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Identifiability issues in dynamic stress–strength modeling

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Abstract In many real-life scenarios, system reliability depends on dynamic stress–strength interference, where strength degrades and stress accumulates concurrently over time. In some other cases, shocks appear at random time points, causing damage which is only effective at the instant of shock arrival. In this paper, we consider the identifiability problem of a system under deterministic strength degradation and stochastic damage due to shocks arriving according to a homogeneous Poisson process. We provide conditions under which the models are identifiable with respect to lifetime data only. We also consider current status data and suggest to collect additional information and discuss the issues of model identifiability under different data configurations.

Keywords Poisson process · Cumulative damage · Identifiability · Strength degradation · Current status data · Shock arrival process

1 Introduction

The stress–strength model is widely used in mechanical engineering (Gupta et al 1999), aerospace engineering (Guttman et al 1988), seismic risk assessment (Kaplan 1981),

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medicine (Wilcoxon 1945), psychology (Simonoff et al 1986), and various other allied fields for reliability calculation. An item fails whenever stress on the item equals or exceeds its strength. Traditionally, estimation of reliability of a stochastic system with random strength (Y) and subject to random stress (X) has been addressed as the problem of estimating $P(Y > X)$ (Kotz et al 2003). In many important applications, for example, in the area of disaster risk management, including assessment of seismic risk (Kaplan 1981; Cornell 1968), the strength of the system degrades possibly by corrosion, fatigue, ageing, etc., which may be reasonably described by a deterministic curve, $s(t)$, say (Gertbsbakh and Kordonskiy 1969). In addition, the stress is accumulation $X(t)$ of random damages due to shocks arriving at random time points according to a point process $N(t)$ (Kapur and Lamberson 1977, p-192). See Bhuyan and Dewanji (2015) for details, regarding such a stochastic system and calculation of corresponding reliability. In the following, we refer to this as the 'cumulative damage model'. In some other cases, shocks appear at random time points, causing damage which is only effective at the instant of shock arrival (Nakagawa and Osaki 1974), referred to as 'non-cumulative damage model'. Fracture of brittle materials, such as glasses (Esary et al 1973), and semiconductor parts that have failed by some overcurrent or fault voltage (Nakagawa 2007, p-21) are real-life examples of such models for fixed threshold or strength. Examples, such as the impact forces on vehicle wheels due to road bumps, and the forces on building structure due to wind are appropriate real-life scenarios for such models (Xue and Yang 1997) with both strength and stress being time dependent.

To explain natural random phenomena arising in real-life scenarios, stochastic modeling based on the system mechanism involves certain classes of probability distributions and its associated parameters, which may lead to identifiability problem. There has not been much work on identifiability issues when both stress and strength are time dependent. Problem of non-identifiability of life distributions arising out of stochastic shock models is of fundamental importance (Puri 1977). Clifford (1972) and Esary et al (1973) mentioned the issue of identifiability problem for a stochastic shock model, where shocks arrive according to a Poisson process with intensity μ, and the probability of surviving first k shocks is denoted by $\bar{P}_k$, independent of time t. The reliability function of such a system is given by $R(t) = \sum_{k=0}^{\infty} \bar{P}_k e^{-\mu t}(\mu t)^k/k!$. Puri (1983) discussed the identifiability problems in detail for the above-mentioned shock model. Clifford (1972) has emphasized its seriousness by means of numerical examples producing conflicting predictions in the presence of identifiability problems. In this paper, we discuss identifiability issues for a system under deterministic strength degradation and stochastic damages caused by shocks arriving according to a homogeneous Poisson process.

Note that the failure time, for the non-cumulative damage model, is the arrival time of the first such shock, when the corresponding damage equals or exceeds the strength at that time. Let $N(t)$ denote the point process representing the number of shocks arriving by time t and the damages due to successive shocks be denoted by $X_1, X_2, X_3, \ldots$. The reliability function $R(t)$ at time t is then formally defined as

$$\begin{aligned} R(t) &= P[T > t] \\ &= P[X_i < s(\tau_i),\ \text{for } i = 1, \ldots, N(t)], \end{aligned}$$

where $\tau_1, \tau_2 \ldots$ denote the successive shock arrival times, T denotes the failure time, and $s(t)$ is the strength at time t. When the shock arrival process is Poisson with intensity λ, and independent of the iid damages $X_1, X_2, \ldots$, the reliability function reduces to (see Xue and Yang 1997),

$$R(t) = \exp\left\{-\lambda \int_0^t \{1 - F(s(\tau))\}\mathrm{d}\tau\right\}, \tag{1}$$

where $F(\cdot)$ is the distribution function of X_1. We assume that (i) $s(t)$ is non-increasing, (ii) $s(t) > 0$, for all $t > 0$, (iii) $\lim_{t\to\infty} s(t) = 0$, and (iv) $s(t)$ is continuous. One popular choice is the exponential degradation model given by $s(t) = a\exp(-bt)$.

Remark 1.1 Note that the model may not be identifiable if $s(t)$ is not continuous. For example, let us consider $s(t) = \{90 - t\}I(t < 30) + \max\{80 - t, 0\}I(t \geq 30)$ and $X_1, X_2, \ldots$ are the successive independent damages from the common distribution having equal mass only at 40, 54, and 95 with the successive shocks arriving according to a Poisson process with intensity 1. Let us consider another system with the same strength function, the same intensity of the shock arrival process, and the successive damages $Y_1, Y_2, \ldots$ having the common distribution with equal mass only at 40, 56, and 95. Then, the reliability function, given by (1), for these two systems is equal for all $t > 0$.

Under the same set of assumptions on $s(t)$, the reliability function for the cumulative damage model is given by (see Bhuyan and Dewanji 2014)

$$R(t) = P\,[\,T > t\,] = P\left[\sum_{i=1}^{N(t)} X_i < s(t)\right]. \tag{2}$$

Assuming the shock arrival process to be Poisson with intensity λ and independent of the iid damages $X_1, X_2, \ldots$, the reliability function $R(t)$ of (2) is

$$R(t) = e^{-\lambda t} + \sum_{n=1}^{\infty} F^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n/n!, \tag{3}$$

where $F^{(n)}(s(t)-) = P[\sum_{1=1}^{n} X_i < s(t)]$. Under the condition that $F(s(0)) = 0$, the reliability function, given by either of (3) and (1), reduces to $R(t) = \exp\{-\lambda t\}$; that is, the lifetime variable follows exponential distribution with mean $1/\lambda$, and hence, we do not have any identifiability issue. We do not consider such trivial cases for further investigation. Note that the example in Remark 1.1 also serves to show that, if $s(t)$ is not continuous, the reliability model, given by (3), is not identifiable.

In this paper, we discuss identifiability issues with respect to different data configurations for the stress–strength interference with known strength function. We provide conditions under which the non-cumulative damage model, given by (1), is identifiable with respect to (i) failure time data, (ii) current status data, and (iii) current status data with number of shocks. Similarly, we provide conditions under which the cumulative damage model, given by (3), is identifiable with respect to (i) failure time data, (ii) failure time data with failure type, (iii) current status data, (iv) current status data with cumulative stress, (v) current status data with number of shocks, and (vi) current status with number of shocks and accumulated stress. We first discuss the problem of model identifiability with respect to failure time data in Sect. 2. In Sect. 3, we investigate model identifiability with failure time data and failure type. Next, we investigate model identifiability with current status data and with additional information in Sect. 4. We conclude with some discussion in Sect. 5.

2 Identifiability with failure time data

In this section, we first discuss the identifiability issues with only failure time data with known $s(t)$ and no further assumption. Then, we make additional assumptions on the dynamic stress–strength modeling and investigate the identifiability issue for both non-cumulative and cumulative damage models with known $s(t)$.

Let us first consider the non-cumulative damage model, where $X_1, X_2, \ldots$ are iid damages from $F(\cdot)$, due to shocks arriving according to a Poisson process with intensity λ. For another such system, suppose the successive iid damages $Y_1, Y_2, \ldots$ are from a common distribution $H(\cdot)$, due to shocks arriving according to a Poisson process with intensity μ. If $F(x) = (1 - \mu/\lambda) + (\mu/\lambda)H(x)$, then the reliability function of both the systems are equal for all $t > 0$; that is, from (1),

$$\begin{aligned} R(t) &= \exp\left\{-\lambda \int_0^t \{1 - F(s(\tau))\}\mathrm{d}\tau\right\} \\ &= \exp\left\{-\mu \int_0^t \{1 - H(s(\tau))\}\mathrm{d}\tau\right\}. \end{aligned} \tag{4}$$

Now, we consider the cumulative damage model. Let $X(t) = \sum_{i=1}^{N_1(t)} X_i$ be the cumulative damage at time t due to shocks arriving according to a Poisson process $N_1(t)$ with intensity λ. For another system, let $Y(t) = \sum_{i=1}^{N_2(t)} Y_i$ be the cumulative damage at time t due to shocks arriving according to a Poisson process $N_2(t)$ with intensity μ. Note that two non-negative random variables, V and W, are said to be stochastically equivalent if $P[V < x] = P[W < x]$, for all $x \geq 0$, and we write $V =_{st} W$. Similarly, we define strict stochastic ordering, if $P[W \geq x] \geq P[V \geq x]$, for all $x \geq 0$ and $P[W \geq x_0] > P[V \geq x_0]$ for some $x_0 \geq 0$, and denote it by $V <_{st} W$ (see Shaked and Shanthikumar 2007, for more details). If $X(t) =_{st} Y(t)$, for all $t > 0$, then the reliability functions for these two different systems are the same; that is, $P\,[X(t) < s(t)] = P\,[Y(t) < s(t)]$, for all $t > 0$. Note that the characteristic function of $X(t)$ is given by $\phi_{X(t)}(u) = \exp\{\lambda t(\phi_X(u) - 1)\}$, where $\phi_X(u)$ is the characteristic function of X_1 (Ross 1996, p-82). Similarly, the characteristic function of $Y(t)$ is given by $\phi_{Y(t)}(u) = \exp\{\mu t(\phi_Y(u) - 1)\}$, where $\phi_Y(u)$ is the characteristic function of Y_1. Now, equating $\phi_{X(t)}$ and $\phi_{Y(t)}$, we get

$$\phi_X(u) = \frac{\mu}{\lambda}\phi_Y(u) + \left(1 - \frac{\mu}{\lambda}\right). \tag{5}$$

Assume $\mu < \lambda$ without loss of generality. Applying Gil–Pelaez inversion formula (Gil-Pelaez 1951) on both sides of (5), we get

$$\begin{aligned} F(x) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Im\left(\frac{e^{-iux}\frac{\mu}{\lambda}\phi_Y(u)}{u}\right)\mathrm{d}u - \frac{1}{\pi}\int_0^\infty Im\left(\frac{e^{-iux}\left(1-\frac{\mu}{\lambda}\right)}{u}\right)\mathrm{d}u \\ &= \left(1 - \frac{\mu}{\lambda}\right)\frac{1}{2} - \left(1 - \frac{\mu}{\lambda}\right)\frac{1}{\pi}\int_0^\infty Im\left(\frac{e^{-iux}}{u}\right)\mathrm{d}u \\ &\quad + \frac{\mu}{2\lambda} - \frac{\mu}{\pi\lambda}\int_0^\infty Im\left(\frac{e^{-iux}\phi_Y(u)}{u}\right)\mathrm{d}u \\ &= \left(1 - \frac{\mu}{\lambda}\right) + \frac{\mu}{\lambda}H(x), \end{aligned} \tag{6}$$

for all $x \geq 0$, where $F(\cdot)$ and $H(\cdot)$ are the distribution functions of X_1 and Y_1, respectively. Therefore, from (6), one can conclude that the model is not identifiable if $F(x) = (1 - \mu/\lambda) + (\mu/\lambda)H(x)$. One can easily see that the expected accumulated damage for both of these systems are equal at any time t. It is due to the fact that the system, which suffers from more frequent shocks with rate λ, accumulates damages with distribution function having some mass at 0. Nakagawa and Osaki (1974) provided interpretation of such shock models which may not necessarily incur any damage to the system by citing real-life examples. Non-identifiability for these two models with fixed threshold (that is, $s(t)$ is independent of t) has been discussed by Clifford (1972) and Esary et al (1973). Therefore, the model is not identifiable with damages due to the successive shocks belonging to a family of distribution $\Pi = \{G\colon G(x) = 0, \text{ for all } x < 0\}$, even with a known strength function $s(t)$. Quite naturally, it is interesting to investigate whether these models are identifiable under the condition that the damages due to successive shocks belong to a family of distributions $\Pi_p = \{G\colon G(0) = p\} \subset \Pi$, for some $0 \leq p < 1$.

We first consider the non-cumulative damage model and assume that the damage distribution belongs to Π_p, for some $0 \leq p < 1$. From (4), we get the identity

$$\int_0^t \lambda\{1 - F(s(\tau))\}\mathrm{d}\tau = \int_0^t \mu\{1 - H(s(\tau))\}\mathrm{d}\tau,$$

for all $t > 0$. Since $F(\cdot)$ and $H(\cdot)$ are right-continuous and $s(\tau)$ is non-increasing and continuous, using Lemma 2 (see the Appendix), $F(s(\tau))$ and $H(s(\tau))$ are left-continuous functions of τ. Then, using Lemma 1 (see the Appendix) with $h(\tau) = 1$, we get

$$\lambda\{1 - F(s(\tau))\} = \mu\{1 - H(s(\tau))\}, \tag{7}$$

for all $\tau \geq 0$. Since $F(0) = H(0) = p$ for some $0 \leq p < 1$, taking limit $\tau \to \infty$ in both sides of (7), we get $\lambda = \mu$. Now, putting $\lambda = \mu$ in (7), we get $F(x) = H(x)$ for all $0 \leq x \leq s(0)$. Hence, the model is identifiable. Note that the damage distribution is not anyway identifiable in the region $(s(0), \infty)$.

Remark 2.1 If $s(t) = s$, for all $t \geq 0$ (that is, strength of the system remains fixed over time), then the non-cumulative damage model is not identifiable with failure time data, since the two distinct choices (λ, F) and (μ, H), satisfying $\lambda = \mu[1 - H(s)]/[1 - F(s)]$, lead to (7).

This identifiability issue for the cumulative damage model, under the assumption that the successive damages belong to Π_p, for some $0 \leq p < 1$, remains an open problem. However, with some more restriction on the class Π_p, we prove that the model is identifiable. Let us consider a class of discrete distributions Π^d with the following properties. If $F \in \Pi^d$, then the set $D = \{x\colon F(x-) \neq F(x)\}$ of mass points of F is a non-empty closed set and all $x \in D$ are isolated points. For example, Poisson, Binomial, Geometric, etc. belong to Π^d.

Now, we consider a subclass $\Pi^d_{p_o} = \Pi_{p_o} \bigcap \Pi^d$, where $\Pi_{p_o} = \{G(\cdot)\colon G(0) = p_o > 0\}$. Note that $\Pi^d_{p_o} \subset \Pi^d$. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are two iid sequences of successive damages from the respective common distributions $F(\cdot)$ and $H(\cdot)$ belonging to $\Pi^d_{p_o}$, for some $p_o > 0$, due to shocks arriving according to Poisson processes $N_1(t)$ and $N_2(t)$ with intensities λ and μ, respectively. We consider the identity

$$\begin{aligned} & P\left[X(t) < s(t)\right] = P\left[Y(t) < s(t)\right] \\ \Longrightarrow\ & P\left[\sum_{i=1}^{N_1(t)} X_i < s(t)\right] = P\left[\sum_{i=1}^{N_2(t)} Y_i < s(t)\right] \\ \Longrightarrow\ & \sum_{n=0}^{\infty} F^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n/n! = \sum_{n=0}^{\infty} H^{(n)}(s(t)-)e^{-\mu t}(\mu t)^n/n!, \qquad (8) \end{aligned}$$

for all $t > 0$. We write $z_0 = \min\{x > 0\colon x \in \{x_i, y_i : , i = 1, 2, \ldots\}\}/2$, where x_i's and y_i's are as in the proof of Theorem 1 (see the Appendix). Note that $F(x-) = H(x-) = p_0 > 0$ for all $0 < x \leq z_0$, and hence, $F^{(n)}(x-) = H^{(n)}(x-) = p_0^n$, for all $0 < x \leq z_0$ and for all $n = 1, 2, \ldots$. Suppose, if possible, $\lambda \neq \mu$. Since $s(t)$ is a continuous function and $\lim_{t\to\infty} s(t) = 0$, there exists $t_1 > 0$, such that $s(t_1) \in (0, \min\{z_0, s(0)\})$ and, then, $F^{(n)}(s(t_1)-) = H^{(n)}(s(t_1)-) = p_0^n$ for all $n = 1, 2, \ldots$. Now, from (8) with $t = t_1$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p_0^n e^{-\lambda t_1}(\lambda t_1)^n/n! &= \sum_{n=0}^{\infty} p_0^n e^{-\mu t_1}(\mu t_1)^n/n! \\ &\Longrightarrow\ e^{-\lambda t_1(1-p_0)} = e^{-\mu t_1(1-p_0)} \\ &\Longrightarrow\ \lambda = \mu, \end{aligned}$$

which is a contradiction. Therefore, we assume $\lambda = \mu$.

Now, by Theorem 1 with $\alpha = s(0) > 0$, either $F(x-) = H(x-)$ for all $0 < x \leq s(0)$, or there exist $u_0 \in (0, s(0))$, such that, without loss of generality, $F(u_0-) > H(u_0-)$ and $F^{(n)}(u_0-) \geq H^{(n)}(u_0-)$ for all $n = 2, 3, \ldots$. Since

$s(t)$ is a continuous function, there exists $t_2 > 0$, such that $s(t_2) = u_0$. Therefore, $F(s(t_2)-) > H(s(t_2)-)$ and $F^{(n)}(s(t_2)-) \geq H^{(n)}(s(t_2)-)$, for all $n = 2, 3, \ldots$. Hence, $\sum_{n=0}^{\infty} F^{(n)}(s(t_2)-)e^{-\lambda t_2}(\lambda t_2)^n/n! > \sum_{n=0}^{\infty} H^{(n)}(s(t_2)-)e^{-\lambda t_2}(\lambda t_2)^n/n!$, which contradicts (8). Therefore, $F(x-) = H(x-)$, for all $0 < x \leq s(0)$.

Remark 2.2 Following similar argument, but without requiring Theorem 1, one can easily prove, from (7), that the non-cumulative damage model with $F(\cdot)$ belonging to $\Pi^d_{p_o}$, for some $p_o > 0$, is identifiable. This is expected as $\Pi^d_{p_0} \subset \Pi_{p_0}$.

2.1 Identifiability with failure time data and known damage distribution

Note that, when the damage distribution is known (i.e., $F(x) = H(x)$, for all $x \geq 0$), using (7), we have $\lambda = \mu$, and hence, the non-cumulative damage model is identifiable with known strength function.

For the cumulative damage model, note that $P[X(t) < s(t))] = P[Y(t) < s(t)]$ means

$$\begin{aligned} & P\left[\sum_{i=1}^{N_1(t)} X_i < s(t)\right] = P\left[\sum_{i=1}^{N_2(t)} X_i < s(t)\right] \\ \Longrightarrow\ & \sum_{n=0}^{\infty} F^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n/n! = \sum_{n=0}^{\infty} F^{(n)}(s(t)-)e^{-\mu t}(\mu t)^n/n! \\ \Longrightarrow\ & E_{N_1(t)}[F^{(N_1(t))}(s(t)-)] = E_{N_2(t)}[F^{(N_2(t))}(s(t)-)], \end{aligned} \tag{9}$$

for all $t > 0$. To show $\lambda = \mu$, we first consider $F \in \Pi^d$ with mass points $x_1 < x_2 < \cdots$. Since $s(t)$ is a continuous function and $\lim_{t\to\infty} s(t) = 0$, there exists $t_3 > 0$, such that $s(t_3) = z$, where $z = \frac{x_1}{2} I(x_1 > 0) + \frac{x_2}{2} I(x_1 = 0)$. Now, from (9) with $t = t_3$, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} q^n e^{-\lambda t_3}(\lambda t_3)^n/n! = \sum_{n=0}^{\infty} q^n e^{-\mu t_3}(\mu t_3)^n/n! \\ \Longrightarrow\ & e^{-\lambda t_3(1-q)} = e^{-\mu t_3(1-q)} \\ \Longrightarrow\ & \lambda = \mu, \end{aligned}$$

where $q = F(s(t_3)-)$.

Now, we consider that $F(x-)$ is strictly increasing function of x and show $\lambda = \mu$. Suppose, if possible, $\lambda < \mu$. Hence, $N_1(t) <_{st} N_2(t)$, for any fixed $t > 0$. Then, using Lemmas 3 and 4 (see the Appendix) with $\eta(N_i(t)) = F^{(N_i(t))}(s(t)-)$, for $i = 1, 2$, we get $E_{N_1(t)}[F^{(N_1(t))}(s(t)-)] > E_{N_2(t)}[F^{(N_1(t))}(s(t)-)]$, which contradicts (9). Hence, $\lambda = \mu$.

Remark 2.3 Note that this result holds for a wider class of models for the shock arrival process, in which any two distinct processes $N_1(t)$ and $N_2(t)$ in the class satisfy, without loss of generality, $N_1(t) <_{st} N_2(t)$, for any fixed $t > 0$.

2.2 Identifiability with failure time data and known shock arrival process

Note that when the shock arrival process is known (i.e., $\lambda = \mu$), using (7), we get $F(x) = H(x)$, for all $0 \leq x \leq s(0)$, and hence, the non-cumulative damage model is identifiable with known strength function.

Now, we consider the cumulative damage model and provide sufficient conditions for identifiability of the damage distribution. We start with the following identity.

$$\sum_{n=0}^{\infty} F^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n/n! = \sum_{n=0}^{\infty} H^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n/n!$$
$$\implies \sum_{n=0}^{\infty} [F^{(n)}(s(t)-) - H^{(n)}(s(t)-)]e^{-\lambda t}(\lambda t)^n/n! = 0, \tag{10}$$

for all $t > 0$. Now, we define a class of distributions $\Pi^C \subset \Pi$ with the following properties. If $G \in \Pi^C$, then (i) $G(x)$ is continuous for all $x > 0$ and (ii) $G(x)$ is strictly increasing function for $x > 0$; in addition, if $G, H \in \Pi^C$ and G and H are different, then the set $E_{G,H} = \{x > 0 \colon G(x-) = H(x-)\}$ is a closed set. Note that a standard parametric family $F_\theta(\cdot)$ of continuous life distributions (for example, Exponential, Weibull, Gamma, Log-normal, Pareto, etc.) may be identified with Π^C. Then, the set $E_{G,H}$, with G and H corresponding to F_{θ_1} and F_{θ_2}, respectively, for $\theta_1 \neq \theta_2$, is the set of all those t_0 satisfying $t_0 = F_{\theta_1}^{-1}(F_{\theta_2}(t_0))$. This set is found to be empty or finite for some standard families. For example, it is empty for exponential distributions, whereas it is a singleton set for Weibull or Log-normal distributions. There are many life distributions which can be expressed in terms of power series expansion. It is interesting to investigate the identifiability issues for this class of damage distributions also.

Now, we assume that $F, H \in \Pi^d$ or $F, H \in \Pi^C$. Using Theorems 1 or 2 or Corollary 1 (see the Appendix) and the continuity of the strength function $s(t)$, there exists t_0, such that $\sum_{n=0}^{\infty} [F^{(n)}(s(t_0)-) - H^{(n)}(s(t_0)-)]e^{-\lambda t_0}(\lambda t_0)^n/n! > 0$, which contradicts (10). Hence, $F(x-) = H(x-)$, for $0 < x \leq s(0)$.

3 Identifiability with failure time and failure type

Note that the system fails at the time of arrival of a shock under the non-cumulative damage model and under the cumulative damage model with constant strength. However, when strength degrades with time, under the cumulative damage model, there are two different types of failures either due to strength degradation below the existing level of accumulated stress, or due to arrival of a shock resulting in the increased cumulative stress equalling or exceeding the strength at that time. In this section, we consider only cumulative damage model and investigate the identifiability issues with respect to failure time and failure type.

Let us denote the type of failure by Δ, which takes value 1 if failure of the system is due to the damage of an arriving shock causing increased cumulative stress, and 0 otherwise. To investigate identifiability issue, let us consider the joint probability of $\{T \in (t-h, t], \Delta = i\}$, for $i = 0, 1$, and $h > 0$, as given by

$$P\left[T \in (t-h,t], \Delta = 1\right] = P\left[X(t-h) < s(t) \leq X(t)\right] \tag{11}$$

and

$$P\left[T \in (t-h,t], \Delta = 0\right] = P\left[X(t-h) \in [s(t), s(t-h))\right]. \tag{12}$$

This means that the joint distribution of failure time and failure type depends on the marginal distribution of $X(t)$ and the joint distribution of $\{X(t-h), X(t)\}$ for $h > 0$. Note that the joint characteristic function of $\{X(t-h), X(t)\}$ is given by

$$\begin{aligned}
&\phi_{X(t-h),X(t)}(u_1, u_2)\\
&= E\left[\exp\{iu_1X(t-h) + iu_2X(t)\}\right]\\
&= E\left[\exp\{i(u_1+u_2)X(t-h) + iu_2\{X(t) - X(t-h)\}\}\right]\\
&= E\left[\exp\{i(u_1+u_2)X(t-h)\}\right] E\left[\exp\{iu_2X(h)\}\right]\\
&= \phi_{X(t-h)}(u_1+u_2)\phi_{X(h)}(u_2),
\end{aligned}$$

where $\phi_{X(t)}(u)$ is the characteristic function of $X(t)$, as given in Sect. 2. Note that we have used the property of stationarity and independent increments for $X(t)$ (Snyder and Miller 1991, p-180), that can be proved easily, for this derivation. We have seen in Sect. 2 that the damage distributions $F, H \in \Pi$, related by $F(x) = (1-\frac{\mu}{\lambda}) + \frac{\mu}{\lambda}H(x)$ for all x, lead to the same characteristic function $\phi_{X(t)}(u)$. Hence, this choice of $F(\cdot)$ and $H(\cdot)$ also leads to the same joint characteristic function of $\{X(t-h), X(t)\}$, for $h > 0$. Therefore, the additional information on failure type does not resolve the identifiability problem existing with the failure time data. Now, adding (11) and (12) gives $P[T \in (t-h, t]]$, for all $h > 0$ and $t > 0$. Therefore, the sufficient conditions for model identifiability with failure time data, as discussed in Sect. 2, also provide model identifiability with failure time and failure type data, as expected.

4 Identifiability with current status data

Often, in practice, one inspects the system at some random time point and collects relevant information along with the current status of the system. Here, we consider the inspection time to be a random variable, denoted by U, with known distribution function $\Psi(\cdot)$, which is assumed to be continuous and strictly increasing with the corresponding density function $\psi(\cdot)$. We also assume that U is independent of the shock generating process $N(t)$, and the corresponding damages $X_1, X_2, \ldots$. We observe the inspection time $U = t$ and the current status of the system, denoted by $D(t)$, at the given inspection time $U = t$. If the system is working at a given inspection time $U = t$, then $D(t)$ takes value 1, say, and 0, otherwise. We consider the joint distribution of the observed random variables $\{U, D(U)\}$, in particular, $P[D(U) = j, U \leq v]$. As in Sect. 2, we consider the two models, one with Poisson shock arrival rate λ and damage distribution $F(\cdot)$ and the other with Poisson shock arrival rate μ and damage distribution $H(\cdot)$, leading to the same value for $P[D(U) = j, U \leq v]$ for $j = 0, 1$ and all $v > 0$. Under the non-cumulative damage model and with $j = 1$, we get, using (4),

$$\int_0^v \exp\left[-\lambda \int_0^t \{1 - F(s(\tau))\} \, d\tau\right] \psi(t)dt$$
$$= \int_0^v \exp\left[-\mu \int_0^t \{1 - H(s(\tau))\} \, d\tau\right] \psi(t)dt$$

for all $v > 0$. Applying Lemma 1 twice, we get the identity given by (7) for all $\tau \geq 0$. Therefore, the identifiability issues are again exactly the same as those discussed in Sect. 2 for failure time data. For example, the model is non-identifiable in general, since the two distinct choices (λ, F) and (μ, H), satisfying $F(x) = (1 - \frac{\mu}{\lambda}) + \frac{\mu}{\lambda} H(x)$, for all x, lead to the same value for $P[D(u) = j, U \leq v]$, for $j = 0, 1$ and all $v > 0$.

Next, under the cumulative damage model and with $j = 1$, we have

$$\int_0^v P\left[X(t) < s(t)\right] \psi(t)dt = \int_0^v P\left[Y(t) < s(t)\right] \psi(t)dt,$$

for all $v > 0$. Since $P\left[X(t) < s(t)\right] - P\left[Y(t) < s(t)\right]$ is a right-continuous function of t (see Result 1 in the Appendix), using Lemma 1, we get $P\left[X(t) < s(t)\right] = P\left[Y(t) < s(t)\right]$, for all $t \geq 0$. Hence, the identifiability issues are exactly the same as those discussed in Sect. 2 for failure time data, as under the non-cumulative damage model.

4.1 Current status data and cumulative stress

This section concerns only with the cumulative damage model. The joint distribution of the inspection time U, the current status $D(U)$, and the cumulative stress X(U) is given by

$$P\left[D(U) = j, X(U) < x, U \leq v\right] = \int_0^v P\left[D(t) = j, X(t) < x | U = t\right] \psi(t)dt$$
$$= \int_0^v P\left[D(t) = j, X(t) < x\right] \psi(t)dt,$$

for $v > 0$, $x > 0$, and $j = 0, 1$. As before, we consider the two models (λ, F) and (μ, H) with the cumulative stress denoted by $X(\cdot)$ and $Y(\cdot)$, respectively, and investigate equality of the above probability under the two models. In particular, with $j = 0$, we have, for all $v > 0$ and $x \geq s(t)$,

$$\int_0^v P\left[s(t) \leq X(t) < x\right] \psi(t)dt = \int_0^v P\left[s(t) \leq Y(t) < x\right] \psi(t)dt.$$

Since $P\left[s(t) \leq X(t) < x\right] - P\left[s(t) \leq Y(t) < x\right]$ is a right-continuous function of t (see Result 1 in the Appendix), using Lemma 1, we get

$$P\left[s(t) \leq X(t) < x\right] = P\left[s(t) \leq Y(t) < x\right],$$

for all $t \geq 0$, and $x \geq s(t)$. Therefore, the identifiability issue is similar to those for the failure time data, as discussed in Sect. 2. For example, if $F(x) = (1 - \mu/\lambda) + (\mu/\lambda)H(x)$, then $X(t) =_{st} Y(t)$, for all $t \geq 0$, and, hence,

$$P\left[s(t) \leq X(t) < x\right] = P\left[s(t) \leq Y(t) < x\right], \tag{13}$$

for all $t \geq 0$ and $0 < s(t) \leq x$. Therefore, the additional information on cumulative stress does not resolve the identifiability problem with respect to failure time data.

4.2 Current status data and number of shocks

We now consider information on the number of shocks arriving up to the inspection time along with the current status. Therefore, we consider the joint distribution of $\{U, D(U), N(U)\}$ as given by

$$\begin{aligned} P\left[D(U) = j, N(U) = n, U \leq v\right] &= \int_0^v P\left[D(t) = j, N(t) = n | U = t\right] \psi(t) \mathrm{d}t \\ &= \int_0^v P\left[D(t) = j, N(t) = n\right] \psi(t) \mathrm{d}t, \end{aligned}$$

for $v > 0$, $n = 0, 1, \ldots$, and $j = 0, 1$. As before, we consider the two models (λ, F) and (μ, H) and investigate equality of the above probability. Under the non-cumulative damage model, with $j = 1$, we get

$$\begin{aligned} &\int_0^v \frac{\exp\{-\lambda t\}}{n!} \left\{\int_0^t \lambda F(s(\tau)) \mathrm{d}\tau\right\}^n \psi(t) \mathrm{d}t \\ &= \int_0^v \frac{\exp\{-\mu t\}}{n!} \left\{\int_0^t \mu H(s(\tau)) \mathrm{d}\tau\right\}^n \psi(t) \mathrm{d}t, \end{aligned}$$

for all $v > 0$, and for all $n = 0, 1, \ldots$. Applying Lemma 1, we get

$$\frac{\exp\{-\lambda t\}}{n!} \left\{\int_0^t \lambda F(s(\tau)) \mathrm{d}\tau\right\}^n = \frac{\exp\{-\mu t\}}{n!} \left\{\int_0^t \mu H(s(\tau)) \mathrm{d}\tau\right\}^n, \tag{14}$$

for all $t \geq 0$ and $n = 0, 1, \ldots$. Now, putting $n = 0$ in both sides of (14), we get $\mu = \lambda$. Again, putting $n = 1$ and $\mu = \lambda$ in (14), and applying Lemma 1, we have $F(s(\tau)) = H(s(\tau))$, for all $\tau \geq 0$. Therefore, $F(x) = H(x)$, for all $0 \leq x \leq s(0)$, and the model is identifiable. Most importantly, there is no restriction on the class of damage distributions as required for the failure time data and current status data.

Next, we consider the cumulative damage model. With $j = 1$, we have

$$\int_0^v F^{(n)}(s(t)-) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \psi(t) \mathrm{d}t = \int_0^v H^{(n)}((t)-) \frac{e^{-\mu t}(\mu t)^n}{n!} \psi(t) \mathrm{d}t,$$

for all $v > 0$, and $n = 0, 1, \ldots$. Since $F^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n - H^{(n)}(s(t)-)e^{-\mu t}(\mu t)^n$ is a right-continuous function (see Result 1), using Lemma 1, we have

$$F^{(n)}(s(t)-)\frac{e^{-\lambda t}(\lambda t)^n}{n!} = H^{(n)}(s(t)-)\frac{e^{-\mu t}(\mu t)^n}{n!}, \quad (15)$$

for all $t \geq 0$, and $n = 0, 1, \ldots$. Now, putting $n = 0$ in both sides of (15), we get $\mu = \lambda$. Again putting $n = 1$ and $\mu = \lambda$ in (15), we get $F(s(t)-) = H(s(t)-)$, for all $t \geq 0$. Therefore, $F(x-) = H(x-)$, for all $0 < x \leq s(0)$, and the model is identifiable. In addition, no restriction on the class of damage distributions is required unlike with failure time data and current status data only.

4.3 Current status, number of shocks, and accumulated stress

This section also concerns only with the cumulative damage model. The joint distribution of $\{U, D(U), N(U), X(U)\}$ is given by

$$\begin{aligned} &P[D(U) = j, X(U) < x, N(U) = n, U \leq v] \\ &= \int_0^v P[D(t) = j, X(t) < x, N(t) = n | U = t]\, \psi(t)\mathrm{d}t \\ &= \int_0^v P[D(t) = j, X(t) < x, N(t) = n]\, \psi(t)\mathrm{d}t \\ &= \int_0^v P[D(t) = j, X(t) < x | N(t) = n]\, P[N(t) = n]\, \psi(t)\mathrm{d}t, \end{aligned}$$

for all $v > 0$, $x > 0$, $n = 0, 1, \ldots$, and $j = 0, 1$. To investigate the equality of this joint probability under the two models $\{\lambda, F\}$ and $\{\mu, H\}$, as before, we have, with $j = 0$,

$$\int_0^v F^*(x, n, t)\frac{e^{-\lambda t}(\lambda t)^n}{n!}\psi(t)\mathrm{d}t = \int_0^v H^*(x, n, t)\frac{e^{-\mu t}(\mu t)^n}{n!}\psi(t)\mathrm{d}t,$$

for all $v > 0$, $x \geq s(t)$, and $n = 1, 2 \ldots$, where $F^*(n, x, t) = P[s(t) \leq X(t) < x | N_1(t) = n] = F^{(n)}(x-) - F^{(n)}(s(t)-)$ and $H^*(n, x, t) = P[s(t) \leq Y(t) < x | N_2(t) = n] = H^{(n)}(x-) - H^{(n)}(s(t)-)$ are right-continuous functions of t (see Result 1). Applying Lemma 1, we get

$$F^*(x, n, t) = \left(\frac{\mu}{\lambda}\right)^n e^{-(\mu-\lambda)t} H^*(x, n, t), \quad (16)$$

for all $t \geq 0$, $x \geq s(t)$, and $n = 1, 2, \ldots$. Now, without loss of generality, suppose $\mu > \lambda$. Putting $n = 1$ and taking limit $t \to \infty$ in both sides of (16), we get $F(x-) = \lim_{t\to\infty} F(s(t)-)$, for all $x > 0$, which contradicts the fact that $F(\cdot)$ is a distribution function. Hence, we have $\mu = \lambda$. Now put $n = 1$, $\mu = \lambda$, and take $x \to \infty$ in both sides of (16) to get $F(s(t)-) = H(s(t)-)$, for all $t \geq 0$. Therefore,

$F(x-) = H(x-)$, for all $0 < x \leq s(0)$. In fact, again putting $n = 1$, $\mu = \lambda$, and $F(s(t)-) = H(s(t)-)$ in (16), we get $F(x-) = H(x-)$, for all $x > 0$. The model is, therefore, completely identifiable.

Remark 4.1 Even if the strength function $s(t)$ is unknown, one can readily prove its identifiability. If $s_1(t)$ and $s_2(t)$ denote the strength functions corresponding to the models (λ, F) and (μ, H), respectively, we then have, following the same approach, $F(s_1(t)-) = H(s_2(t)-)$, for all $t \geq 0$. This gives, from (16), with $n = 1$ and $\mu = \lambda$, $F(x-) = H(x-)$, for all $x > 0$. Assuming the damage distribution to be strictly increasing, we have $s_1(t) = s_2(t)$, for all $t \geq 0$.

5 Concluding remarks

Dynamic stress–strength interference, where stress varies over time and strength degrades concurrently, can be used to analyse a wide range of mechanical and natural phenomena. Stochastic mechanisms which are used to model such natural phenomena involve a family of distributions of the observed random variables with associated unknown parameters. In many situations, the model may not be identifiable with respect to the observed random variables. Identifiability problems must be resolved before one attempt to draw any inference based on the stochastic model under consideration. Importance of scientific investigation on the identifiability issues of stochastic shock models has been emphasized by several authors in the past. This paper makes an attempt in that direction by considering different data configurations which may be available from experimental data under the dynamic stress–strength interference accounting for both stochastic damages due to shocks and deterministic strength degradation in a single model.

Non-identifiability of the model under consideration has been discussed with respect to failure time data and current status data. Conditions for model identifiability have been provided in the same context. It is seen that the identifiability issues with failure time data, or current status data, in the presence of additional information like type of failure and accumulated stress are the same as those with only failure time data. However, considering information on the number of shocks, it is observed that problem of non-identifiability for both the non-cumulative and the cumulative damage models is resolved under the assumption that the strength function $s(t)$ is known. Interestingly, the cumulative damage model is completely identifiable, even for an unknown strength function $s(t)$, with information on accumulated stress, number of shocks, and current status. Quite naturally, the models under consideration are identifiable with respect to any further information which may be available from continuous monitoring of the system. In most of the discussions on the identifiability issue, the damage distribution has been kept as arbitrary (that is, not assumed to belong to any particular family of parametric distributions). As a result, it is identifiable only up to $s(0)$, if at all. On the other hand, if a parametric distribution is assumed for the successive damages, then the associated parameter(s) will be identifiable, if at all, and the damage distribution will be identifiable in the whole range.

As remarked in Sect. 1, continuity of $s(t)$ is required for identifiability of both the non-cumulative and the cumulative damage models. In addition, if $\lim_{t\to\infty} s(t) =$

$c > 0$, under the non-cumulative damage model and with failure time only, the damage distribution is identifiable in the range $[c, s(0)]$. Our proofs of the identifiability results for the cumulative model, and with failure time only, do not hold if $c > 0$. However, interestingly, with current status data and information on number of shocks, the damage distribution is identifiable in the range $[c, s(0)]$ under both the non-cumulative and the cumulative damage models.

If the damage distribution and the deterministic strength function are both unknown, then both the cumulative and non-cumulative damage models are non-identifiable, except in the case of Sect. 4.3. The following example clarifies this non-identifiability problem further. Let us consider $s_1(t) = AB^t$ and $s_2(t) = \frac{C}{1+t}$, $A, C > 0, 0 < B < 1$. If $Y_1 = \frac{C \log(B)}{\log(X_1)+\log(B/A)}$, then it is easy to check that $F(s_1(t)-) = H(s_2(t)-)$, for all $t \geq 0$, where $F(\cdot)$ and $H(\cdot)$ are distribution functions of X_1 and Y_1, respectively. One can ensure that Y_1 is a non-negative random variable by choosing any distribution for X_1 with support $\{x : x \in (0, B/A)\}$. As for example, suppose X_1 follows an Uniform (0.3, 0.4) distribution with $s_1(t) = 0.5(0.5)^t$ and $s_2(t) = 1/(1+t)$. Then, with $Y_1 = \log(0.5)/\log(X_1)$, $F(s_1(t)-) = H(s_2(t)-)$ and the reliability functions of the two systems are exactly equal for both the cumulative and the non-cumulative damage models. Therefore, the damage distribution and the deterministic strength function will be confounded. This happens even when the damage distribution belongs to a specific family and the strength function has a specific form. In the industrial applications, engineers may often be able to provide some physical knowledge about the nature of damage distribution and strength degradation. In addition, empirical evidence from past experiments may help to identify the parametric forms of the damage distribution and the deterministic strength function; however, the associated parameters may be unknown. In such cases, parameters associated with damage distribution will be confounded with the unknown parameters of the strength function $s(t)$ and, hence, these parameters will not be identifiable individually. The confounding between $F(\cdot)$ and $s(t)$ is due to the result that $F(s(t)) = H(s^*(t))$, where $H(\cdot)$ is the cdf of cX with some $c > 0$ and $s^*(t) = cs(t)$. For example, assuming $F(\cdot)$ and $H(\cdot)$ to be Exponential distributions with mean β and $c\beta$, respectively, and taking $s(t) = AB^t$ and $s^*(t) = cAB^t$, leads to $F(s(t)) = H(s^*(t)) = 1 - \exp\{-\frac{A}{\beta}B^t\}$. This non-identifiability remains and, therefore, $s(t)$ may be assumed known when considering estimation of $F(\cdot)$ and λ. However, in many real-life scenarios, initial strength of the system is known, which may sometime resolve such identifiability problem.

Once such identifiability problem is detected, there are two different remedies (Puri 1983). If one or more of the parameters involved in this stress–strength interference is known a priori, then one can hope that the other parameters do not suffer from the non-identifiability problem. The same can happen if there exists any plausible relationship among the parameters. Another way is to observe additional information and consider the corresponding joint probability distribution for further analysis, as discussed in the previous sections. Note that the model identifiability, as discussed in this work, only indicates estimability of the model parameters and/or the corresponding damage distribution. One needs to develop a method of estimating such quantities. For example, from (7), the non-cumulative damage model is identifiable with failure time data when the damage distribution belongs to Π_p, for some $0 \leq p < 1$, and is otherwise arbitrary.

It is, however, not clear how to estimate the arbitrary damage distribution F and the shock arrival rate λ based on failure time data. This methodological development is to be taken up in future work. On the other hand, if a parametric distribution is assumed for F, say F_θ, then we have a parametric failure time model with hazard rate given by $\lambda \times [1 - F_\theta(s(t))]$ which can be analyzed using standard maximum likelihood method.

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6 Appendix

Lemma 1 *Suppose $g(\cdot)$ is a right-continuous or left-continuous function, $h(0) \geq 0$, and $h(t) > 0$, for all $t > 0$, then $\int_0^v g(t)h(t)dt = 0$, for all $v > 0$, implies $g(t) = 0$, for all $t \geq 0$.*

Proof Let us first suppose that $g(t)$ is a right-continuous function. In addition, suppose, if possible, $g(t_0) \neq 0$, for some $t_0 \geq 0$. Without loss of generality, let us consider $g(t_0) > 0$. Then, there exists $\delta > 0$, such that $g(t) > 0$, for all $t \in [t_0, t_0 + \delta)$. Then, from the following equation:

$$\int_0^{t_0+\delta} g(t)h(t)\mathrm{d}t = \int_0^{t_0} g(t)h(t)\mathrm{d}t + \int_{t_0}^{t_0+\delta} g(t)h(t)\mathrm{d}t.$$

we have $\int_{t_0}^{t_0+\delta} g(t)h(t)\mathrm{d}t = 0$, which is a contradiction, since $h(t) > 0$ for all $t > 0$, and $g(t) > 0$, for all $t \in [t_0, t_0 + \delta)$. Hence, $g(t) = 0$, for all $t \geq 0$. The proof is similar when $g(t)$ is a left-continuous function. □

Lemma 2 *If f is a right-continuous function and g is a non-increasing left-continuous function, then $f \circ g$ is left-continuous.*

Proof Let $\{x_n\}$ be a sequence, such that $x_n \to x$ as $n \to \infty$ and $x_n \leq x$, for all $n = 1, 2, \ldots$. Then, since g is left-continuous, $g(x_n) \to g(x)$. In addition, since g is a non-increasing function, $g(x_n) \geq g(x)$, for all $n = 1, 2, \ldots$. Therefore, $f(g(x_n)) \to f(g(x))$, since f is a right-continuous function. From sequential characterization of continuity (Rudin 1976, Theorem 4.2, p-84), we conclude that $f \circ g$ is left-continuous. □

Result 1 *If f is a left-continuous function and g is non-increasing right-continuous function, then $f \circ g$ is right-continuous.*

Proof Similar to the proof of Lemma 2 (see Bhuyan and Dewanji 2014). □

Lemma 3 *If V and W are two non-negative random variables, such that $V <_{st} W$, and $\eta(\cdot) \geq 0$ is a strictly decreasing function, then $E[\eta(V)] > E[\eta(W)]$.*

Proof We first show that $\eta(W) <_{st} \eta(V)$. We know that $P[W > x] \geq P[V > x]$, for all $x \geq 0$, and $P[W > x_0] > P[V > x_0]$, for some $x_0 \geq 0$. Now $P[\eta(W) < x] = P[W > \eta^{-1}(x)] \geq P[V > \eta^{-1}(x)] = P[\eta(V) < x]$, for all $x \geq 0$, and $P[\eta(W) < \eta(x_0)] = P[W > x_0] > P[V > x_0] = P[\eta(V) < \eta(x_0)]$. Hence, $\eta(W) <_{st} \eta(V)$.

Since $\eta(W)$ and $\eta(V)$ are non-negative random variables, we can write $E[\eta(V)] - E[\eta(W)] = \int_0^\infty \{P[\eta(V) > x] - P[\eta(W) > x]\}\mathrm{d}x$. We know that $P[\eta(V) > x] - P[\eta(W) > x]$ is a right-continuous function and $P[\eta(V) > y_0] - P[\eta(W) > y_0] > 0$ for some $y_0 \geq 0$. Therefore, there exists $\delta > 0$, such that $P[\eta(V) > x] - P[\eta(W) > x] > 0$ for all $y_0 \leq x < y_0 + \delta$. Hence, $E[\eta(V)] > E[\eta(W)]$. □

Lemma 4 *Suppose X is a non-negative random variable and $G(x-) = P(X < x)$ is a strictly increasing function of $x > 0$, then $G^{(n)}(x-)$ is a strictly decreasing function of n, for all $x > 0$, where $n = 0, 1, \ldots$.*

Proof We prove this result by the method of induction. Let us first fix some arbitrary $x_0 > 0$. Since $G(x-)$ is strictly increasing in x and $\lim_{x\to\infty} G(x-) = 1$, $G(x_0-) < G^{(0)}(x_0-) = 1$. Now

$$\begin{aligned} G^{(2)}(x_0-) &= \int_{[0,x_0)} G(x_0 - t-)dG(t) \\ &< \int_{[0,x_0)} G(x_0-)dG(t) \\ &= \{G(x_0-)\}^2 \\ &\leq G(x_0-). \end{aligned}$$

By definition, $G^{(2)}(x-)$ is also strictly increasing in $x > 0$, since $G(x-)$ is. Similarly, it is easy to see that $G^{(n)}(x-)$ is also strictly increasing in $x > 0$, for all $n = 1, 2, \ldots$. Then

$$\begin{aligned} G^{(n+1)}(x_0-) &= \int_{[0,x_0)} G^{(n)}(x_0 - t-)dG(t) \\ &< \int_{[0,x_0)} G^{(n)}(x_0-)dG(t) \\ &= G^{(n)}(x-)G(x_0-) \\ &\leq G^{(n)}(x_0-). \end{aligned}$$

Therefore, by induction, we conclude that $G^{(n)}(x-)$ is a strictly decreasing function of n, for all $x > 0$. □

Theorem 1 *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are two sequences of iid non-negative random variables with the common cdf $F \in \Pi^d$ and $H \in \Pi^d$, respectively. If $X_1 \neq_{st} Y_1$, then, for all $\alpha > 0$, either there exists $u_0 \in (0, \alpha)$, such that, without loss of generality, $P[X_1 < u_0] > P[Y_1 < u_0]$ and $P[\sum_{i=1}^n X_i < u_0] \geq P[\sum_{i=1}^n Y_i < u_0]$ for all $n = 2, 3, \ldots$, or for all $u \in (0, \alpha]$, $P[Y_1 < u] = P[X_1 < u]$.*

Proof Fix $\alpha > 0$. Let X_1 and Y_1 take values $x_1, x_2, \ldots$, and $y_1, y_2, \ldots$, respectively. Let us write $v_0 = \inf\{x \geq 0 \colon P(X_1 < x) \neq P(Y_1 < x)\}$. Note that, since the mass points of both F and H are isolated, this v_0 is a mass point of either F or H, but does not satisfy $P[X_1 < v_0] \neq P[Y_1 < v_0]$. Therefore, v_0 is the first point, where the masses of F and H differ and all the mass points of F and H, smaller than v_0, are common having equal mass. Suppose $z_1, \ldots, z_k$ are the common mass points of F and H, smaller than v_0. If $v_0 \geq \alpha$, then $P[Y_1 < u] = P[X_1 < u]$ for all $u \in (0, \alpha]$. If $v_0 < \alpha$, then we define $u_0 = v_0 + w_0$, where $w_0 = \left[\left\{\min\left\{(v_0, \infty) \bigcap \{x_i, y_i : , i = 1, 2, \ldots\}\right\} \wedge \alpha\right\} - v_0\right]/2$. Note that the set $\{x_i, y_i : i = 1, 2, \ldots\}$ has no limit point, and hence, $(v_0, \infty) \bigcap \{x_i, y_i : , i = 1, 2, \ldots\}$ is a closed and non-empty set. Hence, the minimum is well-defined. Since $u_0 > v_0$, we have $P[X_1 < u_0] \neq P[Y_1 < u_0]$. We assume, without loss of generality, $P[X_1 < u_0] > P[Y_1 < u_0]$. Therefore, $P[X_1 = v_0] > P[Y_1 = v_0]$. Let us consider the set $S = \{z_1, \ldots, z_k, v_0\}$. Then,

$$
\begin{aligned}
P\left[\sum_{i=1}^{n} X_i < u_0\right] &= \sum_{\{(l_1,\ldots,l_n) \colon l_i \in S, l_1+\cdots+l_n < u_0\}} \prod_{i=1}^{n} P[X_i = l_i] \\
&\geq \sum_{\{(l_1,\ldots,l_n) \colon l_i \in S, l_1+\cdots+l_n < u_0\}} \prod_{i=1}^{n} P[Y_i = l_i] \\
&= P\left[\sum_{i=1}^{n} Y_i < u_0\right],
\end{aligned}
$$

for all $n = 2, 3, \ldots$. Hence, the proof. □

Theorem 2 *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are two sequences of iid non-negative random variables with the common cdf $F(\cdot) \in \Pi^C$ and $G(\cdot) \in \Pi^C$, respectively. If $X_1 \neq_{st} Y_1$, then there exists $x_0 > 0$, such that, for all $u \in (0, x_0)$, without loss of generality, $P[X_1 < u] > P[Y_1 < u]$ and $P[\sum_{i=1}^{n} X_i < u] \geq P[\sum_{i=1}^{n} Y_i < u]$, for all $n = 2, 3, \ldots$.*

Proof If $E_{G,H}$, as defined above, is empty, then, without loss of generality, we have $P[X_1 < x] > P[Y_1 < x]$ for all $x > 0$; that is, $X_1 <_{st} Y_1$. Now, by Theorem 1.A.3 of (Shaked and Shanthikumar 2007, p-6), we get $\sum_{i=1}^{n} X_i \leq_{st} \sum_{i=1}^{n} Y_i$ for all $n = 2, 3, \ldots$ and, hence, $P[\sum_{i=1}^{n} X_i < x] \geq P[\sum_{i=1}^{n} Y_i < x]$ for all $x > 0$ and for all $n = 2, 3, \ldots$.

If $E_{G,H}$ is non-empty, let us write $x_0 = \min\{x : x \in E_{G,H}\} > 0$. Since $E_{G,H}$ is a closed set, this minimum x_0 exists. Let us define random variables X_i^* and Y_i^*, $i = 1, 2, \ldots$, with probability distributions defined as $P[X_i^* < x] = P[X_i < x]/P[X_i \leq x_0]$ and $P[Y_i^* < x] = P[Y_i < x]/P[Y_i \leq x_0]$, respectively, for all $0 < x \leq x_0$, and $P[X_i^* < x] = P[Y_i^* < x] = 1$ for all $x > x_0$. Note that $P[X_1 \leq x_0] = P[X_1 < x_0] = P[Y_1 < x_0] = P[Y_1 \leq x_0] > 0$. Since $P[X_1 < x] - P[Y_1 < x]$ is a continuous function, using Theorem 4.23 of (Rudin 1976, p-93), without loss of generality, we have $P[X_1 < x] > P[Y_1 < x]$ for all $x \in (0, x_0)$.

Hence, $P[X_1^* < x] > P[Y_1^* < x]$ for all $x \in (0, x_0)$, that is $X_1^* <_{st} Y_1^*$. Now, by Theorem 1.A.3 of (Shaked and Shanthikumar 2007, p-6), we get $\sum_{i=1}^n X_i^* \leq_{st} \sum_{i=1}^n Y_i^*$ for all $n = 2, 3, \ldots$, and hence, $P[\sum_{i=1}^n X_i^* < x] \geq P[\sum_{i=1}^n Y_i^* < x]$ for all $x > 0$, and for all $n = 2, 3, \ldots$. Note that this theorem is not applicable on the original variables, since $P[X_1 < x] > P[Y_1 < x]$ for $x \in (0, x_0)$ only, not on the entire support. Now, $P[\sum_{i=1}^n X_i < x] = \{P[X_1 \leq x_0]\}^n P[\sum_{i=1}^n X_i^* < x]$ and $P[\sum_{i=1}^n Y_i < x] = \{P[Y_1 \leq x_0]\}^n P[\sum_{i=1}^n Y_i^* < x]$, for all $0 < x \leq x_0$ and for all $n = 2, 3, \ldots$. Therefore, $P[\sum_{i=1}^n X_i < x] \geq P[\sum_{i=1}^n Y_i < x]$ for all $x \in (0, x_0)$, and for all $n = 2, 3, \ldots$. Hence, the proof. □

Corollary 1 *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are two sequences of iid non-negative random variables with the common distributions $P[X_1 \leq x] = \sum_{j=0}^{\infty} \alpha_j x^j$ and $P[Y_1 \leq x] = \sum_{j=0}^{\infty} \beta_j x^j$, respectively. If $X_1 \neq_{st} Y_1$, then there exists $x_0 > 0$, such that, for all $u \in (0, x_0)$, without loss of generality, $P[X_1 < u] > P[Y_1 < u]$ and $P[\sum_{i=1}^n X_i < u] \geq P[\sum_{i=1}^n Y_i < u]$, for all $n = 2, 3, \ldots$.*

Proof Let us consider the set $A = \{x > 0 \colon P(X_1 < x) = P(Y_1 < x)\}$. Note that $P[X_1 < x]$ and $P[Y_1 < x]$ are continuous and strictly increasing functions for all $x > 0$.

If $A = \phi$, then A is closed. If $A \neq \phi$ and the set of limit points of A is also non-empty, then by Theorem 8.5 of (Rudin 1976, p-177), we have $\alpha_j = \beta_j$, for all $j = 0, 1, \ldots$. This is a contradiction to the fact that $X_1 \neq_{st} Y_1$. Therefore, we consider that A has no limit point. This implies A is a closed set.

For both the cases, distributions of X_1 and Y_1 belong to Π^C, and the Corollary 1 follows from Theorem 2. □

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