

Additional aspects of the generalized linear-fractional branching process

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Abstract We derive some additional results on the Bienyamé–Galton–Watson-branching process with θ -linear fractional branching mechanism, as studied by Sagitov and Lindo (Branching Processes and Their Applications. Lecture Notes in Statistics—Proceedings, 2016). This includes the explicit expression of the limit laws in both the subcritical cases and the supercritical cases with finite mean, and the long-run behavior of the population size in the critical case, limits laws in the supercritical cases with infinite mean when the θ process is either regular or explosive, and results regarding the time to absorption, an expression of the probability law of the θ -branching mechanism involving Bell polynomials, and the explicit computation of the stochastic transition matrix of the θ process, together with its powers.

Keywords Bienyamé–Galton–Watson-branching process · θ -linear-fractional-branching mechanism · Population growth · Yaglom limits · Powers of probability transition matrix

1 Introduction

Recently, in Sagitov and Lindo (2015), a family of branching mechanisms involving explosions was introduced: the so-called θ -linear-fractional family. It fixes the reproduction law of some specific Bienyamé–Galton–Watson-branching processes (Harris

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1963), and it is given in terms of its probability generating function (pgf). This pgf family has the remarkable invariance under iterated composition property, so that in principle, the law of the population size at each generation can be computed. This family extends the classical linear-fractional model (obtained when $\theta = 1$), whose study dates back to Schröder, (Harris 1963, p. 9; Schröder 1871). This makes computation of important statistical quantities of great interest quite explicit. In this construction $\theta \in [-1, 1]$, with very special properties for the cases $\theta \in \{-1, 0, 1\}$ when θ is an integer. We shall revisit this θ family and give some additional results, among which:

- the expression of the limit laws in the subcritical cases and supercritical cases with finite mean, solving respectively the associated Schröder and Poincaré functional equations;
- the long-run behavior of the population size in the critical case;
- limit laws in the supercritical cases with infinite mean when either the θ process is regular or explosive;
- information on the time to absorption defined as the infimum of the times to extinction and explosion;
- an expression of the probability mass distribution of the θ ;
- branching mechanism, alternative to the one given in Proposition 4 of Sagitov and Lindo (2015), using of Faà di Bruno formulae and Bell polynomials;
- the explicit computation of the stochastic transition matrix of the associated Bienyamé–Galton–Watson θ -branching processes, together with its powers. This gives some access to the resolvent of such processes as a key ingredient to compute passage time statistics, hitting probabilities, etc.

We end up this work by a short section of examples, where the following problem of concrete interest is addressed: what is the probability that given the θ -branching process has not yet gone extinct at some given generation, its extinction time be infinite with a large probability close to 1. We do some computations in the special cases $\theta \in \{-1, 0, 1\}$.

2 Generalities on Bienyamé–Galton–Watson (BGW)-branching processes

We start with generalities on such BGW processes, including the case displaying finite-time explosion (Sagitov and Lindo 2015).

2.1 The pgf approach

Consider a discrete-time Bienyamé–Galton–Watson-branching process (Harris 1963), whose reproduction law is given by the (sub-)probability law $\mathbf{P}(M = m) =: \pi(m)$, $m \geq 0$ for the number M of offspring per capita. We assume $\pi(0) > 0$, so that the process can go extinct. We let $\phi(z) = \mathbf{E}(z^M) = \sum_{m \geq 0} \pi(m)z^m$ be the probability generating function of M and we assume $\phi(1) \leq 1$.

With $N_n(1)$ the number of individuals alive at generation n given $N_0 = 1$, we have

$$\mathbf{E} \left(z^{N_n(1)} \right) := \mathbf{E} \left(z^{N_n} \mid N_0 = 1 \right) = \phi^{on} (z), \tag{1}$$

where $\phi^{on}(z)$ is the n th composition of $\phi(z)$ with itself,¹. Similarly, if $N_n(i)$ is the number of individuals alive at generation n given there are $N_0 = i$ independent founders, we clearly get

$$\mathbf{E} \left(z^{N_n(i)} \right) := \mathbf{E} \left(z^{N_n} \mid N_0 = i \right) = \phi^{on} (z)^i. \tag{2}$$

We shall also let

$$\tau_{i,j} = \inf (n \geq 1 : N_n = j \mid N_0 = i),$$

the first hitting time of state $j \neq i$ given $N_0 = i \neq 0$.

- If $\phi(1) < 1$, there is a positive probability $1 - \phi(1) =: \pi(\infty)$ that $M = \infty$ (explosion is made possible even at the first branching step): following [Sagitov and Lindo \(2015\)](#), we shall speak of an explosive or non-regular process.
- If $\phi(1) = 1$ (regular case), depending on $\mu := \mathbf{E}(M) \leq 1$ [i.e., the (sub-)critical case] or $\mu > 1$ (supercritical case): the process $N_n(1)$ goes extinct with probability 1 or goes extinct with probability $\rho < 1$, where ρ is the smallest fixed point solution in $[0, 1]$ to $\phi(\rho) = \rho$, respectively. In the latter case, the distribution of the time to extinction $\tau_{1,0}$ is given by

$$\mathbf{P} (\tau_{1,0} \leq n) = \mathbf{P} (N_n(1) = 0) = \phi^{on} (0),$$

and the process explodes with probability $\bar{\rho} := 1 - \rho$, but not in finite time: only state $\{0\}$ is absorbing. Clearly also, if there are i independent founders instead of simply 1

$$\mathbf{P} (\tau_{i,0} \leq n) = \mathbf{P} (N_n(i) = 0) = \phi^{on} (0)^i.$$

- If $\phi(1) < 1$ (explosive case), $\mu := \mathbf{E}(M) = \infty$, because there is a positive probability $1 - \phi(1)$ that $M = \infty$. Notice that

$$\phi' (1) = \mathbf{E} (M \cdot \mathbf{1}_{\{M < \infty\}}) = \sum_{m \geq 1} m \pi (m),$$

if this quantity exists (is finite). If $\phi(1) < 1$, state $\{\infty\}$ should be added to the state-space $\mathbb{N}_0 = \{0, 1, \dots\}$ of $N_n(i)$ and then both states $\{0, \infty\}$ are absorbing. In this

¹ Throughout this work, a pgf will, therefore, be a function ϕ which is absolutely monotone on $(0, 1)$ with all non-negative derivatives of any order there, obeying $\phi(1) \leq 1$.

supercritical case, $\rho < 1$ always, and both the time to extinction $\tau_{1,0}$ and the time to explosion $\tau_{1,\infty}$ of $N_n(1)$ are finite with positive probability, now with

$$\begin{cases} \mathbf{P}(\tau_{1,0} \leq n) = \mathbf{P}(N_n(1) = 0) = \phi^{on}(0) \xrightarrow{n \rightarrow \infty} \rho = \mathbf{P}(\tau_{1,0} < \infty). \\ \mathbf{P}(\tau_{1,\infty} > n) = \mathbf{P}(N_n(1) < \infty) = \phi^{on}(1) \xrightarrow{n \rightarrow \infty} \rho = \mathbf{P}(\tau_{1,\infty} = \infty). \end{cases} \quad (3)$$

Thus, ρ and $\bar{\rho}$ are now also the probabilities that $\tau_{1,0} < \infty$ and $\tau_{1,\infty} < \infty$, respectively. We thus have

$$\begin{cases} \mathbf{P}(n < \tau_{1,0} < \infty) = \rho - \phi^{on}(0), \\ \mathbf{P}(n < \tau_{1,\infty} < \infty) = \bar{\rho} - (1 - \phi^{on}(1)) = \phi^{on}(1) - \rho, \text{ and,} \\ \mathbf{P}(n < \tau_1 < \infty) = \mathbf{P}(0 < N_n(1) < \infty) = \phi^{on}(1) - \phi^{on}(0), \end{cases} \quad (4)$$

where we defined the global absorption time $\tau_1 := \tau_{1,0} \wedge \tau_{1,\infty}$. Clearly also, with $\tau_i := \tau_{i,0} \wedge \tau_{i,\infty}$

$$\begin{cases} \mathbf{P}(n < \tau_{i,0} < \infty) = \rho^i - \phi^{on}(0)^i, \\ \mathbf{P}(n < \tau_{i,\infty} < \infty) = (1 - \rho^i) - (1 - \phi^{on}(1)^i) = \phi^{on}(1)^i - \rho^i, \text{ and,} \\ \mathbf{P}(n < \tau_i < \infty) = \mathbf{P}(0 < N_n(i) < \infty) = \phi^{on}(1)^i - \phi^{on}(0)^i. \end{cases} \quad (5)$$

Suppose a supercritical situation for which the extinction probability of $N_n(i)$ is smaller than 1 (always the case if $\phi(1) < 1$). Of concrete interest is then the probability that, given the process $N_n(i)$ has not yet gone extinct at generation n , the extinction time of the process will be finite, namely

$$\mathbf{P}(\tau_{i,0} < \infty \mid N_n(i) > 0) = \mathbf{P}(\tau_{i,0} < \infty \mid \tau_{i,0} > n).$$

We get

$$\mathbf{P}(\tau_{i,0} < \infty \mid N_n(i) > 0) = \frac{\rho^i - \phi^{on}(0)^i}{1 - \phi^{on}(0)^i},$$

and the larger n , the smaller this probability because $\phi^{on+1}(0) > \phi^{on}(0)$. There is thus a value n_c of n for which, with probability c close to 1

$$1 - \mathbf{P}(\tau_{i,0} < \infty \mid N_{n_c}(i) > 0) = \frac{1 - \rho^i}{1 - \phi^{on_c}(0)^i} = c \text{ (= say 0.99).} \quad (6)$$

This is the probability that some population with i founders, still alive at generation n_c , will never go extinct.

2.2 The transition matrix approach

A Bienaymé–Galton–Watson process is a time-homogeneous Markov chain with denumerable state-space $\mathbb{N}_0 := \{0, 1, \dots\}$. Its stochastic transition matrix is P , with entries $P(i, j) = [z^j]\phi(z)^i = \mathbf{P}(N_1(i) = j)$ (with $[z^j]\phi(z)^i$ meaning the z^j coefficient of the pgf $\phi(z)^i$). When there is explosion and in the supercritical cases, an interesting problem arises when conditioning N_n either on extinction or on explosion. This may be understood as follows:

The harmonic column vector \mathbf{h} , solution to $P\mathbf{h} = \mathbf{h}$, is given by its coordinates $h(i) = \rho^i, i \geq 0$, because $\sum_{j \geq 0} P(i, j)\rho^j = \phi(\rho)^i = \rho^i$. Letting $D_{\mathbf{h}} := \text{diag}(h(0), h(1), \dots)$, introduce the stochastic matrix $P_{\mathbf{h}}$ given by a Doob transform (Norris 1998; Rogers and Williams 1994, p. 327): $P_{\mathbf{h}} = D_{\mathbf{h}}^{-1} P D_{\mathbf{h}}$ or $P_{\mathbf{h}}(i, j) = h(i)^{-1} P(i, j) h(j) = P(i, j)\rho^{j-i}, i, j \geq 0$. Note $h(N_n(i)) = \rho^{N_n(i)}$ is a martingale because $\mathbf{E}(h(N_n(i))) = \phi^{on}(\rho)^i = \rho^i = h(i) = h(N_0(i))$. Then, $P_{\mathbf{h}}$ is the transition matrix of N_n conditioned on almost sure extinction. Equivalently, when conditioning N_n on almost sure extinction, one is led to a regular subcritical BGW process with new branching mechanism $\phi_0(z) = \phi(\rho z)/\rho$, satisfying $\phi_0(1) = 1$ and $\phi'_0(1) = \phi'(\rho) < 1$. Indeed, $\phi_0(z) = \sum_{j \geq 0} P_{\mathbf{h}}(1, j)z^j$. Upon iterating, we get $\phi_0^{on}(z) = \phi^{on}(\rho z)/\rho$.

Similarly, when conditioning N_n on almost sure explosion, one is led to an explosive supercritical BGW process with new Harris–Sevastyanov-branching mechanism $\phi_{\infty}(z) = [\phi(\rho + \bar{\rho}z) - \rho]/\bar{\rho}$, satisfying $\phi_{\infty}(0) = 0$ and $\phi_{\infty}(1) = (\phi(1) - \rho)/\bar{\rho} < 1$. Upon iterating, we have $\phi_{\infty}^{on}(z) = [\phi^{on}(\rho + \bar{\rho}z) - \rho]/\bar{\rho}$.

The second largest eigenvalue of P is $\gamma = \phi'(\rho) < 1$. The corresponding eigenvector \mathbf{u} obeys $P\mathbf{u} = \gamma\mathbf{u}$ with $u(i) = i\rho^{i-1}, i \geq 1$, because $\sum_{j \geq 1} P(i, j)j\rho^{j-1} = \phi'(\rho)i\phi(\rho)^{i-1} = \gamma i\rho^{i-1}$. Conditioning N_n on never hitting $\{0, \infty\}$ in the remote future is given by the Q process with stochastic transition matrix $Q = \gamma^{-1} D_{\mathbf{u}}^{-1} P D_{\mathbf{u}}$ or $Q(i, j) = \gamma^{-1} u(i)^{-1} P(i, j) u(j) = \gamma^{-1} \rho^{j-i} i^{-1} P(i, j) j, i, j \geq 1$ (see Lambert 2010; Sagitov and Lindo 2015, Section 6 in the θ -special case).

There are classes of discrete-branching processes for which the pgf $\phi^{on}(z)$ of $N_n(1)$ is exactly computable, thereby making the above computations concrete and somehow explicit.

3 The θ -linear-fractional branching mechanism model (Sagitov and Lindo 2015)

With $|\theta| \leq 1, a, b > 0$, and $z_c \geq 1$, we shall consider the θ -linear-fractional branching mechanism model, namely

$$\begin{cases} \phi(z) = z_c - (a(z_c - z)^{-\theta} + b)^{-1/\theta} \text{ or} \\ (z_c - \phi(z))^{-\theta} = a(z_c - z)^{-\theta} + b, \end{cases} \tag{7}$$

and for those values of $z_c \geq 1$ and $a, b > 0$ for which ϕ is a pgf with $\phi(1) \leq 1$. The case $\theta = 0$ will be considered in (11).

3.1 The boundary cases $\theta = \pm 1$

The boundary cases $\theta = \pm 1$ deserve a special treatment that we shall first evacuate.

- When $\theta = 1$, $\phi(z) = z_c - (a(z_c - z)^{-1} + b)^{-1}$ is an homographic map. Assuming $a + b > 1$ and introducing the probabilities $p_0 = 1/(a + b)$, $q = a/(a + b)$, with $p_0 + q_0 = 1$ and $p + q = 1$, this is also ($a = q/p_0$, $b = p/p_0$)

$$\frac{1}{z_c - \phi(z)} = \frac{q}{p_0} \frac{1}{z_c - z} + \frac{p}{p_0}.$$

Note $\phi(z_c) = z_c$ but z_c is not the convergence radius of ϕ , which is $z_c + q/p$.

– In the particular case $z_c = 1$, we have the two following interpretations for $\phi(z)$:

Proposition 1 (i) When $z_c = 1$, $\phi(z) = q_0 + p_0(qz)/(1 - pz)$, the classical form of the simple linear-fractional model. This pgf is the one of a random variable M obtained as $M \stackrel{d}{=} G \cdot B$ (equality in law), where B is Bernoulli (p_0) distributed, independent of G , a geometric ($1/q$) distributed random variable.

(ii) When $z_c = 1$ and if $b < 1$, we also have

$$\phi(z) = \frac{1 + (1 - z) \frac{b-1}{a}}{1 + (1 - z) \frac{b}{a}},$$

which can be put in the alternative form

$$\phi(z) = \frac{\beta(\beta_0 + \alpha_0 z)}{1 - \alpha(\beta_0 + \alpha_0 z)},$$

while defining the probabilities $\alpha_0 = (1 - b)/a$, $\alpha = b$ and $\beta_0 = 1 - \alpha_0$, $\beta = 1 - \alpha$. This $\phi(z)$ is thus the pgf of the random variable:

$$M \stackrel{d}{=} \sum_{k=1}^G B_k,$$

where G now is geometric ($1/\beta$) distributed, independent of the sequence of independent and identically distributed $(B_k)_{k \geq 1}$, with B_1 Bernoulli (α_0) distributed. M is thus a Bernoulli-thinned version of G in the sense of [Steutel and van Harn \(1979\)](#).

We have $\mu := \mathbf{E}(M) = \phi'(1) = p_0/q = 1/a$ and

$$\begin{cases} \phi^{on}(z) = 1 - (a_n(1 - z)^{-1} + b_n)^{-1} \text{ where} \\ a_n = a^n \text{ and } b_n = b(1 + a + \dots + a^{n-1}) \end{cases}.$$

Depending on $a > 1$, $a = 1$, or $a < 1$, the corresponding branching process is subcritical, critical, or supercritical. In the supercritical case $a = q/p_0 < 1$, the extinction probability is $\rho = q_0/p < 1$.

– If now $z_c > 1$, the additional constraints $\phi(0) \in (0, 1)$ and $\phi(1) \leq 1$ impose $p_0 < q + pz_c \leq p + p_0$. This family is of interest, because its n th iterate is explicit, also homographic, with

$$\begin{cases} \phi^{on}(z) = z_c - (a_n(z_c - z)^{-1} + b_n)^{-1} \text{ where} \\ a_n = a^n \text{ and } b_n = b(1 + a + \dots + a^{n-1}). \end{cases} \tag{8}$$

Thus, for instance, if $z_c > 1$, and $q + pz_c < p + p_0$

$$\mathbf{P}(n < \tau_1 < \infty) = \phi^{on}(1) - \phi^{on}(0) = \frac{a_n}{(a_n + b_n z_c)(a_n + b_n(z_c - 1))},$$

with

$$\mathbf{P}(n < \tau_1 < \infty) \underset{n \rightarrow \infty}{\sim} \begin{cases} \frac{(a-1)^2}{(a-1+bz_c)(a-1+b(z_c-1))} a_n^{-1} & \text{if } a > 1 \\ \frac{(a-1)^2}{b^2 z_c(z_c-1)} a^n & \text{if } a < 1 \\ \frac{1}{b^2 z_c(z_c-1)} n^{-2} & \text{if } a = 1 \end{cases} .$$

When $a = 1$ ($p_0 = q$ and $q_0 = p$), the tails of τ_1 are no longer asymptotically geometric, rather they are power law with tail index 2.

- When $\theta = -1$, $\phi(z) = az + z_c(1 - a) - b$ is the affine map and, if $\phi(1) = 1$, the corresponding branching process is the regular death process as each individual can only either die or survive upon splitting. With $\pi(1) = a$, $\pi(0) = z_c(1 - a) - b = 1 - \pi(1)$, $\phi(z) = \pi(1)z + \pi(0)$ and the corresponding branching process is subcritical, always, with mean $\mu = \pi(1) = a < 1$. With $\pi_n(0) + \pi_n(1) = 1$, we have

$$\phi^{on}(z) = \pi_n(0) + \pi_n(1)z, \quad \text{where } \pi_n(1) = \pi(1)^n .$$

If $\phi(1) < 1$, the corresponding branching process is an explosive process, where each individual can either die, survive, or give birth to infinitely many descendants on splitting. The additional constraints $\phi(0) \in (0, 1)$ and $\phi(1) < 1$ impose $\pi(1) = a \in (0, 1)$, $\pi(0) = z_c(1 - a) - b < 1 - \pi(1) = 1 - a$, thus $(z_c - 1)(1 - a) < b$. This family is of interest, because its n th iterate is again explicit

$$\begin{cases} \phi^{on}(z) = z_c - (a_n(z_c - z) + b_n) \text{ with} \\ a_n = a^n \text{ and } b_n = b(1 + a + \dots + a^{n-1}) = b \frac{1-a^n}{1-a}, \end{cases} \tag{9}$$

and again in the same class of affine maps. With $\pi_n(0) + \pi_n(1) < 1$, this is also

$$\phi^{on}(z) = \pi_n(0) + \pi_n(1)z, \quad \text{where } \pi_n(0) = \pi(0) \frac{1 - a^n}{1 - a} \text{ and } \pi_n(1) = \pi(1)^n .$$

We have

$$\begin{aligned} \mathbf{P}(N_n(1) < \infty) &= \pi_n(0) + \pi_n(1) = \pi(0) \frac{1 - a^n}{1 - a} + \pi(1)^n \\ &\xrightarrow{n \rightarrow \infty} \mathbf{P}(N_\infty(1) < \infty) := \pi(0) / (1 - a) < 1. \end{aligned}$$

$\mathbf{P}(N_n(1) = \infty)$ is an increasing sequence. The relative rate of approach of $\mathbf{P}(N_n(1) = \infty)$ to its limiting value decays geometrically with

$$\frac{\mathbf{P}(N_\infty(1) = \infty) - \mathbf{P}(N_n(1) = \infty)}{\mathbf{P}(N_\infty(1) = \infty)} = a^n.$$

Note $\mathbf{P}(n < \tau_1 < \infty) = \phi^{on}(1) - \phi^{on}(0) = a^n$, an exact geometric distribution.

3.2 The case $\theta \in (-1, 1)$

Although we deal here with the case $\theta \in (-1, 1)$, we, somehow abusively, extend the range of the parameter set to its boundary whenever it causes no particular problem.

– With $\theta \in (-1, 1)$, $a, b > 0$, and $z_c = \sup\{z > 0 : \phi(z) < \infty\} \geq 1$, let us reconsider $\phi(z)$ as defined by (1). Note now $\phi(z_c) \leq z_c$ ($= z_c$ if $\theta \in (0, 1]$) and $z_c > 1$ could produce, $\phi(1) < 1$, the explosion opportunity. This family is of interest, because its n th iterate is also explicit with (if $\theta \neq 0$):

$$\begin{cases} \phi^{on}(z) = z_c - (a_n(z_c - z)^{-\theta} + b_n)^{-1/\theta} \text{ where} \\ a_n = a^n \text{ and } b_n = b(1 + a + \dots + a^{n-1}), \end{cases} \tag{10}$$

and it is in the same class as ϕ , although for a different set of parameters a, b (an invariance under iteration property).

The case $\theta = 0$ is defined by continuity from the case $\theta \in (-1, 1) \setminus \{0\}$ while observing

$$\begin{aligned} \phi(z) &= z_c - (a(z_c - z)^{-\theta} + (1 - a)(z_c - \rho)^{-\theta})^{-1/\theta} \\ &\xrightarrow{|\theta| \rightarrow 0} z_c - (z_c - \rho)^{1-a} (z_c - z)^a, \end{aligned} \tag{11}$$

with $\phi(1) < 1$ if $z_c > 1$. Notice that if $z_c = 1$, $\phi(1) = 1$ and $\mu = \infty$ (the only regular case with infinite mean).

There are three cases, depending on $\mu := \mathbf{E}(M) < 1, = 1$ or > 1 :

- (A): subcritical cases:
 - (i) If $\theta \in (0, 1]$, $z_c = 1, a > 1, b > 0$, then $\mu = a^{-1/\theta} < 1$. Again, if $\theta = 1$, $\phi(z) = q_0 + p_0 q z / (1 - pz)$ with $p_0 = 1/(a + b), p = b/(a + b)$, the classical form of the 1-fractional model as the composition of a Bernoulli(p_0) pgf with the one of a geometric (p/q) pgf.
 - (ii) If $\theta \in (-1, 1]$, $z_c > 1, a \in (0, 1)$ and $b = (1 - a)(z_c - 1)^{-\theta}$, then $\mu = a < 1$.
 - (iii) If $\theta = -1, z_c = 1$ and $a \in (0, 1)$, then $\mu = a < 1$.

- (B): critical case ($\mu = 1$): this situation occurs only when $\theta \in (0, 1]$, $z_c = 1$, $a = 1, b > 0$.
- (C): supercritical case ($\infty \geq \mu > 1$): $\theta \in (-1, 1]$, $z_c \geq 1, a \in (0, 1), b = (1 - a)(z_c - \rho)^{-\theta}$ where equivalently $\rho = z_c - ((1 - a)/b)^{1/\theta}$ is the extinction probability of the process, as the smallest solution in the interval $[0, 1]$ to $\phi(\rho) = \rho$ with $\rho \in (0, 1)$. We have $a = \phi'(\rho)$ In the supercritical case with $z_c > 1$, then $\mu = \infty$ because in this case,

$$\phi(1) = z_c - (a(z_c - 1)^{-\theta} + (1 - a)(z_c - \rho)^{-\theta})^{-1/\theta} < 1,$$

and $M = \infty$ with a positive probability.

In general, we have $\phi'(1) = a(a + b(z_c - 1)^\theta)^{-(\theta+1)/\theta} = a(a + (1 - a)(\frac{z_c - 1}{z_c - \rho})^\theta)^{-(\theta+1)/\theta}$ which coincides with μ if $z_c = 1$. We conclude that in the supercritical case with $z_c = 1$

$$\mu = \begin{cases} \infty & \text{if } \theta \in (-1, 0], a \in (0, 1) \\ a^{-1/\theta} & \text{if } \theta \in (0, 1], a \in (0, 1) \end{cases}.$$

In the first case,

- if $\theta \in (-1, 0), a \in (0, 1)$ then $\mu = \infty$ as a result of finite-time explosion because $\phi(1) = 1 - ((1 - a)(1 - \rho)^{-\theta})^{-1/\theta} < 1$ (explosive case).
- if $\theta = 0, a \in (0, 1), \mu = \infty$ even though $\phi(1) = 1$ (the only regular case with infinite mean).

Remarks (i) To the subset of models (A) to (B), we have added the special affine case $\theta = -1$ with $z_c = 1$. If $z_c > 1$, the affine model is supercritical with $\mu = \infty$, because the branching event $M = \infty$ has a positive probability. The special case $\theta = 0$ is supercritical with $\mu = \infty$ both when $z_c = 1$ and $z_c > 1$. The special case $\theta = 1$ corresponds to the standard linear-fractional model and its criticality status has been included in the above classification.

(ii) Due to the invariance under iterated composition of the θ family of pgfs, it holds that $[\phi^{on}]^{-1}(z) = \phi^{o(-n)}(z)$: the inverse function of $\phi^{on}(z)$ simply is $\phi^{o(-n)}(z)$, obtained while substituting $-n$ to n in $\phi^{on}(z)$, (a time-reversal property).

4 Limit laws

We shall investigate different limit laws concerning cases (A) to (C).

4.1 Limit laws (subcritical/critical and supercritical with finite mean cases)

- **Subcritical case with $\mu < 1$**

In the subcritical case, considering the population size, given it is positive, gives rise to a limiting random variable as the generation number goes to infinity. This limiting random variable is known as the quasi-stationary Yaglom limit (Yaglom 1947).

In our context, there are three different cases where this situation can occur: (A)/(i). In this case, with $\theta \in (0, 1]$, $z_c = 1$, $a > 1$, $b > 0$ and $\phi(z) = 1 - (a(1 - z)^{-\theta} + b)^{-1/\theta}$, $N_n \mid N_n > 0 \xrightarrow{d} N_\infty$ where N_∞ is a random variable with value in $\mathbb{N}_0 = \{1, 2, \dots\}$ whose pgf $\phi_\infty(z) := \mathbf{E}(z^{N_\infty}) = \sum_{l \geq 1} \pi_\infty(l)z^l$ obeys the Schröder functional equation:

$$\bar{\phi}_\infty(\phi(z)) = \mu \bar{\phi}_\infty(z), \bar{\phi}_\infty(z) = 1 - \phi_\infty(z), \mu = a^{-1/\theta}. \tag{12}$$

Note $\phi(z) = \bar{\phi}_\infty^{-1}(\mu \bar{\phi}_\infty(z))$ and thus $\phi^{on}(z) = \bar{\phi}_\infty^{-1}(\mu^n \bar{\phi}_\infty(z))$.

Proposition 2 With $\alpha = \frac{a-1}{a+b-1}$ and $\beta = \frac{b}{a+b-1}$ ($\alpha + \beta = 1$), we find the pgf of the Yaglom quasi-stationary limit N_∞ as

$$\phi_\infty(z) = 1 - \frac{1 - z}{(\alpha + \beta(1 - z)^\theta)^{1/\theta}}, \tag{13}$$

obeying $\phi_\infty(0) = 0$, $\phi_\infty(1) = 1$ and with mean $\mu_\infty := \phi'_\infty(1) = \alpha^{-1/\theta} = (\frac{a-1}{a+b-1})^{-1/\theta}$.

If in particular $\theta = 1$

$$\phi_\infty(z) = \frac{z}{1 + \frac{\beta}{\alpha}(1 - z)} = \frac{\alpha z}{1 - \beta z}$$

is the pgf of a geometric random variable with mean $1 + \beta/\alpha = 1/\alpha$. Thus, $\pi_\infty(l) = \mathbf{P}(N_\infty = l) = \alpha\beta^{l-1}$, $l \geq 1$, decays geometrically fast.

Corollary 3 Defining $\bar{\pi}_\infty(k) := \sum_{l > k} \pi_\infty(l)$

$$\bar{\pi}_\infty(k) \underset{k \uparrow \infty}{\sim} -\frac{1}{\Gamma(-\theta)} \frac{\beta}{\theta \alpha^{1+1/\theta}} k^{-(1+\theta)},$$

displaying power law tails with index $1 + \theta$ if $\theta \in (0, 1)$: N_∞ only has moments of order strictly less than $1 + \theta$.

Proof If $\theta \in (0, 1)$, the tail pgf of N_∞ is

$$\frac{1 - \phi_\infty(z)}{1 - z} = (\alpha + \beta(1 - z)^\theta)^{-1/\theta},$$

and the proof follows from Tauberian theorem, observing

$$(\alpha + \beta(1 - z)^\theta)^{-1/\theta} \underset{z \downarrow 1}{\sim} \mu_\infty \left(1 - \frac{\beta}{\alpha\theta}(1 - z)^\theta\right).$$

□

(A)/(ii). In the subcritical case (A)/(ii), with $\theta \in (-1, 1] \setminus \{0\}$, $z_c > 1$, $a \in (0, 1)$, $b = (1 - a)(z_c - 1)^{-\theta}$. Here, with $\phi(1) = 1$ (a regular case)

$$\phi(z) = z_c - (a(z_c - z)^{-\theta} + (1 - a)(z_c - 1)^{-\theta})^{-1/\theta} \quad \text{and} \quad \mu = \phi'(1) = a < 1. \tag{14}$$

Let $h(z) = z_c - z = h^{-1}(z)$, $g(z) = z^{-\theta}$ and $f(z) = g(h(z)) = (z_c - z)^{-\theta}$. The above equation is also (Hoppe 1980)

$$f(\phi(z)) = af(z) + (1 - a)f(1).$$

Let us look for an invertible function $A(z)$ with inverse $B(x) = A^{-1}(x)$, such that $\phi(z) = B(\mu A(z)) = B(aA(z))$. Combining the two equations, we should have

$$\begin{aligned} f \circ B(aA(z)) &= af(z) + (1 - a)f(1) \quad \text{or} \\ f \circ B(ax) &= af \circ B(x) + (1 - a)f(1), \end{aligned}$$

leading to an affine solution $f \circ B(x) = \alpha x + \beta$ with $\beta = f(1)$ and α left undetermined so far. We get

$$\begin{aligned} B(x) &= f^{-1}(\alpha x + f(1)) = z_c - (\alpha x + f(1))^{-1/\theta} \\ A(z) &= B^{-1}(z) = \frac{1}{\alpha} \left((z_c - z)^{-\theta} - (z_c - 1)^{-\theta} \right). \end{aligned}$$

We thus have $\phi_\infty(z) = 1 - A(z) = 1 - \frac{1}{\alpha} \left((z_c - z)^{-\theta} - (z_c - 1)^{-\theta} \right)$ with $\phi_\infty(1) = 1$. Imposing $\phi_\infty(0) = 0$ yields $\alpha = z_c^{-\theta} - (z_c - 1)^{-\theta}$ and so

Proposition 4

$$\phi_\infty(z) = 1 - \left(\frac{(z_c - z)^{-\theta} - (z_c - 1)^{-\theta}}{z_c^{-\theta} - (z_c - 1)^{-\theta}} \right) = \frac{1 - (1 - z/z_c)^{-\theta}}{1 - (1 - 1/z_c)^{-\theta}} \tag{15}$$

is the searched pgf of the unique Yaglom limit N_∞ in this case study. It has finite mean $\phi'_\infty(1)$ (and moments) and $\mathbf{P}(N_\infty = k)$ is asymptotically equivalent to $k^{\theta-1} z_c^{-k}$ with both power law and geometrically decaying factors.

The case $\theta = 0$ is finally obtained by continuity.

Corollary 5 If $\theta = 0$, we get a logarithmic pgf for N_∞ as a result of

$$\phi_\infty(z) = \frac{1 - (1 - z/z_c)^{-\theta}}{1 - (1 - 1/z_c)^{-\theta}} \xrightarrow{|\theta| \rightarrow 0} \frac{-\log(1 - z/z_c)}{-\log(1 - 1/z_c)}, \tag{16}$$

with mean $\phi'_\infty(1) = -\frac{1}{(z_c - 1) \log(1 - 1/z_c)} > 1$.

(A)/(iii). In the subcritical case (A)/(iii) (pure death case with $\phi(z) = \pi(0) + \pi(1)z$ and $\mu = \pi(1) < 1$, $N_n | N_n > 0 \xrightarrow{d} N_\infty$, where simply $N_\infty = 1$, whose pgf $\phi_\infty(z) := \mathbf{E}(z^{N_\infty}) = z$ clearly obeys the Schröder functional equation:

$$\bar{\phi}_\infty(\phi(z)) = \mu \bar{\phi}_\infty(z), \bar{\phi}_\infty(z) = 1 - z.$$

Obviously, $\phi(z) = \bar{\phi}_\infty^{-1}(\mu \bar{\phi}_\infty(z))$ and thus $\phi^{on}(z) = \bar{\phi}_\infty^{-1}(\mu^n \bar{\phi}_\infty(z)) = 1 - \mu^n(1 - z)$ as required.

- **Critical case with $\mu = 1$:** This concerns the case (B) when $\theta \in (0, 1]$, $z_c = 1$, $a = 1, b > 0$. We have

$$\begin{aligned} \phi(z) &= 1 - ((1 - z)^{-\theta} + b)^{-1/\theta} \\ \phi^{on}(z) &= 1 - ((1 - z)^{-\theta} + nb)^{-1/\theta}. \end{aligned}$$

This is a regular case with $\phi(1) = 1$.

Proposition 6 *The process goes extinct with probability 1, but it takes a long time to do so. Indeed*

$$\begin{aligned} \mathbf{P}(\tau_{1,0} > n) &= 1 - \phi^{on}(0) = (1 + nb)^{-1/\theta} \sim (nb)^{-1/\theta}, \\ \mathbf{P}(\tau_{i,0} > n) &= 1 - \phi^{on}(0)^i \sim i (nb)^{-1/\theta}, \text{ for large } n, \end{aligned}$$

with persistent heavy tails, non-geometric.

The pgf of $N_n(1)$ conditioned on $N_n(1) > 0$ is

$$\frac{\phi^{on}(z) - \phi^{on}(0)}{1 - \phi^{on}(0)},$$

therefore

$$\begin{aligned} \mathbf{E}(N_n(1) | N_n(1) > 0) &= (1 + nb)^{1/\theta} \sim b^{1/\theta} n^{1/\theta}, \\ \mathbf{E}(N_n(i) | N_n(i) > 0) &\sim i b^{1/\theta} n^{1/\theta}, \text{ for large } n, \end{aligned}$$

with slow algebraic growth of order $n^{1/\theta}$ in n . A direct computation shows that

$$\phi''(z) = \frac{b(\theta + 1)(1 - z)^{\theta-1}}{(1 + b(1 - z)^\theta)^{1/\theta+2}}.$$

Because $\phi''(1) = 2b < \infty$ only when $\theta = 1$, it holds (Harris 1963; Athreya and Ney 1972) that, if $\theta = 1$, $\mathbf{E}(N_n(1) | N_n(1) > 0) \sim nb$ and

$$\mathbf{P}\left(\frac{N_n(1)}{nb} > x | N_n(1) > 0\right) \xrightarrow[n \rightarrow \infty]{} e^{-x}, x > 0.$$

• **Regular supercritical case with $\mu < \infty$.**

In the supercritical case (C) for which $\mu = a^{-1/\theta} < \infty$ ($z_c = 1, \theta \in (0, 1], a \in (0, 1)$), $\mu^{-n} N_n \xrightarrow{d} W$, where $W \geq 0$ is a random variable with value in $\mathbb{R}_+ = [0, \infty)$, whose Laplace–Stieltjes transform $\phi_W(\lambda) := \mathbf{E}(e^{-\lambda W}), \lambda \geq 0$, obeys the Poincaré functional equation:

$$\phi_W(\mu\lambda) = \phi(\phi_W(\lambda)). \tag{17}$$

Note $\phi(z) = \phi_W(\mu\phi_W^{-1}(z))$ and thus $\phi^{\circ n}(z) = \phi_W(\mu^n\phi_W^{-1}(z))$.

Proposition 7 *With $\alpha = \frac{1-a}{a+b-1} > 0$ and $\beta = \frac{b}{a+b-1} > 0$ ($\beta - \alpha = 1$), if $z_c = 1$, the Laplace–Stieltjes transform of the asymptotic growth rate W of $\mu^{-n} N_n$ is*

$$\phi_W(\lambda) = 1 - \lambda\alpha^{1/\theta} (\beta\lambda^\theta + 1)^{-1/\theta} = \rho + \bar{\rho} \left(1 - (1 + \beta^{-1}\lambda^{-\theta})^{-1/\theta} \right). \tag{18}$$

The extinction probability is $\phi_W(\infty) = \rho = 1 - (\frac{\alpha}{\beta})^{1/\theta}$ and W has an atom at $r = 0$ with mass ρ . We have $\phi_W(0) = 1$ and the mean of W is $\mu_W := -\phi'_W(0) = \alpha^{1/\theta}$.

For general supercritical BGW processes, the limiting W given $W > 0$ is known to be infinitely divisible in some but not all cases (Biggins and Shanbhag 1981). We do not know if $W | W > 0$ here in (18) is infinitely divisible or not.

Corollary 8 *If $\theta = 1$*

$$\phi_W(\lambda) = 1 - \lambda\alpha (\beta\lambda + 1)^{-1} = \frac{\lambda + 1}{\beta\lambda + 1} = \frac{1}{\beta} + \left(1 - \frac{1}{\beta} \right) \frac{1}{1 + \beta\lambda}$$

is the Laplace–Stieltjes transform of an exponential random variable with an atom at 0 with mass $\rho = 1/\beta$ and mean $\bar{\rho}\beta = \frac{1-a}{a+b-1} = \alpha$. In addition, $\mathbf{P}(W > r | W > 0) \sim e^{-r/\beta}$ decays exponentially fast.

Furthermore, using Feller (1971, p. 445),

Corollary 9 *If $\theta \in (0, 1)$, $\phi_W(\lambda) \sim \rho + \bar{\rho}(1 - \beta^{1/\theta}\lambda)$ as λ is close to 0, meaning exponential tails again, now with $\mathbf{P}(W > r | W > 0) \underset{r \rightarrow \infty}{\sim} e^{-r/\beta^{1/\theta}}$. As λ is close to ∞ , $\phi_W(\lambda) \sim \rho + \bar{\rho}(\beta\theta)^{-1}\lambda^{-\theta}$, meaning heavy algebraic left tails $\mathbf{P}(W \leq r | W > 0)r \rightarrow 0 \sim (\beta\theta)^{-1}r/\theta/\Gamma(1 + \theta)$.*

4.2 Limit laws (supercritical with infinite mean cases)

There are two different regimes, depending on $\mu = \infty$ resulting or not from finite-time explosion:

- **Regular case.** If $z_c = 1$, the infinite mean case $\mu = \infty$ occurs when $\theta = 0$, $a \in (0, 1)$. In such a case, $\phi(z) = 1 - (1 - \rho)^{1-a}(1 - z)^a$ and $\phi(1) = 1$ (no finite-time explosion). With $E(1)$ a standard mean 1 exponential random variable

$$a^n \log(1 + N_n(1)) \xrightarrow{a.s.} W = \begin{cases} 0 & \text{with probability } \rho \\ E(1) & \text{with probability } \bar{\rho} \end{cases}, \text{ as } n \rightarrow \infty \quad (19)$$

and conditionally given $N_n(1)$ does not go extinct, $N_n(1)$ grows at double exponential speed.

The pgf of $N_n(1)$ given explosion indeed is

$$\phi_\infty^{on}(z) = 1 - (1 - z)^{a^n},$$

and the above statement follows from the martingale proof of Hénard (2015, proposition 3.8), adapted to the discrete time context. Similar regular models with infinite offspring mean were recently studied in Huillet (2016).

Remark It can be checked that, with $\log_a b = \log b / \log a$ and $A(z) = 1 - \log_{1-\rho}(1 - z)$, $z < 1$,

$$\phi(z) = A^{-1}(aA(z)), \text{ so that } \phi^{on}(z) = A^{-1}(a^n A(z)). \quad (20)$$

This is an alternative way to see that such a branching model is ‘integrable’.

- **Explosive case.** If (i) $z_c > 1$ and $\theta \in (-1, 1]$, $a \in (0, 1)$, $b = (1 - a)(z_c - \rho)^{-\theta}$ with $\rho \in (0, 1)$ or (ii) if $z_c = 1$ and $\theta \in (-1, 0)$, $a \in (0, 1)$ and $b = (1 - a)(1 - \rho)^{-\theta}$, where $\rho \in (0, 1)$, then $N_n(1)$ can be infinite even in the first iteration step (finite-time explosion). What only matters in this context is the time $\tau_{1,\infty}$ to explosion and also $\tau_1 = \tau_{1,0} \wedge \tau_{1,\infty}$, as well as τ_i . We get

Proposition 10 (i) *When $\theta \in (-1, 1]$, $z_c > 1$, $a \in (0, 1)$, $b = (1 - a)(z_c - \rho)^{-\theta}$ with $\rho \in (0, 1)$, leading to $\mu = \infty$, we have, for instance*

$$\mathbf{P}(n < \tau_1 < \infty) = \phi^{on}(1) - \phi^{on}(0) \underset{n \rightarrow \infty}{\sim} \left(\frac{1 - a}{b}\right)^{1+1/\theta} ((z_c - 1)^\theta - z_c^\theta) a^n,$$

showing that τ_1 is tail equivalent to a geometric random variable. Similarly

$$\mathbf{P}(n < \tau_i < \infty) = \phi^{on}(1)^i - \phi^{on}(0)^i \underset{n \rightarrow \infty}{\sim} i \left(\frac{1 - a}{b}\right)^{1+i/\theta} ((z_c - 1)^\theta - z_c^\theta) a^n.$$

(ii) *If $z_c = 1$ and $\theta \in (-1, 0)$, $a \in (0, 1)$ and $b = (1 - a)(1 - \rho)^{-\theta}$, where $\rho \in (0, 1)$, we have*

$$\mathbf{P}(n < \tau_i < \infty) = \phi^{on}(1)^i - \phi^{on}(0)^i \underset{n \rightarrow \infty}{\sim} -i \frac{(1 - \rho)^{-(1+i/\theta)}}{\theta} a^n,$$

still with the tail equivalence to a geometric random variable.

5 Powers of the θ -process transition matrix obtained by iteration

So far, we dealt with this θ family of pgfs for the reproduction law. It remains to compute the probability mass function to which they are associated. A related question is to compute the stochastic transition matrix of the θ -branching processes together with its powers in time. We shall now address these points. We shall start with the cases $\theta \in (-1, 1) \setminus \{0\}$ before addressing the special cases $\theta \in \{-1, 0, 1\}$.

5.1 The case $\theta \in (-1, 1) \setminus \{0\}$

- We start with **the reproduction law**. Let $\phi(z) = z_c - (a(z_c - z)^{-\theta} + b)^{-1/\theta}$ be a θ -pgf with $\phi(z) \leq 1$. We first wish to compute the associate probability mass distribution: $\pi(k) = [z^k]\phi(z)$. Introduce $\phi_c(z) := z_c^{-1}\phi(z_c z)$, so with $\phi_c(z) = 1 - (a(1 - z)^{-\theta} + bz_c^\theta)^{-1/\theta}$ (this operation is meaningful of course only if $z_c > 1$). $\phi_c(z)$ is a new pgf because $\phi_c(1) = z_c^{-1}\phi(z_c) \leq 1$. We have $\pi_c(k) = [z^k]\phi_c(z) = z_c^{k-1}\pi(k)$, so one can work with ϕ_c as well. We also have $\phi_c(z) = f \circ g(z)$ with $g(z) = 1 - (1 - z)^{-\theta}$ and $f(z) = 1 - (a + bz_c^\theta - az)^{-1/\theta}$. This allows to compute $\pi(k)$ by Faa di Bruno formula for the composition of Taylor series. First, we have $\pi(0) = z_c(1 - (a + bz_c^\theta)^{-1/\theta})$. By Faa di Bruno formula (Comtet 1970, Tome 1, p. 149), then

Proposition 11

$$\pi(k) = \frac{1}{k!z_c^{k-1}} \sum_{l=1}^k f_l B_{k,l}(g_\bullet), \quad k \geq 1, \tag{21}$$

where f_l are the Taylor coefficients of $f(z)$, $B_{k,l}(g_\bullet)$ is the Bell polynomials in the indeterminate $g_\bullet := (g_1, g_2, \dots)$, and g_k s being the Taylor coefficients of $g(z)$. The Bell polynomials are defined by

$$B_{k,l}(g_\bullet) = \frac{k!}{l!} [z^k] g(z)^l,$$

with the boundary conditions

$$B_{k,0}(g_\bullet) = B_{0,l}(g_\bullet) = 0, \quad k, l \geq 1 \quad \text{and} \quad B_{0,0}(g_\bullet) := 1, \quad \text{and}$$

$$B_{k,1}(g_\bullet) = g_k \quad \text{and} \quad B_{k,k}(g_\bullet) = g_1^k.$$

This computation of $\pi(k)$ is in agreement with Proposition 4 of Sagitov and Lindo but our representation and its proof, inspired from Faa di Bruno formulae and making use of Bell polynomials, are different. We now list some properties concerning the coefficients f_l and $B_{k,l}(g_\bullet)$. We first recall that (Comtet 1970)

$$B_{k,l}(x_\bullet) = k! \sum_{j \geq 1}^* \prod_{c_j} \frac{1}{c_j!} \left(\frac{x_j}{j!}\right)^{c_j},$$

the latter star summations running over the integers c_j obeying $\sum_{j \geq 1} c_j = l$ and $\sum_{j \geq 1} j c_j = k \geq l$.

We note now that, with $[a]_l = a(a + 1) \cdots (a + l - 1)$ the ascending factorial with $[a]_0 := 1$, $C = (a + bz_c^\theta)^{-1/\theta}$ and $D = (a + bz_c^\theta)/a = C^{-\theta}/a$, $f(z) = 1 - C(1 - z/D)^{-1/\theta}$ with Taylor coefficients:

$$f_0 = 1 - C \quad \text{and} \quad f_l = -\frac{C}{\theta} [1 + 1/\theta]_{l-1} D^{-l} = -CD^{-l} [1/\theta]_l, \quad l \geq 1. \tag{22}$$

For the case $g(z) = 1 - (1 - z)^{-\theta}$, it holds $g_\bullet = -\theta[1 + \theta]_{\bullet-1} = -[\theta]_\bullet$. Because $g_1 = -\theta$ and $g_{m+1} = g_m(m + \theta)$, $m \geq 1$, it follows that the Bell coefficients $B_{k,l}(g_\bullet)$ for this function g obey a simple 3-term recursion:

$$B_{k+1,l}(g_\bullet) = -\theta B_{k,l-1}(g_\bullet) + (k + l\theta) B_{k,l}(g_\bullet), \quad k, l \geq 1. \tag{23}$$

For instance, $B_{1,1}(g_\bullet) = -\theta$ leading to $\pi(1) = f_1 B_{1,1}(g_\bullet) = CD^{-1}$, $B_{2,1}(g_\bullet) = (1 + \theta)B_{1,1}(g_\bullet) = -\theta(1 + \theta)$, leading to $\pi(2) = \frac{1}{2z_c}(f_1 B_{2,1}(g_\bullet) + f_2 B_{2,2}(g_\bullet)) = \frac{1}{2z_c}(1 + \theta)CD^{-2}(D - 1), \dots$ The formulae (21), (22), and (23) completely characterize the $\pi(k)$ s. $B_{k,l}(g_\bullet)$ constitute generalized Stirling numbers studied in Charalambides and Singh (1988).

Remark If $\theta = -1/L$, where $L > 1$ is an integer, $f(z) = 1 - C(1 - z/D)^L$ is a polynomial of degree L in z so $f_l = 0$ if $l > L$ which largely simplifies (21). Furthermore, in this case, $g_k = L^{-k} \prod_{l=1}^{k-1} (lL - 1)$. If $L = 2$, $g_k = 2^{-2(k-1)}(2k - 3)!/(k - 2)!$.

- **The transition matrix and its powers.** We now first wish to compute $P_{a,b}(i, j) = [z^j]\phi(z)^i = z_c^{i-j}[z^j]\phi_c(z)^i$, the transition matrix of the θ -branching process, where its dependence on the parameters (a, b) has been emphasized. We have $\phi_c(z)^i = f_i \circ g(z)$, still with $g(z) = 1 - (1 - z)^{-\theta}$ and now with $f_i(z) := [1 - (a + bz_c^\theta - az)^{-1/\theta}]^i$. So with $f_{i,k}$, $k \geq 1$, the Taylor coefficients of $f_i(z)$, we similarly get

Proposition 12

$$P_{a,b}(i, j) = \frac{z_c^{i-j}}{j!} \sum_{k=1}^j f_{i,k} B_{j,k}(g_\bullet). \tag{24}$$

We note that $f_i(z) = h_i(f(z))$ where $h_i(z) = (1 - C + Cz)^i$ and $f(z) := 1 - (1 - z/D)^{-1/\theta}$, so that with $h_{i,l} = \frac{i!}{(i-l)!}(1 - C)^{i-l}C^l$ ($= 0$ if $l > i$) and with f_\bullet given from (22) as $f_l = -[1/\theta]_l D^{-l}$, $l \geq 1$, by Faà di Bruno formula again

$$f_{i,0} = (1 - C)^i \quad \text{and} \quad f_{i,k} = \sum_{l=1}^{k \wedge i} h_{i,l} B_{k,l}(f_\bullet). \tag{25}$$

Note $\pi(j) = P_{a,b}(1, j)$ as required. To obtain now $P_{a,b}^n(i, j)$, the (i, j) entry of the n th power of $P_{a,b}$, we just need to substitute $(a_n = a^n, b_n = b(1 + a + \dots + a^{n-1}))$ to (a, b) , so it simply holds

$$P_{a,b}^n(i, j) = P_{a_n, b_n}(i, j), \tag{26}$$

taking advantage of the invariance under iteration of the θ family when $\theta \in (-1, 1) \setminus \{0\}$. We note that the dependence on n in $P_{a_n, b_n}(i, j)$ is only in the coefficients $f_{i,k}$ in (24), through C and D . To emphasize this point, we shall also write

Corollary 13

$$P_{a,b}^n(i, j) = P_{a_n, b_n}(i, j) = \frac{z_c^{i-j}}{j!} \sum_{k=1}^j f_{i,k}^{(n)} B_{j,k}(g_\bullet), \tag{27}$$

where $f_{i,k}^{(n)}$ is obtained from $f_{i,k}$ in (25) while substituting $(a_n = a^n, b_n = b(1 + a + \dots + a^{n-1}))$ to (a, b) in the expressions of $C = (a + bz_c^\theta)^{-1/\theta}$ and $D = (a + bz_c^\theta)/a$.

It remains to discuss the special integral cases for θ .

5.2 The case $\theta = 0$

We recall that $\phi(z) = z_c - \lambda(z_c - z)^a$, where $\lambda = (z_c - \rho)^{1-a}$ and ρ obey $\phi(\rho) = \rho$. With $\phi_c(z) = 1 - \lambda z_c^{a-1}(1 - z)^a$ and $\pi_c(k) = [z^k]\phi_c(z)$, $\pi(k) = \pi_c(k)/z_c^{k-1}$ with $\pi_c(k) = -\lambda z_c^{a-1}[-a]_k/k!$. Next, with $\lambda_c := \lambda z_c^{a-1}$

$$\phi_c(z)^i = (1 - \lambda_c(1 - z)^a)^i = (1 - \lambda_c + \lambda_c(1 - (1 - z)^a))^i = h_i \circ g(z),$$

with $g(z) = 1 - (1 - z)^a$ and $h_i(z) = (1 - \lambda_c + \lambda_c z)^i$. With $g_\bullet = -[-a]_\bullet$ and $h_{i,k} = \frac{i!}{(i-k)!} (1 - \lambda_c)^{i-k} \lambda_c^k$, we thus get similarly

$$P_{a,\lambda}(i, j) = \frac{z_c^{i-j}}{j!} \sum_{k=1}^j h_{i,k} B_{j,k}(g_\bullet) \quad \text{and} \quad P_{a,\lambda}^n(i, j) = P_{a_n, \lambda_n}(i, j), \tag{28}$$

where $a_n = a^n$ ($a \in (0, 1)$) and $\lambda_n = \lambda^{(1-a^n)/(1-a)}$. $B_{j,k}(g_\bullet)$ also obey a three terms recursion of the type (23) with $-a$ substituted to θ . Note $\pi(j) = P_{a,\lambda}(1, j)$ as required.

5.3 The case $\theta = 1$

With $\phi(z) = z_c - (a(z_c - z)^{-1} + b)^{-1}$, we wish to compute $\pi(k) = [z^k]\phi(z)$ with $\pi(0) = z_c(a + b - 1)/(a + b)$ in the first place. Introduce $\phi_c(z) = z_c^{-1}\phi(z_c z)$, so with $\phi_c(z) = 1 - (a(1 - z)^{-1} + bz_c)^{-1}$. We have $\phi_c(z) = f \circ g(z)$ with $g(z) := (1 - z)^{-1} - 1$ and $f(z) = 1 - (a + bz_c + az)^{-1} = 1 - C(1 + z/D)^{-1}$, where $C = (a + bz_c)^{-1}$ and $D = (a + bz_c)/a$. Let f_j be the Taylor coefficients of $f(z)$ and g_k be the Taylor coefficients of $g(z)$. By Faa di Bruno formula

$$\pi(k) = \frac{1}{k!z_c^{k-1}} \sum_{l=1}^k f_l B_{k,l}(g_\bullet), \quad k \geq 1,$$

with $g_k = k!$ and $f_0 = 1 - C$ and $f_l = (-1)^{l-1} C D^{-l} l!$, $l \geq 1$. We have $B_{k,l}(\bullet!) = \binom{k-1}{l-1} \frac{k!}{l!}$, so

$$\pi(k) = \frac{C}{z_c^{k-1}} \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} D^{-l} = C D^{-1} \left(\frac{1 - D^{-1}}{z_c} \right)^{k-1}, \quad k \geq 1. \tag{29}$$

Next

$$P_{a,b}(i, j) = z_c^{i-j} [z^j] \phi_c(z)^i.$$

We have $\phi_c(z)^i = f_i \circ g(z)$ still with $g(z) = (1 - z)^{-1} - 1$ and now with $f_i(z) = [1 - C(1 + z/D)^{-1}]^i$. Therefore, with $f_{i,k}$, $k \geq 1$, the Taylor coefficients of $f_i(z)$ and with $B_{j,k}(\bullet!) = \binom{j-1}{k-1} \frac{j!}{k!}$, we get similarly

$$P_{a,b}(i, j) = \frac{z_c^{i-j}}{j!} \sum_{k=1}^j f_{i,k} B_{j,k}(g_\bullet). \tag{30}$$

It remains to compute the $f_{i,k}$ s. We note that $f_i(z) = h_i(f(z))$, where $h_i(z) = (1 - C - Cz)^i$ and $f(z) = (1 + z/D)^{-1} - 1$, so that with $h_{i,l} = \frac{i!}{(i-l)!} (1 - C)^{i-l} (-C)^l$ and with f_\bullet given by $f_l = (-D)^{-l} l!$, $l \geq 1$, by Faa di Bruno formula again

$$f_{i,0} = (1 - C)^i \quad \text{and} \quad f_{i,k} = \sum_{l=1}^{k \wedge i} h_{i,l} B_{k,l}(f_\bullet).$$

Now, $B_{k,l}(f_\bullet) = (-D)^{-k} B_{k,l}(\bullet!) = (-D)^{-k} \binom{k-1}{l-1} \frac{k!}{l!}$ and

$$f_{i,k} = k! (1 - C)^i D^{-k} \sum_{l=1}^{k \wedge i} (-1)^{k-l} \binom{i}{l} \binom{k-1}{l-1} \left(\frac{C}{1 - C} \right)^l. \tag{31}$$

Exchanging the summation over k and l in (30) and applying the binomial identity (keeping in mind $D^{-1} = aC$)

$$P_{a,b}(i, j) = z_c^{i-j} (1 - C)^i (1 - D^{-1})^j \sum_{l=1}^{i \wedge j} \binom{i}{l} \binom{j-1}{l-1} \left(\frac{C}{1 - C} \frac{D^{-1}}{1 - D^{-1}} \right)^l. \tag{32}$$

To obtain now $P_{a,b}^n(i, j)$, (i, j) entry of the n th power of $P_{a,b}$, we just need to substitute $(a_n = a^n, b_n = b(1 + a + \dots + a^{n-1}))$ to (a, b) in (C, D) , so it simply holds

$$P_{a,b}^n(i, j) = P_{a_n, b_n}(i, j),$$

where $P_{a,b}(i, j)$ is given by (32). The resulting expression generalizes Proposition 2.2 of Klebaner et al. (2007).

5.4 The case $\theta = -1$ (Greenwood model)

Here, $\phi(z) = az + z_c(1 - a) - b$. We get $\pi(1) = a, \pi(0) = z_c(1 - a) - b \leq 1 - \pi(1)$. We have

$$P(i, j) = [z^j] \phi(z)^i = \binom{i}{j} \pi(0)^{i-j} \pi(1)^j,$$

and

$$P^n(i, j) = [z^j] \phi^{on}(z)^i = \binom{i}{j} \pi_n(0)^{i-j} \pi_n(1)^j,$$

where $\pi_n(1) = \pi(1)^n$ and $\pi_n(0) = \pi(0) \frac{1 - \pi(1)^n}{1 - \pi(1)}$. Both P and P^n have binomial entries with $P^n(i, i) = \pi(1)^{ni}$. If $\pi(0) + \pi(1) = 1$ (the regular case), $\pi_n(0) + \pi_n(1) = 1$ and P^n is stochastic. If $\pi(0) + \pi(1) < 1$ (the explosive case), $\pi_n(0) + \pi_n(1) < 1$ and P^n is sub-stochastic. To make it stochastic, we can add state $\{\infty\}$ to the state-space and assume that it is absorbing. We can thus complete P to make it stochastic while considering $P(i, \infty) = 1 - \sum_{j=0}^i \binom{i}{j} \pi(0)^{i-j} \pi(1)^j = 1 - (\pi(0) + \pi(1))^i$ and $P(\infty, \infty) = 1$. If $\phi(1) = 1$, such regular pure death process was recently considered by Möhle (2017), revisiting the Greenwood model of infectiousness (Greenwood 1931).

5.5 Resolvent of the θ -linear-fractional processes

With $\delta_{i,j}$ the Kronecker delta, for $i, j \geq 1$, we also obtain the resolvent of $N_n(i)$ as

$$g_{i,j}(z) := \delta_{i,j} + \sum_{n \geq 1} z^n P^n(i, j). \tag{33}$$

In particular

$$g_{i,i}(z) = 1 + \sum_{n \geq 1} z^n P^n(i, i).$$

Note $g_{i,j}(1) = \delta_{i,j} + \mathbf{E}(\sum_{n \geq 1} \mathbf{1}_{\{N_n(i)=j\}})$, the expected value of the time spent on state j starting from i is the Green kernel.

Proposition 14 Using (27), with $F_{i,k}(z) := \sum_{n \geq 1} z^n f_{i,k}^{(n)}$, we get the following tricky expression for the resolvent:

$$g_{i,j}(z) := \delta_{i,j} + \sum_{n \geq 1} z^n P_{a_n, b_n}(i, j) = \delta_{i,j} + \frac{z^c}{j!} \sum_{k=1}^j F_{i,k}(z) B_{j,k}(g_\bullet). \tag{34}$$

These quantities are fundamental to compute pgfs of important quantities, such as passage times. It holds, for example, that $\mathbf{E}(z^{\tau_{i,j}}) = g_{i,j}(z)/g_{j,j}(z)$, where

$$\tau_{i,j} = \inf (n \geq 1 : N_n(i) = j) \tag{35}$$

is the first passage time to state $j \neq i$ of N_n given $N_0 = i$ (Norris 1998; Woess 2009). In particular, $\mathbf{P}(\tau_{i,j} < \infty) = g_{i,j}(1)/g_{j,j}(1)$ are the hitting probabilities of state j starting from i . Furthermore, with

$$\tau_{i,i}^* = \inf (n \geq 1 : N_n(i) = i), \tag{36}$$

the first return time to state i of $N_n(i)$, it holds by renewal arguments that $\mathbf{E}(z^{\tau_{i,i}^*}) = 1 - 1/g_{i,i}(z)$ (Norris 1998). In particular, $\mathbf{P}(\tau_{i,i}^* < \infty) = 1 - 1/g_{i,i}(1)$. Therefore, for example, the mean return time to state i given $\tau_{i,i}^* < \infty$ is

$$\mathbf{E}(\tau_{i,i}^* | \tau_{i,i}^* < \infty) = \frac{g'_{i,i}(1)}{g_{i,i}(1)(g_{i,i}(1) - 1)}, \tag{37}$$

whenever this quantity exists.

Let us briefly sketch what this says for the simplest Greenwood model example when $\theta = -1$: first, $P^n(i, i) = \pi_n(1)^j$ leading to $g_{i,i}(z) = 1 + \sum_{n \geq 1} z^n \pi(1)^{ni} = 1/(1 - z\pi(1)^i)$. Therefore, $\mathbf{E}(z^{\tau_{i,i}^*}) = z\pi(1)^i$, translating the fact that $\tau_{i,i}^* = 1$ with probability $\pi(1)^i$, $= \infty$ with probability $1 - \pi(1)^i$ (the no return to i event if in the first step, one of the i founders moved to one of the absorbing states, 0 or ∞). In addition, in the regular case $\pi(0) = 1 - \pi(1)$

$$\begin{aligned} \mathbf{P}(\tau_{i,j} < \infty) &= g_{i,j}(1) / g_{j,j}(1) \\ &= (1 - \pi(1)^j) \left(1 + \sum_{n \geq 1} \binom{i}{j} (1 - \pi(1)^n)^{i-j} \pi(1)^{nj} \right), \end{aligned}$$

which, upon developing $(1 - \pi(1)^n)^{i-j}$ and summing over n is Proposition 1.1 and Theorem 1.2 of Möhle (2017).

6 One illustrative example

As an illustrative application of the previous results, let us look for the value of n for which a supercritical process as in (C) will nearly never (with large probability c) go extinct as soon as $N_n(i) > 0$. It is given by (6)

$$c = \frac{1 - \rho^i}{1 - \phi^{on}(0)^i} =: 1 - \epsilon.$$

When ϵ is small, it leads to

$$\rho - \phi^{on}(0) \approx \epsilon \cdot \frac{1 - \rho^i}{i\rho^{i-1}}. \tag{38}$$

The condition for this Taylor expansion to be valid is given² by $\rho - \phi^{on}(0) \ll \rho/(i - 1)$ or alternatively

$$\epsilon \ll \frac{\rho^i}{1 - \rho^i}.$$

The relation (38) shows that for small ϵ , having i founders amounts simply to multiply ϵ by a factor that only depends on the value of the fixed point ρ and the number i . Let us now use the explicit form of the supercritical θ -linear-fractional pgfs. In that case, it holds that

$$a^n = \frac{(z_c - \phi^{on}(z))^{-\theta} - (z_c - \rho)^{-\theta}}{(z_c - z)^{-\theta} - (z_c - \rho)^{-\theta}}.$$

A Taylor expansion of $(z_c - \phi^{on}(z))^{-\theta}$ for small $\rho - \phi^{on}(0)$ yields

$$-\theta \frac{\rho - \phi^{on}(z)}{\rho - z_c} \approx a^n \left(1 - \left(\frac{z_c - z}{z_c - \rho} \right)^{-\theta} \right),$$

the Taylor expansion validity condition being $|\rho - \phi^{on}(z)| \ll |(z_c - \rho)/\theta|$. Combined with the previous result, we get

$$a^n \approx \epsilon \cdot \frac{1 - \rho^i}{i\rho^i} \cdot (-\theta) \frac{1 - \alpha}{1 - \alpha^{-\theta}},$$

with $\alpha := z_c/(z_c - \rho) > 1$; the validity conditions are

$$\epsilon \ll \frac{\rho^i}{1 - \rho^i} \quad \text{and} \quad \epsilon \ll \frac{i\rho^i}{|\theta|(\alpha - 1)}. \tag{39}$$

This shows that the searched value of n for which a supercritical process as in (C) will nearly never go extinct is approximately the sum of three terms:

- one, related to the required accuracy, that is the logarithm of ϵ in base a . In particular, to have a result ten times more precise, one has to wait $|\log_a(10)|$ more steps,

² It is assumed here that $i > 1$. If $i = 1$, the condition on $\rho - \phi^{on}(0)$ is no longer valid, but the one on ϵ still is.

- one, related to the number i of founders, which also depends on the parameters a and ρ ,
- one, related to the model parameters only, which depends on a , θ , and z_c and ρ through α .

For instance, taking $a = 0.63$ (so that $a^5 \approx 0.1$), $\rho = 0.7$, $z_c = 1$, we get:

- when $\theta = +1$ and $i = 1$, 9 generations are needed if the population is to survive with a probability $1 - 10^{-2}$. 5 more generations will increase this probability to $1 - 10^{-3}$, and another 5 to $1 - 10^{-4}$.
- when $\theta = +1$, with an uncertainty $\epsilon = 10^{-4}$ and eight founders, the time to wait decreases to 16 generations. With thirteen founders, it decreases further to 13 generations. Notice that $\rho^{19} \approx 0.001$, so one has to be careful not to get out of the range of (39).
- with one founder and an uncertainty 10^{-4} , 19 generations are needed for $\theta = 1$, 20 generations for the limit $\theta = 0$ and 21 for $\theta = -1$.

In all these special cases, we conclude that if extinction is to occur, it occurs rapidly or nearly never.

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