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On the identifiability of start-up demonstration mixture models

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Abstract In start-up demonstration testing, the performance of the unit on successive start-ups is taken into account and several different types of decision criteria (most of them are inspired by the theory of runs and scans) for accepting or rejecting the unit have been introduced. Although the use of a start-up demonstration test assumes the existence of units of lower quality, when the estimation of the respective probability comes up, there is still much work to be done. Therefore, in this paper, we study binary start-up demonstration tests, assuming that we have at hand two different types of units with potentially different probabilities of successful start-up. In this case, the waiting time distributions are expressed as two-component mixture models and their identifiability is discussed. Finally, an estimation method based on the EM algorithm for the model parameters is described and some numerical examples are presented to illustrate the methods developed here.

Keywords Start-up demonstration tests · i.i.d. binary trials · Waiting time distributions · Mixture models · Identifiability · Maximum likelihood method · EM algorithm

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1 Introduction

Start-up demonstration testing is a technique for assessing the quality/reliability of a unit (manufactured unit) through its performance on successive start-ups. This form of testing could be useful for testing the reliability of units such as lawn mowers, batteries, power generators, fire alarm systems, etc. (e.g. Balakrishnan et al. 1997; Smith and Griffith 2005; Antzoulakos et al. 2009; Yue et al. 2010; Gera 2010, 2011; Yalcin and Eryilmaz 2012; Zhao 2014; Balakrishnan et al. 2014a).

In the related literature, several different types of decision criteria for accepting or rejecting the unit have been studied. Assuming that the outcomes of the individual start-ups are binary random variables (i.e., successful or failed start-up), we can use, for example, the model proposed by Hahn and Gage (1983) based on which the unit under test is accepted if a specified number of consecutive successful start-ups is observed (CS model); see also Viveros and Balakrishnan (1993) for further details on this model. One more example is the CSTF model introduced by Balakrishnan and Chan (2000), in which the unit under test is accepted if a number of consecutive successful start-ups is observed before a total number of failures; otherwise, the unit is rejected (see also Martin 2004, 2008). Therefore, some criteria take into account only the successful start-ups (such as their frequency, run or scan occurrences), some others take into account only the failed start-ups and some both. A more realistic model in start-up demonstration testing is generated by considering the family of multistate tests (e.g. Smith and Griffith 2011) where more than one type of successful start-ups and/or failed start-ups are allowed. Note also that the probabilistic aspects of decision criteria used in the literature are strictly related to the study of reliability systems, the theory of runs and scans, the more general theory of pattern waiting times and the theory of discrete distributions [see, for example, Aki et al. 1996; or the monographs by Johnson et al. (1992), Balakrishnan and Koutras (2002) and Fu and Lou (2003)]; for a recent review on the area of start-up demonstration tests, interested readers may also refer to the recent discussion article by Balakrishnan et al. (2014a).

Hence, the use of a start-up demonstration test actually assumes the existence of units of lower quality, in the testing procedure. However, in the literature of the binary start-up demonstration tests, the estimation of model parameters is carried out under the assumption that the available data consist of units with the same success probability (no matter if they have been accepted or rejected). Therefore, according to the underlying principle of this testing procedure, a more realistic approach must be adopted with regard to the estimation of model parameters. The present work offers a new direction in this regard; specifically, we assume that we have at hand two different types of units, type A and type B, with probabilities of successful start-ups p_{A} and p_{B}, respectively. Furthermore, we assume that the probability of selecting a unit of type A for conducting a series of start-up tests on it equals π, with $1-\pi$ being the corresponding probability of selecting a unit of type B. Consequently, the corresponding waiting time distributions can be expressed as a two-component mixture.

One of the aims of the present study is to develop suitable statistical inference (point and interval estimation methods) for the model parameters. However, since our set-up results in the use of mixture models, before proceeding with the estimation of the probabilities p_{A}, p_{B} and π, we have to consider the identifiability issue of the proposed mixture model which is clearly a critical issue as it would provide conditions under which the estimation would be feasible.

The rest of the paper is organized as follows. In Sect. 2, after presenting some introductory material, we introduce the mixture model; the identifiability of the mixture model is studied in Sect. 3 while the necessary steps for estimating the model parameters, using the EM algorithm, are described in Sect. 4. A simulation study is carried out in Sect. 5 and finally, some concluding remarks are made in Sect. 6.

2 The mixture model

Recently, Balakrishnan et al. (2014a, b) studied the class of binary and multistate start-up demonstration tests under a general/unified framework by using families of sets that parallel the nature and role of the minimal path and cut sets found in statistical reliability theory; for the properties of minimal path and cut sets, an interested reader may refer to the classical monograph by Barlow and Proschan (1981). Adopting the aforementioned approach, we shall illustrate now how one can use a binary random variable ϕ that parallels the structure function of a reliability system, to describe the length of a binary start-up demonstration test.

Let us start by introducing the families $P = \{P_j : j = 1, \ldots, M_n\}$ and $C = \{C_j : j = 1, \ldots, N_n\}$, in such a way that a unit will be accepted if the outcomes of the trials $i \in P_j$ (for at least one $j \in \{1, \ldots, M_n\}$) are successful start-ups while no other set of trials $i \in C_j$ (for all $j \in \{1, \ldots, N_n\}$), has resulted in failed start-ups. Then, we define the binary (0–1) functions, $\phi_0(\mathbf{X}_n)$, $\phi_1(\mathbf{X}_n)$ and $\phi(\mathbf{X}_n)$, as follows:

$$\phi_0(\mathbf{X}_n) = \prod_{j=1}^{N_n} \left(1 - \prod_{i \in C_j} (1 - X_i)\right), \quad \phi_1(\mathbf{X}_n) = \prod_{j=1}^{M_n} \left(1 - \prod_{i \in P_j} X_i\right)$$

and

$$\phi(\mathbf{X}_n) = \phi_0(\mathbf{X}_n)\phi_1(\mathbf{X}_n),$$

where $\mathbf{X}_n = (X_1, \ldots, X_n) \in \{0, 1\}^n$, and

$$X_i = \begin{cases} 1, & \text{if the outcome of the } i\text{-th start-up is a success} \\ 0, & \text{if the outcome of the } i\text{-th start-up is a failure,} \end{cases}$$

for $i = 1, 2, \ldots$ Denoting by X the test length, i.e., the number of trials until termination (i.e., by accepting or rejecting the unit), we can see that $\phi(\mathbf{X}_n) = 1$ if and only if $X > n$ and so

$$P(X > n) = E(\phi(\mathbf{X}_n)).$$

Therefore, assuming independent and identically distributed (i.i.d.) binary trials, the tail probabilities for the test length can be expressed as

$$S(n; p) = P(X > n; p) = \sum_{i=0}^{n} c_{ni}\ p^i (1-p)^{n-i},$$

where the coefficients c_{ni} are given by

$$c_{ni} = \sum_{\mathbf{x}_n \in \{0,1\}^n : \mathbf{x}_n \mathbf{x}_n' = i} \phi(\mathbf{x}_n), \tag{1}$$

for $i = 0, \ldots, n$. Note that the quantities c_{ni} in (1) denote the number of ways in which i successes could be allocated into n positions such that neither acceptance nor rejection criteria are met in the first n trials.

In this work, we assume that there is a proportion π of, say, highly reliable units (type A unit), with correspondingly high probability of a successful start-up p_{A}; the remaining proportion of units $(1-\pi)$ are of low quality (type B unit), having correspondingly a low probability of a successful start-up p_{B}. Then, the tail probabilities for the test length can be expressed as

$$P(X > n; \pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \pi S(n; p_{\mathrm{A}}) + (1-\pi) S(n; p_{\mathrm{B}}). \tag{2}$$

Although not explicitly displayed in formula (2), the nature of the quantities c_{ni} given by (1) plays a significant role in the identifiability of the mixture model, discussed in the next section.

3 Model identifiability

The identifiability of finite mixtures is the subject of many papers and monographs; see, for example, Teicher (1963), Yakowitz and Spragins (1968) and McLachlan and Peel (2004). Let us denote by $\mathcal{S}$ the class

$$\mathcal{S} = \{S(n; p) : p \in (0, 1)\},$$

i.e., the class of tail probabilities for the test length, for a specific binary start-up demonstration test, with success probability p; denote also by $\mathcal{H}$ the class of all two-component mixtures of $\mathcal{S}$, i.e.,

$$\mathcal{H} = \{\pi S(n; p_{\mathrm{A}}) + (1-\pi) S(n; p_{\mathrm{B}}) : p_{\mathrm{A}}, p_{\mathrm{B}}, \pi \in (0, 1) \text{ and } p_{\mathrm{A}} \neq p_{\mathrm{B}}\}.$$

The above class of mixtures will be identifiable if the relation

$$\pi S(n; p_{\mathrm{A}}) + (1-\pi) S(n; p_{\mathrm{B}}) = a S(n; q_{\mathrm{A}}) + (1-a) S(n; q_{\mathrm{B}}),$$

for all n implies that

$$\pi = a,\ p_{\mathrm{A}} = q_{\mathrm{A}},\ p_{\mathrm{B}} = q_{\mathrm{B}} \text{ or } \pi = 1-a,\ p_{\mathrm{A}} = q_{\mathrm{B}},\ p_{\mathrm{B}} = q_{\mathrm{A}}.$$

Recall that the test length X is an integer positive random variable with $X \in \{n_{\min}, \dots, n_{\max}\}$, where in some cases $n_{\max} = \infty$. A first comment before studying the identifiability of the model is that, since our mixture model contains three parameters, it is necessary that the start-up demonstration test must be such that $n_{\max} - n_{\min} \geq 3$. A very useful theorem by Yakowitz and Spragins (1968) provides (in its two-component version) a necessary and sufficient condition for a class $\mathcal{H}$ of two-component mixtures of $\mathcal{S}$ to be identifiable; this condition calls for $\mathcal{S}$ to be a linearly independent set over the field of real numbers, i.e.,

$$\psi_1 S(n; p_{\mathrm{A}}) + \psi_2 S(n; p_{\mathrm{B}}) = 0, \quad \text{for every } n \text{ and } p_{\mathrm{A}} \neq p_{\mathrm{B}} \Rightarrow \psi_1 = \psi_2 = 0.$$

Hence, if there exist two real values n_1 and n_2 such that the determinant of the matrix

$$\begin{pmatrix} S(n_1; p_{\mathrm{A}}) & S(n_1; p_{\mathrm{B}}) \\ S(n_2; p_{\mathrm{A}}) & S(n_2; p_{\mathrm{B}}) \end{pmatrix} \tag{3}$$

does not vanish, then $\mathcal{H}$ is identifiable (see also Teicher 1963). Without loss of generality, we assume that $0 < p_{\mathrm{B}} < p_{\mathrm{A}} < 1$ and by setting $n_2 = n_{\min} - 1$, $n_1 = n$ we conclude that a sufficient condition for $\mathcal{H}$ to be identifiable is the existence of an n such that

$$\sum_{i=0}^{n} c_{ni} [p_{\mathrm{A}}^i (1 - p_{\mathrm{A}})^{n-i} - p_{\mathrm{B}}^i (1 - p_{\mathrm{B}})^{n-i}] \neq 0, \tag{4}$$

where c_{ni} are as defined in (1).

Suppose now that we choose n such that the probability $\sum_{i=0}^{n} c_{ni} p_{\mathrm{A}}^i (1 - p_{\mathrm{A}})^{n-i}$ is different than 0 which also implies that $\sum_{i=0}^{n} c_{ni} p_{\mathrm{B}}^i (1 - p_{\mathrm{B}})^{n-i}$ is different than 0. In this case the coefficients c_{ni}, $i = 0, \dots, n$ could not vanish simultaneously. Let m denote the number of non-vanishing c_{ni}'s. In the simplest case $m = 1$, when there exists a unique $k \in \{0, \dots, n\}$ such that k successes can be allocated into n positions with neither acceptance nor rejection criteria being met, condition (4) is true when

$$\left(\frac{p_{\mathrm{A}}}{p_{\mathrm{B}}}\right)^k \neq \left(\frac{1 - p_{\mathrm{B}}}{1 - p_{\mathrm{A}}}\right)^{n-k}.$$

If $k = 0$ or $k = n$, then (4) is true (i.e., the mixture model is identifiable) since it reduces to

$$\left(\frac{1 - p_{\mathrm{B}}}{1 - p_{\mathrm{A}}}\right)^n \neq 1 \quad \text{or} \quad \left(\frac{p_{\mathrm{A}}}{p_{\mathrm{B}}}\right)^n \neq 1,$$

which holds true for all p_{A} and p_{B}, with $p_{\mathrm{A}} \neq p_{\mathrm{B}}$. In the more general case when $m = 1$ and $k \in \{0, \dots, n\}$, condition (4) does not hold true if and only if

$$\frac{\log\left(\frac{1 - p_{\mathrm{B}}}{1 - p_{\mathrm{A}}}\right)}{\log\left(\frac{1 - p_{\mathrm{B}}}{1 - p_{\mathrm{A}}}\right) + \log\left(\frac{p_{\mathrm{A}}}{p_{\mathrm{B}}}\right)} = \frac{k}{n}. \tag{5}$$

Therefore, we can state the following theorem:

Theorem 1 *Suppose X is the test length of a start-up demonstration test, with $X \in D = \{n_{\min}, \ldots, n_{\max}\}$. If there is $n \in D$ and $0 \leq k \leq n$ such that $c_{nk} > 0$ and $c_{ni} = 0, \forall i \neq k$, then the class $\mathcal{H}$ is identifiable, provided (5) does not hold true.*

Note that the fraction shown on the left-hand side of (5) can take any value between 0 and 1 as $p_{\mathrm{A}} \neq p_{\mathrm{B}}$ varies in the interval (0, 1); hence, it can be said that the probability of having this identity to be true is zero.

As an illustration, consider the CSTF model where the unit under test is accepted if a number of consecutive successful start-ups, say c, is observed before a total number of failures, say d; otherwise, the unit is rejected. We have $X \in D = \{\min\{c, d\}, \ldots, cd\}$ and

$$c_{nk} = \begin{cases} 0, & \text{if } k \leq n - d \\ N(n, k, c - 1), & \text{if } k > n - d, \end{cases} \quad \text{for } n \geq \max\{c, d\},$$

where

$$N(n, m_2, m_1 - 1) = \sum_{j=0}^{[m_2/m_1]} (-1)^j \binom{n - m_2 + 1}{j} \binom{n - jm_1}{n - m_2}, \quad m_1 - 1 \leq m_2 \leq n,$$

denotes the number of ways in which m_2 1's can be distributed in n distinct places with at most $m_1 - 1$ consecutive 1's (see, for example, Balakrishnan and Koutras 2002). Selecting n as $n = cd - 1$ it can be readily verified that the only non-zero c_{nk} arises for $k = d(c-1)$. Note also that if $d = 1$, then $c_{nn} = 1$ and $c_{ni} = 0, i = 1, \ldots, n-1$, for every n.

It is true that the acceptance and rejection criteria of most start-up demonstration tests are based on the appearance of a total number of successes/failures and/or a number of consecutive successes/failures. This category encompasses the models CS, CSTF, TSTF, CSCF, CSTS, TSCSTF, CSTFCF, TSTFCF, TSCSCF, TSCSTFCF, R-TSCSTFCF, R-CSCF and others (e.g. Balakrishnan et al. 2014a). For example, according to:

- the TSCSTFCF model, the unit under test is accepted if a total number of successful trials, say d_1, or a specified number of consecutive successful trials, say c_1, is observed, before a total number of failures, say d_2, and a specified number of consecutive failures, say c_2; otherwise, the unit is rejected;
- the R-TSCSTFCF model, the unit under test is accepted if r_1 non overlapping success runs of size c_1, or d_1 successful trials occur prior to r_2 non-overlapping failure runs of size c_2, or d_2 failed trials; otherwise, the unit is rejected.

For some start-up demonstration test models the test length can take arbitrarily large values (i.e., $n_{\max} = \infty$). On the other hand, the minimum of the test length (i.e., $n_{\min}$) can be attained by meeting either one of the rejection or one of the acceptance criteria. The models studied in the literature so far require the appearance of a number of consecutive failures or successes for having $X = n_{\min}$; this means that the probability of the event $X = n_{\min}$ would be equal to $p^{n_{\min}}$ (if the minimum is attained only by accepting the unit) or $(1-p)^{n_{\min}}$ (if the minimum is attained only by rejecting the unit) or $p^{n_{\min}} + (1-p)^{n_{\min}}$ (if the minimum is attained by accepting or rejecting the unit). Typically, the rejection of a unit requires a number of consecutive/total failures much less than the number of successes for the acceptance of the unit; hence, the case $P(X = n_{\min}) = (1-p)^{n_{\min}}$ is what is usually encountered. The next theorem provides a useful sufficient condition for the identifiability of the proposed mixture model.

Theorem 2 *Consider a start-up demonstration test with i.i.d. binary trials where the minimum of the test length (i.e., $X = n_{\min}$) cannot be attained by both rejecting or accepting a unit; assume also that the probability $P(X = n_{\min})$ is equal to $p^{n_{\min}}$ (if the minimum is attained only by accepting the unit) or $(1-p)^{n_{\min}}$ (if the minimum is attained only by rejecting the unit). Then, the class of all two-component mixtures generated by this model is identifiable.*

Proof Let us use the determinant criterion on matrix (3) by choosing $n_1 = n_{\min}$ and $n_2 = n_{\min} - 1$. Suppose the minimum is attained only by accepting the unit; then, for $p_{\mathrm{A}} \neq p_{\mathrm{B}}$ we have $1 - p_{\mathrm{A}}^{n_{\min}} \neq 1 - p_{\mathrm{B}}^{n_{\min}}$ and the determinant of matrix (3) does not vanish; therefore, the mixture model is identifiable.

Finally, suppose the minimum is attained only by rejecting the unit; in this case, from $p_{\mathrm{A}} \neq p_{\mathrm{B}}$ we conclude that $1-(1-p_{\mathrm{A}})^{n_{\min}} \neq 1-(1-p_{\mathrm{B}})^{n_{\min}}$, and the mixture model is identifiable since the determinant of matrix (3) does not vanish. □

Let us now consider the family *finite consecutive/total start-up models* (FCT models) whose members possess the following two properties: (a) they have acceptance and rejection criteria based on the appearance of a total number of successes/failures and/or a number of consecutive successes/failures, and (b) their test length is a bounded random variable. This family encompasses many of the popular models of the start-up demonstration test literature, for example, CSTF, TSTF, CSTS, TSCSTF, CSTFCF, TSTFCF, TSCSCF, TSCSTFCF, R-TSCSTFCF and R-CSCF.

For the FCT models, we shall now prove the next general result.

Theorem 3 *The class of two-component mixtures generated by a FCT model is identifiable provided (5) does not hold true.*

Proof According to Theorem 1, it suffices to prove the existence of n such that $c_{nk} \neq 0$, for only one $k \in \{0, \ldots, n\}$; we shall prove that this is the case for $n = n_{\max} - 1$. To achieve this, let us assume that for $n = n_{\max} - 1$ there exist $k_1 < k_2$ such that $c_{nk_1} > 0$ and $c_{nk_2} > 0$; this means that there is at least one way of placing k_1 or k_2 ones in n positions such that neither acceptance nor rejection criterion will be met.

In one of these ways, with k_1 ones, we may have observed $X_n = 0$ or $X_n = 1$; let us next examine separately the following cases that may arise:

(a) if $X_n = 0$, then by setting $X_{n+1} = 1$, the test will not terminate; this is true because the occurrence of $k_1 + 1$ ones does not allow any of the acceptance criteria to be realized since $k_2 > k_1$ and $c_{nk_2} > 0$. This means that $P(X > n_{\max}) > 0$ (contradiction);

(b) if $X_n = 1$, then observe first that the number of zeros among the first n trials is $n - k_1 > 0$ (since $k_1 < k_2 \leq n = n_{\max} - 1$); if $n - k_1 = 1$, then $k_2 = n$ and the occurrence of n ones does not allow any criterion to be realized. In this case, if $X_1 = \cdots = X_n = 1$, $X_{n+1} = 0$, the test will not terminate (contradiction);
(c) suppose $X_n = 1$, $n - k_1 \geq 2$ and there exist at least two successive zeros; then, by placing one 1 between these two successive zeros, the resulting $1 \times (n + 1)$ vector makes the termination infeasible at the $(n + 1)$-th trial (contradiction);
(d) suppose $X_n = 1$, $n - k_1 \geq 2$ with no two successive zeros; then, $X_1 = 0$ or $X_1 = 1$. If $X_1 = 0$, then the vector which is derived by transferring the zero found at the first place to the end and setting $X_{n+1} = 1$, makes the termination infeasible (contradiction). If $X_1 = 1$, by replacing one zero with one and reordering, we can have a vector with $k_1 + 1$ ones with which the test will not terminate. In this vector we must have at least two consecutive ones (if $n > 1$); then by placing one zero between them, the resulting $1 \times (n + 1)$ vector makes the termination infeasible at the $(n + 1)$-th trial (contradiction).

Therefore, for any FCT model, on choosing $n = n_{\max} - 1$, we have only one non-zero c_{ni}. This completes the proof. □

Hence, from the above proof, for any FCT model and $n = n_{\max} - 1$, there exists only one $k \in \{0, \ldots, n\}$ such that $c_{nk} \neq 0$; this means that only the appearance of k successes could make the test not to terminate at $n = n_{\max} - 1$'th trial. Following similar steps, we can verify that if for a family of start-up demonstration tests we have $c_{nk_1} > 0$ and $c_{nk_2} > 0$ with $k_1 > k_2$, then $c_{n+1,k_1+1} > 0$; a direct conclusion from this observation is that if $n = n_{\max} - 2$, then the number of non-zero c_{ni}'s is either 1 or 2. The next result offers a useful sufficient condition for the identifiability of a mixture model.

Proposition 1 *Suppose X is the test length of a start-up demonstration test, with $X \in D = \{n_{\min}, \ldots, n_{\max}\}$. Assume also that there exist $n \in D$, $k \geq 0$ and $\mu > 0$, such that:*

(a) $c_{nk} > 0$ *and* $c_{n+\mu,k+\mu} > 0$;
(b) $c_{ni} = 0, \forall i \neq k$ *and* $c_{n+\mu,i} = 0, \forall i \neq k + \mu$.

Then, the class of two-component mixtures generated by this model is identifiable.

Proof Suppose the following two conditions hold true simultaneously:

$$\left(\frac{p_{\mathrm{A}}}{p_{\mathrm{B}}}\right)^{k} = \left(\frac{1 - p_{\mathrm{B}}}{1 - p_{\mathrm{A}}}\right)^{n-k} \quad \text{and} \quad \left(\frac{p_{\mathrm{A}}}{p_{\mathrm{B}}}\right)^{k+\mu} = \left(\frac{1 - p_{\mathrm{B}}}{1 - p_{\mathrm{A}}}\right)^{n+\mu-k-\mu}.$$

Then, it can be easily proved that $p_{\mathrm{A}} = p_{\mathrm{B}}$; therefore, at least one of the above equalities is not true and so (4) holds for $n_1 = n$ or $n_1 = n + \mu$. □

The next result is of the same nature as Proposition 1 and offers new sufficient conditions for the identifiability of the mixture model. The proof can be carried out by following exactly the same reasoning as that of Proposition 1 and, therefore, it is not presented here for the sake of conciseness.

Proposition 2 *Suppose X is the test length of a start-up demonstration test, with $X \in D = \{n_{\min}, \ldots, n_{\max}\}$. Assume also that there exist $n_1, n_2 \in D$ ($n_1 \neq n_2$), $k \geq 0$ such that:*

(a) $c_{n_1 k} > 0$ *and* $c_{n_2,k} > 0$;
(b) $c_{n_1 i} = c_{n_2 i} = 0, \forall i \neq k$.

Then, the class of two-component mixtures generated by this model is identifiable.

Restricting our attention again to the family of FCT models, we can state the following theorem which covers the case where the maximum value of the test length is an even number. In this case the respective class of two-component mixture is identifiable, no matter whether condition (5) is true or not.

Theorem 4 *If $n_{\max}$ is an even number, then the class of two-component mixtures of a FCT model is identifiable. If instead, $n_{\max}$ is an odd number and the unique k for which $c_{n_{\max}-1,k} > 0$ is not equal to $(n_{\max} - 1)/2$, then the class of two-component mixtures of a FCT model is identifiable, as well.*

Proof In view of Theorem 2, if the minimum of the test length (i.e., $X = n_{\min}$) cannot be attained by both rejecting or accepting a unit, then the class of all two-component mixtures of this model is identifiable. Therefore, we should only turn our attention to the case where $P(X = n_{\min}) = (1 - p)^{n_{\min}} + p^{n_{\min}}$.

Suppose the determinant of matrix (3) is zero for $n_1 = n_{\min}$ and $n_2 = n_{\min} - 1$ (with $p_A > p_B$), and for $n_1 = n_{\max} - 1$ and $n_2 = n_{\min} - 1$. Then, the following two equalities are true:

$$(1 - p_A)^{n_{\min}} + p_A^{n_{\min}} = (1 - p_B)^{n_{\min}} + p_B^{n_{\min}}$$

and

$$\left(\frac{p_A}{p_B}\right)^k = \left(\frac{1 - p_B}{1 - p_A}\right)^{n_{\max}-1-k},$$

where k is such that $c_{n_{\max}-1,k} > 0$. Taking advantages of the monotonicity of the function

$$f(x) = x^n + (1 - x)^n, \quad n > 1, \quad x \in (0, 1),$$

we can state that the first equality holds only for $p_A = p_B$ or $p_A = 1 - p_B$. Since we have assumed that $p_A > p_B$, only the solution $p_A = 1 - p_B$ is allowed; this also means that $p_B < 1/2$. Replacing now $p_A = 1 - p_B$ in the second equality, we have

$$\left(\frac{1 - p_B}{p_B}\right)^k = \left(\frac{1 - p_B}{p_B}\right)^{n_{\max}-1-k}.$$

Assuming that $n_{\max}$ is an even number, we have $n_{\max} - 1 - k > k$ or $n_{\max} - 1 - k < k$; from either case, we gain $p_B = 1/2 = p_A$ (contradiction). Therefore, one of the two equalities does not hold and the model is identifiable. Obviously, we could also come to the same conclusion if $n_{\max}$ is an odd number with $n_{\max} - 1 - k \neq k$. □

Before closing this section, it is worth paying more attention to the CSTF model, which is one of the well-studied models in the literature; the next theorem states that the mixture of this model is always identifiable.

Theorem 5 *The class of two-component mixtures of the CSTF model is identifiable.*

Proof Suppose the unit under test is accepted if c consecutive successful start-ups are observed before a total number of d failures; otherwise, the unit is rejected ($X \in D = \{\min\{c, d\}, \ldots, cd\}$). We know that a non-zero c_{nk} exists if and only if

$$n - k \leq d - 1 \quad \text{and} \quad \frac{k}{n - k + 1} \leq c - 1;$$

therefore,

$$n - d + 1 \leq k \leq \frac{(n+1)(c-1)}{c}.$$

Note also that

$$\frac{(n+1)(c-1)}{c} - (n - d + 1) \geq 0 \Leftrightarrow n \leq cd - 1,$$

as expected. Furthermore, we have

$$\frac{(n+1)(c-1)}{c} - (n - d + 1) \leq 1 \Leftrightarrow n \geq c(d-1) - 1,$$

which means that for every n with $c(d-1) - 1 \leq n \leq cd - 1$ the number of non-zero c_{nk}'s is 1; note that there are $cd - 1 - c(d-1) + 1 = c$ such n's. If $c = 1$, then the identifiability of the mixture model can be proved couching on the discussion done before Theorem 1 or Theorem 2 and the fact that $n_{\max} - n_{\min} \geq 3$. If $c \geq 2$, then it can be easily verified that we have $c_{n_{\max}-1, d(c-1)} > 0$ and $c_{n_{\max}-2, d(c-1)-1} > 0$; hence, due to Proposition 1, the class of two-component mixtures is again identifiable. □

4 Statistical inference

In this section, we discuss the estimation problem using the maximum likelihood estimation (MLE) and the EM algorithm. Suppose r independent start-up procedures are carried out on r different units. We shall next discuss three different scenarios for the observed/available data. More specifically, let us assume that the data collected from the r independent start-up testing procedures, include the following:

Scenario I: Only the test length, i.e., n_i, $i = 1, \ldots, r$;

Scenario II: the test length and the outcome of the testing procedure; in this case the available data contain the pairs (n_i, y_i), $i = 1, \ldots, r$, with $y_i = 1$ if the ith unit was accepted and $y_i = 0$, otherwise;

Scenario III: the test length, the outcome and the number of successes; in this case, we have observed the triplets (n_i, s_i, y_i), $i = 1, \ldots, r$, where s_i is the number of successes in the ith testing unit and n_i, s_i are as for Scenario II.

Then, the likelihood function for Scenario I is given by

$$L_{\mathrm{I}}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \prod_{i=1}^{r} f(n_i; \pi, p_{\mathrm{A}}, p_{\mathrm{B}})$$

with

$$f(n; \pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \pi f(n; p_{\mathrm{A}}) + (1-\pi) f(n; p_{\mathrm{B}})$$

and $f(n; p) = S(n-1; p) - S(n; p)$; equivalently, $f(n; p)$ can be written as

$$f(n; p) = \sum_{k=0}^{n} v_{nk} p^k (1-p)^{n-k},$$

where

$$v_{nk} = \sum_{\boldsymbol{x}_n \in \{0,1\}^n : \boldsymbol{x}_n \boldsymbol{x}_n{}' = k} (1 - \phi(\boldsymbol{x}_n))\phi(\boldsymbol{x}_{n-1})$$

and $\boldsymbol{x}_{n-1}$ stands for a vector of size $n-1$ containing the first $n-1$ components of $\boldsymbol{x}_n$ (see also Balakrishnan et al. 2014a, b).

Similarly, the likelihood function for Scenario II has the following form:

$$L_{\mathrm{II}}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \prod_{i=1}^{r} f(n_i, y_i; \pi, p_{\mathrm{A}}, p_{\mathrm{B}})$$

with

$$f(n, y; \pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \pi f(n, y; p_{\mathrm{A}}) + (1-\pi) f(n, y; p_{\mathrm{B}}),$$

where

$$f(n, 1; p) = \sum_{k=0}^{n} b_{nk} p^k (1-p)^{n-k}, \quad f(n, 0; p) = \sum_{k=0}^{n} d_{nk} p^k (1-p)^{n-k},$$

and

$$b_{nk} = \sum_{\boldsymbol{x}_n \in \{0,1\}^n : \boldsymbol{x}_n \boldsymbol{x}_n{}' = k} \phi_0(\boldsymbol{x}_n)(1 - \phi_1(\boldsymbol{x}_n))\phi_1(\boldsymbol{x}_{n-1}), \quad k = 0, \ldots, n,$$

$$d_{nk} = \sum_{\boldsymbol{x}_n \in \{0,1\}^n : \boldsymbol{x}_n \boldsymbol{x}_n{}' = k} \phi_1(\boldsymbol{x}_n)(1 - \phi_0(\boldsymbol{x}_n))\phi_0(\boldsymbol{x}_{n-1}), \quad k = 0, \ldots, n.$$

In Scenario III, the likelihood function is given by

$$L_{\mathrm{III}}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \prod_{i=1}^{r} f(n_i, s_i, y_i; \pi, p_{\mathrm{A}}, p_{\mathrm{B}})$$

with

$$f(n, s, y; \pi, p_{\mathrm{A}}, p_{\mathrm{B}}) = \pi f(n, s, y; p_{\mathrm{A}}) + (1-\pi) f(n, s, y; p_{\mathrm{B}})$$

where

$$f(n, s, 1; p) = b_{ns} p^s (1-p)^{n-s}, \quad f(n, s, 0; p) = d_{ns} p^s (1-p)^{n-s}.$$

Clearly, a direct maximization of L_{I} $(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$, L_{II} $(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$ or L_{III} $(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$ would provide us the MLEs of the model parameters $(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$ and then using them along with the observed information matrix, we could also have the asymptotic confidence intervals. Alternatively, we can employ the EM algorithm where the complete data log-likelihood for Scenario I is given by

$$\begin{aligned} l_{\mathrm{I}}^{c}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}) &= \log(\pi) \sum_{i=1}^{r} Z_i + \log(1-\pi) \sum_{i=1}^{r} (1 - Z_i) \\ &\quad + \sum_{i=1}^{r} [Z_i \log f(n_i; p_{\mathrm{A}}) + (1 - Z_i) \log f(n_i; p_{\mathrm{B}})], \end{aligned}$$

with

$$Z_i = \begin{cases} 1, & \text{if the } i\text{th unit is of type A (i.e., its success probability equals} p_{\mathrm{A}}), \\ 0, & \text{if the } i\text{th unit is of type B (i.e., its success probability equals} p_{\mathrm{B}}), \end{cases}$$

for $i = 1, \ldots, r$. In the E-step, the expected value of the complete data log-likelihood function is computed with respect to the expected value of Z_i, given both the current values of the parameter and the observed data. Taking into account that

$$E[Z_i | \text{observed data}] = E[Z_i | X = n_i] = \frac{\pi f(n_i; p_{\mathrm{A}})}{\pi f(n_i; p_{\mathrm{A}}) + (1-\pi) f(n_i; p_{\mathrm{B}})},$$

we may write

$$\begin{aligned} Q_I(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}; \boldsymbol{\theta}^{(t)}) &= E[l_{\mathrm{I}}^{c}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}) | \text{observed data}] \\ &= \log(\pi) \sum_{i=1}^{r} \theta_i^{(t)} + \log(1-\pi) \sum_{i=1}^{r} (1 - \theta_i^{(t)}) \\ &\quad + \sum_{i=1}^{r} [\theta_i^{(t)} \log f(n_i; p_{\mathrm{A}}) + (1 - \theta_i^{(t)}) \log f(n_i; p_{\mathrm{B}})] \end{aligned}$$

where $\boldsymbol{\theta}^{(t)} = (\theta_1^{(t)}, \ldots, \theta_r^{(t)})$ and

$$\theta_i^{(t)} = \frac{\pi^{(t)} f(n_i; p_{\mathrm{A}}^{(t)})}{\pi^{(t)} f(n_i; p_{\mathrm{A}}^{(t)}) + (1-\pi^{(t)}) f(n_i; p_{\mathrm{B}}^{(t)})}, \quad i = 1, \ldots, r,$$

with $\pi^{(t)}$, $p_{\mathrm{A}}^{(t)}$ and $p_{\mathrm{B}}^{(t)}$ being the parameter estimates at the t-th iteration step. Similarly, in Scenario II, we have

$$\begin{aligned} Q_{\mathrm{II}}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}; \boldsymbol{\theta}^{(t)}) &= \log(\pi) \sum_{i=1}^{r} \theta_i^{(t)} + \log(1-\pi) \sum_{i=1}^{r} (1-\theta_i^{(t)}) \\ &\quad + \sum_{i=1}^{r} [\theta_i^{(t)} \log f(n_i, y_i; p_{\mathrm{A}}) + (1-\theta_i^{(t)}) \log f(n_i, y_i; p_{\mathrm{B}})] \end{aligned}$$

with

$$\theta_i^{(t)} = \frac{\pi^{(t)} f(n_i, y_i; p_{\mathrm{A}}^{(t)})}{\pi^{(t)} f(n_i, y_i; p_{\mathrm{A}}^{(t)}) + (1-\pi^{(t)}) f(n_i, y_i; p_{\mathrm{B}}^{(t)})},$$

while in Scenario III, we have

$$\begin{aligned} Q_{\mathrm{III}}(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}; \boldsymbol{\theta}^{(t)}) &= \log(\pi) \sum_{i=1}^{r} \theta_i^{(t)} + \log(1-\pi) \sum_{i=1}^{r} (1-\theta_i^{(t)}) \\ &+ \sum_{i=1}^{r} [\theta_i^{(t)} \log f(n_i, s_i, y_i; p_{\mathrm{A}}) + (1-\theta_i^{(t)}) \log f(n_i, s_i, y_i; p_{\mathrm{B}})] \end{aligned}$$

with

$$\theta_i^{(t)} = \frac{\pi^{(t)} f(n_i, s_i, y_i; p_{\mathrm{A}}^{(t)})}{\pi^{(t)} f(n_i, s_i, y_i; p_{\mathrm{A}}^{(t)}) + (1-\pi^{(t)}) f(n_i, s_i, y_i; p_{\mathrm{B}}^{(t)})}.$$

In the M-step, the maximization of the expected value of the complete data log-likelihood is carried out and the new set of parameter estimates are obtained; the E-step and the M-step are continued iteratively until a convergence criterion is met. To compute the asymptotic confidence intervals for model parameters through the EM algorithm, one may use the method of missing information principle Louis (1982); analytically, denoting by $B(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$ the negative of the second derivatives matrix of $l^c(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$ and by $S(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$ the gradient vector of $l^c(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})$, the observed Fisher information matrix will be given by $I(\hat{\pi}, \hat{p}_{\mathrm{A}}, \hat{p}_{\mathrm{B}})$, where $\hat{\pi}, \hat{p}_{\mathrm{A}}, \hat{p}_{\mathrm{B}}$ is the EM estimate of $\pi, p_{\mathrm{A}}, p_{\mathrm{B}}$ and

$$\begin{aligned} I(\pi, p_{\mathrm{A}}, p_{\mathrm{B}}) =& E\,[B(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})] - E[S(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})' S(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})] \\ &+ E[S(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})]' E[S(\pi, p_{\mathrm{A}}, p_{\mathrm{B}})]. \end{aligned}$$

5 Numerical results

The method of inference developed in the previous section will be evaluated by means of a Monte Carlo simulation study. First, we generate 100 random samples of sizes $r = 200, 400$ and 600; i.e., for each of the 100 iterations, we assume that 200 or 400 or 600 units from the same population are tested. In Tables 1, 2 and 3, we consider three cases, concerning the values of the probabilities π, p_{A} and p_{B}; we assume that: (a) the probability of successful start-up for a high quality unit (type A unit) is $p_{\mathrm{A}} = 0.9$, whereas the probability of successful start-up for a low quality unit (type B unit) is $p_{\mathrm{B}} = 0.2$; the proportion of type A unit is taken to be $\pi = 0.7$; (b) the proportion of type A unit and its success probability are $\pi = 0.7$ and $p_{\mathrm{A}} = 0.85$, respectively, while $p_{\mathrm{B}} = 0.4$; and (c) $\pi = 0.8$, $p_{\mathrm{A}} = 0.95$ and $p_{\mathrm{B}} = 0.5$. Our attention is on the CSTF model which is one of the well-studied start-up demonstration tests; we study the cases where $c = 4, 5$ and $d = 1, 2$ (for the identifiability of this model, see Theorem 5).

The starting values of the EM algorithm can be selected by a grid search in the area $[0, 1]^3$ and then the choice that corresponds to the largest likelihood will be chosen as the final estimation. The accuracy of our inference procedure is assessed using the sample mean of the estimates (mean), the sample standard error (SE), and the root mean square error (RMSE) given by

$$\mathrm{RMSE} = \sqrt{\frac{1}{r}\sum_{i=1}^{r}(\hat{p}_i - p)^2},$$

where $p = \pi$ or p_{A} or p_{B} and $\hat{p}_i$ are the respective estimates.

Based on our numerical study, the following remarks can be made for Scenario I:

- the more distinct the two categories of units (i.e., type A and B) are, the better accuracy is gained in terms of Bias, SE and RMSE; this can be seen in Table 1 where the case $\pi = 0.70$, $p_{\mathrm{A}} = 0.90$, $p_{\mathrm{B}} = 0.20$ offers the more accurate estimates. When the difference between p_{A} and p_{B} becomes smaller (see the second case), less accurate results are seen;
- the estimation of p_{B} almost always gives the largest SE and RMSE, whilst p_{A} gives the smaller;
- the largest values of SE and RMSE for p_{B} are found in the last case ($\pi = 0.8$, $p_{\mathrm{A}} = 0.95$ and $p_{\mathrm{B}} = 0.5$); this is mainly due to the fact that in this case the proportion of units of type B is the smallest;
- the results between $c = 4, d = 1$ and $c = 5, d = 1$ are quite comparable, while $c = 5, d = 2$ gives much higher accuracy; this is because of the larger test lengths, i.e., available information;
- larger sample sizes decrease the Bias, SE and RMSE.

The above remarks also hold true for Scenarios II and III. Moreover, moving from Scenario I to Scenario II or III, i.e., increasing the available information, the accuracy of the estimates increases as well; the accuracy in Scenario II and III are quite comparable. As expected, for $c = 5, d = 2$ and Scenario II or III, the results are very precise. Furthermore, our numerical study reveals that a grid search in larger areas slightly

Table 1 CSTF model

Scenario I	$c=4, d=1$			$c=5, d=1$			$c=5, d=2$		
	Mean	SE	RMSE	Mean	SE	RMSE	Mean	SE	RMSE
True parameters[a]: $\pi = 0.70$, $p_A = 0.90$, $p_B = 0.20$									
$r = 200$									
$\hat{\pi}$	0.687	0.093	0.009	0.680	0.078	0.006	0.699	0.044	0.002
$\hat{p}_A$	0.905	0.039	0.002	0.907	0.027	0.0007	0.899	0.017	0.0003
$\hat{p}_B$	0.197	0.134	0.018	0.201	0.128	0.016	0.200	0.052	0.003
$r = 400$									
$\hat{\pi}$	0.691	0.080	0.006	0.692	0.061	0.004	0.699	0.027	0.001
$\hat{p}_A$	0.904	0.033	0.001	0.905	0.020	0.0004	0.901	0.012	10^{-4}
$\hat{p}_B$	0.197	0.113	0.013	0.194	0.101	0.010	0.202	0.038	0.001
$r = 600$									
$\hat{\pi}$	0.692	0.068	0.005	0.695	0.051	0.003	0.698	0.021	10^{-4}
$\hat{p}_A$	0.904	0.029	0.001	0.903	0.017	0.0003	0.902	0.009	10^{-4}
$\hat{p}_B$	0.198	0.090	0.008	0.193	0.086	0.007	0.205	0.030	0.001
True parameters[b]: $\pi = 0.70$, $p_A = 0.85$, $p_B = 0.40$									
$r = 200$									
$\hat{\pi}$	0.699	0.148	0.022	0.664	0.156	0.025	0.691	0.105	0.011
$\hat{p}_A$	0.856	0.051	0.003	0.861	0.044	0.0021	0.853	0.036	0.001
$\hat{p}_B$	0.341	0.169	0.032	0.369	0.179	0.033	0.383	0.111	0.013
$r = 400$									
$\hat{\pi}$	0.684	0.145	0.021	0.663	0.143	0.022	0.699	0.076	0.006
$\hat{p}_A$	0.860	0.046	0.002	0.862	0.038	0.002	0.851	0.026	0.001
$\hat{p}_B$	0.357	0.150	0.024	0.377	0.158	0.025	0.391	0.073	0.005
$r = 600$									
$\hat{\pi}$	0.685	0.136	0.018	0.658	0.138	0.021	0.694	0.066	0.004
$\hat{p}_A$	0.859	0.045	0.002	0.863	0.038	0.002	0.853	0.022	0.001
$\hat{p}_B$	0.364	0.134	0.019	0.395	0.143	0.020	0.397	0.067	0.005
True parameters[c]: $\pi = 0.80$, $p_A = 0.95$, $p_B = 0.50$									
$r = 200$									
$\hat{\pi}$	0.809	0.107	0.011	0.784	0.120	0.014	0.790	0.074	0.005
$\hat{p}_A$	0.947	0.033	0.001	0.950	0.029	0.001	0.952	0.014	0.0002
$\hat{p}_B$	0.426	0.242	0.063	0.426	0.245	0.065	0.498	0.110	0.012
$r = 400$									
$\hat{\pi}$	0.795	0.106	0.011	0.789	0.106	0.011	0.794	0.051	0.003
$\hat{p}_A$	0.952	0.029	0.001	0.952	0.025	0.001	0.951	0.010	10^{-4}
$\hat{p}_B$	0.439	0.202	0.044	0.441	0.183	0.037	0.504	0.080	0.006
$r = 600$									
$\hat{\pi}$	0.798	0.106	0.011	0.788	0.104	0.011	0.795	0.044	0.002
$\hat{p}_A$	0.952	0.029	0.001	0.953	0.024	0.001	0.950	0.008	10^{-4}
$\hat{p}_B$	0.433	0.186	0.039	0.454	0.167	0.030	0.504	0.067	0.004

Grid search in the area:
[a] $(\pi, p_A, p_B) \in [0.6, 0.8] \times [0.75, 0.95] \times [0.1, 0.3]$ (step size 0.2)
[b] $(\pi, p_A, p_B) \in [0.6, 0.8] \times [0.75, 0.95] \times [0.3, 0.5]$ (step size 0.2)
[c] $(\pi, p_A, p_B) \in [0.7, 0.9] \times [0.79, 0.99] \times [0.4, 0.6]$ (step size 0.2)

Table 2 CSTF model

Scenario II	c = 4, d = 1			c = 5, d = 1			c = 5, d = 2		
	Mean	SE	RMSE	Mean	SE	RMSE	Mean	SE	RMSE
True parameters[a]: $\pi = 0.70$, $p_A = 0.90$, $p_B = 0.20$									
$r = 200$									
$\hat{\pi}$	0.692	0.081	0.007	0.693	0.072	0.005	0.700	0.043	0.002
$\hat{p}_A$	0.901	0.029	0.001	0.901	0.023	0.001	0.900	0.014	0.0002
$\hat{p}_B$	0.195	0.127	0.016	0.185	0.121	0.015	0.197	0.051	0.003
$r = 400$									
$\hat{\pi}$	0.694	0.063	0.004	0.704	0.053	0.003	0.700	0.027	0.001
$\hat{p}_A$	0.901	0.022	0.0003	0.900	0.015	0.0002	0.900	0.010	10^{-4}
$\hat{p}_B$	0.204	0.104	0.011	0.177	0.089	0.008	0.199	0.037	0.001
$r = 600$									
$\hat{\pi}$	0.696	0.054	0.003	0.702	0.044	0.002	0.699	0.021	10^{-4}
$\hat{p}_A$	0.901	0.017	0.0003	0.900	0.014	0.0002	0.901	0.008	10^{-4}
$\hat{p}_B$	0.200	0.084	0.007	0.186	0.080	0.007	0.203	0.030	0.001
True parameters[b]: $\pi = 0.70$, $p_A = 0.85$, $p_B = 0.40$									
$r = 200$									
$\hat{\pi}$	0.693	0.139	0.019	0.664	0.150	0.024	0.693	0.086	0.007
$\hat{p}_A$	0.854	0.041	0.002	0.860	0.035	0.001	0.853	0.027	0.001
$\hat{p}_B$	0.336	0.178	0.035	0.366	0.172	0.030	0.387	0.095	0.009
$r = 400$									
$\hat{\pi}$	0.691	0.132	0.017	0.673	0.123	0.016	0.699	0.066	0.004
$\hat{p}_A$	0.854	0.039	0.001	0.856	0.029	0.001	0.851	0.021	0.0004
$\hat{p}_B$	0.356	0.153	0.025	0.379	0.149	0.022	0.393	0.070	0.005
$r = 600$									
$\hat{\pi}$	0.704	0.117	0.014	0.678	0.114	0.013	0.696	0.061	0.004
$\hat{p}_A$	0.850	0.033	0.001	0.856	0.028	0.001	0.853	0.019	0.0004
$\hat{p}_B$	0.354	0.131	0.019	0.391	0.128	0.016	0.396	0.065	0.004
True parameters[c]: $\pi = 0.80$, $p_A = 0.95$, $p_B = 0.50$									
$r = 200$									
$\hat{\pi}$	0.790	0.111	0.012	0.770	0.115	0.014	0.804	0.045	0.002
$\hat{p}_A$	0.952	0.026	0.001	0.954	0.025	0.001	0.949	0.012	10^{-4}
$\hat{p}_B$	0.440	0.229	0.056	0.457	0.218	0.049	0.492	0.081	0.007
$r = 400$									
$\hat{\pi}$	0.778	0.110	0.012	0.771	0.096	0.010	0.803	0.033	0.001
$\hat{p}_A$	0.956	0.025	0.001	0.955	0.019	0.0004	0.949	0.009	10^{-4}
$\hat{p}_B$	0.454	0.197	0.041	0.477	0.169	0.029	0.499	0.058	0.003
$r = 600$									
$\hat{\pi}$	0.775	0.105	0.012	0.777	0.093	0.009	0.798	0.025	0.001
$\hat{p}_A$	0.957	0.024	0.001	0.955	0.018	0.0004	0.950	0.007	10^{-4}
$\hat{p}_B$	0.470	0.169	0.029	0.479	0.152	0.023	0.508	0.042	0.002

Grid search in the area:
[a] $(\pi, p_A, p_B) \in [0.6, 0.8] \times [0.75, 0.95] \times [0.1, 0.3]$ (step size 0.2)
[b] $(\pi, p_A, p_B) \in [0.6, 0.8] \times [0.75, 0.95] \times [0.3, 0.5]$ (step size 0.2)
[c] $(\pi, p_A, p_B) \in [0.7, 0.9] \times [0.79, 0.99] \times [0.4, 0.6]$ (step size 0.2)

Table 3 CSTF model

Scenario III	$c = 4, d = 1$			$c = 5, d = 1$			$c = 5, d = 2$		
	Mean	SE	RMSE	Mean	SE	RMSE	Mean	SE	RMSE
True parameters[a]: $\pi = 0.70$, $p_A = 0.90$, $p_B = 0.20$									
$r = 200$									
$\hat{\pi}$	0.692	0.081	0.007	0.693	0.072	0.005	0.700	0.043	0.002
$\hat{p}_A$	0.901	0.029	0.001	0.901	0.023	0.001	0.900	0.014	0.0002
$\hat{p}_B$	0.195	0.127	0.016	0.185	0.121	0.015	0.197	0.051	0.003
$r = 400$									
$\hat{\pi}$	0.699	0.061	0.004	0.704	0.053	0.003	0.700	0.027	0.001
$\hat{p}_A$	0.900	0.022	0.001	0.900	0.015	0.0002	0.900	0.010	10^{-4}
$\hat{p}_B$	0.194	0.105	0.011	0.177	0.089	0.008	0.199	0.037	0.001
$r = 600$									
$\hat{\pi}$	0.699	0.052	0.003	0.702	0.044	0.002	0.699	0.021	10^{-4}
$\hat{p}_A$	0.900	0.017	0.0003	0.900	0.014	0.0002	0.901	0.008	10^{-4}
$\hat{p}_B$	0.194	0.083	0.007	0.186	0.080	0.007	0.203	0.030	0.001
True parameters[b]: $\pi = 0.70$, $p_A = 0.85$, $p_B = 0.40$									
$r = 200$									
$\hat{\pi}$	0.693	0.139	0.019	0.664	0.150	0.024	0.693	0.087	0.008
$\hat{p}_A$	0.854	0.041	0.002	0.860	0.035	0.001	0.853	0.028	0.001
$\hat{p}_B$	0.336	0.178	0.035	0.366	0.172	0.030	0.387	0.095	0.009
$r = 400$									
$\hat{\pi}$	0.691	0.131	0.017	0.673	0.123	0.016	0.699	0.066	0.004
$\hat{p}_A$	0.854	0.039	0.001	0.856	0.029	0.001	0.851	0.021	0.0004
$\hat{p}_B$	0.356	0.152	0.025	0.379	0.149	0.022	0.393	0.070	0.005
$r = 600$									
$\hat{\pi}$	0.704	0.117	0.014	0.678	0.113	0.013	0.696	0.061	0.004
$\hat{p}_A$	0.850	0.033	0.001	0.856	0.028	0.001	0.853	0.019	0.0004
$\hat{p}_B$	0.354	0.132	0.019	0.391	0.127	0.016	0.396	0.065	0.004
True parameters[c]: $\pi = 0.80$, $p_A = 0.95$, $p_B = 0.50$									
$r = 200$									
$\hat{\pi}$	0.790	0.111	0.012	0.770	0.115	0.014	0.803	0.044	0.002
$\hat{p}_A$	0.952	0.026	0.001	0.954	0.025	0.001	0.949	0.012	10^{-4}
$\hat{p}_B$	0.440	0.229	0.056	0.457	0.218	0.049	0.492	0.080	0.006
$r = 400$									
$\hat{\pi}$	0.825	0.066	0.005	0.815	0.051	0.003	0.803	0.033	0.001
$\hat{p}_A$	0.946	0.016	0.0003	0.947	0.011	10^{-4}	0.949	0.009	10^{-4}
$\hat{p}_B$	0.410	0.178	0.040	0.437	0.139	0.023	0.499	0.059	0.003
$r = 600$									
$\hat{\pi}$	0.822	0.058	0.004	0.814	0.050	0.003	0.798	0.024	0.001
$\hat{p}_A$	0.946	0.014	0.0002	0.948	0.011	10^{-4}	0.950	0.007	10^{-4}
$\hat{p}_B$	0.425	0.147	0.027	0.446	0.122	0.018	0.509	0.042	0.002

Grid search in the area:
[a] $(\pi, p_A, p_B) \in [0.6, 0.8] \times [0.75, 0.95] \times [0.1, 0.3]$ (step size 0.2)
[b] $(\pi, p_A, p_B) \in [0.6, 0.8] \times [0.75, 0.95] \times [0.3, 0.5]$ (step size 0.2)
[c] $(\pi, p_A, p_B) \in [0.7, 0.9] \times [0.79, 0.99] \times [0.4, 0.6]$ (step size 0.2)

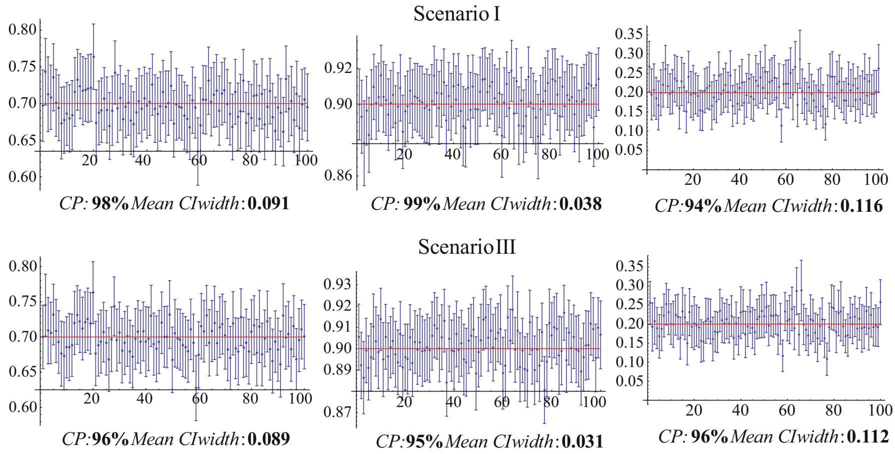

Fig. 1 95 % confidence intervals (CI) for $r = 600$, $c = 5$, $d = 2$, $\pi = 0.7$, $p_{\mathrm{A}} = 0.9$, $p_{\mathrm{B}} = 0.2$, with the respective observed coverage probabilities (CP) and mean width of the CI

increases Bias, SE and RMSE, but this negative effect can be balanced by larger sample sizes; a grid search around the moment estimates (which can be numerically approximated) could also be a good strategy.

Finally, in Fig. 1, we have the 95 % confidence intervals for $r = 600$, $c = 5$, $d = 2$, with $\pi = 0.7$, $p_{\mathrm{A}} = 0.9$, $p_{\mathrm{B}} = 0.2$ and Scenarios I and III. Obviously, the observed coverage probabilities in Scenario III are closer to their nominal levels than those for Scenario I; also note that the sample mean of the width of confidence intervals is slightly smaller in Scenario III.

6 Concluding remarks

In this article, we have provided a new direction for the estimation problem in the context of start-up demonstration theory; specifically, we have studied the two-component mixture models for waiting times used in start-up demonstration theory, by offering general sufficient conditions for the identifiability of these models. This study has focused on the case of i.i.d. binary trials and the results in Sect. 3 have covered all the start-up demonstration models studied in the literature. The necessary steps for the implementation of the EM algorithm have also been described in detail while the numerical study carried out has shown that there is good accuracy in the proposed estimation procedure, for at least large sample sizes and/or groups of units with quite distinct characteristics.

Future work could include the study of identifiability of μ-component mixture models, with $\mu > 2$. Relaxing the i.i.d. condition and/or dealing with multistate trials will also be of great importance. We are currently working on these issues and hope to report our findings in a future paper.

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References

Aki, S., Balakrishnan, N., Mohanty, S. G. (1996). Sooner and later waiting time problems for success and failure runs in higher order Markov dependent trials. *Annals of the Institute of Statistical Mathematics*, *48*, 773–787.

Antzoulakos, D. L., Koutras, M. V., Rakitzis, A. C. (2009). Start-up demonstration tests based on run and scan statistics. *Journal of Quality Technology*, *41*, 1–12.

Balakrishnan, N., Chan, P. S. (2000). Start-up demonstration tests with rejection of units upon observing d failures. *Annals of the Institute of Statistical Mathematics*, *52*, 184–196.

Balakrishnan, N., Koutras, M. V. (2002). *Runs and scans with applications*. New York: Wiley.

Balakrishnan, N., Mohanty, S. G., Aki, S. (1997). Start-up demonstration tests under Markov dependence model with corrective actions. *Annals of the Institute of Statistical Mathematics*, *49*, 155–169.

Balakrishnan, N., Koutras, M. V., Milienos, F. S. (2014a). Start-up demonstration tests: models, methods and applications, with some unifications. *Applied Stochastic Models in Business and Industry*, *30*, 373–413 (with discussion).

Balakrishnan, N., Koutras, M. V., Milienos, F. S. (2014b). Some binary start-up demonstration tests and associated inferential methods. *Annals of the Institute of Statistical Mathematics*, *66*, 759–787.

Barlow, R., Proschan, F. (1981). *Statistical theory of reliability and life testing*. New York: Holt, Reinhart and Winston.

Fu, J. C., Lou, W. Y. W. (2003). *Distribution theory of runs and patterns and its applications: A finite Markov chain imbedding approach*. Singapore: World Scientific.

Gera, A. E. (2010). A new start-up demonstration test. *IEEE Transactions on Reliability*, *59*, 128–131.

Gera, A. E. (2011). A general model for start-up demonstration tests. *IEEE Transactions on Reliability*, *60*, 295–304.

Hahn, G. J., Gage, J. B. (1983). Evaluation of a start-up demonstration test. *Journal of Quality Technology*, *15*, 103–106.

Johnson, N. L., Kotz, S., Kemp, A. W. (1992). *Univariate discrete distributions* (2nd ed.). New York: Wiley.

Martin, D. E. K. (2004). Markovian start-up demonstration tests with rejection of units upon observing d failures. *European Journal of Operational Research*, *155*, 474–486.

Martin, D. E. K. (2008). Application of auxiliary Markov chains to start-up demonstration tests. *European Journal of Operational Research*, *184*, 574–582.

McLachlan, G., Peel, D. (2004). *Finite mixture models*. Hoboken: Wiley.

Smith, M. L., Griffith, W. S. (2005). Start-up demonstration tests based on consecutive successes and total failures. *Journal of Quality Technology*, *37*, 186–198.

Smith, M. L., Griffith, W. S. (2011). Multistate start-up demonstration tests. *International Journal of Reliability, Quality and Safety Engineering*, *18*, 99–117.

Teicher, H. (1963). Identifiability of finite mixtures. *The Annals of Mathematical Statistics*, *34*, 1265–1269.

Viveros, R., Balakrishnan, N. (1993). Statistical inference from start-up demonstration test data. *Journal of Quality Technology*, *25*, 119–130.

Yakowitz, S. J., Spragins, J. D. (1968). On the identifiability of finite mixtures. *The Annals of Mathematical Statistics*, *39*, 209–214.

Yalcin, F., Eryilmaz, S. (2012). Start-up demonstration test based on total successes and total failures with dependent start-ups. *IEEE Transactions on Reliability*, *61*, 227–230.

Yue, P., Cui, L., Wang, L. (2010). Improvement on start-up demonstration test. In *Electrical and control engineering (international conference on electrical and control engineering, ICECE), Wuhan, China* (pp. 3819-3822).

Zhao, X. (2014). On generalized start-up demonstration tests. *Annals of Operations Research*, *212*, 225–239.