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On the coverage probabilities of parametric confidence bands for continuous distribution and quantile functions constructed via confidence regions for a location-scale parameter

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Abstract In parametric statistics, confidence bands for continuous distribution (quantile) functions may be constructed by unifying the graphs of all distribution (quantile) functions corresponding to parameters lying in some confidence region. It is then desirable that the coverage probabilities of both, band and region, coincide, e.g., to prevent from wide and less informative bands or to transfer the property of unbiasedness; this is ensured if the confidence region is exhaustive. Properties and representations of exhaustive confidence regions are presented. In location-scale families, the property of some confidence region to be exhaustive depends on the boundedness of the supports of the distributions in the family. For unbounded, one-sided bounded and bounded supports, characterizations of exhaustive confidence regions are derived. The results are useful to decide whether the trapezoidal confidence regions based on the standard pivotal quantities are exhaustive and may serve to construct exhaustive confidence regions in (log-)location-scale models.

Keywords Comprehensive convex hull · Confidence band · Confidence region · Coverage probability · Location-scale family · Simultaneous confidence intervals

1 Introduction

In applications of parametric statistical models as, for instance, lifetime experiments in reliability, one is often interested in estimating the unknown cumulative distribution

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function (cdf) of some continuous quantity based on a given data set, rather than in estimating the underlying parameters themselves. For instance, the task to be solved might be to derive a random set in the (x, y)-plane formed like a band which contains the entire true cdf with some specified probability. In the literature, these sets are usually referred to as confidence bands, and several procedures are proposed for their construction by making use of the assumed parametric structure of the model in different sense. One method is presented by Kanofsky and Srinivasan (1972) using so-called Kolmogorov–Smirnov type statistics which are defined as the supremum norm of the difference of the true and estimated cdf, where the latter is obtained by plug-in of some estimator for the parameter. If such a statistic then forms a pivotal quantity, it may serve to construct a confidence band for the true cdf of a desired level upon choosing respective quantiles of its distribution. The resulting confidence band is centered around the estimated cdf and has the same vertical width all over the abscissa, even in the tails of the distribution. Hence, it will include points in the plane which do not lie on any graph of some cdf being entirely contained in the confidence band. Thus, this approach will usually require some subsequent procedure to remove these points from the confidence band. The detailed method along with results for the normal distribution can be found in the articles of Kanofsky and Srinivasan (1972) and Srinivasan and Wharton (1973, 1976). The case of the Weibull distribution is studied by Srinivasan and Wharton (1975).

In this paper, we focus on the approach of Kanofsky (1968a, b) to the construction of confidence bands for continuous cdfs, which is based on the use of (joint) confidence regions for the underlying parameters. Here, the confidence band for the true cdf is obtained by unifying the graphs of all cdfs corresponding to parameters lying in the confidence region for the true parameter. In contrast to the approach via Kolmogorov–Smirnov type statistics, the method does not require any 'cut-away-procedure' applied to the resulting confidence band. Moreover, it is evident from the construction that the confidence band will contain the true cdf at least with the probability that the true parameter lies in the confidence region. It is then of some interest to guarantee equal coverage probabilities of both, confidence band and region, which brings along several pleasant properties as we will point out in Sect. 2.1 in detail. Obviously, this is met if, with probability 1, no parameter lying outside the confidence region corresponds to a cdf whose graph is entirely covered by the confidence band, in the case of which the confidence region is called 'exhaustive' (see Kanofsky 1968b). In general, i.e., without additional structural model assumptions, it will be difficult to decide whether a given confidence region is exhaustive or not. In the location-scale (l-s) family, however, this problem is manageable, and there are results in the literature concerning the general or particular l-s families which are summarized in what follows.

For the l-s family generated by the cdf F, say, Kanofsky (1968b) states that convex confidence regions will often be exhaustive and that this property depends on the strict monotonicity of F: for the class of normal distributions with support $\text{supp}(F) = \mathbb{R} = (-\infty, \infty)$, every convex confidence region is exhaustive, whereas this is not generally the case for distribution families with range parameters as, e.g., the two-parameter exponential distribution. These two particular l-s families are extensively studied by Kanofsky (1968a) and Srinivasan et al. (1975), where goodness criteria of confidence bands are also discussed. Based on some elliptically shaped confidence region, Cheng and Iles (1983) construct confidence bands for the cdf in the l-s family, and the normal and Gumbel distribution are treated in detail. Therein, at the end of Sect. 3.1, it is stated that the considered confidence region is exhaustive. The proof of this statement, however, is essentially based on the fact that $\text{supp}(F) = \mathbb{R}$ (as it is the case for the two examples), which seems to be a missing assumption, here. The authors, moreover, discuss the construction of confidence bands for the cdf of the log-normal and Weibull distributions, which form particular log-location-scale families. In the study by Cheng and Iles (1988), the method is extended to the construction of one-sided confidence bands. With a focus on the construction of confidence bands for the quantile function (qf), Satten (1995) gives geometrical arguments why convex and compact confidence regions are exhaustive in the l-s family, which require the additional assumption that $\text{supp}(F) = \mathbb{R}$. It is also described how to obtain a confidence band for the cdf from one for the qf. In Czarnowska and Nagaev (2001, Section 3), confidence bands for the qf in the l-s family are derived which are obtained from (compact) confidence regions with minimum Lebesgue measure (among all confidence regions based on the same pivotal quantity). In case of $\text{supp}(F) = \mathbb{R}$, the authors prove that these (optimal) confidence regions are exhaustive provided that they are convex. Moreover, they state that if $\text{supp}(F)$ has at least one finite boundary point, this conclusion is not true any longer and the exact confidence level of the band then coincides with that of a certain convex superset of the (optimal) confidence region. These findings are illustrated by means of the normal, two-parameter exponential and continuous uniform distribution for which explicit expressions are presented (see Czarnowska and Nagaev 2001, Sections 4–6). Following the approach of Cheng and Iles (1983, 1988), Jeng and Meeker (2001) compare a variety of confidence bands for the cdf in the l-s family based on confidence regions obtained from either Wald or likelihood ratio statistics, where censored data, bootstrap methods and the relation to the corresponding confidence bands for the qf are also discussed (cf. Escobar et al. 2009). Result 2 in Appendix A, therein, states that convex confidence regions are exhaustive in the l-s family, where the assumption $\text{supp}(F) = \mathbb{R}$ is implicitly used in the proof. Likewise, this additional assumption is required in the work of Hong et al. (2010) who show that the convex hull of a compact confidence region is exhaustive in this l-s family (see the proof in the appendix, therein). Finally, we refer to Frey et al. (2009) for the construction of a confidence band with minimum area for the cdf of a normal distribution, the method described may also be applied to other l-s families.

Based on the existing literature on confidence bands for the cdf and qf in the l-s family, there seems to be a need for clarification which confidence regions are indeed exhaustive guaranteeing equal coverage probabilities of the associated bands, and how this property depends on the support of F, the specified standard cdf in the family. This is the main scope of this article, the remaining part of which is organized as follows.

In Sect. 2, the concept of the exhaustive (confidence) set is introduced for a parametric family of continuous cdfs. The methodology along with motivational aspects for the construction of confidence bands for the cdf via confidence regions for the parameters is presented (Sect. 2.1). Elementary properties of exhaustive sets are shown (Sect. 2.2), and a result is established which enables to make use of the findings for the construction of confidence bands for qfs as well (Sect. 2.3). In Sect. 3, characterizations of exhaustive sets for the l-s family are derived, where a case distinction turns out to be required regarding the boundedness of the supports of the cdfs in the family. An example is presented first which demonstrates that a (compact and) convex set is not exhaustive in the l-s family, in general. We then give a rigorous proof of the known fact that compact and convex sets are exhaustive if the support of the cdfs in the l-s family is the full real line (Sect. 3.1) and, moreover, provide characterizations of exhaustive sets subject to the case of one- or two-sided bounded supports (Sects. 3.2 and 3.3). In Sect. 4, we summarize our findings and illustrate the main results by means of the standard trapezoidal confidence regions for the l-s parameter.

2 Confidence bands for a continuous cdf based on confidence regions for the parameters

Let $\Theta \subseteq \mathbb{R}^k$ be a non-empty set of parameters (k-dimensional vectors) and $\mathfrak{F} = \{F_{\boldsymbol{\vartheta}} : \boldsymbol{\vartheta} \in \Theta\}$ be a parametric family of continuous cdfs (on $\mathbb{R}$). For $\boldsymbol{\vartheta} \in \Theta$, the support of $F_{\boldsymbol{\vartheta}}$ is denoted by $\mathrm{supp}(F_{\boldsymbol{\vartheta}}) = \{x \in \mathbb{R} : F_{\boldsymbol{\vartheta}}(x+\delta) > F_{\boldsymbol{\vartheta}}(x-\delta)\ \forall \delta > 0\}$. Throughout the manuscript, we assume that

$$\text{the mapping } \boldsymbol{\vartheta} \mapsto F_{\boldsymbol{\vartheta}}(x) \text{ on } \Theta \text{ is continuous for every } x \in \mathbb{R}. \tag{1}$$

Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables with cdf $F_{\boldsymbol{\vartheta}}$ for some unknown $\boldsymbol{\vartheta} \in \Theta$. We denote by $P_{\boldsymbol{\vartheta}}$ the distribution of $\boldsymbol{X} = (X_1, \ldots, X_n)$ which is defined on the Borel space $(\mathbb{R}^n, \mathcal{B}^n)$. The elements of the sample space (realizations of $\boldsymbol{X}$) are indicated by small letters $\boldsymbol{x} = (x_1, \ldots, x_n)$.

The aim is now to find a (random) set in the (x, y)-plane based on $\boldsymbol{X}$ which contains the entire true (unknown) cdf with some given probability. To this end, we follow the approach of Kanofsky (1968a, b) which is based on the availability of some confidence region for the underlying parameter $\boldsymbol{\vartheta}$. The general method and related notation is introduced in the following subsection.

2.1 The method: definitions and motivation

First, a precise definition of a confidence region for $F_{\boldsymbol{\vartheta}}$ in the (x, y)-plane and its (exact) confidence level is required.

Definition 1 (i). A function $B : \mathbb{R}^n \to \mathrm{Pot}(\mathbb{R}^2) = \{D : D \subseteq \mathbb{R}^2\}$ is called a confidence region for $F_{\boldsymbol{\vartheta}}$ if $\{\boldsymbol{x} \in \mathbb{R}^n : \mathrm{graph}\, F_{\boldsymbol{\vartheta}} \subseteq B(\boldsymbol{x})\} \in \mathcal{B}^n$ for all $\boldsymbol{\vartheta} \in \Theta$, where $\mathrm{graph}\, F_{\boldsymbol{\vartheta}} = \{(x, F_{\boldsymbol{\vartheta}}(x)) : x \in \mathbb{R}\}$ denotes the graph of $F_{\boldsymbol{\vartheta}}$.

(ii). A confidence region B for $F_{\boldsymbol{\vartheta}}$ is said to have (exact) confidence level $p \in (0, 1)$ if

$$P_{\boldsymbol{\vartheta}}\left(\{\boldsymbol{x} \in \mathbb{R}^n : \mathrm{graph}\, F_{\boldsymbol{\vartheta}} \subseteq B(\boldsymbol{x})\}\right) \overset{(=)}{\geq} p \qquad \forall\, \boldsymbol{\vartheta} \in \Theta.$$

Now, we define the concept of the (confidence) band (see Kanofsky 1968a, b).

Definition 2 (i). A set $B \subseteq \mathbb{R}^2$ is called a band if $\{y \in \mathbb{R} : (x, y) \in B\}$ forms an interval for every $x \in \mathbb{R}$.

(ii). A confidence region B for $F_{\boldsymbol{\vartheta}}$ is called a confidence band (for $F_{\boldsymbol{\vartheta}}$) if, for every $\boldsymbol{\vartheta} \in \Theta$, $B(\boldsymbol{x})$ is a band in the sense of (i) for $P_{\boldsymbol{\vartheta}}$-almost all ($P_{\boldsymbol{\vartheta}}$-a.a.) $\boldsymbol{x} \in \mathbb{R}^n$.

If some confidence region for the true parameter $\boldsymbol{\vartheta}$ is available, an intuitive approach to the construction of a confidence band for the true cdf $F_{\boldsymbol{\vartheta}}$ is to unify the graphs of all cdfs corresponding to parameters in the confidence region. This method is formalized in the following definition (cf. Kanofsky 1968a, b).

Definition 3 (i). For $C \subseteq \Theta$, let

$$B_C = \bigcup_{\boldsymbol{\vartheta} \in C} \operatorname{graph} F_{\boldsymbol{\vartheta}} = \bigcup_{\boldsymbol{\vartheta} \in C} \{(x, F_{\boldsymbol{\vartheta}}(x)) : x \in \mathbb{R}\}. \tag{2}$$

If C is path-connected, i.e., if any two points $a, b \in C$ can be joined in C via a continuous function $f : [0, 1] \to C$ with $f(0) = a$ and $f(1) = b$, B_C is said to be the band based on C.

(ii). Let the confidence region $C : \mathbb{R}^n \to \operatorname{Pot}(\Theta)$ be path-connected, i.e., for every $\boldsymbol{\vartheta} \in \Theta$, $C(\boldsymbol{x})$ is path-connected for $P_{\boldsymbol{\vartheta}}$-a.a. $\boldsymbol{x} \in \mathbb{R}^n$. Then $B_C : \mathbb{R}^n \to \operatorname{Pot}(\mathbb{R}^2) : \boldsymbol{x} \mapsto B_{C(\boldsymbol{x})}$ is said to be the confidence band based on C if it meets the measurability condition in Definition 1 (i).

The assumption that C ($C(\boldsymbol{x})$) is path-connected (for $P_{\boldsymbol{\vartheta}}$-a.a. $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{\vartheta} \in \Theta$) in combination with property (1) ensures that B_C is, in fact, a (confidence) band according to Definition 2 (see Kanofsky 1968b).

Let us compare the coverage probabilities of some path-connected confidence region C, say, with those of the corresponding band B_C. By construction, $\boldsymbol{\vartheta} \in C(\boldsymbol{x})$ implies graph $F_{\boldsymbol{\vartheta}} \subseteq B_{C(\boldsymbol{x})}$ for every $\boldsymbol{x} \in \mathbb{R}^n$ which yields that

$$P_{\tilde{\boldsymbol{\vartheta}}}\left(\{\boldsymbol{x} \in \mathbb{R}^n : \operatorname{graph} F_{\boldsymbol{\vartheta}} \subseteq B_{C(\boldsymbol{x})}\}\right) \geq P_{\tilde{\boldsymbol{\vartheta}}}\left(\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\vartheta} \in C(\boldsymbol{x})\}\right) \tag{3}$$

for all $\boldsymbol{\vartheta}, \tilde{\boldsymbol{\vartheta}} \in \Theta$. In particular, if C has confidence level $p \in (0, 1)$, then B_C has confidence level p and may thus be considered a conservative band.

However, there are different reasons why one may also be interested in guaranteeing that

$$\operatorname{graph} F_{\boldsymbol{\vartheta}} \subseteq B_{C(\boldsymbol{x})} \quad \Rightarrow \quad \boldsymbol{\vartheta} \in C(\boldsymbol{x}) \qquad \text{for every } \boldsymbol{x} \in \mathbb{R}^n \tag{4}$$

which then leads to equality in (3). First, without imposing condition (4) it may happen that, although choosing a suitable (exact) confidence level for C, B_C has confidence level (too) close to 1, so that the method might produce wide and thus less informative bands for $F_{\boldsymbol{\vartheta}}$ which are unusable in applications. Second, provided that condition (4) is satisfied, the property of unbiasedness of C will transfer to B_C (cf. Kanofsky 1968b). Here, C (B_C) is said to be unbiased if for every (false) parameter $\boldsymbol{\vartheta} \neq \tilde{\boldsymbol{\vartheta}}$ the $P_{\tilde{\boldsymbol{\vartheta}}}$-coverage probability of $\boldsymbol{\vartheta}$ (graph $F_{\boldsymbol{\vartheta}}$) in (3) is bounded from above by the (exact) confidence level of C (B_C). Furthermore, when constructing confidence regions by use of pivotal quantities, one is in the desirable situation that the $P_{\boldsymbol{\vartheta}}$-coverage probability of $\boldsymbol{\vartheta}$ is the same for all $\boldsymbol{\vartheta} \in \Theta$ as it will be the case for the confidence band and graph $F_{\boldsymbol{\vartheta}}$ if condition (4) is met. Finally, assumption (4) will make the measurability condition in Definition 3 (ii) superfluous, since the sets $\{\boldsymbol{x} \in \mathbb{R}^n : \text{graph } F_{\boldsymbol{\vartheta}} \subseteq B_{C(\boldsymbol{x})}\}$ and $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\vartheta} \in C(\boldsymbol{x})\}$ then coincide, and the latter is a borel set by definition of a confidence region.

We are aiming at deriving necessary and sufficient conditions on the confidence region for the parameter under which statement (4) is true. For this purpose, we introduce the concept of the exhaustive set (cf. Kanofsky 1968b).

Definition 4 (i). A set $C \subseteq \Theta$ is called exhaustive (for $\mathfrak{F}$) if $C = C^*$, where

$$C^* = \{\boldsymbol{\vartheta} \in \Theta : \text{graph } F_{\boldsymbol{\vartheta}} \subseteq B_C\}. \tag{5}$$

(ii). A confidence region $C : \mathbb{R}^n \to \text{Pot}(\Theta)$ is called exhaustive (for $\mathfrak{F}$) if, for every $\boldsymbol{\vartheta} \in \Theta$, $C(\boldsymbol{x})$ is exhaustive for $\mathfrak{F}$ in the sense of (i) for $P_{\boldsymbol{\vartheta}}$-a.a. $\boldsymbol{x} \in \mathbb{R}^n$.

Remark 1 By definition, $\emptyset$ and Θ are exhaustive sets and $C \subseteq C^*$ ($C(\boldsymbol{x}) \subseteq (C(\boldsymbol{x}))^*$, $\boldsymbol{x} \in \mathbb{R}^n$), for every set (confidence region) C.

Now, if some confidence region C is path-connected and exhaustive, B_C will form a confidence band satisfying equality in (3) along with the above pleasant properties, since the statements $\boldsymbol{\vartheta} \in C(\boldsymbol{x})$ and graph $F_{\boldsymbol{\vartheta}} \subseteq B_{C(\boldsymbol{x})}$ are then equivalent for $P_{\tilde{\boldsymbol{\vartheta}}}$-a.a. $\boldsymbol{x} \in \mathbb{R}^n$, $\tilde{\boldsymbol{\vartheta}} \in \Theta$. In particular, this is the case if $C(\boldsymbol{x})$ is path-connected and exhaustive for all $\boldsymbol{x} \in \mathbb{R}^n$. In what follows, it will be convenient to focus on the deterministic case (pointwise analysis of confidence regions) fading probabilistic aspects into the background.

2.2 Properties and representations of exhaustive sets

We continue by showing some elementary properties of exhaustive sets, first.

Lemma 1 *Let $C, C_i \subseteq \Theta$ for $i \in I$, where I denotes an arbitrary index set. Then holds*:

(i) $(C^*)^* = C^*$, *i.e., C^* is exhaustive.*
(ii) *If $C_1 \subseteq C_2$, then $C_1^* \subseteq C_2^*$.*
(iii) *If C_i is exhaustive for all $i \in I$, then $\cap_{i \in I} C_i$ is exhaustive.*
(iv) $C^* = \bigcap_{D \supseteq C, D = D^*} D$, *i.e., C^* is the smallest exhaustive superset of C.*

Proof (i). From formulas (2) and (5) along with Remark 1, we obtain $B_C = B_{C^*}$ and thus $(C^*)^* = C^*$.

(ii). Since $C_1 \subseteq C_2$ implies that $B_{C_1} \subseteq B_{C_2}$, the statement is obvious from formula (5).

(iii). Statement (ii) together with the assumption yields that $\left(\bigcap_{i \in I} C_i\right)^* \subseteq C_j^* = C_j$ for all $j \in I$ and thus $\left(\bigcap_{i \in I} C_i\right)^* \subseteq \bigcap_{i \in I} C_i$. The equality of both sets then follows by Remark 1.

(iv). With $A = \bigcap_{D \supseteq C, D = D^*} D$, statements (i)–(iii) and Remark 1 yield $C^* \subseteq A^* = A \subseteq C^*$. ◻

Important examples of exhaustive sets are the lower and upper coverings in the points of the plane, which are introduced in Kanofsky (1968b).

Definition 5 The sets $L(x, y) = \{\boldsymbol{\vartheta} \in \Theta : F_{\boldsymbol{\vartheta}}(x) \leq y\}$ and $U(x, y) = \{\boldsymbol{\vartheta} \in \Theta : F_{\boldsymbol{\vartheta}}(x) \geq y\}$ are called lower and upper covering in $(x, y) \in \mathbb{R}^2$. Moreover, $\mathcal{L} = \{L(x, y) : (x, y) \in \mathbb{R}^2\}$ and $\mathcal{U} = \{U(x, y) : (x, y) \in \mathbb{R}^2\}$ denote the families of all lower and upper coverings, respectively.

Lemma 2 *Every set in $\mathcal{L} \cup \mathcal{U}$ is exhaustive.*

Proof Let $(x, y) \in \mathbb{R}^2$ and $L = L(x, y)$. By Remark 1, $L \subseteq L^*$. If $\boldsymbol{\vartheta} \notin L$, then $F_{\boldsymbol{\vartheta}}(x) > y$ which implies that graph $F_{\boldsymbol{\vartheta}} \not\subseteq B_L$ (see formula (2)) and thus $\boldsymbol{\vartheta} \notin L^*$. Hence, $L = L^*$. Analogously, the statement can be shown for upper coverings. □

There exists a useful representation of C^* as the intersection of particular lower and upper coverings (cf. Kanofsky 1968b).

Lemma 3 *Let $C \subseteq \Theta$ be path-connected and compact. Then we have $C^* = \bigcap_{x \in \mathbb{R}} L(x, M_x) \cap U(x, m_x)$ with $m_x = \min_{\boldsymbol{\vartheta} \in C} F_{\boldsymbol{\vartheta}}(x)$ and $M_x = \max_{\boldsymbol{\vartheta} \in C} F_{\boldsymbol{\vartheta}}(x)$.*

Proof First, since C is compact, m_x and M_x are well defined in virtue of property (1) with $(x, m_x), (x, M_x) \in B_C$ by formula (2) for every $x \in \mathbb{R}$.

Let $\boldsymbol{\vartheta} \in \Theta$ satisfy $F_{\boldsymbol{\vartheta}}(x) \in [m_x, M_x]$ for all $x \in \mathbb{R}$. Since B_C is a band, $\{y \in \mathbb{R} : (x, y) \in B_C\}$ forms an interval which contains m_x and M_x and necessarily the whole interval $[m_x, M_x]$ for $x \in \mathbb{R}$. Hence, $(x, F_{\boldsymbol{\vartheta}}(x)) \in B_C$ for $x \in \mathbb{R}$, and, therefore, $\boldsymbol{\vartheta} \in C^*$.

On the other hand, the intersection of coverings is exhaustive by application of Lemma 2 and Lemma 1 (iii) and contains C and, by Lemma 1 (iv), C^*. □

From Lemma 3 we immediately obtain the following characterization result.

Corollary 1 *Let $C \subseteq \Theta$ be path-connected and compact. Then, C is exhaustive if and only if (iff) C is an intersection of lower and upper coverings.*

Proof If C is of the mentioned form, it is exhaustive by Lemma 2 and Lemma 1 (iii). Vice versa, if $C = C^*$, the assertion follows from Lemma 3. □

2.3 Bands for the qf

In applications, one might also be interested in constructing a confidence band for the true qf $F_{\boldsymbol{\vartheta}}^{-1}(y) = \inf\{x \in \mathbb{R} : F_{\boldsymbol{\vartheta}}(x) \geq y\}$, $y \in (0, 1)$, which then provides simultaneous confidence intervals for all quantiles of the underlying distribution. To this end, given some path-connected confidence region for the parameters, one may proceed as in (2) and unify the graphs of all qfs corresponding to parameters lying in the confidence region. Again, the question arises whether the resulting band for the qf has the same coverage probabilities and, in particular, the same (exact) confidence level as the confidence region. This is subject of the following theorem which points out a useful relation of the bands for the cdf and qf based on the same confidence region.

Theorem 1 *Let $C \subseteq \Theta$ be path-connected and compact. Moreover, let the support of $F_{\boldsymbol{\vartheta}}$ be a (not necessarily finite) interval for all $\boldsymbol{\vartheta} \in \Theta$. Then,*

$$\begin{aligned} & \textit{graph}\ F_{\boldsymbol{\vartheta}}^{-1} \subseteq Q_C \quad \Leftrightarrow \quad \textit{graph}\ F_{\boldsymbol{\vartheta}} \subseteq B_C, \qquad (6)\\ \textit{where} \qquad & Q_C = \bigcup_{\boldsymbol{\vartheta} \in C} \textit{graph}\ F_{\boldsymbol{\vartheta}}^{-1} = \bigcup_{\boldsymbol{\vartheta} \in C} \{(y, F_{\boldsymbol{\vartheta}}^{-1}(y)) : y \in (0,1)\}. \end{aligned}$$

In particular, C is exhaustive for $\{F_{\boldsymbol{\vartheta}}^{-1} : \vartheta \in \Theta\}$ iff it is exhaustive for $\mathfrak{F}$.

Proof By assumption, we have for every $\boldsymbol{\vartheta} \in \Theta$ that $F_{\boldsymbol{\vartheta}}^{-1}$ is continuous and coincides with the usual inverse function of $F_{\boldsymbol{\vartheta}}$ restricted to the interval $(F_{\boldsymbol{\vartheta}}^{-1}(0+), F_{\boldsymbol{\vartheta}}^{-1}(1-))$, where $F_{\boldsymbol{\vartheta}}^{-1}(0+) = \lim_{y \searrow 0} F_{\boldsymbol{\vartheta}}^{-1}(y)$ and $F_{\boldsymbol{\vartheta}}^{-1}(1-) = \lim_{y \nearrow 1} F_{\boldsymbol{\vartheta}}^{-1}(y)$. For $\boldsymbol{\vartheta} \in \Theta$, we, therefore, conclude that

$$\begin{aligned} & \text{graph}\ F_{\boldsymbol{\vartheta}}^{-1} \subseteq Q_C \\ \Leftrightarrow \quad & \forall\, y \in (0,1) \ \ \exists\, \tilde{\boldsymbol{\vartheta}} \in C \ : \ F_{\tilde{\boldsymbol{\vartheta}}}^{-1}(y) = F_{\boldsymbol{\vartheta}}^{-1}(y) \\ \Leftrightarrow \quad & \forall\, y \in (0,1) \ \ \exists\, \tilde{\boldsymbol{\vartheta}} \in C \ : \ F_{\tilde{\boldsymbol{\vartheta}}}(F_{\boldsymbol{\vartheta}}^{-1}(y)) = F_{\boldsymbol{\vartheta}}(F_{\boldsymbol{\vartheta}}^{-1}(y)) \quad (= y) \\ \Leftrightarrow \quad & \forall\, x \in (F_{\boldsymbol{\vartheta}}^{-1}(0+), F_{\boldsymbol{\vartheta}}^{-1}(1-)) \ \ \exists\, \tilde{\boldsymbol{\vartheta}} \in C \ : \ F_{\tilde{\boldsymbol{\vartheta}}}(x) = F_{\boldsymbol{\vartheta}}(x) \qquad (7)\\ \Leftarrow \quad & \forall\, x \in \mathbb{R} \ \ \exists\, \tilde{\boldsymbol{\vartheta}} \in C \ : \ F_{\tilde{\boldsymbol{\vartheta}}}(x) = F_{\boldsymbol{\vartheta}}(x) \qquad (8)\\ \Leftrightarrow \quad & \text{graph}\ F_{\boldsymbol{\vartheta}} \subseteq B_C, \end{aligned}$$

and what is left to show is that (7) implies (8). To this end, we assume statement (7) to be true. For brevity, let $s = F_{\boldsymbol{\vartheta}}^{-1}(0+)$ and $t = F_{\boldsymbol{\vartheta}}^{-1}(1-)$. Since $F_{\boldsymbol{\vartheta}}$ is continuous, $F_{\boldsymbol{\vartheta}}(s) = 0$ and $F_{\boldsymbol{\vartheta}}(t) = 1$, which implies that $F_{\boldsymbol{\vartheta}}(x) = 0$ for $x \leq s$ and $F_{\boldsymbol{\vartheta}}(x) = 1$ for $x \geq t$. Thus, we have to show that for all $x \leq s$ ($x \geq t$) there exists $\tilde{\boldsymbol{\vartheta}} \in C$ with $F_{\tilde{\boldsymbol{\vartheta}}}(x) = 0$ ($F_{\tilde{\boldsymbol{\vartheta}}}(x) = 1$). A sufficient condition for this property is the existence of $\boldsymbol{\eta}, \boldsymbol{\zeta} \in C$ with $F_{\boldsymbol{\eta}}(s) = 0$ and $F_{\boldsymbol{\zeta}}(t) = 1$ which is guaranteed by the compactness of C as we will see in the following.

Let x_n, $n \in \mathbb{N}$, be a sequence in (s, t) with $x_n \nearrow t$ for $n \to \infty$. By assumption, there exists $\tilde{\boldsymbol{\vartheta}}_n \in C$ with $F_{\tilde{\boldsymbol{\vartheta}}_n}(x_n) = F_{\boldsymbol{\vartheta}}(x_n)$ for all $n \in \mathbb{N}$. Since C is compact, there exists $\boldsymbol{\zeta} \in C$ and a subsequence $\tilde{\boldsymbol{\vartheta}}_{n_k}$, $k \in \mathbb{N}$, with $\tilde{\boldsymbol{\vartheta}}_{n_k} \to \boldsymbol{\zeta}$ for $k \to \infty$. Then,

$$\begin{aligned} & F_{\tilde{\boldsymbol{\vartheta}}_{n_k}}(x_{n_k}) = F_{\boldsymbol{\vartheta}}(x_{n_k}) \to F_{\boldsymbol{\vartheta}}(t) = 1 \qquad \text{for } k \to \infty, \\ \text{and} \qquad & F_{\boldsymbol{\zeta}}(t) = \lim_{k \to \infty} F_{\tilde{\boldsymbol{\vartheta}}_{n_k}}(t) \geq \lim_{k \to \infty} F_{\tilde{\boldsymbol{\vartheta}}_{n_k}}(x_{n_k}) = 1, \end{aligned}$$

using property (1). Thus, $F_{\boldsymbol{\zeta}}(t) = 1$. The existence of $\boldsymbol{\eta} \in C$ with $F_{\boldsymbol{\eta}}(s) = 0$ follows analogously. Hence, the equivalence in (6) is shown from which the last statement of the theorem is obvious. □

Theorem 1 allows for using the derived results, in particular those of Sect. 3, for the construction of confidence bands for qfs as well.

3 Confidence bands for the continuous cdf in the l-s family

We now turn to the discussion of exhaustive sets in the particular framework of the l-s family.

Let the statistical model and the sample situation be given as in the beginning of Sect. 2 along with the assumptions made there. In addition, we assume that $\Theta = \mathbb{R} \times (0, \infty)$ and that, for $\boldsymbol{\vartheta} = (\mu, \sigma) \in \Theta$,

$$F_{\boldsymbol{\vartheta}}(x) = F\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}, \tag{9}$$

for some continuous cdf $F = F_{(0,1)}$ (independent of μ and σ), i.e., $\mathfrak{F} = \{F_{\boldsymbol{\vartheta}} : \boldsymbol{\vartheta} \in \Theta\}$ forms a l-s family. F will be referred to as the standard cdf in $\mathfrak{F}$.

Examples of l-s families with a continuous standard cdf are the (univariate) normal distributions with mean μ and standard deviation σ, the two-parameter exponential distributions with initial point μ and scale parameter σ, and the (continuous) uniform distributions with initial point μ and range σ. We will see that, in each case, the family of exhaustive sets will be different, and that this finding is connected with the character of the support of the standard (or any other) cdf in $\mathfrak{F}$ (see Kanofsky 1968b). Here, the crucial point is whether $\operatorname{supp}(F)$ is (left- or right-) bounded or not, whereas the exact boundary values themselves, which if they exist may depend on the choice of F, are not critical to the same extent. Indeed, when $\mathfrak{F}$ is parametrized as in (9) with standard cdf $F_{(\mu_0,\sigma_0)}$, say, the family of the corresponding exhaustive sets may be obtained via application of the (bijective) parameter transformation $(\mu, \sigma) \mapsto (\mu - \mu_0\sigma/\sigma_0, \sigma/\sigma_0)$ to the exhaustive sets for $\mathfrak{F}$ with standard cdf F (and vice versa). Upon assuming that the support of the standard cdf F of the l-s family forms an interval, we may, therefore, restrict ourselves to the three cases that $\operatorname{supp}(F)$ is given by either $(-\infty, \infty)$, $[0, \infty)$, or $[-1, 1]$. Note that the case $\operatorname{supp}(F) = (-\infty, 0]$ can be reduced to the one with left-bounded support by considering the cdf of $-X_1$ which lies in the l-s family with standard cdf $G(x) = 1 - F(-x)$, $x \in \mathbb{R}$.

Before we give explicit results and characterizations of exhaustive sets in the three cases, we present an example which highlights that a compact and convex set will not be exhaustive, in general.

Example 1 Let $X_1, \ldots, X_n$ be iid random variables following the two-parameter exponential distribution with location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$, and cdf $F_{\boldsymbol{\vartheta}}(x) = 1 - \exp\{-(x - \mu)/\sigma\}$, $x > \mu$, where $\boldsymbol{\vartheta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times (0, \infty)$. Denoting by $x_{(1)} \leq \cdots \leq x_{(n)}$ the ordered components of $\boldsymbol{x}$, the maximum likelihood estimator $(\hat{\mu}, \hat{\sigma})$ of (μ, σ) is given by $\hat{\mu}(\boldsymbol{x}) = x_{(1)}$ and $\hat{\sigma}(\boldsymbol{x}) = (\sum_{i=1}^{n} x_{(i)} - n x_{(1)})/n$, where $\hat{\mu}$ and $\hat{\sigma}$ are independent (see, e.g., Balakrishnan and Basu 1995, Subsection 6.3.1). We consider the (random) line $C : \mathbb{R}^n \to \operatorname{Pot}(\Theta)$ defined by

$$C(\boldsymbol{x}) = \left\{\boldsymbol{\vartheta} \in \Theta : \quad \mu = \hat{\mu}(\boldsymbol{x}) - b\,\hat{\sigma}(\boldsymbol{x}) + \sigma\ , \ a\,\hat{\sigma}(\boldsymbol{x}) \leq \sigma \leq b\,\hat{\sigma}(\boldsymbol{x})\right\}$$

for $\boldsymbol{x} \in \mathbb{R}^n$, where a and b are fixed numbers with $0 < a < b$. For $\boldsymbol{x} \in \mathbb{R}^n$, the end points of $C(\boldsymbol{x})$ are given by $(\mu_L(\boldsymbol{x}), \sigma_L(\boldsymbol{x})) = (\hat{\mu}(\boldsymbol{x}) - (b - a)\hat{\sigma}(\boldsymbol{x}), a\hat{\sigma}(\boldsymbol{x}))$ and

$(\mu_U(\boldsymbol{x}), \sigma_U(\boldsymbol{x})) = (\hat{\mu}(\boldsymbol{x}), b\hat{\sigma}(\boldsymbol{x}))$. Since $(\hat{\mu} - \mu + \sigma)/\hat{\sigma}$ is continuously distributed, $P_{\boldsymbol{\vartheta}}(\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\vartheta} \in C(\boldsymbol{x})\}) = 0$ for all $\boldsymbol{\vartheta} \in \Theta$. Now, we introduce the random rectangle $\tilde{C} : \mathbb{R}^n \to \text{Pot}(\Theta)$ via

$$\begin{aligned}\tilde{C}(\boldsymbol{x}) &= [\mu_L(\boldsymbol{x}), \mu_U(\boldsymbol{x})] \times [\sigma_L(\boldsymbol{x}), \sigma_U(\boldsymbol{x})] \\ &= \left\{\boldsymbol{\vartheta} \in \Theta : \quad 0 \leq \frac{\hat{\mu}(\boldsymbol{x}) - \mu}{\hat{\sigma}(\boldsymbol{x})} \leq b - a\,, \ \frac{1}{b} \leq \frac{\hat{\sigma}(\boldsymbol{x})}{\sigma} \leq \frac{1}{a}\right\} \end{aligned} \tag{10}$$

for $\boldsymbol{x} \in \mathbb{R}^n$. By definition, $C(\boldsymbol{x}) \subseteq \tilde{C}(\boldsymbol{x})$ and thus $B_{C(\boldsymbol{x})} \subseteq B_{\tilde{C}(\boldsymbol{x})}$ for every $\boldsymbol{x} \in \mathbb{R}^n$. In fact, both bands coincide for every $\boldsymbol{x} \in \mathbb{R}^n$ as we will demonstrate in the following. Let $(x, y) \in B_{\tilde{C}(\boldsymbol{x})}$ for some $\boldsymbol{x} \in \mathbb{R}^n$. Then, there exists $\boldsymbol{\vartheta} \in \tilde{C}(\boldsymbol{x})$ with $F_{\boldsymbol{\vartheta}}(x) = y$. From the structure of $F_{\boldsymbol{\vartheta}}$, it is obvious that $F_{(\mu_1,\sigma_1)} \geq F_{(\mu_2,\sigma_2)}$ if $\mu_1 \leq \mu_2$ and $\sigma_1 \leq \sigma_2$. Since $C(\boldsymbol{x})$ is path-connected and property (1) is met, $B_{C(\boldsymbol{x})}$ is a band so that $\{z \in \mathbb{R} : (x, z) \in B_{C(\boldsymbol{x})}\}$ forms an interval (see Definition 2). Therefore, necessarily, $y = F_{\boldsymbol{\vartheta}}(x) \in [F_{(\mu_U(\boldsymbol{x}),\sigma_U(\boldsymbol{x}))}(x), F_{(\mu_L(\boldsymbol{x}),\sigma_L(\boldsymbol{x}))}(x)] \subseteq \{z \in \mathbb{R} : (x, z) \in B_{C(\boldsymbol{x})}\}$. Hence, $(x, y) \in B_{C(\boldsymbol{x})}$ and thus $B_{\tilde{C}(\boldsymbol{x})} \subseteq B_{C(\boldsymbol{x})}$ which yields $B_{C(\boldsymbol{x})} = B_{\tilde{C}(\boldsymbol{x})}$, i.e., equality of the bands based on $C(\boldsymbol{x})$ and $\tilde{C}(\boldsymbol{x})$.

As a consequence, we obtain for $\boldsymbol{\vartheta} \in \Theta$ that

$$\begin{aligned} P_{\boldsymbol{\vartheta}}\left(\{\boldsymbol{x} \in \mathbb{R}^n : \text{graph}\, F_{\boldsymbol{\vartheta}} \subseteq B_{C(\boldsymbol{x})}\}\right) &= P_{\boldsymbol{\vartheta}}\left(\{\boldsymbol{x} \in \mathbb{R}^n : \text{graph}\, F_{\boldsymbol{\vartheta}} \subseteq B_{\tilde{C}(\boldsymbol{x})}\}\right) \\ &\geq P_{\boldsymbol{\vartheta}}\left(\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\vartheta} \in \tilde{C}(\boldsymbol{x})\}\right), \end{aligned} \tag{11}$$

where the latter probability does not depend on $\boldsymbol{\vartheta}$ by construction of $\tilde{C}$ via the pivotal quantities $(\hat{\mu} - \mu)/\hat{\sigma}$ and $\hat{\sigma}/\sigma$ (see (10) and Antle and Bain 1969). Thus, if $\tilde{C}$ has exact confidence level $p \in (0, 1)$, which may be chosen arbitrarily close to 1 by appropriate choices of a and b, then B_C has confidence level p. For instance, by setting $a = 0.75$ and $b = 1.2$, (11) yields that B_C almost has exact confidence level 0.95, whereas $P_{\boldsymbol{\vartheta}}(\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\vartheta} \in C(\boldsymbol{x})\}) = 0$ for all $\boldsymbol{\vartheta} \in \Theta$. Later, we will see that $\tilde{C}(\boldsymbol{x})$ is exhaustive as the comprehensive convex hull of $C(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^n$ and, hence, even equality holds true in (11).

3.1 The case supp$(F) = \mathbb{R}$

We start with the case that the support of the standard cdf (and thus of every cdf) in $\mathfrak{F}$ is $\mathbb{R}$, which implies that F is strictly increasing. Our aim is to give a formal proof of that C^* then coincides with the convex hull of C provided that C is path-connected and compact. In particular, under these assumptions on C, convexity turns out to be a necessary and sufficient condition for C to be exhaustive. Former proofs and remarks on this result can be found in the studies by Kanofsky (1968b), Cheng and Iles (1983), Czarnowska and Nagaev (2001), and Jeng and Meeker (2001). Beyond that, the role of the convex hull in this context is discussed by Satten (1995) and Hong et al. (2010). We give a proof here, since it indicates which arguments will be problematic in the case, where supp(F) has (at least) one finite boundary point.

First, it is well known that the lower and upper coverings (see Definition 5) coincide with particular half-spaces (in $\mathbb{R} \times (0, \infty)$) under the assumption that $\mathfrak{F}$ forms a l-s family.

Definition 6 For $a, b, c \in \mathbb{R}$, we introduce a half-space in $\mathbb{R}^2$ as $H(a, b, c) = \left\{(\mu, \sigma) \in \mathbb{R}^2 : a\mu + b\sigma \leq c\right\}$ and the corresponding half-space in Θ via $\tilde{H}(a, b, c) = H(a, b, c) \cap \Theta$.

Lemma 4 *For $a, b, c \in \mathbb{R}$, we find*

$$\tilde{H}(a, b, c) = \begin{cases} U(c/a, F(b/a)) & \textit{for} \quad a > 0, \\ L(c/a, F(b/a)) & \textit{for} \quad a < 0, \end{cases}$$

$$\textit{and, hence,} \qquad \{\emptyset, \Theta\} \cup \{\tilde{H}(a, b, c) : a \neq 0, b, c \in \mathbb{R}\} = \mathcal{L} \cup \mathcal{U}. \qquad (12)$$

Proof Let $a > 0$. Since F is strictly increasing, we find

$$a\mu + b\sigma \leq c \quad \Leftrightarrow \quad \frac{c/a - \mu}{\sigma} \geq \frac{b}{a} \quad \Leftrightarrow \quad F_{\vartheta}\left(\frac{c}{a}\right) \geq F\left(\frac{b}{a}\right). \qquad (13)$$

Hence, $\tilde{H}(a, b, c) = U(c/a, F(b/a))$. The statement for $a < 0$ follows analogously. These identities then establish the equality in (12). □

Hence, every lower or upper covering forms a half-space in Θ but not vice versa. The following corollary is essential for the subsequent theorem.

Corollary 2 *$\tilde{H}(a, b, c)$ is exhaustive if $a \neq 0$.*

Proof The statement is obvious from Lemma 4 and Lemma 2. □

As usual, we define the convex hull $\mathrm{ch}(C)$ of a set C as the smallest convex set that contains C which then suffices the representation $\mathrm{ch}(C) = \bigcap_{D \supseteq C, D \text{ convex}} D$. Now, we give the main result of this subsection.

Theorem 2 *Let $C \subseteq \Theta$ be path-connected and compact. Then,*

(i) *$C^* = ch(C)$,*
(ii) *C is exhaustive iff C is convex.*

Proof (i). Since $C \subseteq C^*$ and $C \subseteq \mathrm{ch}(C)$, it is sufficient to show that C^* is convex and $\mathrm{ch}(C)$ is exhaustive (see Lemma 1 (iv)). First, from Lemma 3 and formula (12) in Lemma 4, C^* is convex as an intersection of convex sets.

In the following, we show that $\tilde{C} = \mathrm{ch}(C)$ is exhaustive. By definition, $\tilde{C}$ is convex, and it is, moreover, a compact subset of Θ, since C is compact (see Rockafellar 1970, Theorem 17.2, p. 158). The separation theorem (see Rockafellar 1970, Corollary 11.4.1, p. 98) then yields that, for every $\vartheta = (\mu, \sigma) \in \mathbb{R}^2 \setminus \tilde{C}$, there exists $a_{\vartheta}, b_{\vartheta}, c_{\vartheta} \in \mathbb{R}$, and $\varepsilon_{\vartheta} > 0$ in such a way that

$$a_{\vartheta}\,\tilde{\mu} + b_{\vartheta}\,\tilde{\sigma} + \varepsilon_{\vartheta} < c_{\vartheta} < a_{\vartheta}\,\mu + b_{\vartheta}\,\sigma - \varepsilon_{\vartheta} \qquad \forall\, (\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}. \qquad (14)$$

If $a_\vartheta = 0$, we switch to another separating line with $\tilde{a}_\vartheta > 0$ as follows. Since $\tilde{C}$ is bounded, there exists a constant $M > 0$ with $|\tilde{\mu}| \leq M$ for all $(\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}$. For $\tilde{a}_\vartheta = \min\{\varepsilon_\vartheta/M, \varepsilon_\vartheta/|\mu|\} > 0$, where $\varepsilon_\vartheta/0 = \infty$, we obtain that

$$\begin{aligned} & \tilde{a}_\vartheta\, \tilde{\mu} + b_\vartheta\, \tilde{\sigma} \leq \tilde{a}_\vartheta\, M + b_\vartheta\, \tilde{\sigma} \leq b_\vartheta\, \tilde{\sigma} + \varepsilon_\vartheta \\ \text{and} \qquad & b_\vartheta\, \sigma - \varepsilon_\vartheta \leq b_\vartheta\, \sigma - \tilde{a}_\vartheta\, |\mu| \leq \tilde{a}_\vartheta\, \mu + b_\vartheta\, \sigma, \end{aligned} \tag{15}$$

and, hence, using (14) with $a_\vartheta = 0$,

$$\tilde{a}_\vartheta\, \tilde{\mu} + b_\vartheta\, \tilde{\sigma} < c_\vartheta < \tilde{a}_\vartheta\, \mu + b_\vartheta\, \sigma \qquad \forall\, (\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}. \tag{16}$$

From (14) and (16), we, therefore, have the representation

$$\tilde{C} = \tilde{C} \cap \Theta = \bigcap_{\vartheta \in \mathbb{R}^2 \setminus \tilde{C} \,:\, a_\vartheta \neq 0} \tilde{H}(a_\vartheta, b_\vartheta, c_\vartheta) \cap \bigcap_{\vartheta \in \mathbb{R}^2 \setminus \tilde{C} \,:\, a_\vartheta = 0} \tilde{H}(\tilde{a}_\vartheta, b_\vartheta, c_\vartheta),$$

and $\tilde{C}$ is exhaustive by application of Lemma 1 (iii) and Corollary 2.

(ii). The equivalence follows directly from statement (i). □

As an example, Theorem 2 may directly be applied to the standard confidence region considered in the work of Kanofsky (1968a) for the parameter (μ, σ) of the normal distribution which is convex and thus exhaustive.

3.2 The case supp$(F) = [0, \infty)$

We continue by considering the case that the support of F is given by the non-negative real axis on which F then strictly increases. As in Sect. 3.1, our aim is to establish the relation of C and C^* and to derive a necessary and sufficient condition for C to be exhaustive under the assumption that C is path-connected and compact. With regard to the latter, Example 1 has shown that convexity alone does not fill this role. However, by utilizing the concept of the comprehensive set, we will present a characterization result on exhaustive sets in what follows, which still admits a simple geometric description.

First, we identify the half-spaces which correspond to lower and upper coverings in the actual case.

Lemma 5 *For $a, b, c \in \mathbb{R}$, we find*

$$\tilde{H}(a, b, c) = \begin{cases} U(c/a, F(b/a)) & \textit{for} \quad a > 0,\ b > 0, \\ L(c/a, F(b/a)) & \textit{for} \quad a < 0,\ b \leq 0, \end{cases}$$

and, hence,

$$\{\emptyset, \Theta\} \cup \{\tilde{H}(a, b, c) \,:\, a, b > 0, c \in \mathbb{R} \;\; \textit{or} \;\; a < 0, b \leq 0, c \in \mathbb{R}\} = \mathcal{L} \cup \mathcal{U}. \tag{17}$$

Proof Let $a > 0$ and $b > 0$. Since F is strictly increasing on $[0, \infty)$ and $b/a > 0$, the equivalences in (13) remain true and thus $\tilde{H}(a, b, c) = U(c/a, F(b/a))$. For $a < 0$ and $b < 0$, we have again $b/a > 0$ and, hence,

$$a\mu + b\sigma \leq c \quad \Leftrightarrow \quad \frac{c/a - \mu}{\sigma} \leq \frac{b}{a} \quad \Leftrightarrow \quad F_{\boldsymbol{\vartheta}}\left(\frac{c}{a}\right) \leq F\left(\frac{b}{a}\right), \tag{18}$$

which means that $\tilde{H}(a, b, c) = L(c/a, F(b/a))$. Moreover, the equivalences in (18) are also true if $a < 0$ and $b = 0$, so that $\tilde{H}(a, 0, c) = L(c/a, 0)$. From these findings then follows the identity in (17). □

When comparing representations (12) and (17), we notice that less half-spaces correspond to lower or upper coverings in the actual case. Inspecting the proof of Lemma 5, this finding is connected with the reverse implication in the last equivalence in (13) and (18), respectively, which fails if a and b do not have the same sign and thus $F(b/a) = 0$. We give a sufficient condition for a half-space in Θ to be exhaustive when $\mathrm{supp}(F) = [0, \infty)$.

Corollary 3 *$\tilde{H}(a, b, c)$ is exhaustive if $a > 0$ and $b \geq 0$ or $a < 0$ and $b \leq 0$.*

Proof If $a > 0$ and $b > 0$ or $a < 0$ and $b \leq 0$, the statement follows directly from Lemma 5 and Lemma 2. Hence, let $a > 0$ and $b = 0$. Moreover, let $\boldsymbol{\vartheta} = (\mu, \sigma) \in \tilde{H}(a, 0, c)^*$. In particular, graph $F_{\boldsymbol{\vartheta}} \subseteq B_{\tilde{H}(a,0,c)}$ then implies that for every $x > c/a$ there exists some $\boldsymbol{\vartheta}_x = (\mu_x, \sigma_x) \in \tilde{H}(a, 0, c)$ with $F((x - \mu)/\sigma) = F_{\boldsymbol{\vartheta}}(x) = F_{\boldsymbol{\vartheta}_x}(x) = F((x - \mu_x)/\sigma_x) > 0$, since $\mu_x \leq c/a$. Thus, $\mu \leq c/a$ and, therefore, $\boldsymbol{\vartheta} \in \tilde{H}(a, 0, c)$, i.e., $\tilde{H}(a, 0, c)$ is exhaustive. □

We introduce the concept of the comprehensive set in the following definition (cf. Peters 2015, p. 379).

Definition 7 A set $C \subseteq \mathbb{R}^2$ is said to be comprehensive if $x \leq z \leq y$ with $x, y \in C$ implies that $z \in C$, where $\leq$ is used in componentwise sense.

In analogy to the convex hull, the comprehensive convex hull $\mathrm{cch}(C)$ of a set C is defined as the smallest comprehensive and convex set that contains C. Since arbitrary intersections of comprehensive sets are again comprehensive, $\mathrm{cch}(C)$ coincides with the intersection of all both comprehensive and convex supersets of C. In Figure 1, an example of a comprehensive and convex set is depicted along with the comprehensive convex hull of an ellipse.

In preparation for the main result of this subsection, we prove two lemmas.

Lemma 6 *Let $C \subseteq \mathbb{R}^2$, $C^+ = \{\boldsymbol{\eta} \in \mathbb{R}^2 : \exists\, \boldsymbol{\vartheta} \in C \text{ with } \boldsymbol{\vartheta} \leq \boldsymbol{\eta}\}$, and $C^- = \{\boldsymbol{\zeta} \in \mathbb{R}^2 : \exists\, \boldsymbol{\vartheta} \in C \text{ with } \boldsymbol{\vartheta} \geq \boldsymbol{\zeta}\}$. Then the following properties hold true*:

(i) *If C is convex, then C^+ and C^- are comprehensive and convex, and $\mathrm{cch}(C) = C^+ \cap C^-$.*
(ii) *If C is compact, then C^+ and C^- are closed.*

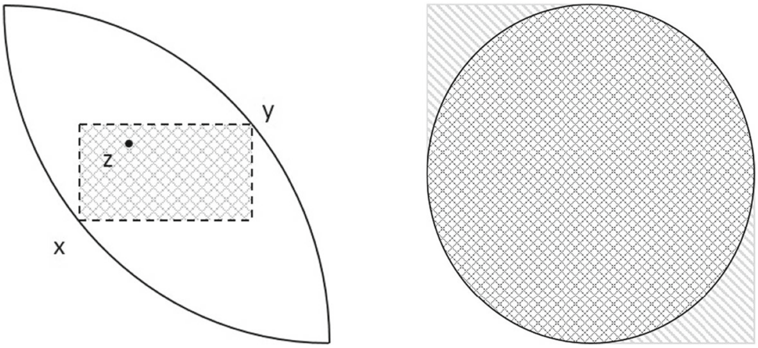

Fig. 1 Example of a comprehensive and convex set (*left-hand side*) and the comprehensive convex hull of an ellipse (*right-hand side*)

Proof (i). Let $\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} \in C^+$ and $\alpha \in [0, 1]$. Then there exist $\boldsymbol{\vartheta}, \tilde{\boldsymbol{\vartheta}} \in C$ with $\boldsymbol{\eta} \geq \boldsymbol{\vartheta}$ and $\tilde{\boldsymbol{\eta}} \geq \tilde{\boldsymbol{\vartheta}}$. Since C is convex, we obtain $\alpha\boldsymbol{\eta} + (1-\alpha)\tilde{\boldsymbol{\eta}} \geq \alpha\boldsymbol{\vartheta} + (1-\alpha)\tilde{\boldsymbol{\vartheta}} \in C$. Therefore, $\alpha\boldsymbol{\eta} + (1-\alpha)\tilde{\boldsymbol{\eta}} \in C^+$ so that C^+ is shown to be convex. From the definition of C^+, it is obvious that $x \leq z \leq y$ with $x, y \in C^+$ implies that $z \in C^+$. Hence, C^+ is comprehensive. In the same way, it can be shown that C^- is comprehensive and convex.

It is left to show that $\mathrm{cch}(C) = C^+ \cap C^-$. First, $C^+ \cap C^-$ is comprehensive and convex as the intersection of two comprehensive and convex sets, and it contains C, by definition of C^+ and C^-. Thus, $\mathrm{cch}(C) \subseteq C^+ \cap C^-$. Now, let $\boldsymbol{\eta} \in C^+ \cap C^-$. Then there exist $\boldsymbol{\vartheta}, \tilde{\boldsymbol{\vartheta}} \in C \subseteq \mathrm{cch}(C)$ with $\boldsymbol{\vartheta} \leq \boldsymbol{\eta} \leq \tilde{\boldsymbol{\vartheta}}$. Hence, $\boldsymbol{\eta} \in \mathrm{cch}(C)$, since $\mathrm{cch}(C)$ is comprehensive and convex.

(ii). Let C be compact and $\boldsymbol{\eta}_n \in C^+, n \in \mathbb{N}$, with $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$. Then there exist $\boldsymbol{\vartheta}_n \in C$, $n \in \mathbb{N}$, with $\boldsymbol{\eta}_n \geq \boldsymbol{\vartheta}_n$. Since C is compact, there exist $\boldsymbol{\vartheta} \in C$ and a subsequence $\boldsymbol{\vartheta}_{n_k}$, $k \in \mathbb{N}$, with $\boldsymbol{\vartheta}_{n_k} \to \boldsymbol{\vartheta}$. Hence, $\boldsymbol{\eta} = \lim_{k\to\infty} \boldsymbol{\eta}_{n_k} \geq \lim_{k\to\infty} \boldsymbol{\vartheta}_{n_k} = \boldsymbol{\vartheta} \in C$ which yields that $\boldsymbol{\eta} \in C^+$. By the same arguments, it follows that C^- is closed. □

Lemma 7 *For $C \subseteq \mathbb{R}^2$, $cch(C) = cch(ch(C))$.*

Proof Since $C \subseteq \mathrm{ch}(C) \subseteq \mathrm{cch}(\mathrm{ch}(C))$ and $\mathrm{cch}(\mathrm{ch}(C))$ is comprehensive and convex, we obtain that $\mathrm{cch}(C) \subseteq \mathrm{cch}(\mathrm{ch}(C))$. On the other hand, we have that $C \subseteq \mathrm{cch}(C)$ and $\mathrm{cch}(C)$ is convex. Thus, $\mathrm{ch}(C) \subseteq \mathrm{cch}(C)$. Since $\mathrm{cch}(C)$ is comprehensive and convex, $\mathrm{cch}(\mathrm{ch}(C)) \subseteq \mathrm{cch}(C)$. □

By analogy with Theorem 2, we state the main result of this subsection.

Theorem 3 *Let $C \subseteq \Theta$ be path-connected and compact. Then,*

(i) $C^* = cch(C)$,
(ii) *C is exhaustive iff C is comprehensive and convex.*

Proof (i). As in the proof of Theorem 2, it suffices to show that C^* is comprehensive and convex, and $\mathrm{cch}(C)$ is exhaustive, since $C \subseteq C^*$ and $C \subseteq \mathrm{cch}(C)$. First, from Lemma 3 and formula (17) in Lemma 5, C^* can be represented as the intersection of half-spaces $\tilde{H}(a, b, c)$ with $a, b > 0$ or $a < 0, b \leq 0$, which are easily seen to be comprehensive. Thus, C^* is comprehensive and convex as the intersection of both comprehensive and convex sets.

What is left to show is that $\mathrm{cch}(C)$ is exhaustive. To this end, let $\tilde{C} = \mathrm{ch}(C)$. Application of Lemma 7 in combination with Lemma 6 (i) then leads to the representation $\mathrm{cch}(C) = \mathrm{cch}(\tilde{C}) = \tilde{C}^+ \cap \tilde{C}^-$, where $\tilde{C}^+$ and $\tilde{C}^-$ are convex (and comprehensive). Since $\tilde{C}$ is compact as the convex hull of a compact set (see Rockafellar 1970, Theorem 17.2, p. 158), $\tilde{C}^+$ and $\tilde{C}^-$ are closed by application of Lemma 6 (ii). Then, application of the separation theorem by Rockafellar (1970), Corollary 11.4.1, p. 98, on set $\tilde{C}^+$ ensures that for every $\boldsymbol{\vartheta} = (\mu, \sigma) \in \mathbb{R}^2 \setminus \tilde{C}^+$, there exist $a_{\boldsymbol{\vartheta}}^+, b_{\boldsymbol{\vartheta}}^+, c_{\boldsymbol{\vartheta}}^+ \in \mathbb{R}$, and $\varepsilon_{\boldsymbol{\vartheta}}^+ > 0$ such that

$$a_{\boldsymbol{\vartheta}}^+ \tilde{\mu} + b_{\boldsymbol{\vartheta}}^+ \tilde{\sigma} + \varepsilon_{\boldsymbol{\vartheta}}^+ < c_{\boldsymbol{\vartheta}}^+ < a_{\boldsymbol{\vartheta}}^+ \mu + b_{\boldsymbol{\vartheta}}^+ \sigma - \varepsilon_{\boldsymbol{\vartheta}}^+, \qquad \forall\, (\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}^+. \tag{19}$$

Since $(\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}^+$ implies that $(\tilde{\mu} + k, \tilde{\sigma}) \in \tilde{C}^+$ and $(\tilde{\mu}, \tilde{\sigma} + k) \in \tilde{C}^+$ for all $k \geq 0$, we obtain from (19) that $a_{\boldsymbol{\vartheta}}^+ \leq 0$ and $b_{\boldsymbol{\vartheta}}^+ \leq 0$ must necessarily be true. Moreover, we may assume that $a_{\boldsymbol{\vartheta}}^+ < 0$ which can be seen as follows. Suppose that $a_{\boldsymbol{\vartheta}}^+ = 0$. Since $\tilde{C}$ is bounded, there exists a constant $M^+ < 0$ with $\tilde{\mu} \geq M^+$ for all $(\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}^+$. Then, by setting $\tilde{a}_{\boldsymbol{\vartheta}}^+ = \max\{\varepsilon_{\boldsymbol{\vartheta}}^+/M^+, -\varepsilon_{\boldsymbol{\vartheta}}^+/|\mu|\} < 0$, where $\varepsilon_{\boldsymbol{\vartheta}}^+/0 = \infty$, we may follow similar arguments as in (15) and (16) in the proof of Theorem 2 to obtain the desired separating line.

Likewise and by the same arguments, the separation theorem yields that for every $\boldsymbol{\vartheta} = (\mu, \sigma) \in \mathbb{R}^2 \setminus \tilde{C}^-$, there exist $a_{\boldsymbol{\vartheta}}^-, b_{\boldsymbol{\vartheta}}^-, c_{\boldsymbol{\vartheta}}^- \in \mathbb{R}$, and $\varepsilon_{\boldsymbol{\vartheta}}^- > 0$ such that

$$a_{\boldsymbol{\vartheta}}^- \tilde{\mu} + b_{\boldsymbol{\vartheta}}^- \tilde{\sigma} + \varepsilon_{\boldsymbol{\vartheta}}^- < c_{\boldsymbol{\vartheta}}^- < a_{\boldsymbol{\vartheta}}^- \mu + b_{\boldsymbol{\vartheta}}^- \sigma - \varepsilon_{\boldsymbol{\vartheta}}^-, \qquad \forall\, (\tilde{\mu}, \tilde{\sigma}) \in \tilde{C}^-,$$

with $a_{\boldsymbol{\vartheta}}^- > 0$ and $b_{\boldsymbol{\vartheta}}^- \geq 0$, and we finally arrive at the representation

$$\begin{aligned} \mathrm{cch}(C) &= \mathrm{cch}(C) \cap \Theta = (\tilde{C}^+ \cap \Theta) \cap (\tilde{C}^- \cap \Theta) \\ &= \bigcap_{\boldsymbol{\vartheta} \in \mathbb{R}^2 \setminus \tilde{C}^+} \tilde{H}(a_{\boldsymbol{\vartheta}}^+, b_{\boldsymbol{\vartheta}}^+, c_{\boldsymbol{\vartheta}}^+) \cap \bigcap_{\boldsymbol{\vartheta} \in \mathbb{R}^2 \setminus \tilde{C}^-} \tilde{H}(a_{\boldsymbol{\vartheta}}^-, b_{\boldsymbol{\vartheta}}^-, c_{\boldsymbol{\vartheta}}^-), \end{aligned}$$

so that $\mathrm{cch}(C)$ is exhaustive using Lemma 1 (iii) and Corollary 3.

(ii). The statement is obvious from (i). □

For later use in Sect. 3.3, we state the following corollary.

Corollary 4 *Let $C \subseteq \mathbb{R}^2$ be path-connected and compact. Then, there exists a family $H(a_i, b_i, c_i)$, $i \in I$, of half-spaces in $\mathbb{R}^2$ with $a_i > 0, b_i > 0$ or $a_i < 0, b_i < 0$ for all $i \in I$ and $cch(C) = \bigcap_{i \in I} H(a_i, b_i, c_i)$.*

Proof By inspecting the proof of Theorem 3, it is obvious that $\mathrm{cch}(C)$ admits a representation as the intersection of half-spaces $H(a_i, b_i, c_i)$ with $a_i > 0, b_i \geq 0$ or $a_i < 0, b_i \leq 0$ for all $i \in I$. Moreover, since C is compact, one may follow similar arguments to those used there to show that every half-space $H(a_i, 0, c_i)$ with $a_i > 0$

($a_i < 0$) of this intersection may be replaced by $H(a_i, b_i, c_i)$ with an appropriate $b_i > 0$ ($b_i < 0$). □

Theorem 3 may directly be applied, e.g., to show that the standard confidence region discussed in the work of Srinivasan et al. (1975) for the parameter (μ, σ) of the two-parameter exponential distribution is exhaustive.

3.3 The case supp(F) = [−1, 1]

We finally consider the case that the support of F is given by the interval $[-1, 1]$. As in the preceding subsections, we are aiming in deriving a simple representation of C^* and in finding a necessary and sufficient condition for C to be exhaustive, when C is assumed to be path-connected and compact. We will see that the concept of the comprehensive convex hull in combination with an orthogonal transformation will be suitable for this purpose, and, again, geometric interpretations will be near at hand.

In the present case, the lower and upper coverings are given by the following half-spaces in Θ.

Lemma 8 *For $a, b, c \in \mathbb{R}$,*

$$\tilde{H}(a, b, c) = \begin{cases} U(c/a, F(b/a)) & \textit{for} \quad a > 0, \ -a < b \leq a, \\ L(c/a, F(b/a)) & \textit{for} \quad a < 0, \ a < b \leq -a, \end{cases}$$

$$\textit{and} \quad \{\emptyset, \Theta\} \cup \{\tilde{H}(a, b, c) \, : \, a > 0, -a < b \leq a, c \in \mathbb{R} \; \textit{or} \;\; a < 0, a < b \leq -a, c \in \mathbb{R}\} = \mathcal{L} \cup \mathcal{U}. \tag{20}$$

Proof Let $a > 0$ and $-a < b \leq a$. Since F is strictly increasing on $[-1, 1]$ and $-1 < b/a \leq 1$, the equivalences in (13) remain true and thus $\tilde{H}(a, b, c) = U(c/a, F(b/a))$. For $a < 0$ and $a < b \leq -a$, we have $-1 \leq b/a < 1$, and the equivalence in (18) holds true, so that $\tilde{H}(a, b, c) = L(c/a, F(b/a))$. The identity in (20) is then obvious. □

We directly turn to the main result in the actual case.

Theorem 4 *Let $C \subseteq \Theta$ be path-connected and compact, and let*

$$\mathbf{O} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

denote the orthogonal matrix corresponding to a left-hand rotation (of a row vector) by 45 degrees. Then,

(i) $C^* = cch(C\mathbf{O})\,\mathbf{O}' \cap \Theta$,
(ii) *C is exhaustive iff $C\mathbf{O}$ is comprehensive and convex.*

Here, the apostrophe denotes transposition, and $C\mathbf{O} = \{\boldsymbol{\vartheta}\mathbf{O} : \boldsymbol{\vartheta} \in C\}$.

Proof (i). Let the families $\mathcal{H}_2$ and $\mathcal{H}_3$ of half-spaces in $\mathbb{R}^2$ be defined as

$$\begin{aligned}\mathcal{H}_2 &= \{H(a,b,c) : a \geq 0, b > 0, c \in \mathbb{R} \quad \text{or} \quad a < 0, b \leq 0, c \in \mathbb{R}\}\\ \mathcal{H}_3 &= \{H(a,b,c) : a > 0, -a < b \leq a, c \in \mathbb{R} \quad \text{or} \quad a < 0, a < b \leq -a, c \in \mathbb{R}\}.\end{aligned}$$

For every half-space H in $\mathbb{R}^2$ then follows that

$$H \in \mathcal{H}_3 \quad \Leftrightarrow \quad H\,\mathbf{O} \in \mathcal{H}_2, \tag{21}$$

$$\text{since} \qquad \begin{aligned} H(a,b,c) &= \left\{(\mu,\sigma) \in \mathbb{R}^2 : (\mu,\sigma)\,\mathbf{O}\,\mathbf{O}'\begin{pmatrix}a\\b\end{pmatrix} \leq c\right\}\\ &= \left\{(\mu,\sigma) \in \mathbb{R}^2 : (\mu,\sigma)\,\mathbf{O}'\begin{pmatrix}a\\b\end{pmatrix} \leq c\right\}\mathbf{O}'.\end{aligned}$$

and with $(\tilde{a},\tilde{b})' = \mathbf{O}'(a,b)' = (a-b, a+b)'/\sqrt{2}$

$$(\tilde{a} \geq 0, \tilde{b} > 0 \text{ or } \tilde{a} < 0, \tilde{b} \leq 0) \;\Leftrightarrow\; (a > 0, -a < b \leq a \text{ or } a < 0, a < b \leq -a).$$

Now, from Lemma 3 and Lemma 8, there exists a family H_i, $i \in I$, of half-spaces in $\mathbb{R}^2$ with $H_i \in \mathcal{H}_3$ for all $i \in I$ and $C^* = \bigcap_{i\in I}(H_i \cap \Theta) = \left(\bigcap_{i\in I} H_i\right) \cap \Theta$. Thus,

$$C^*\,\mathbf{O} = \left(\bigcap_{i\in I} H_i\,\mathbf{O}\right) \cap \Theta\mathbf{O}, \tag{22}$$

where statement (21) ensures that $H_i\,\mathbf{O} \in \mathcal{H}_2$ for all $i \in I$. Evidently, all half-spaces in $\mathcal{H}_2$ are comprehensive and convex, so that $\bigcap_{i\in I} H_i\,\mathbf{O}$ is comprehensive and convex as the intersection of comprehensive and convex sets. Since $C\mathbf{O} \subseteq C^*\mathbf{O} \subseteq \bigcap_{i\in I} H_i\,\mathbf{O}$ by Remark 1 and formula (22), we obtain $\mathrm{cch}(C\mathbf{O}) \subseteq \bigcap_{i\in I} H_i\,\mathbf{O}$, and, hence, $\mathrm{cch}(C\mathbf{O}) \cap \Theta\mathbf{O} \subseteq C^*\mathbf{O}$ which is equivalent to $\mathrm{cch}(C\mathbf{O})\mathbf{O}' \cap \Theta \subseteq C^*$.

On the other hand, C is path-connected and compact and so is $C\,\mathbf{O}$, since $\mathbf{O}$ induces a continuous mapping. Hence, by application of Corollary 4, there exists a family of half-spaces H_i, $i \in I$, with $H_i \in \mathcal{H}_2$ for all $i \in I$, such that $\mathrm{cch}(C\,\mathbf{O}) = \bigcap_{i\in I} H_i$, and, hence,

$$\mathrm{cch}(C\,\mathbf{O})\,\mathbf{O}' \cap \Theta = \left(\bigcap_{i\in I} H_i\,\mathbf{O}'\right) \cap \Theta = \bigcap_{i\in I}(H_i\,\mathbf{O}' \cap \Theta).$$

From (21), we have that $H_i\,\mathbf{O}' \in \mathcal{H}_3$ for all $i \in I$, and thus $H_i\,\mathbf{O}' \cap \Theta$, $i \in I$, is exhaustive by application of Lemma 8. Consequently, $\mathrm{cch}(C\mathbf{O})\mathbf{O}' \cap \Theta$ is exhaustive by Lemma 1 (iii). Since $C \subseteq \mathrm{cch}(C\mathbf{O})\mathbf{O}' \cap \Theta$, we finally obtain $C^* \subseteq \mathrm{cch}(C\mathbf{O})\mathbf{O}' \cap \Theta$ from Lemma 1 (iv).

(ii). If $C\mathbf{O}$ is comprehensive and convex, statement (i) directly yields that C is exhaustive. Vice versa, if C is exhaustive, then we obtain from statement (i) that $C = C^* = \mathrm{cch}(C\mathbf{O})\mathbf{O}' \cap \Theta$. Since C is a compact subset of (the open set) $\Theta = \mathbb{R} \times (0,\infty)$,

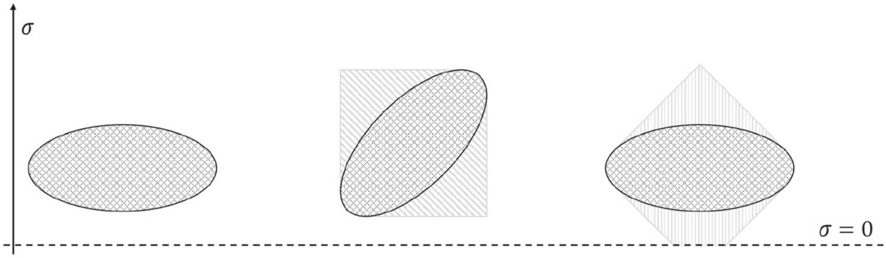

Fig. 2 Graphical illustration of the result of Theorem 4 (i): a set $C \subseteq \Theta$ is left-hand rotated by 45 degrees, then the comprehensive convex hull is formed and rotated backwards, and the intersection of the resulting set with Θ finally gives C^*

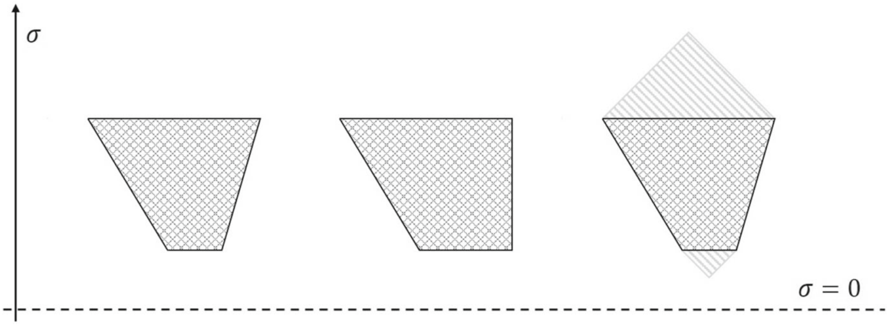

Fig. 3 Graphical illustration of the standard trapezoidal confidence regions $C = \{(\mu, \sigma) \in \mathbb{R} \times (0, \infty) : a \le (\hat{\mu} - \mu)/\sigma \le b, c \le \hat{\sigma}/\sigma \le d\}$, $c, d > 0$, in case of the normal distribution with $a < 0 < b$ (*left*) and two-parameter exponential distribution with $a = 0 < b$ (*middle*) being both exhaustive. For the uniform distribution (mean and half range parametrization), the trapezoid is not exhaustive and the confidence level of the band is given by that of the indicated area (*right*)

this already implies that $\mathrm{cch}(C\mathbf{O})\mathbf{O}' \subseteq \Theta$ and thus $C = \mathrm{cch}(C\mathbf{O})\mathbf{O}'$. Hence, $C\mathbf{O}$ is comprehensive and convex. □

In Figure 2, the finding of Theorem 4 (i) is illustrated, which is applicable, for instance, to the uniform distribution with standard cdf $F(x) = (x+1)/2$, $x \in [-1, 1]$, i.e., with parametrization $F_{\mu,\sigma}(x) = (x - \mu + \sigma)/(2\sigma)$, $x \in [\mu - \sigma, \mu + \sigma]$, where μ is the mean value and σ $\sqrt{3}$-times the standard deviation (or half of the range) of the distribution. One may then proceed as stated at the beginning of Sect. 3 to obtain respective results for a more conventional parametrization, e.g., in terms of the initial point and range of the distribution.

4 Concluding remarks

In a general parametric statistical model, we discuss the coverage probabilities of confidence bands for a continuous cdf (or qf) which are constructed via confidence regions for the parameters. To guarantee that such a confidence band is useful in applications, we focus on the question as to when the coverage probabilities and, in particular, the (exact) confidence level of the band are the same as that of the underlying confidence region, which is the case if the latter is exhaustive. Properties and representations of exhaustive sets are obtained, and an equivalence is established to use the derived results for the construction of bands for qfs as well. In the l-s family, necessary and sufficient conditions for a confidence region to be exhaustive are presented. Here, a case distinction is required regarding the boundedness of the support $\mathrm{supp}(F)$ of the standard cdf F used for the parametrization of the distribution family. In case of (a), $\mathrm{supp}(F) = \mathbb{R}$, a path-connected and compact confidence region is exhaustive iff it is convex, whereas in situation (b), $\mathrm{supp}(F) = [0, \infty)$, it is exhaustive iff it is comprehensive and convex. A similar characterization is addressed for the case (c), $\mathrm{supp}(F) = [-1, 1]$, by utilizing some orthogonal transformation. Note that confidence regions for the l-s parameter (μ, σ) are usually obtained by combining the pivotal quantities $(\hat{\mu} - \mu)/\sigma$ and $\hat{\sigma}/\sigma$ or $(\hat{\mu} - \mu)/\hat{\sigma}$ and $\hat{\sigma}/\sigma$, where $(\hat{\mu}, \hat{\sigma})$ denotes the maximum likelihood estimator of (μ, σ) (see Antle and Bain 1969), and often have trapezoidal shape. For a given l-s family as, e.g., the normal (case (a)), two-parameter exponential (case (b)), or continuous uniform (case (c)) distributions, the results may be applied to verify that these standard confidence regions are exhaustive (cases (a) and (b)) or not exhaustive (case (c)). Moreover, they enable to compute the (exact) confidence levels of confidence bands based on non-exhaustive confidence regions, which are given by the confidence levels of the convex hulls (case (a)), comprehensive convex hulls (case (b)), or other particular supersets (case (c)) of the confidence regions. In Figure 3, these findings are illustrated. Since the characterization results shown have simple geometric interpretations, they may also serve to construct exhaustive confidence regions. Finally, in virtue of appropriate transformations of both, data and band, the findings are also applicable to log-location-scale families, e.g., the log-normal or Weibull distributions.

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