit distributions. In particular, for the MLE, we have the convergence

$$\left(\frac{\hat\vartheta_{1,\varepsilon}-\vartheta_1}{\varepsilon^{1/H_1}},\ldots,\frac{\hat\vartheta_{L,\varepsilon}-\vartheta_L}{\varepsilon^{1/H_L}}\right)\Longrightarrow\left(\hat\xi_1,\ldots,\hat\xi_L\right),$$

where the random variables $\hat\xi_1,\ldots,\hat\xi_L$ are defined by the equations

$$Z_\ell\left(\hat\xi_\ell\right)=\sup_{u\in\mathbb{R}}Z_\ell(u),\qquad \ell=1,\ldots,L,$$

and are independent. Of course, the BEs have the same rate of convergence and their asymptotic distribution is given by

$$\left(\frac{\tilde\vartheta_{1,\varepsilon}-\vartheta_1}{\varepsilon^{1/H_1}},\ldots,\frac{\tilde\vartheta_{L,\varepsilon}-\vartheta_L}{\varepsilon^{1/H_L}}\right)\Longrightarrow\left(\tilde\xi_1,\ldots,\tilde\xi_L\right),$$

where the random variables $\tilde\xi_1,\ldots,\tilde\xi_L$ are defined by

$$\tilde\xi_\ell=\frac{\int_{\mathbb{R}}u_\ell\,Z_\ell(u_\ell)\,\mathrm{d}u_\ell}{\int_{\mathbb{R}}Z_\ell(u_\ell)\,\mathrm{d}u_\ell},\qquad \ell=1,\ldots,L,$$

and are also independent.

3 Misspecification

We are interested in the following problem of misspecification. Suppose that the model of observations chosen by the statistician (*theoretical model*) is

$$\mathrm{d}X_t = M(\vartheta, t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T. \tag{18}$$

The signal $M(\vartheta, t)$ is supposed to be

$$M(\vartheta, t) = a\,|t-\vartheta|^{\kappa}, \qquad 0 \le t \le T,$$

where $\kappa \in \left(0, \frac{1}{2}\right)$ and $\vartheta \in \Theta = (\alpha, \beta)$, $0 < \alpha < \beta < T$.

The observed process (*real model*) is

$$\mathrm{d}X_t = S(\vartheta_0, t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T, \tag{19}$$

where $\vartheta_0 \in \Theta$ is the true value and the function $S(\vartheta, \cdot) \in L_2(0, T)$ is sufficiently smooth.

The (misspecified) likelihood-ratio function constructed on the base of the theoretical model (18) is

$$V\left(\vartheta, X^T\right) = \exp\left\{\frac{1}{\varepsilon^2}\int_0^T \left[M(\vartheta, t) - M(\vartheta_1, t)\right]\mathrm{d}X_t \right.$$
$$\left. - \frac{1}{2\varepsilon^2}\int_0^T \left[M(\vartheta, t)^2 - M(\vartheta_1, t)^2\right]\mathrm{d}t\right\}, \quad \vartheta \in \Theta,$$

where we have to substitute the observations from Eq. (19). Here, ϑ_1 is some fixed value. Therefore, the (pseudo) MLE $\hat{\vartheta}_\varepsilon$ is defined by the equation

$$V\left(\hat{\vartheta}_\varepsilon, X^T\right) = \sup_{\vartheta \in \Theta} V\left(\vartheta, X^T\right). \tag{20}$$

To see the limit of the MLE, we write the likelihood ratio as follows (below we put $M(\vartheta_1, t) = 0$):

$$\begin{aligned}
&\varepsilon^2 \ln V\left(\vartheta, X^T\right) \\
&\quad = \varepsilon \int_0^T M(\vartheta, t)\,\mathrm{d}W_t - \frac{1}{2}\int_0^T \left[M(\vartheta, t)^2 - 2\,M(\vartheta, t)\,S(\vartheta_0, t)\right]\mathrm{d}t \\
&\quad = \varepsilon \int_0^T M(\vartheta, t)\,\mathrm{d}W_t - \frac{1}{2}\left\|M(\vartheta, \cdot) - S(\vartheta_0, \cdot)\right\|^2 + \frac{1}{2}\left\|S(\vartheta_0, \cdot)\right\|^2
\end{aligned}$$

where $\|\cdot\|$ denotes the $L_2(0, T)$ norm. It is easy to verify the convergence

$$\sup_{\vartheta \in \Theta}\left|\varepsilon^2 \ln V\left(\vartheta, X^T\right) + \frac{1}{2}\left\|M(\vartheta, \cdot) - S(\vartheta_0, \cdot)\right\|^2 - \frac{1}{2}\left\|S(\vartheta_0, \cdot)\right\|^2\right| \longrightarrow 0.$$

Suppose that the equation

$$\inf_{\vartheta\in\Theta}\left\|M(\vartheta,\cdot)-S(\vartheta_0,\cdot)\right\|=\left\|M\big(\hat\vartheta,\cdot\big)-S(\vartheta_0,\cdot)\right\|$$

has a unique solution $\hat\vartheta\in\Theta$. Then, we obtain as usual in such situations that the MLE $\hat\vartheta_\varepsilon$ converges to the value $\hat\vartheta$, which minimizes the Kullback–Leibler distance. It is interesting to note that, in general, misspecified case $\hat\vartheta\neq\vartheta_0$, but for some models, we have $\hat\vartheta=\vartheta_0$, i.e., despite having a wrong model, the MLE is consistent. The most interesting for us question is to determine the rate of convergence and the limiting distribution of the MLE.

Introduce the function

$$\Phi(\vartheta,\hat\vartheta)=\left\|M(\vartheta,\cdot)-S(\vartheta_0,\cdot)\right\|^2-\left\|M\big(\hat\vartheta,\cdot\big)-S(\vartheta_0,\cdot)\right\|^2$$

and the following condition of regularity:

Condition $\mathcal{M}$.

1. *The parameter* $\kappa\in\left(0,\frac12\right)$.
2. *For all* $\vartheta_0\in\Theta$, *the function* $S(\vartheta_0,t)$ *is two times continuously differentiable w.r.t.* $t\in[0,T]$.
3. *For all* $\vartheta_0\in\Theta$, *the function* $\Phi(\cdot,\hat\vartheta)$ *has a unique minimum at the point* $\hat\vartheta=\hat\vartheta(\vartheta_0)\in\Theta$.
4. *Its second derivative*

$$\gamma(\hat\vartheta)\equiv\left.\frac{\partial^2\Phi(\vartheta,\hat\vartheta)}{\partial\vartheta^2}\right|_{\vartheta=\hat\vartheta}>0$$

for all $\vartheta_0\in\Theta$.

Let us denote

$$\hat Z(u)=\exp\left\{aW^H(u)-\frac{\gamma(\hat\vartheta)}{4}u^2\right\},\quad u\in\mathbb{R},$$
$$\hat Z^o(u)=\exp\left\{w^H(v)-\frac{v^2}{2}\right\},\qquad v\in\mathbb{R}$$

where $W^H(\cdot)$ and $w^H(\cdot)$ are fBm processes, and define the random variables $\hat\zeta$ and $\hat\zeta_o$ by the relations

$$\hat Z\big(\hat\zeta\big)=\sup_{u\in\mathbb{R}}\hat Z(u)\quad\text{and}\quad\hat Z^o\big(\hat\zeta_o\big)=\sup_{v\in\mathbb{R}}\hat Z^o(v).$$

Note that

$$\hat\zeta=\left(\frac{2a}{\gamma\big(\hat\vartheta\big)}\right)^{\frac{H}{2H-1}}\hat\zeta_o.\tag{21}$$

To verify (21), we change the variable $u = rv$ with $r = (2a)^{\frac{1}{2-H}}\,\gamma\big(\hat\vartheta\big)^{-\frac{1}{2-H}}$ and write

$$
\begin{aligned}
a\,W^H(u) - \frac{\gamma\big(\hat\vartheta\big)}{4}\,u^2 &= a\,W^H(rv) - \frac{\gamma\big(\hat\vartheta\big)\,r^2}{4}\,v^2\\
&= a\,r^H\left(\frac{W^H(rv)}{r^H} - \frac{\gamma\big(\hat\vartheta\big)\,r^{2-H}}{4a}\,v^2\right) = a\,r^H\left(w^H(v) - \frac{v^2}{2}\right),
\end{aligned}
$$

where the fBm $w^H(v) = r^{-H}\,W^H(rv)$.

Theorem 2 *Let the condition $\mathcal{M}$ be fulfilled, then the estimator $\hat\vartheta_\varepsilon$ converges to the value $\hat\vartheta$, its limit distribution is given by*

$$
\frac{\hat\vartheta_\varepsilon - \hat\vartheta}{\varepsilon^{\frac{2}{3-2\kappa}}} \Longrightarrow \hat\zeta, \tag{22}
$$

and for any $p > 0$, we have

$$
\lim_{\varepsilon\to 0}\mathbf{E}_\vartheta\left|\frac{\hat\vartheta_\varepsilon - \hat\vartheta}{\varepsilon^{\frac{2}{3-2\kappa}}}\right|^p = \mathbf{E}_\vartheta\big|\hat\zeta\big|^p = \left(\frac{2a}{\gamma\big(\hat\vartheta\big)}\right)^{\frac{p}{2-H}}\mathbf{E}\big|\hat\zeta_o\big|^p. \tag{23}
$$

Proof Introduce the normalized pseudo-likelihood-ratio process

$$
Z_\varepsilon(u) = \frac{V\big(\hat\vartheta + \varphi_\varepsilon u, X^T\big)}{V\big(\hat\vartheta, X^T\big)}, \qquad u \in \mathbb{U}_\varepsilon = \left(\frac{\alpha - \hat\vartheta}{\varphi_\varepsilon}, \frac{\beta - \hat\vartheta}{\varphi_\varepsilon}\right),
$$

where $\varphi_\varepsilon \to 0$ will be defined later and denote $\vartheta_u = \hat\vartheta + \varphi_\varepsilon u$. Below, we use the same argument as the one used in the preceding section in a similar situation ($u > 0$)

$$
\begin{aligned}
\ln Z_\varepsilon(u) &= \frac{1}{\varepsilon^2}\int_0^T\Big[M\big(\hat\vartheta + \varphi_\varepsilon u, t\big) - M\big(\hat\vartheta, t\big)\Big]\,\mathrm{d}X_t\\
&\quad - \frac{1}{2\varepsilon^2}\int_0^T\Big[M\big(\hat\vartheta + \varphi_\varepsilon u, t\big)^2 - M\big(\hat\vartheta, t\big)^2\Big]\,\mathrm{d}t\\
&= \frac{1}{\varepsilon}\int_0^T\Big[a\,|t - \vartheta_u|^\kappa - a\,\big|t - \hat\vartheta\big|^\kappa\Big]\,\mathrm{d}W_t\\
&\quad - \frac{1}{2\varepsilon^2}\int_0^T\Big[a\,|t - \vartheta_u|^\kappa - a\,\big|t - \hat\vartheta\big|^\kappa\Big]\\
&\qquad\qquad \Big[a\,|t - \vartheta_u|^\kappa + a\,\big|t - \hat\vartheta\big|^\kappa - 2\,S(\vartheta_0, t)\Big]\,\mathrm{d}t\\
&= \frac{a\,\varphi_\varepsilon^{\kappa+\frac{1}{2}}}{\varepsilon}\int_{-\frac{\hat\vartheta}{\varphi_\varepsilon}}^{\frac{T-\hat\vartheta}{\varphi_\varepsilon}}\big[|s-u|^\kappa - |s|^\kappa\big]\,\mathrm{d}W(s) - \frac{1}{2\varepsilon^2}\,\Phi\big(\vartheta_u, \hat\vartheta\big).
\end{aligned}
$$

Let us study the function $\Phi(\vartheta_u,\hat\vartheta)$ for a fixed $u>0$ as $\varphi_\varepsilon\to 0$. We have

$$\begin{aligned}\Phi(\vartheta,\hat\vartheta)&=\int_0^T\left[M(\vartheta,t)-S(\vartheta_0,t)\right]^2\mathrm{d}t-\int_0^T\left[M\left(\hat\vartheta,t\right)-S(\vartheta_0,t)\right]^2\mathrm{d}t\\&=\int_{-\vartheta}^{T-\vartheta}\left[a\left|s\right|^\kappa-S(\vartheta_0,s+\vartheta)\right]^2\mathrm{d}t-\int_0^T\left[M\left(\hat\vartheta,t\right)-S(\vartheta_0,t)\right]^2\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\Phi'_\vartheta(\vartheta,\hat\vartheta)=&\left[a\left|\vartheta\right|^\kappa-S(\vartheta_0,0)\right]^2-\left[a\left|T-\vartheta\right|^\kappa-S(\vartheta_0,T)\right]^2\\&-2\int_{-\vartheta}^{T-\vartheta}\left[a\left|s\right|^\kappa-S(\vartheta_0,s+\vartheta)\right]S'(\vartheta_0,s+\vartheta)\,\mathrm{d}s.\end{aligned}$$

Recall that as $\hat\vartheta\in\Theta$ is the point of minimum of the function $\Phi(\vartheta,\hat\vartheta)$, $\vartheta\in\Theta$, we have the equalities

$$\Phi\left(\hat\vartheta,\hat\vartheta\right)=0\quad\text{and}\quad\Phi'_\vartheta\left(\hat\vartheta,\hat\vartheta\right)=0.$$

Let us write the Taylor expansion

$$\begin{aligned}\Phi\left(\vartheta_u,\hat\vartheta\right)&=\Phi\left(\hat\vartheta,\hat\vartheta\right)+\varphi_\varepsilon u\,\Phi'_\vartheta\left(\hat\vartheta,\hat\vartheta\right)+\frac{\varphi_\varepsilon^2\,u^2}{2}\,\Phi''_\vartheta\left(\hat\vartheta,\hat\vartheta\right)\left(1+o(1)\right)\\&=\frac{\varphi_\varepsilon^2\,u^2}{2}\,\Phi''_\vartheta\left(\hat\vartheta,\hat\vartheta\right)\left(1+o(1)\right)\end{aligned}$$

and study the difference

$$\begin{aligned}\Phi'_\vartheta\left(\vartheta_u,\hat\vartheta\right)-\Phi'_\vartheta\left(\hat\vartheta,\hat\vartheta\right)=&\left[a\left|\vartheta_u\right|^\kappa-S(\vartheta_0,0)\right]^2-\left[a\left|\hat\vartheta\right|^\kappa-S(\vartheta_0,0)\right]^2\\&+\left[a\left|T-\hat\vartheta\right|^\kappa-S(\vartheta_0,T)\right]^2-\left[a\left|T-\vartheta_u\right|^\kappa-S(\vartheta_0,T)\right]^2\\&-2\int_{-\vartheta_u}^{T-\vartheta_u}\left[a\left|s\right|^\kappa-S(\vartheta_0,s+\vartheta_u)\right]S'(\vartheta_0,s+\vartheta_u)\,\mathrm{d}s\\&+2\int_{-\hat\vartheta}^{T-\hat\vartheta}\left[a\left|s\right|^\kappa-S\left(\vartheta_0,s+\hat\vartheta\right)\right]S'\left(\vartheta_0,s+\hat\vartheta\right)\mathrm{d}s.\end{aligned}$$

We have the estimates

$$\begin{aligned}&\left[a\left|\vartheta_u\right|^\kappa-S(\vartheta_0,0)\right]^2-\left[a\left|\hat\vartheta\right|^\kappa-S(\vartheta_0,0)\right]^2\\&\quad=a\left[\left|\hat\vartheta+\varphi_\varepsilon u\right|^\kappa-\left|\hat\vartheta\right|^\kappa\right]\left[a\left|\hat\vartheta+\varphi_\varepsilon u\right|^\kappa+a\left|\hat\vartheta\right|^\kappa-2\,S(\vartheta_0,0)\right]\\&\quad=\frac{2a\kappa}{\hat\vartheta^{1-\kappa}}\left[a\left|\hat\vartheta\right|^\kappa-S(\vartheta_0,0)\right]\varphi_\varepsilon u+O(\varphi_\varepsilon^2\,u^2)\end{aligned}$$

and

$$\left[a\left|T-\hat{\vartheta}\right|^{\kappa}-S(\vartheta_0,T)\right]^2-\left[a\left|T-\hat{\vartheta}-\varphi_\varepsilon u\right|^{\kappa}-S(\vartheta_0,T)\right]^2$$
$$=\frac{2a\kappa}{|T-\hat{\vartheta}|^{1-\kappa}}\left[a\left|T-\hat{\vartheta}\right|^{\kappa}-S(\vartheta_0,T)\right]\varphi_\varepsilon u+O(\varphi_\varepsilon^2\,u^2)$$

since

$$\left|\hat{\vartheta}+\varphi_\varepsilon u\right|^{\kappa}-\left|\hat{\vartheta}\right|^{\kappa}=\left|\hat{\vartheta}\right|^{\kappa}\left(1+\frac{\kappa\varphi_\varepsilon u}{\hat{\vartheta}}\right)-\left|\hat{\vartheta}\right|^{\kappa}+O(\varphi_\varepsilon^2\,u^2)$$
$$=\frac{\kappa\,\varphi_\varepsilon u}{\hat{\vartheta}^{1-\kappa}}+O(\varphi_\varepsilon^2\,u^2).$$

Furthermore, we can write

$$\int_{-\hat{\vartheta}}^{T-\hat{\vartheta}}\left[a\,|s|^{\kappa}-S\big(\vartheta_0,s+\hat{\vartheta}\big)\right]S'\big(\vartheta_0,s+\hat{\vartheta}\big)\,\mathrm{d}s$$
$$-2\int_{-\vartheta_u}^{T-\vartheta_u}\left[a\;,|s|^{\kappa}-S(\vartheta_0,s+\vartheta_u)\right]S'(\vartheta_0,s+\vartheta_u)\,\mathrm{d}s$$
$$=\int_0^T\left|t-\hat{\vartheta}\right|^{\kappa}\left[S'(\vartheta_0,t+\varphi_\varepsilon u)-S'(\vartheta_0,t)\right]\mathrm{d}t$$
$$+\int_0^T\left[S(\vartheta_0,t+\varphi_\varepsilon u)\,S'(t+\varphi_\varepsilon u)-S(\vartheta_0,t)\,S'(\vartheta_0,t)\right]\mathrm{d}t$$
$$+\left(\int_{-\vartheta_u}^{-\hat{\vartheta}}-\int_{T-\vartheta_u}^{T-\hat{\vartheta}}\right)\left[a\,|s|^{\kappa}-S(\vartheta_0,s+\vartheta_u)\right]S'(\vartheta_0,s+\vartheta_u)\,\mathrm{d}s.$$

For the above integrals, we obtain the relations

$$\int_0^T\left|t-\hat{\vartheta}\right|^{\kappa}\left[S'(\vartheta_0,t+\varphi_\varepsilon u)-S'(\vartheta_0,t)\right]\mathrm{d}t$$
$$=\int_0^T\left|t-\hat{\vartheta}\right|^{\kappa}S''(\vartheta_0,t)\,\mathrm{d}t\;\varphi_\varepsilon u+O(\varphi_\varepsilon^2\,u^2),$$
$$\int_0^T\left[S(\vartheta_0,t+\varphi_\varepsilon u)\,S'(\vartheta_0,t+\varphi_\varepsilon u)-S(\vartheta_0,t)\,S'(\vartheta_0,t)\right]\mathrm{d}t$$
$$=\frac{1}{2}\int_0^T\left[S(\vartheta_0,t)^2\right]_t''\,\mathrm{d}t\;\varphi_\varepsilon u+O(\varphi_\varepsilon^2\,u^2),$$
$$\int_{-\vartheta_u}^{-\hat{\vartheta}}\left[a\,|s|^{\kappa}-S(\vartheta_0,s+\vartheta_u)\right]S'(\vartheta_0,s+\vartheta_u)\,\mathrm{d}s$$

$$= \left[a\left|\hat\vartheta\right|^\kappa - S(\vartheta_0,0)\right] S'(\vartheta_0,0)\,\varphi_\varepsilon u + O(\varphi_\varepsilon^2\, u^2),$$

$$\int_{T-\vartheta_u}^{T-\hat\vartheta} \left[a\,|s|^\kappa - S(\vartheta_0, s+\vartheta_u)\right] S'(\vartheta_0, s+\vartheta_u)\,\mathrm{d}s$$

$$= \left[a\,|T-\hat\vartheta|^\kappa - S(\vartheta_0,T)\right] S'(\vartheta_0,T)\,\varphi_\varepsilon u + O(\varphi_\varepsilon^2\, u^2).$$

All these together allow us to write

$$\frac{\Phi'_\vartheta\left(\vartheta_u,\hat\vartheta\right)}{\varphi_\varepsilon u} = \frac{2a\kappa}{\hat\vartheta^{\,1-\kappa}}\left[a\left|\hat\vartheta\right|^\kappa - S(\vartheta_0,0)\right] + \frac{2a\kappa}{|T-\hat\vartheta|^{1-\kappa}}\left[a\left|T-\hat\vartheta\right|^\kappa - S(\vartheta_0,T)\right]$$
$$+\left[a\left|\hat\vartheta\right|^\kappa - S(\vartheta_0,0)\right] S'(\vartheta_0,0) + \left[a\left|T-\hat\vartheta\right|^\kappa - S(\vartheta_0,T)\right] S'(\vartheta_0,T)$$
$$+\,2\int_0^T \left|t-\hat\vartheta\right|^\kappa S''(\vartheta_0,t)\,\mathrm{d}t + \int_0^T \left[S(\vartheta_0,t)^2\right]''_t \mathrm{d}t + O(\varphi_\varepsilon u). \tag{24}$$

Therefore, we obtain the following expression for the second derivative

$$\Phi''_\vartheta(\hat\vartheta,\hat\vartheta) = \lim_{\varphi_\varepsilon\to 0}\frac{\Phi'_\vartheta(\vartheta_u,\hat\vartheta) - \Phi'_\vartheta(\hat\vartheta,\hat\vartheta)}{\varphi_\varepsilon u}$$
$$= \frac{2a\kappa}{\hat\vartheta^{\,1-\kappa}}\left[a\left|\hat\vartheta\right|^\kappa - S(\vartheta_0,0)\right] + \frac{2a\kappa}{|T-\hat\vartheta\hat\vartheta|^{1-\kappa}}\left[a\left|T-\hat\vartheta\right|^\kappa - S(\vartheta_0,T)\right]$$
$$+\left[a\left|\hat\vartheta\right|^\kappa - S(\vartheta_0,0)\right] S'(\vartheta_0,0) + \left[a\left|T-\hat\vartheta\right|^\kappa - S(\vartheta_0,T)\right] S'(\vartheta_0,T)$$
$$+\,2\int_0^T \left|t-\hat\vartheta\right|^\kappa S''(\vartheta_0,t)\,\mathrm{d}t + \int_0^T \left[S(\vartheta_0,t)^2\right]''_t \mathrm{d}t. \tag{25}$$

Now, the log-likelihood ratio has the representation

$$\ln Z_\varepsilon(u) = \frac{a\,\varphi_\varepsilon^{\kappa+\frac12}}{\varepsilon}\, W^H(u)\left(1+o(1)\right) - \frac{\varphi_\varepsilon^2\, u^2}{4\varepsilon^2}\,\Phi''_\vartheta\left(\hat\vartheta,\hat\vartheta\right)\left(1+o(1)\right)$$
$$= \frac{\varphi_\varepsilon^{\kappa+\frac12}}{\varepsilon}\left(a\,W^H(u)\left(1+o(1)\right) - \frac{\varphi_\varepsilon^{\frac32-\kappa}}{\varepsilon}\,\Phi''_\vartheta\left(\hat\vartheta,\hat\vartheta\right)\frac{u^2}{4}\left(1+o(1)\right)\right).$$

Therefore, putting

$$\frac{\varphi_\varepsilon^{\frac32-\kappa}}{\varepsilon} = 1, \qquad \varphi_\varepsilon = \varepsilon^{\frac{2}{3-2\kappa}} \qquad \text{and} \qquad \hat Z_\varepsilon(u) = Z_\varepsilon(u)^{\varepsilon^{\frac{4\kappa-2}{3-2\kappa}}},$$

we obtain the convergence of finite-dimensional distributions

$$\left(\hat{Z}_\varepsilon(u_1),\ldots,\hat{Z}_\varepsilon(u_k)\right) \Longrightarrow \left(\hat{Z}(u_1),\ldots,\hat{Z}(u_k)\right)$$

for any $k = 1, 2, \ldots$

Using the same argument as in the proofs of Lemmas 2–4, we obtain the relations

$$\begin{aligned}
&\Phi\left(\vartheta,\hat{\vartheta}\right) \geq \mu\left(\vartheta-\hat{\vartheta}\right)^2,\\
&\mathbf{E}_{\vartheta_0}\,\hat{Z}_\varepsilon^{1/2}(u) \leq e^{-c\,u^2},\\
&\mathbf{E}_{\vartheta_0}\left[\hat{Z}_\varepsilon^{1/2}(u_2)-\hat{Z}_\varepsilon^{1/2}(u_1)\right]^2 \leq C\,(1+N)\,|u_2-u_1|^2.
\end{aligned}$$

Hence, once more the asymptotic properties of the pseudo-MLE $\hat{\vartheta}_\varepsilon$ follow from the general result (Theorem 21 in Appendix 1) from Ibragimov and Has'minskii (1981). This theorem provides the weak convergence of the measures induced by the random process $\hat{Z}_\varepsilon(\cdot)$ to the measure of the limit process $\hat{Z}(\cdot)$, and therefore, we obtain the convergence of the distribution of the MLE.

Let us remind how the properties of $\hat{\vartheta}_\varepsilon$ are related with the convergence of the stochastic processes $\hat{Z}_\varepsilon(\cdot) \Longrightarrow \hat{Z}(\cdot)$. We have

$$\begin{aligned}
\mathbf{P}_{\vartheta_0}\left(\frac{\hat{\vartheta}_\varepsilon-\hat{\vartheta}}{\varphi_\varepsilon}<x\right) &= \mathbf{P}_{\vartheta_0}\left(\hat{\vartheta}_\varepsilon<\hat{\vartheta}+\varphi_\varepsilon x\right)\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{\vartheta<\hat{\vartheta}+\varphi_\varepsilon x} V\left(\vartheta,X^T\right) > \sup_{\vartheta\geq\hat{\vartheta}+\varphi_\varepsilon x} V\left(\vartheta,X^T\right)\right\}\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{\vartheta<\hat{\vartheta}+\varphi_\varepsilon x} \frac{V\left(\vartheta,X^T\right)}{V\left(\hat{\vartheta},X^T\right)} > \sup_{\vartheta\geq\hat{\vartheta}+\varphi_\varepsilon x} \frac{V\left(\vartheta,X^T\right)}{V\left(\hat{\vartheta},X^T\right)}\right\}\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{u<x,\ u\in\mathbb{U}_\varepsilon} Z_\varepsilon(u) > \sup_{u\geq x,\ u\in\mathbb{U}_\varepsilon} Z_\varepsilon(u)\right\}\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{u<x,\ u\in\mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u) > \sup_{u\geq x,\ u\in\mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u)\right\} = \mathbf{P}_{\vartheta_0}\left(\hat{u}_\varepsilon<x\right),
\end{aligned} \tag{26}$$

where $\hat{u}_\varepsilon = \frac{\hat{\vartheta}_\varepsilon-\hat{\vartheta}}{\varphi_\varepsilon}$ is defined by the relation

$$\hat{Z}_\varepsilon(\hat{u}_\varepsilon) = \sup_{u\in\mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u).$$

Now, from the convergence $\hat{Z}_\varepsilon(\cdot) \Longrightarrow \hat{Z}(\cdot)$, we obtain

$$\mathbf{P}_{\vartheta_0}\left\{\sup_{u<x,\ u\in\mathbb{U}_\varepsilon}\hat{Z}_\varepsilon(u)>\sup_{u\geq x,\ u\in\mathbb{U}_\varepsilon}\hat{Z}_\varepsilon(u)\right\}$$
$$\longrightarrow \mathbf{P}_{\vartheta_0}\left\{\sup_{u<x}\hat{Z}(u)>\sup_{u\geq x}\hat{Z}(u)\right\}=\mathbf{P}_{\vartheta_0}\left(\hat{\zeta}<x\right)$$

(for details, see Theorem 1.10.1 from Ibragimov and Has'minskii 1981). □

4 Some other problems

Of course, it is possible to consider a slightly more general problem of misspecification with the signal

$$S(\vartheta,t)=a\left|t-\vartheta\right|^\kappa\,\mathbb{I}_{\{t<\vartheta\}}+b\left|t-\vartheta\right|^\kappa\,\mathbb{I}_{\{t\geq\vartheta\}}+h(\vartheta,t),$$

where $a^2+b^2>0$ and $h(\vartheta,t)$ is some smooth function of ϑ and t. As usual in singular estimation problems, the limit likelihood ratio $Z(\cdot)$ does not depend on the function $h(\cdot,\cdot)$ and the properties of the pseudo-MLE are quite close to those presented in Theorem 2.

There are other interesting problems of misspecification *cusp vs discontinuous* and *discontinuous vs cusp*, which can be illustrated by the following example. Suppose that we have two signals

$$S(\vartheta,t)=\operatorname{sgn}(t-\vartheta)\left[\left|t-\vartheta\right|^\kappa\,\mathbb{I}_{\{|t-\vartheta|\leq 1\}}+\mathbb{I}_{\{|t-\vartheta|>1\}}\right]$$

where $\kappa\in\left(0,\frac{1}{2}\right)$ and

$$M(\vartheta,t)=\operatorname{sgn}(t-\vartheta).$$

A plot of these two signals is given in Fig. 2.

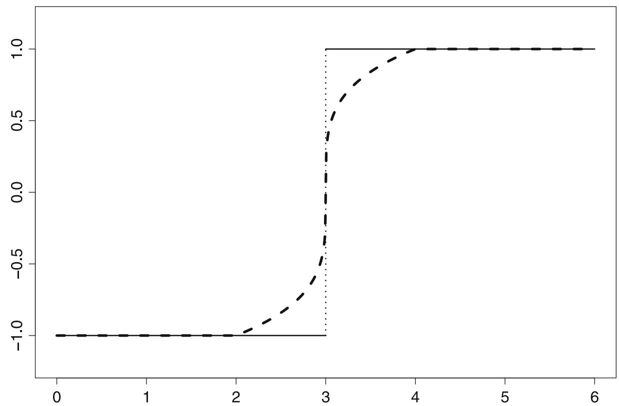

Fig. 2 Signals S (*dashed line*) and M (*continuous line*)

One problem is the estimation of the parameter ϑ in the situation, where $S(\vartheta_0, t)$ is the observed signal and $M(\vartheta, t)$ is the supposed (theoretical) signal. The second problem corresponds to the situation where the observed signal is $M(\vartheta_0, t)$ and the theoretical signal is $S(\vartheta, t)$. Both these problems are studied in a forthcoming paper Chernoyarov et al. (2015).

Let us consider the problem of estimation of the parameter $\kappa_0 \in (k, K)$, $0 < k < K < \infty$ by observations

$$\mathrm{d}X_t = a\,|t-\rho|^{\kappa_0}\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T,$$

where $a > 0$ and $\rho \in (0, T)$ are some known parameters. The likelihood-ratio function is

$$V\left(\kappa, X^T\right) = \exp\left\{\int_0^T \frac{a\,|t-\rho|^{\kappa}}{\varepsilon^2}\,\mathrm{d}X_t - \int_0^T \frac{a^2\,|t-\rho|^{2\kappa}}{2\varepsilon^2}\,\mathrm{d}t\right\}, \qquad \kappa \in (k, K),$$

and the MLE $\hat\kappa_\varepsilon$ is the solution of the equation

$$V\left(\hat\kappa_\varepsilon, X^T\right) = \sup_{\kappa \in (k,K)} V\left(\kappa, X^T\right).$$

This is a regular problem with finite Fisher information

$$\mathbb{I}(\kappa) = a^2 \int_0^T |t-\rho|^{2\kappa}\left(\ln|t-\rho|\right)^2 \mathrm{d}t > 0.$$

It is easy to see that the identifiability condition

$$\inf_{|\kappa-\kappa_0|>\nu} \int_0^T \left[|t-\rho|^{\kappa} - |t-\rho|^{\kappa_0}\right]^2 \mathrm{d}t > 0.$$

is fulfilled for any κ_0 and any $\nu > 0$.

Therefore, the asymptotic normality

$$\frac{\hat\kappa_\varepsilon - \kappa_0}{\varepsilon} \Longrightarrow \mathcal{N}\left(0, \mathbb{I}(\kappa_0)^{-1}\right)$$

follows from the general theorem devoted to the parameter estimation in regular families (see Theorem 3.5.1 from Ibragimov and Has'minskii 1981). Just note that the normalized likelihood ratio

$$Z_\varepsilon^*(v) = \frac{V\left(\kappa_0 + \varepsilon\, v, X^T\right)}{V\left(\kappa_0, X^T\right)}, \qquad v \in \mathbb{V}_\varepsilon = \left(\frac{k-\kappa_0}{\varepsilon}, \frac{K-\kappa_0}{\varepsilon}\right),$$

converges to the process

$$Z^{*}(v)=\exp\left\{v\Delta-\frac{v^{2}}{2}\mathbb{I}(\kappa_{0})\right\},\qquad v\in\mathbb{R},\tag{27}$$

where $\Delta\sim\mathcal{N}\big(0,\mathbb{I}(\kappa_0)\big)$.

It is interesting to consider the problem of estimation of the two-dimensional parameter $\vartheta=(\rho,\kappa)$. The likelihood ratio in this case is

$$V\left(\rho,\kappa,X^{T}\right)=\exp\left\{\int_{0}^{T}\frac{a\left|t-\rho\right|^{\kappa}}{\varepsilon^{2}}\,\mathrm{d}X_{t}-\int_{0}^{T}\frac{a^{2}\left|t-\rho\right|^{2\kappa}}{2\varepsilon^{2}}\,\mathrm{d}t\right\},\qquad\vartheta\in\Theta,$$

where $\Theta=(\alpha,\beta)\times(k,K),0<\alpha<\beta<T,0<k<K<\infty$.

It can be shown that the normalized likelihood ratio

$$Z_{\varepsilon}(u,v)=\frac{V\left(\rho_{0}+\varepsilon^{1/H}\,u,\kappa_{0}+\varepsilon\,v,X^{T}\right)}{V\left(\rho_{0},\kappa_{0},X^{T}\right)}$$

converges to the random process

$$Z(u,v)=Z(u)\,Z^{*}(v)$$

with the processes $Z(\cdot)$ and $Z^{*}(\cdot)$ defined by the expressions (9) and (27), where the fBm $W^{H}(\cdot)$ and the random variable Δ are independent.

The MLE $\hat{\vartheta}_{\varepsilon}=(\hat{\rho}_{\varepsilon},\hat{\kappa}_{\varepsilon})$ is consistent and its components $\hat{\rho}_{\varepsilon}$ and $\hat{\kappa}_{\varepsilon}$ are asymptotically independent and have different limit distributions and different convergence rates:

$$\frac{\hat{\rho}_{\varepsilon}-\rho_{0}}{\varepsilon^{1/H}}\Longrightarrow\hat{\xi}\quad\text{and}\quad\frac{\hat{\kappa}_{\varepsilon}-\kappa_{0}}{\varepsilon}\Longrightarrow\frac{\Delta}{\mathbb{I}(\kappa_{0})}\sim\mathcal{N}\big(0,\mathbb{I}(\kappa_{0})^{-1}\big).\tag{28}$$

The proof follows the mains steps of the proof of the Theorem 1. However, it is rather cumbersome and is not presented here. Similar problems were considered, for example, in Section 5.1 of Kutoyants (1994), and the proof of the convergence (28) can be carried out following the argument presented there.

The case of the three-dimensional parameter $\vartheta=(a,\rho,\kappa)$ can be treated in a similar way.

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References

Chernoyarov, O.V., Kutoyants, Yu.A., Trifonov, A.P. (2015). On misspecification in regularity and properties of estimators (submitted). arXiv:1509.02715.

Dachian, S. (2003). Estimation of cusp location by Poisson observations. *Statistical Inference for Stochastic Processes*, *6*(1), 1–14.

Dachian, S., Kutoyants, Yu A. (2003). On cusp estimation of ergodic diffusion process. *Journal of Statistical Planning and Inference*, *117*, 153–166.

Döring, M. (2015). Asymmetric cusp estimation in regression models. *Statistics*: *A Journal of Theoretical and Applied Statistics*, *49*(6), 1279–1297.
Döring, M., Jensen, U. (2015). Smooth change point estimation in regression models with random design. *Annals of the Institute of Statistical Mathematics*, *67*, 595–619.
Fujii, T. (2010). An extension of cusp estimation problem in ergodic diffusion processes. *Statistics and Probability Letters*, *80*(9–10), 779–783.
Ibragimov, I. A., Has'minskii, R. Z. (1974). Estimation of of a signal parameter in Gaussian white noise. *Problems of Information Transmission*, *10*, 31–46.
Ibragimov, I. A., Has'minskii, R. Z. (1975). Parameter estimation for a discontinuous signal in white Gaussian noise. *Problems of Information Transmission*, *11*, 203–212.
Ibragimov, I. A., Has'minskii, R. Z. (1981). *Statistical Estimation—Asymptotic Theory*. New York: Springer.
Kutoyants, Y. A. (1994). *Identification of Dynamical Systems with Small Noise*. Dordrecht: Kluwer Academic Publisher.
Liptser, R. S., Shiryayev, A. N. (2001). *Statistics of Random Processes, I, II* (2nd ed.). New York: Springer.
Novikov, A., Kordzakhia, N., Ling, T. (2014). On moments of Pitman estimators. *Theory of Probability and its Applications*, *58*(4), 601–614.
Prakasa Rao, B. L. S. (1968). Estimation of the location of the cusp of a continuous density. *The Annals of Mathematical Statistics*, *39*(1), 76–87.
Prakasa Rao, B. L. S. (1985). Asymptotic theory of least squares estimator in a nonregular nonlinear regression model. *Statistics and Probability Letters*, *3*(1), 15–18.
Prakasa Rao, B. L. S. (2004). Estimation of cusp in nonregular nonlinear regression models. *Journal of Multivariate Analysis*, *88*(2), 243–251.
Raimondo, M. (1998). Minimax estimation of sharp change points. *The Annals of Statistics*, *26*(4), 1379–1397.