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A Bayes minimax result for spherically symmetric unimodal distributions

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Abstract We consider Bayesian estimation of the location parameter θ of a random vector X having a unimodal spherically symmetric density $f(\|x-\theta\|^2)$ for a spherically symmetric prior density $\pi(\|\theta\|^2)$. In particular, we consider minimaxity of the Bayes estimator $\delta_\pi(X)$ under quadratic loss. When the distribution belongs to the Berger class, we show that minimaxity of $\delta_\pi(X)$ is linked to the superharmonicity of a power of a marginal associated to a primitive of f. This leads to proper Bayes minimax estimators for certain densities $f(\|x-\theta\|^2)$.

Keywords Bayes estimators · minimax estimators · Spherically symmetric distributions · Location parameter · Unimodal densities · Quadratic loss · Superharmonic priors

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1 Introduction

Let X be a random vector in $\mathbb{R}^p$ with spherically symmetric density

$$f(\|x-\theta\|^2) \tag{1}$$

around an unknown location parameter θ that we wish to estimate. Any estimator δ is evaluated under the squared error loss

$$\|\delta-\theta\|^2, \tag{2}$$

through the corresponding quadratic risk $E_\theta[\|\delta(X)-\theta\|^2]$, where E_θ denotes the expectation with respect to the density in (1). As soon as $E_0[\|X\|^2]<\infty$, the standard estimator X is minimax, and has constant risk (actually, equal to $E_0[\|X\|^2]$), which entails that minimaxity of δ will be obtained by proving that the risk of δ is less than or equal to the risk of X, that is, if $E_\theta[\|\delta(X)-\theta\|^2] \leq E_0[\|X\|^2]$ for any $\theta \in \mathbb{R}^p$ (domination of δ over X being obtained if, furthermore, this inequality is strict for some θ). Note that, as X is admissible for $p \leq 2$, we will assume, in the following, that $p \geq 3$. For a proof of the minimaxity of X in that spherical context, see e.g., Ralescu (2002). For a general discussion of Bayes, minimaxity and admissibility issues, see, e.g., Lehmann and Casella (1998).

In this paper, we consider generalized Bayes estimators of θ for a spherically symmetric prior density, that is, of the form

$$\pi(\|\theta\|^2). \tag{3}$$

As recalled in Fourdrinier and Strawderman (2008), denoting by ∇ the gradient operator, the generalized Bayes estimator is the posterior mean and can be written as

$$\delta_\pi(X) = X + \frac{\nabla M(\|X\|^2)}{m(\|X\|^2)} \tag{4}$$

where

$$m(\|x\|^2) = \int_{\mathbb{R}^p} f(\|x-\theta\|^2)\,\pi(\|\theta\|^2)\,\mathrm{d}\theta \tag{5}$$

is the marginal density and

$$M(\|x\|^2) = \int_{\mathbb{R}^p} F(\|x-\theta\|^2)\,\pi(\|\theta\|^2)\,\mathrm{d}\theta \tag{6}$$

with

$$F(t) = \frac{1}{2}\int_t^\infty f(u)\,\mathrm{d}u \tag{7}$$

for $t \geq 0$.

It is shown in Fourdrinier et al. (2013) that the finiteness risk condition of X, that is, $\mu_2 = E_0[\|X\|^2] < \infty$, is equivalent to $\gamma = \int_{\mathbb{R}^p} F(\|x\|^2)\,\mathrm{d}x < \infty$ (actually,

$\gamma = \mu_2/p$). It is also worth noting that, when the prior density $\pi(\|\theta\|^2)$ in (3) is superharmonic, this condition is sufficient to guarantee the finiteness of the risk of the generalized Bayes estimator $\delta_\pi(X)$ in (4) (see Fourdrinier et al. 2012). We will see, in Sect. 2, that this superharmonicity condition can be weakened to include all unimodal prior densities.

Here, we are interested in minimaxity of generalized Bayes estimators in (4) when the sampling density in (1) belongs to the Berger class, that is, when there exists a positive constant c such that

$$\forall t \geq 0 \quad \frac{F(t)}{f(t)} \geq c \tag{8}$$

(see Berger 1975). Fourdrinier and Strawderman (2008) proved, provided that

$$E_\theta\left[\left\|\frac{\nabla M(\|X\|^2)}{m(\|X\|^2)}\right\|^2\right] < \infty, \tag{9}$$

the risk difference between $\delta_\pi(X)$ and X is bounded above by

$$E_\theta\left[2\,c\,\frac{\Delta M(\|X\|^2)}{m(\|X\|^2)} - 2\,c\,\frac{\nabla M(\|X\|^2)\cdot\nabla m(\|X\|^2)}{m^2(\|X\|^2)} + \frac{\left\|\nabla M(\|X\|^2)\right\|^2}{m^2(\|X\|^2)}\right] \tag{10}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^p$. Thus, the generalized Bayes estimator in (4) will be minimax as soon as, for all $x \in \mathbb{R}^p$,

$$\mathcal{O}_c(\|x\|^2) = 2\,c\,\frac{\Delta M(\|x\|^2)}{m(\|x\|^2)} - 2\,c\,\frac{\nabla M(\|x\|^2)\cdot\nabla m(\|x\|^2)}{m^2(\|x\|^2)} + \frac{\left\|\nabla M(\|x\|^2)\right\|^2}{m^2(\|x\|^2)} \leq 0. \tag{11}$$

Then, thanks to the superharmonicity of $\pi(\|\theta\|^2)$, superharmonicity of M was guaranteed as well and, getting rid of the Laplacian term in (11), they were led to prove that

$$\forall x \in \mathbb{R}^p \quad - 2\,c\,\frac{\nabla M(\|x\|^2)\cdot\nabla m(\|x\|^2)}{m^2(\|x\|^2)} + \frac{\left\|\nabla M(\|x\|^2)\right\|^2}{m^2(\|x\|^2)} \leq 0$$

to demonstrate the above risk difference is nonpositive. Doing that, the scope of Bayes minimax estimators is reduced; for instance, proper Bayes minimax estimators are excluded. Our goal, here, is to provide a different expression for (11) which allows to preserve the Laplacian term and to not impose superharmonicity of the prior.

Throughout this paper we will assume that the functions f and π in (1) and (3) are absolutely continuous. In addition, we will typically assume that π is non constant to avoid the undetermined form 0/0. Note that when π is constant, the generalized Bayes estimator $\delta_\pi(X)$ in (4) corresponds to the usual estimator X which is already minimax.

In Sect. 2, we show that unimodality of the sampling and prior densities in (1) and (3) is the basic condition for the generalized Bayes estimator $\delta_\pi(X)$ in (4) to be a shrinkage estimator (i.e., $\|\delta_\pi(X)\| \leq \|X\|$), which guarantees that it has finite risk as soon as $E_0[\|X\|^2] < \infty$. In Sect. 3, we propose an upper bound for (11) in terms of the marginal M only, which allows to derive sufficient conditions for minimaxity of $\delta_\pi(X)$. In Sect. 4, we show that, if there exists $\beta \leq 1$ such that M^β is superharmonic, then the generalized Bayes estimator in (4) is minimax, retrieving the result of Stein (1981) in the normal case for which $M = m$ and $\beta = 1/2$. For $\beta < 1$, we demonstrate that the generalized Bayes estimator may in fact be proper Bayes. Examples illustrate the theory. Finally, we give technical results in an Appendix.

2 Conditions for Bayes estimators to be shrinkage estimators

In this section, we show that the generalized Bayes estimator $\delta_\pi(X)$ in (4) is a shrinkage estimator as soon as the sampling and prior densities are unimodal and $E_0[\|X\|^2] < \infty$, which guarantees its risk finiteness.

Theorem 1 *Let $f(\|x-\theta\|^2)$ be a sampling density as in (1) such that $E_0[\|X\|^2] < \infty$ and $\pi(\|\theta\|^2)$ a generalized prior density as in (3). Assume that, for any $x \in \mathbb{R}^p$,*

$$\int_{\mathbb{R}^p} \|\theta\|^2 \, f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta < \infty \tag{12}$$

so that the posterior expected value

$$\frac{\int_{\mathbb{R}^p} \theta \, f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta}{\int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta} \tag{13}$$

exists and the posterior risk is finite. We have the following results.

(a) *The generalized Bayes estimator $\delta_\pi(X)$ in (4) can be written as*

$$\delta_\pi(X) = (1 - a(\|X\|^2))\, X, \tag{14}$$

for some function a from $\mathbb{R}_+$ into $\mathbb{R}$.

(b) *If the sampling density $f(\|x-\theta\|^2)$ is unimodal then $a(\|X\|^2) \leq 1$.*

(c) *If the prior density $\pi(\|\theta\|^2)$ is unimodal then $a(\|X\|^2) \geq 0$.*

(d) *If $f(\|x-\theta\|^2)$ and $\pi(\|\theta\|^2)$ are unimodal then $0 \leq 1 - a(\|X\|^2) \leq 1$, so that $\delta_\pi(X)$ is a shrinkage estimator, i.e.,*

$$\|\delta_\pi(X)\| \leq \|X\|, \tag{15}$$

and $\delta_\pi(X)$ has finite risk.

Proof (a) As $\delta_\pi(X)$ is the posterior expected value expressed in (13) and as, according to part (a) of Lemma 4,

$$\int_{\mathbb{R}^p} \theta \, f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta = \Phi(\|X\|^2)\, X$$

for some function Φ, it follows from (5) that $\delta_\pi(X) = \Phi(\|X\|^2)/m(\|X\|^2)\, X$, so that (14) is satisfied with $a(\|X\|^2)) = 1 - \Phi(\|X\|^2)/m(\|X\|^2)$.

(b) It also follows from Lemma 4 and from (13) that, as the denominator of (13) is positive, $1 - a(\|x\|^2)$ has the sign of $\pi(\|\theta\|^2)$, that is, $a(\|x\|^2) \leq 1$.

(c) Applying Lemma 3 with $g(\theta) = \pi(\|\theta\|^2)$, part (c) follows from the representation

$$\frac{\int_{\mathbb{R}^p} (\theta - x)\, f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta}{\int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta} = 2\, \frac{\int_{\mathbb{R}^p} \theta\, F(\|x-\theta\|^2)\, \pi'(\|\theta\|^2)\, \mathrm{d}\theta}{\int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta}$$

and part (b) of Lemma 4, since F is nonincreasing and since $\pi'(\|\theta\|^2) \leq 0$ by unimodality of $\pi(\|\theta\|^2)$.

(d) Part (d) follows from (b) and (c).

3 Risk difference upper bounds and minimaxity

Note that, from the definition of the marginals in (5) and (6), we have

$$\nabla m(\|x\|^2) = \int_{\mathbb{R}^p} (\theta - x)\, (-2\, f')(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta \tag{16}$$

and

$$\nabla M(\|x\|^2) = \int_{\mathbb{R}^p} (\theta - x)\, f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta \tag{17}$$

so that Lemma 4 clearly applies and there exist two functions $\gamma(\|x\|^2)$ and $\Gamma(\|x\|^2)$ such that

$$\nabla m(\|x\|^2) = \gamma(\|x\|^2)\, x \quad \text{and} \quad \nabla M(\|x\|^2) = \Gamma(\|x\|^2)\, x. \tag{18}$$

In the following lemma, we give an expression for the functions $\gamma(\|x\|^2)$ and $\Gamma(\|x\|^2)$. This is essentially Lemma 3.1 of Fourdrinier and Strawderman (2008) where the function H in (17) of Fourdrinier and Strawderman (2008) is $H(r^2, \|x\|^2) = \|x\|^2\, \mu_r(\|x\|^2)/2\, r^{p+1}$, where $\mu_r(\|x\|^2)$ is given in Formula (21).

Lemma 1 *For any $x \in \mathbb{R}^p$,*

$$\gamma(\|x\|^2) = \int_0^\infty \mu_r(\|x\|^2)\, (-2\, f')(r^2)\, \mathrm{d}r \tag{19}$$

and

$$\Gamma(\|x\|^2) = \int_0^\infty \mu_r(\|x\|^2)\, f(r^2)\, \mathrm{d}r \tag{20}$$

where

$$\mu_r(\|x\|^2) = \frac{1}{\|x\|^2} \int_{S_{r,x}} x \cdot (\theta - x)\, \pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta) = \frac{2\, r}{\|x\|^2} \int_{B_{r,x}} x \cdot \theta\, \pi'(\|\theta\|^2)\, \mathrm{d}\theta\,, \tag{21}$$

where $\sigma_{r,x}$ is the uniform measure on the sphere $S_{r,x}$ of radius r and centered at x (see the reminder before Lemma 4 in the Appendix).

In addition, $\mu_r(\|x\|^2) \leq 0$ as soon as the prior $\pi(\|\theta\|^2)$ in (3) is unimodal, so that $\Gamma(\|x\|^2) \leq 0$ and, when the density in (1) is unimodal, $\gamma(\|x\|^2) \leq 0$. Furthermore, when the Laplacian $\Delta\pi(\|\theta\|^2)$ is a nondecreasing function of $\|\theta\|^2$, for any $x \neq 0$, $\mu_r(\|x\|^2)/r^{p+1}$ is a nondecreasing function of r.

We give now an upper bound for $\mathcal{O}_c(\|x\|^2)$ in (11), provided that $\gamma(\|x\|^2)/\,\Gamma(\|x\|^2)$ is appropriately bounded from below. Note that, from (8), it follows that

$$\forall x \in \mathbb{R}^p \quad \frac{M(\|x\|^2)}{m(\|x\|^2)} \geq c. \tag{22}$$

Theorem 2 *Assume that (8) is satisfied and that, for any $x \in \mathbb{R}^p$,*

$$\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} \geq \frac{1}{2\,c}, \tag{23}$$

where the functions $\gamma(\|x\|^2)$ and $\Gamma(\|x\|^2)$ are defined through (18). Then, for any $x \in \mathbb{R}^p$, an upper bound for $\mathcal{O}_c(\|x\|^2)$ in (11) is given by

$$\mathcal{O}_c(\|x\|^2) \leq 2\,c\,\frac{M(\|x\|^2)}{m(\|x\|^2)}\left[\frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} + \left\{\frac{1}{2} - c\,\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)}\right\}\left\|\frac{\nabla M(\|x\|^2)}{M(\|x\|^2)}\right\|^2\right]. \tag{24}$$

Furthermore we have

$$\mathcal{O}_c(\|x\|^2) \leq 2\,c^2\left[\frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} + \left\{\frac{1}{2} - c\,\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)}\right\}\left\|\frac{\nabla M(\|x\|^2)}{M(\|x\|^2)}\right\|^2\right], \tag{25}$$

as soon as the bracketed term in (24) is nonpositive.

Proof Factorizing $\mathcal{O}_c(\|x\|^2)$ in (11) as

$$2\,c\,\frac{M(\|x\|^2)}{m(\|x\|^2)}\left[\frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} - \frac{\nabla M(\|x\|^2)\cdot\nabla m(\|x\|^2)}{M(\|x\|^2)\,m(\|x\|^2)} + \frac{1}{2\,c}\,\frac{\|\nabla M(\|x\|^2)\|^2}{M(\|x\|^2)\,m(\|x\|^2)}\right]$$

and using

$$\nabla M(\|x\|^2)\cdot\nabla m(\|x\|^2) = \Gamma(\|x\|^2)\,\gamma(\|x\|^2)\,\|x\|^2 = \frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)}\,\|\nabla M(\|x\|^2)\|^2,$$

thanks to (18), we have

$$\mathcal{O}_c(\|x\|^2) = 2\,c\,\frac{M(\|x\|^2)}{m(\|x\|^2)}\left[\frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} + \left\{\frac{1}{2\,c} - \frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)}\right\}\frac{\|\nabla M(\|x\|^2)\|^2}{M(\|x\|^2)\,m(\|x\|^2)}\right].$$

Then Inequality (24) immediately follows from (22) and (23).

In the same way, Inequality (25) follows from (22) as soon as the bracketed term in (24) is nonpositive. □

Condition (23) is a condition on the model. Note that, in the normal case with $f(t) \propto \exp(-t/2\sigma^2)$, we have $c = \sigma^2$ and $\gamma(\|x\|^2)/\Gamma(\|x\|^2) = 1/\sigma^2$ so that Condition (23) is clearly satisfied.

The main interest of Inequalities (24) and (25) is that their common bracketed term depends only on the marginal M. Thus, when Condition (23) is satisfied and this bracketed term is nonpositive, the Bayes estimator $\delta_\pi(X)$ is minimax as soon as the marginal M is superharmonic. More precisely, we have the following corollary.

Corollary 1 *Assume that the sampling density $f(\|x - \theta\|^2)$ in (1) and the prior density $\pi(\|\theta\|^2)$ in (3) are unimodal. Under the conditions of Theorem 2, if M is superharmonic, then*

$$\mathcal{O}_c(\|x\|^2) \leq 2\,c^2\, \frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} \leq 0$$

so that the Bayes estimator $\delta_\pi(X)$ in (4) is minimax. In particular, if the prior $\pi(\|\theta\|^2)$ in (3) is superharmonic, then $\delta_\pi(X)$ is minimax.

Proof Thanks to (d) in Theorem 1, $\delta_\pi(X)$ has finite risk. From (24) and (23), it follows that

$$\begin{aligned}\mathcal{O}_c(\|x\|^2) &\leq 2\,c\, \frac{M(\|x\|^2)}{m(\|x\|^2)}\, \frac{\Delta M(\|x\|^2)}{M(\|x\|^2)}\\ &\leq 2\,c^2\, \frac{\Delta M(\|x\|^2)}{M(\|x\|^2)},\end{aligned}$$

according to (22) since, by superharmonicity assumption, $\Delta M(\|x\|^2) \leq 0$. Then the corollary is immediate. □

As noted above, Fourdrinier and Strawderman (2008) studied minimaxity for estimation of θ for spherically symmetric distributions. In particular, their main result (Theorem 3.1) implied minimaxity under the following conditions:

I) conditions on $\pi(\|\theta\|^2)$: the prior $\pi(\|\theta\|^2)$ is superharmonic (and hence unimodal) and its Laplacian $\Delta(\pi(\|\theta\|^2))$ is nondecreasing in $\|\theta\|^2$;
II) conditions on f: $f(\|x - \theta\|^2)$ is unimodal, $f'(t)/f(t)$ is nondecreasing, $F(t)/f(t) \geq c > 0$ and (with a change of variable $r^2 = t$)

$$\frac{\int_0^\infty (-2\,f'(r^2))\,r^{p+1}\,\mathrm{d}r}{\int_0^\infty f(r^2)\,r^{p+1}\,\mathrm{d}r} \geq \frac{1}{2\,c}. \tag{26}$$

In the course of the proof (after Eq. 22), they essentially demonstrate that

$$\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} \geq \frac{\int_0^\infty (-2\,f'(r^2))\,r^{p+1}\,\mathrm{d}r}{\int_0^\infty f(r^2)\,r^{p+1}\,\mathrm{d}r} \geq \frac{1}{2\,c}. \tag{27}$$

Hence, it follows that Condition (23) of this paper is satisfied as soon as (26) holds and that minimaxity follows from (27) and Theorem 2. However, Fourdrinier and Strawderman (2008) require the above monotonicity conditions on $\pi(\|\theta\|^2)$, on $\Delta(\pi(\|\theta\|^2))$ and on $f'(t)/f(t)$, and these are essential in establishing (27).

Theorem 2, in contrast, requires only $F(t)/f(t) \geq c > 0$ and $\gamma(\|x\|^2)/\Gamma(\|x\|^2) \geq 1/2\,c$ along with superharmonicity of $\pi(\|\theta\|^2)$, and makes no monotonicity assumption on $F(t)/f(t)$ (and hence on $f'(t)/f(t)$). Therefore, its potential applicability is broader than Fourdrinier and Strawderman (2008).

The next corollary gives a class of examples where minimaxity follows from Theorem 2, but where Fourdrinier and Strawderman (2008) is not applicable in those cases where $\psi(t) = F(t)/f(t)$ is not monotone.

Corollary 2 *Let*

$$f(t) \propto \frac{1}{\psi(t)} \exp\left(-\frac{1}{2}\int^t \frac{1}{\psi(u)}\,\mathrm{d}u\right) \tag{28}$$

where, for a given $c > 0$, $\psi(t)$ is a differentiable function such that $0 < c \leq \psi(t) \leq 2\,c < \infty$ and $\psi'(t) \geq -1/2$ for all $t \geq 0$. Assume that the prior density $\pi(\|\theta\|^2)$ is superharmonic and that $\Delta(\pi(\|\theta\|^2))$ is a nondecreasing function of $\|\theta\|^2$. Then the Bayes estimator $\delta_\pi(X)$ in (4) is minimax.

Proof From (28) we may assume, without loss of generality, $\int_{\mathbb{R}^p} f(\|x-\theta\|^2)\,\mathrm{d}x = 1$ since $f(t) \leq K\,\exp(-t/4\,c)$ for some constant $K > 0$. Note that the condition $\psi'(t) \geq -1/2$ suffices to imply $f'(t) \leq 0$, so that the density $f(\|x-\theta\|^2)$ is unimodal, since

$$f'(t) \propto -\frac{\psi'(t) + 1/2}{\psi^2(t)}\,\exp\left(-\frac{1}{2}\int^t \frac{1}{\psi(u)}\,\mathrm{d}u\right).$$

Note also that, as $f(t) = -2\,F'(t)$ and, for any $t_0 \geq 0$, $\lim_{t\to\infty}\int_{t_0}^t 1/\psi(u)\,\mathrm{d}u = \infty$ by the boundedness property of ψ , we have

$$F(t) \propto \exp\left(-\frac{1}{2}\int^t \frac{1}{\psi(u)}\mathrm{d}u\right)$$

so that

$$\frac{F(t)}{f(t)} = \psi(t) > 0.$$

Now, by construction, $F(t)/f(t) \geq c$ and $f(t)/F(t) \geq 1/2\,c$ which implies, according to Corollary 4, $\gamma(\|x\|^2)/\Gamma(\|x\|^2) \geq 1/2\,c$ since unimodality of $\pi(\|\theta\|^2)$ is guaranteed by its superharmonicity and since $\Delta(\pi(\|\theta\|^2))$ is a nondecreasing function of $\|\theta\|^2$. Finally minimaxity of $\delta_\pi(X)$ follows from Corollary 1. □

As noted above, when $\psi(t)$ is chosen to be nonmonotone in Corollary 2, minimaxity does not follow from Fourdrinier and Strawderman (2008). Here is an example of such a function.

Choose $\psi(t) = (t^2 - 2\alpha t + \beta)^{-1} + \gamma$ with $\alpha \geq 0$, $\beta - \alpha^2 \geq 3/2$ and $\gamma > 0$. As $\beta - \alpha^2 > 0$, the polynomial in the expression of $\psi(t)$ is positive. It follows that $\psi(t)$ is well defined and is positive for all $t \geq 0$ (note that $\gamma > 0$). In addition $\psi(t)$ is nondecreasing for $0 \leq t \leq \alpha$ and nonincreasing for $t \geq \alpha$, and hence nonmonotone, since $\psi'(t) = -2(t - \alpha)(t^2 - 2\alpha + \beta)^{-2}$. Furthermore, as $\psi(0) = 1/\beta + \gamma$, $\psi(\alpha) = 1/(\beta - \alpha^2) + \gamma$ and $\lim_{t\to\infty} \psi(t) = \gamma$, Condition (8) is satisfied with $c > 0$ such that $c \leq \gamma$ and $1/(\beta - \alpha^2) + \gamma \leq 2c$ (which implies $c \geq 1/(\beta - \alpha^2)$). Finally, we have unimodality of the density $f(\|x - \theta\|^2)$ since, considering the case where $t \geq \alpha$ (when $t \leq \alpha$, $\psi'(t) \geq 0$) and setting $u = t - \alpha$ and $\delta = \beta - \alpha^2$, it can be seen that $\psi'(t) \geq -1/2$ is equivalent to $(u^2 + \delta)^2 - 4u \geq 0$, which is satisfied for $u \geq 0$ since $\delta \geq 3/4$. Indeed $(u^2 + \delta)^2 - 4u \geq 0$ for $\delta \geq -u^2 + 2\sqrt{u}$ and $-u^2 + 2\sqrt{u}$ has a maximum at $u = (1/2)^{2/3}$ which equals $3/2^{4/3}$ which is smaller than $3/2$.

Explicit examples of densities $f(\|x - \theta\|^2)$ can also be directly derived (up to proportionality) that satisfy the conditions of Corollary 2. Indeed, an alternative expression for $f(t)$ is $f(t) = \varphi'(t)\exp(-\varphi(t)/2)$ where $\varphi(t) = \int^t 1/\psi(u)\,\mathrm{d}u$ with $\lim_{t\to\infty} \varphi(t) = \infty$, so that $F(t)/f(t) = 1/\varphi'(t)$. Then the requirements on φ are that $\varphi'(t)$ is positive, but not monotone, and, more precisely, that $1/2c \leq \varphi'(t) \leq 1/c$. As for unimodality of $f(\|x - \theta\|^2)$, the condition is $\varphi''(t) - 1/2\,\{\varphi'(t)\}^2 < 0$.

As an example, consider

$$\varphi(t) = a\,t + b\,\log(1 + t) + \frac{b}{1 + t}$$

where a and b are two positive constants. Then $\lim_{t\to\infty} \varphi(t) = \infty$ and

$$\varphi'(t) = a + \frac{b}{1 + t} - \frac{b}{(1 + t)^2} = a + \frac{b}{1 + t}\left(1 - \frac{1}{1 + t}\right).$$

From the second expression of $\varphi'(t)$ it is easily seen that φ' is not monotone and that $0 < a \leq \varphi'(t) \leq a + b/4 = \varphi'(1)$, which implies that $\varphi(t) \geq \varphi(0) = b > 0$. Hence we can form

$$f(t) = \left\{a + \frac{b}{1 + t}\left(1 - \frac{1}{1 + t}\right)\right\} \exp\left(-\frac{1}{2}\,a\,t + b\,\log(1 + t) + \frac{b}{1 + t}\right)$$

with the appropriate requirements on φ (also with $1/2c \leq a$ and $a + b/4 \leq 1/c$). Finally, the unimodality condition is satisfied for $b < a^2/2$ since

$$\varphi''(t) = -\frac{b}{(1 + t)^2} + \frac{2\,b}{(1 + t)^3} = -\frac{b}{(1 + t)^2}\left(1 - \frac{2}{1 + t}\right) < b$$

implies $\varphi''(t) - \{\varphi'(t)\}^2/2 < b - a^2/2 < 0$.

Here is another class of examples for which the generating function f is modified so that $\psi(t) = F(t)/f(t)$ becomes nonmonotone, but where Theorem 2 implies minimaxity. We note once more that, in each of the above examples (and the one which follows), minimaxity of the generalized Bayes estimator follows from Corollary 1 of this paper, but does not follow from the results of Fourdrinier and Strawderman (2008).

We assume the pair (f, F) is such that $\psi(u) \geq c$ and $\psi(u)$ is continuous, and is strictly monotone increasing on the interval $[a, b]$ where $0 < a < b < \infty$. Define

$$\psi_1(t) = \begin{cases} \psi(t) & \text{for } 0 \leq t \leq a \text{ and } b \leq t < \infty; \\ \psi^*(t) & \text{for } a < t < b, \end{cases}$$

where $\psi_1(t)$ is continuous (so $\psi^*(a) = \psi(a)$ and $\psi^*(b) = \psi(b)$) and so that $\psi^*(t) \geq c$, but $\psi^*(t)$ is not monotone (this can be done in infinitely many ways including a linear decrease on $]a, d[$ from $\psi^*(a) = \psi(a)$ to $\psi^*(d) = c$ and a linear increase on $]d, b[$ from $\psi^*(d) = c$ to $\psi^*(b) = \psi(b)$ for any $d \in]a, b[$). Then define (for some $k > 0$ chosen so that $\int_{\mathbb{R}^p} f_1(\|x - \theta\|^2)\, \mathrm{d}x = 1$)

$$F_1(t) = k \exp\left(-\frac{1}{2} \int^t \frac{1}{\psi_1(u)}\, \mathrm{d}u\right)$$

and

$$f_1(t) = k \frac{1}{\psi_1(t)} \exp\left(-\frac{1}{2} \int^t \frac{1}{\psi_1(u)}\, \mathrm{d}u\right).$$

Hence $F_1(t)/f_1(t) = \psi_1(t) \geq c$, but $F_1(t)/f_1(t)$ is not monotone increasing (since $\psi_1(t)$ is not monotone increasing).

Now note we may choose $\psi_1(t)$, for any $\epsilon > 0$ such that $1-\epsilon \leq f_1(t)/f(t) \leq 1+\epsilon$ and $1 - \epsilon \leq F_1(t)/F(t) \leq 1 + \epsilon$. Hence, by Corollary 4, setting, with $y = \|x\|^2$, $L(r, y) = \left|\int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}A\sigma_{r,x}(\theta)\right|$, we have

$$\begin{aligned} \frac{\gamma_1(y)}{\Gamma_1(y)} &= E_x^*\left[\frac{f_1(R^2)}{F_1(R^2)}\right] \\ &= \frac{\int_0^\infty f_1(r^2)\, L(r, y)\, \mathrm{d}r}{\int_0^\infty F_1(r^2)\, L(r, y)\, \mathrm{d}r} \\ &= \frac{\int_0^\infty \frac{f_1(r^2)}{f(r^2)}\, f(r^2)\, L(r, y)\, \mathrm{d}r}{\int_0^\infty \frac{F_1(r^2)}{F(r^2)}\, F(r^2)\, L(r, y)\, \mathrm{d}r} \\ &\geq \frac{1 - \varepsilon}{1 + \varepsilon} \frac{\int_0^\infty f(r^2)\, L(r, y)\, \mathrm{d}r}{\int_0^\infty F(r^2)\, L(r, y)\, \mathrm{d}r} \\ &= \frac{1 - \varepsilon}{1 + \varepsilon} \frac{\gamma(y)}{\Gamma(y)}. \end{aligned}$$

Hence, if $\gamma(y)/\Gamma(y) > 1/2c$, it is possible to choose f_1 such that $\gamma_1(y)/\Gamma_1(y) > 1/2c$. Therefore, if $X \sim f_1(\|x - \theta\|^2)$, minimaxity of the generalized Bayes estimator corresponding to $\pi(\|\theta\|^2)$ satisfying the conditions of Corollary 2 is guaranteed. Since

ψ_1 is not monotone, minimaxity does not follow from Fourdrinier and Strawderman (2008).

It is worth noticing that, if $f(t)$ is a variance mixture of normals, $F(t)/f(t) = \psi(t)$ is always strictly monotone increasing provided the mixing distribution is not degenerate at a point. In addition, we note that Theorem 3.1 of Fourdrinier and Strawderman (2008) may be applied to $f(t)$ such that $f(t)/(-f'(t))$ is monotone increasing, and that this implies $F(t)/f(t)$ is monotone increasing. The results of this paper also apply to $f_1(t)$ constructed above, while the results of Fourdrinier and Strawderman (2008) do not.

4 Superharmonicity of a power of M and minimaxity

The development in this section is in the spirit of Stein (1981) where he demonstrated, in the normal case, that, if $m^{1/2}(\cdot)$ is superharmonic, then the corresponding generalized Bayes estimator is minimax. Our results center on proving minimaxity when $M^{\beta}(\|\cdot\|^2)$ is superharmonic for some $1/2 \le \beta \le 1$.

It will be relevant to bound from above the bracketed terms in (24) and in (25) using a lower bound α for $\gamma(\|x\|^2)/\Gamma(\|x\|^2)$ greater than or equal to $1/2c$. Given such a bound, an upper bound for $\mathcal{O}_c(\|x\|^2)$ in (11) may involve the Laplacian of a certain power of the marginal $M(\|x\|^2)$. This is specified in the following theorem.

Theorem 3 *Assume that there exists $\alpha \in \mathbb{R}_+$ such that, for any $x \in \mathbb{R}^p$,*

$$\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} \ge \alpha \ge \frac{1}{2\,c}. \tag{29}$$

Then, for any $x \in \mathbb{R}^p$, an upper bound for $\mathcal{O}_c(\|x\|^2)$ in (11) is given by

$$\mathcal{O}_c(\|x\|^2) \le \frac{2\,c}{\beta}\,\frac{M^{1-\beta}(\|x\|^2)}{m(\|x\|^2)}\,\Delta M^{\beta}(\|x\|^2), \tag{30}$$

where $\beta = 3/2 - c\,\alpha$.

Proof Thanks to (29), Inequality (23) is satisfied so that we are under the conditions of Theorem 2 . Then it follows from (24) that

$$\begin{aligned}\mathcal{O}_c(\|x\|^2) &\le 2\,c\,\frac{M(\|x\|^2)}{m(\|x\|^2)}\left[\frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} + \left\{\frac{1}{2} - c\,\alpha\right\}\left\|\frac{\nabla M(\|x\|^2)}{M(\|x\|^2)}\right\|^2\right]\\ &= 2\,c\,\frac{M(\|x\|^2)}{m(\|x\|^2)}\left[\frac{\Delta M(\|x\|^2)}{M(\|x\|^2)} + \{\beta - 1\}\left\|\frac{\nabla M(\|x\|^2)}{M(\|x\|^2)}\right\|^2\right]\\ &= \frac{2\,c}{\beta}\,\frac{M^{1-\beta}(\|x\|^2)}{m(\|x\|^2)}\,\Delta M^{\beta}(\|x\|^2), \end{aligned} \tag{31}$$

according to the definition of β and to Lemma 6. $\square$

It follows from Theorem 3 that a sufficient condition for $\delta_\pi(X)$ in (4) to be minimax is that M^β is superharmonic. Note that, as Condition (29) implies that $\beta \leq 1$, superharmonicity of M^β is a weaker condition than superharmonicity of the marginal M itself. When this is the case, the following corollary provides another upper bound for $\mathcal{O}_c(\|x\|^2)$.

Corollary 3 *Under the conditions of Theorem 3, if $M^\beta(\|\cdot\|^2)$ is superharmonic, then*

$$\mathcal{O}_c(\|x\|^2) \leq \frac{2\,c^2}{\beta}\,\frac{\Delta M^\beta(\|x\|^2)}{M^\beta(\|x\|^2)} \leq 0 \tag{32}$$

so that the Bayes estimator in (4) is minimax.

Proof The proof is similar to the proof of Corollary 1. □

In the normal case where $f(t) = 1/(2\,\pi)^{p/2}\,e^{-t/2}$, we have $c = 1$ and $M = m$. Then Inequalities (30) and (32) which give an upper bound for $\mathcal{O}_c(\|x\|^2)$ are in fact equalities for $\beta = 1/2$, that is,

$$\mathcal{O}_c(\|x\|^2) = 4\,\frac{\Delta m^{1/2}(\|x\|^2)}{m^{1/2}(\|x\|^2)}.$$

This is the result of Stein (1981).

Not only is superharmonicity of $M^\beta(\|\cdot\|^2)$ weaker than that of $M(\|\cdot\|^2)$ but, for $1/2 \leq \beta \leq 1$, there is another possible benefit; namely $M^\beta(\|\cdot\|^2)$ may be superharmonic and simultaneously $M(\|\cdot\|^2)$ (and hence $\pi(\|\cdot\|^2)$) may be proper if p is sufficiently large. This allows the possibility, which will be demonstrated in the example below, that $M(\|\cdot\|^2)$, and hence $\pi(\|\cdot\|^2)$, leads to a proper Bayes minimax estimator which is automatically admissible.

In particular, if

$$M(\|x\|^2) = \left(\frac{1}{b + \|x\|^2}\right)^{(p-2)/2\beta}$$

and $1/2 \leq \beta \leq 1$, then, for $p > 2/(1-\beta)$, it is straightforward to demonstrate that $M^\beta(\|\cdot\|^2)$ is superharmonic and $M(\|\cdot\|^2)$ is also integrable (proper). Note, for $\beta = 1/2$, the above gives $p > 4$ and, for $\beta = 3/4$, $p > 8$, but, for $\beta = 1$, there is no p for which $M(\|\cdot\|^2)$ is integrable (as was pointed out more generally in Fourdrinier et al. 1998). This example is intended to be illustrative of the point that superharmonicity of $M^\beta(\|\cdot\|^2)$ may be useful. This particular $M(\|\cdot\|^2)$ may not correspond to any choice of sampling density or generalized prior.

The following minimaxity result illustrates the applicability of Corollary 3 when the sampling density in (1) and the prior density in (3) are variance mixtures of normals, that is,

$$f(t) = \int_0^\infty \frac{1}{(2\,\pi\,v)^{p/2}}\,\exp\!\left(-\frac{t}{2v}\right) \mathrm{d}G(v), \tag{33}$$

where G is the distribution of a random variable V on $\mathbb{R}_+$ and when

$$\pi(\|\theta\|^2) = \int_0^\infty \frac{1}{(2\pi t)^{p/2}} \exp\left(-\frac{\|\theta\|^2}{2t}\right) h(t)\, \mathrm{d}t \tag{34}$$

for a certain (possibly improper) mixing density h.

Theorem 4 *Assume that* $0 < a \leq V \leq b < \infty$ *such that*

$$\frac{a}{b} \geq \frac{3}{2} - \beta \tag{35}$$

with $1/2 < \beta \leq 1$ *and that the distribution G of V has a density g such that* $g(u-t)$ *has nondecreasing monotone likelihood ratio in t with respect to the parameter u. Assume also that, for any* $v \in \mathbb{R}_+$,

$$\lim_{t\to\infty} (v+t)^{-p/2}\, h(t) = 0 \tag{36}$$

and that $-(t+b)\, h'(t)/h(t)$ *can be decomposed as* $l_1(t) + l_2(t)$ *where* $0 \leq l_1(t) \leq A$ *and is nondecreasing while* $0 \leq l_2(t) \leq B$.

Then the Bayes estimator $\delta_\pi(X)$ *in* (4) *is minimax provided*

$$\left[1 - (1-\beta)\,\frac{a}{b}\right] A + B \leq (1-\beta)\,\frac{p-2}{2}. \tag{37}$$

Proof First note that c in (8) can be expressed as

$$c = \frac{\int_a^b v^{1-p/2}\, \mathrm{d}G(v)}{\int_a^b v^{-p/2}\, \mathrm{d}G(v)}$$

[see (33) in Fourdrinier and Strawderman 2008]. Hence Condition (35) implies $1/b \geq (3/2-\beta)/c$ so that, as according to Lemma 8, for any $x \in \mathbb{R}^p$, $\gamma(\|x\|^2)/\Gamma(\|x\|^2)$ is an expectation of V^{-1} and as V is bounded from above by b,

$$\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} \geq \frac{1}{b} \geq \frac{3/2-\beta}{c}. \tag{38}$$

Therefore (29), is satisfied with $\alpha = (3/2-\beta)/c$ since $\beta \leq 1$ (see comment after the proof of Theorem 3).

Now we will show that $\Delta M^\beta(\|x\|^2) \leq 0$ so that, according to Corollary 3, the Bayes estimator $\delta_\pi(X)$ in (4) is minimax. By Lemma 9, this is equivalent to

$$Q + R \leq (1-\beta)\,\frac{p-2}{2}, \tag{39}$$

where

$$Q = -h(0) \left[\frac{\int_0^\infty v^{1-p/2} \exp\left(\frac{-s}{v}\right) \mathrm{d}G(v)}{\int_0^\infty J_{1+p/2}(v)\, v \,\mathrm{d}G(v)} - (1-\beta) \frac{\int_0^\infty v^{2-p/2} \exp\left(\frac{-s}{v}\right) \mathrm{d}G(v)}{\int_0^\infty J_{p/2}(v)\, v\,\mathrm{d}G(v)} \right] \tag{40}$$

and

$$R = \frac{\int_0^\infty \int_0^\infty -\frac{h'(t)}{(v+t)^{p/2}} \exp\left(\frac{-s}{v+t}\right) \mathrm{d}t\, v\,\mathrm{d}G(v)}{\int_0^\infty J_{1+p/2}(v)\, v\,\mathrm{d}G(v)} \\ -(1-\beta) \frac{\int_0^\infty \int_0^\infty -\frac{h'(t)}{(v+t)^{p/2-1}} \exp\left(\frac{-s}{v+t}\right) \mathrm{d}t\, v\,\mathrm{d}G(v)}{\int_0^\infty J_{p/2}(v)\, v\,\mathrm{d}G(v)} \tag{41}$$

with

$$J_k(v) = \int_0^\infty (t+v)^{-k} \exp\left(\frac{-s}{t+v}\right) h(t)\,\mathrm{d}t.$$

Inequality (39) will follow from showing that

$$Q \leq 0 \tag{42}$$

and

$$R \leq (1-\beta)\frac{p-2}{2}. \tag{43}$$

Using the expression of Q in (40), Inequality (42) can be written as

$$1-\beta \leq \frac{\int_0^\infty v^{1-p/2} \exp\left(\frac{-s}{v}\right) \mathrm{d}G(v)}{\int_0^\infty v^{2-p/2} \exp\left(\frac{-s}{v}\right) \mathrm{d}G(v)} \frac{\int_0^\infty J_{p/2}(v)\, v\,\mathrm{d}G(v)}{\int_0^\infty J_{1+p/2}(v)\, v\,\mathrm{d}G(v)}. \tag{44}$$

As $\beta \geq 1/2$ so that $1-\beta \leq 1/2$, we will show that the right-hand side of (44) can be bounded from below by $1/2$. Note first that, in the right-hand side of (44), as $a < V < b$,

$$\frac{\int_0^\infty v^{1-p/2} \exp\left(\frac{-s}{v}\right) \mathrm{d}G(v)}{\int_0^\infty v^{2-p/2} \exp\left(\frac{-s}{v}\right) \mathrm{d}G(v)} = E^*[V^{-1}] \geq \frac{1}{b} \tag{45}$$

where E^* is the expectation with respect to a density proportional to $v^{2-p/2} \exp(-s/v)$ with respect to G.

Expressing the second ratio in the right-hand side of (44) as

$$\begin{aligned}
\frac{\int_0^\infty J_{p/2}(v)\, v \,\mathrm{d}G(v)}{\int_0^\infty J_{p/2+1}(v)\, v \,\mathrm{d}G(v)} &= \frac{\int_0^\infty \ v^{p/2} J_{p/2}(v)\, v^{1-p/2} \,\mathrm{d}G(v)}{\int_0^\infty \ v^{p/2+1}\, J_{p/2+1}(v)\, v^{-p/2} \,\mathrm{d}G(v)} \\
&\geq \frac{\int_0^\infty \ v^{p/2+1} J_{p/2+1}(v)\, v^{1-p/2} \,\mathrm{d}G(v)}{\int_0^\infty \ v^{p/2+1}\, J_{p/2+1}(v)\, v^{-p/2} \,\mathrm{d}G(v)} \\
&= E^{**}[V]
\end{aligned} \tag{46}$$

where E^{**} is the expectation with respect to a density proportional to $J_{p/2+1}(v)\, v$ with respect to G and where the above inequality follows from

$$v^k\, J_k(v) \geq v^{k+1}\, J_{k+1}(v)$$

applied with $k = p/2$. As $V \geq a$, (45) and (46) give that the right-hand side of (42) is bounded from below by a/b, and hence, by $1/2$ since, by assumption $2a \geq b$. Thus (42), is obtained.

In terms of the density g we can see that N in (41) is expressed as

$$R \ = \ E^*_{p/2+1}\left[-\frac{h^{'}(T)}{h(T)}\,(V+T)\right] - (1-\beta)E^*_{p/2}\left[-\frac{h^{'}(T)}{h(T)}\,(V+T)\right] \tag{47}$$

where E^*_k is the expectation with respect to the density

$$f_k(t,v) \ \propto \ \left(\frac{1}{t+v}\right)^k \exp\left(\frac{-s}{t+v}\right)\, h(t)\, v\, g(v). \tag{48}$$

Using the decomposition in the statement of the theorem, Equality (47) becomes

$$\begin{aligned}
R = E^*_{p/2+1}&\left[l_1(T)\frac{V+T}{b+T} + l_2(T)\frac{V+T}{b+T}\right] \\
&-(1-\beta)E^*_{p/2}\left[l_1(T)\frac{V+T}{b+T} + l_2(T)\frac{V+T}{b+T}\right] \\
&\leq E^*_{p/2+1}[l_1(T) + B] - (1-\beta)\, E^*_{p/2}\left[l_1(T)\frac{a}{b}\right]
\end{aligned} \tag{49}$$

since l_1 and l_2 are nonnegative and $0 < a \leq V \leq b < \infty$.

Now note that the density in (48) has nonincreasing monotone likelihood ratio in $v + t$ with respect to k since

$$\frac{f_{k+1}(t,v)}{f_k(t,v)} = \frac{1}{v+t}.$$

As l_1 is nondecreasing and, according to Lemma 7 (note that $v\, g(v)$ has the same monotone likelihood property as $g(v)$), the density of T given $T + V$ does not depend on k and has monotone likelihood ratio in $T + V$. Hence (49) becomes

$$
\begin{aligned}
R &\leq E^*_{p/2}\left[l_1(T) + B\right] - (1-\beta)\,\frac{a}{b}\,E^*_{p/2}\left[l_1(T)\right] \\
&= E^*_{p/2}\left[\left\{1 - (1-\beta)\,\frac{a}{b}\right\}\,l_1(T)\right] + B \\
&\leq \left\{1 - (1-\beta)\,\frac{a}{b}\right\}\,A + B
\end{aligned}
\tag{50}
$$

since $l_1(T) \leq A$, and therefore, according to (37), we obtain (43), which gives with (42) the desired result. □

Theorem 4 is still valid for $\beta = 1/2$, the case where the distribution of V is degenerate, and hence, has no density.

Here is an example illustrating Theorem 4. It gives generalized and proper Bayes minimax estimators corresponding to Strawderman-type priors (see Fourdrinier et al. 1998) for the class of mixtures of normal sampling distributions of Theorem 4. The dimension cut-off between proper Bayes minimax and generalized Bayes minimax estimators depends on $\beta \geq 3/2 - a/b$ and varies between $p = 5$, for $\beta = 1/2$, to ∞, for $\beta = 1$.

Example 1 Let the sampling mixing density g be any density on $[a, b]$ such that $g(u - t)$ has nondecreasing monotone likelihood ratio as in Theorem 4 (e.g., the uniform distribution on $[a, b]$). Let the prior mixing density h be

$$
h(t) \propto \frac{1}{(b+t)^A} \tag{51}
$$

with $A \geq 0$ so that $-(t+b)\,h'(t)/h(t) \equiv A$. Choosing $l_1(t) \equiv A$ and $l_2(t) \equiv B = 0$ in Theorem 4 implies minimaxity for the corresponding generalized (or proper) Bayes estimator provided $\beta \geq 3/2 - a/b$, $1/2 < \beta \leq 1$ and, by (37),

$$
\left[1 - (1-\beta)\,\frac{a}{b}\right] A \leq (1-\beta)\,\frac{p-2}{2},
$$

or, taking $\beta = 3/2 - a/b$,

$$
A \leq \frac{a/b - 1/2}{1 - (a/b - 1/2)(a/b)}\;\frac{p-2}{2}. \tag{52}
$$

Note that the mixing density (51) is proper as soon as $A > 1$. By (52), this is possible provided $1/2 \leq a/b < 1$ and

$$
p > 2 + 2\,\frac{1 - (a/b - 1/2)(a/b)}{a/b - 1/2}. \tag{53}
$$

For $a/b = 1$, which corresponds to $\beta = 1/2$, minimaxity follows for $p > 4$. Since, in this case, the mixing distribution is degenerate, this corresponds to Strawderman's result Strawderman (1971) for the normal distribution. It also corresponds to Stein's result (1981) that implies minimaxity in the normal case provided $m^{1/2}$ is superharmonic (recall that $m = M$ in the normal case).

If $a/b = 3/4$, then $\beta = 3/4$ and (53) holds for $p > 8.5$ so that proper Bayes minimaxity results by a choice of $A > 1$ for $p \geq 9$. This corresponds to superharmonicity of $M^{3/4}$. If $a/b = 1/2$, the right-hand side of (53) is infinite and Theorem 4 does not lead to proper Bayes minimaxity for any dimension. Note, however, that (52) gives generalized Bayes minimax estimators for all $p \geq 3$ provided $1/2 \leq a/b \leq 1$.

As a last remark, note that, for $A = 0$, the mixing density in (51) is constant, which corresponds to $\pi(\|\theta\|^2) = 1/\|\theta\|^{p-2}$ the fundamental harmonic function.

5 Concluding remarks

We have studied Bayes minimax estimators for the case of a spherically symmetric unimodal distribution under squared error loss. A main result is that spherically symmetric superharmonic priors with nondecreasing Laplacian lead to minimaxity for unimodal densities which satisfy $0 < c < F/f < 2c < \infty$. This paper also implies the main result in Fourdrinier and Strawderman (2008) as well, but goes well beyond that paper in terms of generality as indicated by the above result when $\psi(t) = F(t)/f(t)$ is nonmonotone. We also extend the scope of Fourdrinier and Strawderman (2008) in that proper Bayes minimax estimators are produced for certain variance mixtures of normal densities. This possibility arises when M^{β} is superharmonic for some β in the interval $(1/2, 1)$, and when the dimension, p, is larger than $2/(1-\beta)$.

In the process of proving the minimaxity findings, we develop some technical results which may be of independent interest. In particular, Lemmas 3, 4, 5 and 8 may have useful applications in shrinkage estimation problems for spherically symmetric distributions.

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6 Appendix

We recall below a Stein-type lemma in the framework of spherically symmetric distributions whose proof can be found in Fourdrinier and Strawderman (2008).

Lemma 2 *Let X be a random vector in $\mathbb{R}^p$ with density as in (1) and let h be a weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}^p$. Then*

$$E_\theta[(X-\theta)\cdot h(X)] = E_\theta\left[\frac{F(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\,\mathrm{div} h(X)\right], \tag{54}$$

where F is defined in (7), provided these expectations exist.

Lemma 2 leads directly to the following result.

Lemma 3 *Let g be a weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}$. Then, for f in* (1) *and F in* (7)*, we have*

$$\int_{\mathbb{R}^p} (\theta - x)\, f(\|x-\theta\|^2)\, g(\theta)\, \mathrm{d}\theta = \int_{\mathbb{R}^p} \nabla_\theta g(\theta)\, F(\|x-\theta\|^2)\, \mathrm{d}\theta$$
$$= -\int_{\mathbb{R}^p} \nabla_\theta F(\|x-\theta\|^2)\, g(\theta)\, \mathrm{d}\theta, \qquad (55)$$

provided that one of these integrals exists. In addition, when the function f is absolutely continuous, we have

$$\int_{\mathbb{R}^p} (\theta - x)\, (-2\, f')(\|x-\theta\|^2)\, g(\theta)\, \mathrm{d}\theta = \int_{\mathbb{R}^p} \nabla_\theta g(\theta)\, f(\|x-\theta\|^2)\, \mathrm{d}\theta$$
$$= -\int_{\mathbb{R}^p} \nabla_\theta f(\|x-\theta\|^2)\, g(\theta)\, \mathrm{d}\theta. \qquad (56)$$

Proof We will only prove (55) since this result only relies on the absolute continuity of F. Let $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $\theta = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p$ and $1 \le i \le p$. Note that

$$(\theta_i - x_i)\, g(\theta) = (\theta - x) \cdot h^{(i)}(\theta)$$

where $h^{(i)}(\theta) = (0, \ldots, 0, g(\theta), 0, \ldots, 0)$ is the vector in $\mathbb{R}^p$ whose all components are equal to 0 with the exception of the ith component which is equal to $g(\theta)$. Then we have

$$\int_{\mathbb{R}^p} (\theta_i - x_i)\, f(\|x-\theta\|^2)\, g(\|\theta\|)\, \mathrm{d}\theta = E_x\Big[(\theta - x)\cdot h^{(i)}(\theta)\Big]$$
$$= E_x\left[\mathrm{div}_\theta h^{(i)}(\theta)\, \frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\right]$$
$$= E_x\left[\frac{\partial g(\theta)}{\partial \theta_i}\, \frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\right]$$
$$= \int_{\mathbb{R}^p} \frac{\partial g(\theta)}{\partial \theta_i}\, F(\|x-\theta\|^2)\, \mathrm{d}\theta \qquad (57)$$

where, for the second equality, Lemma 2 is applied with $h = h^{(i)}$ (the role of x and θ being interchanged) and, for the third equality, the fact that $\mathrm{div}_\theta h^{(i)}(\theta) = \partial g(\theta)/\partial\theta_i$ is used. This gives the first equality in (55). Finally, the second equality in (55) is derived noticing that $(\theta - x)\, f(\|x-\theta\|^2) = -\nabla_\theta F(\|x-\theta\|^2)$. □

A main feature of Lemma 3 is that the second equality in (55) is valid under weak assumptions on g and F while, in the literature, stronger assumptions are needed. Thus, in Fourdrinier et al. (2012), the function $\theta \longmapsto F\left(\|x-\theta\|^2\right)$ belongs to a functional space close to the Schwarz space. Here, only the weak differentiability of g is needed.

The next two results are used in Sects. 2 and 3. We use the following notation. For $x \in \mathbb{R}^p$ and $r \geq 0$, $U_{r,x}$ and $\sigma_{r,x}$ are, respectively, the uniform distribution and the uniform measure on the sphere $S_{r,x} = \{\theta \in \mathbb{R}^p \,/ \|\theta - x\| = r\}$ of radius r and centered at x. They are related by the following property. If γ is a Lebesgue integrable function then

$$\begin{aligned}\int_{\mathbb{R}^p} \gamma(\theta)\,\mathrm{d}\theta &= \int_0^\infty \int_{S_{r,x}} \gamma(\theta)\,\mathrm{d}\sigma_{r,x}\,\mathrm{d}r \\ &= \int_0^\infty \sigma_{r,x}(S_{r,x}) \int_{S_{r,x}} \gamma(\theta)\,\mathrm{d}U_{r,x}\,\mathrm{d}r \end{aligned} \tag{58}$$

with

$$\sigma_{r,x}(S_{r,x}) = \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}.$$

It follows from (58) that, if $V_{r,x}$ is the uniform distribution on the ball $B_{r,x} = \{\theta \in \mathbb{R}^p \,/ \|\theta - x\| \leq r\}$ of radius r and centered at x and if λ is the Lebesgue measure on $\mathbb{R}^p$, we have

$$\begin{aligned}\int_{B_{r,x}} \gamma(\theta)\,\mathrm{d}V_{r,x}(\theta) &= \frac{1}{\lambda(B_{r,x})} \int_0^r \int_{S_{\tau,x}} \gamma(\theta)\,\mathrm{d}\sigma_{\tau,x}(\theta)\,\mathrm{d}\tau \\ &= \frac{p}{r^p} \int_0^r \int_{S_{\tau,x}} \gamma(\theta)\,\mathrm{d}U_{\tau,x}(\theta)\,\tau^{p-1}\,\mathrm{d}\tau. \end{aligned} \tag{59}$$

Lemma 4 *Let $x \in \mathbb{R}^p$ be fixed and let Θ be a random vector in $\mathbb{R}^p$ with a spherically symmetric distribution around x. Let g be a function from $\mathbb{R}_+$ into $\mathbb{R}$. Denote by E_x the expectation with respect to the distribution of Θ.*

(a) *If, for $r \geq 0$, Θ has the uniform distribution $U_{r,x}$ on the sphere $S_{r,x}$ of radius r and centered at x, then there exists a function G from $\mathbb{R}_+$ into $\mathbb{R}$ such that*

$$E_x\left[g(\|\Theta\|^2)\,\Theta\right] = G(\|x\|^2)\,x, \tag{60}$$

provided this expectation exists. Therefore (60), is valid for any spherically symmetric distribution.

(b) *If Θ has a unimodal spherically symmetric density $f(\|\theta - x\|^2)$ (f is nonincreasing) and if the function g is nonnegative then the function G in (60) is nonnegative.*

Proof We will use the orthogonal decomposition $\Theta = x + U = x + \alpha + \beta$ with U spherically symmetric around 0, $\alpha \in \triangle_x$ and $\beta \in \triangle_x^\perp$ where $\triangle_x$ denotes the linear space spanned by x and $\triangle_x^\perp$ is its orthogonal subspace in $\mathbb{R}^p$. We have

$$E_x\Big[\Theta\, g(\|\Theta\|^2)\Big] = A(x) + B(x) \tag{61}$$

where

$$A(x) = E_0[(x+\alpha)\, g(\|x+\alpha\|^2 + \|\beta\|^2)]$$

and

$$B(x) = E_0\big[E_0[\beta\, g(\|x+\alpha\|^2 + \|\beta\|^2)|\alpha]\big] = 0,$$

since $\beta|\alpha$ is spherically symmetric around 0. Setting $\alpha = Z\, x/\|x\|$, we have

$$A(x) = x\, G(\|x\|^2)$$

where

$$G(\|x\|^2) = E_0\left[E_0\left[\left(1+\frac{Z}{\|x\|}\right) g\left(\|x\|^2\left(1+\frac{Z}{\|x\|}\right)^2 + \|\beta\|^2\right)\middle|\beta\right]\right].$$

This proves the first part.

Now assume that the density $f(\|\theta - x\|^2)$ is unimodal and consider

$$\begin{aligned} E_0&\left[\left(1+\tfrac{Z}{\|x\|}\right) g\left(\|x\|^2\left(1+\tfrac{Z}{\|x\|}\right)^2 + \|\beta\|^2\right)\middle|\beta, \left(1+\tfrac{Z}{\|x\|}\right)^2 = y^2\right] \\ &= g\big(\|x\|^2\, y^2 + \|\beta\|^2\big)\, E_0\left[\left(1+\tfrac{Z}{\|x\|}\right)\middle|\beta, \left(1+\tfrac{Z}{\|x\|}\right)^2 = y^2\right]. \end{aligned}$$

To finish the proof it suffices to show that the above conditional expectation is non-negative which can be seen noticing that

$$\begin{aligned} E_0&\left[\left(1+\frac{Z}{\|x\|}\right)\middle|\beta, \left(1+\frac{Z}{\|x\|}\right)^2 = y^2\right] \\ &= |y|\, \frac{f\big(\{[-1+|y|]^2\, \|x\|^2\} + \|\beta\|^2\big) - f\big(\{[-1-|y|]^2\, \|x\|^2\} + \|\beta\|^2\big)}{f\big(\{[-1+|y|]^2\, \|x\|^2\} + \|\beta\|^2\big) + f\big(\{[-1-|y|]^2\, \|x\|^2\} + \|\beta\|^2\big)} \geq 0, \end{aligned}$$

by monotonicity of f. □

Versions of Lemma 4 have been often used in the literature (for instance, in Cellier et al. 1995 and in Fourdrinier et al. 2012) when dealing with spherical densities. Here, we provide an extension to the entire class of spherically symmetric distributions.

Lemma 5 *If the prior $\pi(\|\theta\|^2)$ in (3) is unimodal and if its Laplacian $\Delta(\pi(\|\theta\|^2))$ is a nondecreasing function of $\|\theta\|^2$ then, for any $r \geq 0$ and for any $x \in \mathbb{R}^p$,*

$$\int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta) = \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1} \int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{r,x}(\theta) \leq 0. \quad (62)$$

Proof Under the conditions of the lemma, Fourdrinier and Strawderman (2008) showed that the sphere mean $\int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{r,x}(\theta)$ is a nondecreasing function of r and that the ball mean $\int_{B_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}V_{r,x}(\theta)$ is nonpositive (which may also be derived from Lemma 4). Therefore, for any $r \geq 0$,

$$2\,\|x\|^2\,\pi'(\|x\|^2) = x \cdot \nabla\pi(\|x\|^2) \leq \int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{r,x}(\theta)$$

and, as $\pi'(\|x\|^2) \leq 0$ by unimodality of $\pi(\|\theta\|^2)$, $\int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{r,x}(\theta)$ is nonpositive in a neighborhood of 0. If there exists $r_0 > 0$ such that

$$\delta = \int_{S_{r_0,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{r_0,x}(\theta) > 0,$$

then, using (59), we have

$$\begin{aligned}
0 &\geq \int_{B_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}V_{r,x}(\theta) \\
&= \frac{p}{r^p}\Bigg(\int_0^{r_0} \tau^{p-1} \int_{S_{\tau,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{\tau,x}(\theta)\, \mathrm{d}\tau \\
&\quad + \int_{r_0}^{r} \tau^{p-1} \int_{S_{\tau,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}U_{\tau,x}(\theta)\, \mathrm{d}\tau\Bigg) \\
&\geq \frac{p}{r^p}\left(2\,\|x\|^2\,\pi'(\|x\|^2) \int_0^{r_0} \tau^{p-1}\, \mathrm{d}\tau + \delta \int_{r_0}^{r} \tau^{p-1}\, \mathrm{d}\tau\right) \\
&= \frac{1}{r^p}\left(2\, r_0^p\, \|x\|^2\, \pi'(\|x\|^2) + \delta\, (r^p - r_0^p)\right).
\end{aligned}$$

As this last quantity goes to δ when r goes to infinity, the nonpositivity of the ball mean $\int_{B_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}V_{r,x}(\theta)$ is contradicted and (62) follows. □

The sphere mean in (62) occurs in Bayesian analysis such as in Fourdrinier and Strawderman (2008). The interest of Lemma 5 is that its sign may be controled although the integrand term changes sign.

Corollary 4 *Under the conditions of Lemma 5, for any $y \in \mathbb{R}_+$, the ratio of the functions defined in (19) and (20) equals, for any $x \in \mathbb{R}^p$ and for $y = \|x\|^2$,*

$$\frac{\gamma(y)}{\Gamma(y)} = E_x^*\left[\frac{f(R^2)}{F(R^2)}\right] \tag{63}$$

where E_x^* is the expectation with respect to a density proportional to

$$F(r^2)\left|\int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta)\right|,$$

provided the function π is not constant.

Proof According to (18), we can write, for any $x \in \mathbb{R}^p$,

$$x \cdot \nabla m(\|x\|^2) = \|x\|^2 \, \gamma(\|x\|^2)$$

and

$$x \cdot \nabla M(\|x\|^2) = \|x\|^2 \, \Gamma(\|x\|^2)$$

so that

$$\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} = \frac{x \cdot \nabla m(\|x\|^2)}{x \cdot \nabla M(\|x\|^2)}. \tag{64}$$

Now, by absolute continuity of f and F, interchange of the gradient and the integral sign is valid so that, according to the expressions of m and M in (5) and (6), (64) becomes

$$\begin{aligned}\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} &= \frac{x \cdot \int_{\mathbb{R}^p} \nabla f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta)}{x \cdot \int_{\mathbb{R}^p} \nabla F(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta)} \\ &= \frac{x \cdot \int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \nabla\pi(\|\theta\|^2)\, \mathrm{d}\theta)}{x \cdot \int_{\mathbb{R}^p} F(\|x-\theta\|^2)\, \nabla\pi(\|\theta\|^2)\, \mathrm{d}\theta)},\end{aligned} \tag{65}$$

by Lemma 3. Hence, by linearity of the inner product and integrating over the sphere, it follows from (65) that

$$\begin{aligned}\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} &= \frac{\int_0^\infty f(r^2) \int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta)\mathrm{d}r}{\int_0^\infty F(r^2) \int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta)\mathrm{d}r} \\ &= \frac{\int_0^\infty f(r^2) \left| \int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta) \right| \mathrm{d}r}{\int_0^\infty F(r^2) \left| \int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, \mathrm{d}\sigma_{r,x}(\theta) \right| \mathrm{d}r},\end{aligned} \tag{66}$$

since, according to (62), $\int_{S_{r,x}} x \cdot \nabla\pi(\|\theta\|^2)\, d\sigma_{r,x}(\theta)$ has constant sign. Hence the result follows. □

The next lemma is used in the proof of Theorem 3.

Lemma 6 *Let φ be a function from $\mathbb{R}^p$ into $\mathbb{R}$ such that its Laplacian exists on $\mathbb{R}^p$ and let $\beta \in \mathbb{R}$. Then, for any $x \in \mathbb{R}^p$, we have*

$$\Delta\varphi^\beta(x) = \beta\, \varphi^\beta(x) \left[\frac{\Delta\varphi(x)}{\varphi(x)} + (\beta - 1) \left\| \frac{\nabla\varphi(x)}{\varphi(x)} \right\|^2 \right]. \tag{67}$$

Proof For any $x \in \mathbb{R}^p$, we have

$$\begin{aligned}\Delta\varphi^\beta(x) &= \operatorname{div}\big(\nabla\varphi^\beta(x)\big)\\ &= \operatorname{div}\Big(\beta\,\varphi^{\beta-1}(x)\,\nabla\varphi(x)\Big)\\ &= \beta\,\Big\{\varphi^{\beta-1}(x)\operatorname{div}(\nabla\varphi(x)) + \nabla\varphi^{\beta-1}(x)\cdot\nabla\varphi(x)\Big\}\\ &= \beta\,\Big\{\varphi^{\beta-1}(x)\,\Delta\varphi(x) + (\beta-1)\,\varphi^{\beta-2}(x)\,\nabla\varphi(x)\cdot\nabla\varphi(x)\Big\}\\ &= \beta\,\varphi^\beta(x)\left\{\frac{\Delta\varphi(x)}{\varphi(x)} + (\beta-1)\,\frac{\|\nabla\varphi(x)\|^2}{\varphi^2(x)}\right\}.\end{aligned}$$

□

The next lemma is used in the proof of Theorem 4.

Lemma 7 *Let V and T two random variables such that (V, T) has density $f_{(V,T)}$ of the form*

$$f_{(V,T)}(v,t) = q(t+v)\,h(t)\,g(v) \tag{68}$$

for some functions q, h and g.

Let $U = V + T$. Then the density of T given U does not depend on q.

If $g(u-t)$ has nondecreasing monotone likelihood ratio in t with respect to u and if λ is a nondecreasing function then the conditional expectation $E[\lambda(T)\,|\,U = u]$ is nondecreasing in u.

Proof Clearly (U, T) has density $f_{(U,T)}$ given by

$$f_{(U,T)}(u,t) = f_{(V,T)}(u-t,t)$$

so that the conditional density of T given $U = u$ can be expressed as

$$f_T(t\,|\,U=u) = \frac{f_{(U,T)}(u,t)}{\int f_{(U,T)}(u,t')\,\mathrm{d}t'} = \frac{f_{(V,T)}(u-t,t)}{\int f_{(V,T)}(u-t',t')\,\mathrm{d}t'}.$$

Using (68) this becomes

$$f_T(t\,|\,U=u) = \frac{q(u)\,h(t)\,g(u-t)}{\int q(u)\,h(t')\,g(u-t')\,\mathrm{d}t'} = \frac{h(t)\,g(u-t)}{\int h(t')\,g(u-t')\,\mathrm{d}t'}.$$

Hence the first result follows.

Now, according to the monotone likelihood property of g, for fixed $u_1 < u_2$,

$$\frac{f_T(t\,|\,U=u_2)}{f_T(t\,|\,U=u_1)} \propto \frac{g(u_2-t)}{g(u_1-t)}$$

is nondecreasing in t. Hence, as λ is a nondecreasing function, $E[\lambda(T)\,|\,U = u]$ is nondecreasing in u. □

The two following lemmas are used in the proof of Theorem 4.

Lemma 8 *Assume that the sampling density in* (1) *is a variance mixture of normals as in* (33), *that is,*

$$f(t) = \int_0^\infty \frac{1}{(2\pi v)^{p/2}} \exp\left(-\frac{t}{2v}\right) \mathrm{d}G(v), \tag{69}$$

where G is the distribution of a random variable V on $\mathbb{R}_+$. Denote by E the expectation with respect to G. Assume also that the prior density is as in (3) *and is unimodal, that is, $\pi'(\|\theta\|^2) \le 0$. For $\lambda > 0$ and for a random variable W having a noncentral chi-squared distribution $\chi_p^2(\lambda)$ with p degrees of freedom and noncentral parameter λ, let*

$$H'(\lambda, V) = \frac{\partial}{\partial \lambda} E_\lambda[\pi(V\,W)]. \tag{70}$$

Then, for any $y \in \mathbb{R}_+$, the ratio of the functions defined in (19) *and* (20) *equals*

$$\frac{\gamma(y)}{\Gamma(y)} = \frac{E\left[\frac{1}{V}\,|H'(\frac{y}{2V}, V)|\right]}{E\left[|H'(\frac{y}{2V}, V)|\right]} = E_y^*[V^{-1}] \tag{71}$$

where E_y^* is the expectation defined through the second equality in (71).

Proof According to (18), we can write, for any $x \in \mathbb{R}^p$,

$$2\,\|x\|^2\, m'(\|x\|^2) = x \cdot \nabla m(\|x\|^2) = \|x\|^2\, \gamma(\|x\|^2)$$

and

$$2\,\|x\|^2\, M'(\|x\|^2) = x \cdot \nabla M(\|x\|^2) = \|x\|^2\, \Gamma(\|x\|^2)$$

so that

$$\frac{\gamma(\|x\|^2)}{\Gamma(\|x\|^2)} = \frac{m'(\|x\|^2)}{M'(\|x\|^2)}. \tag{72}$$

Now, thanks to (5) and to (69), we have

$$\begin{aligned} m(\|x\|^2) &= \int_{\mathbb{R}^p} \int_0^\infty \frac{1}{(2\pi v)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2v}\right) \mathrm{d}G(v)\, \pi(\|\theta\|^2)\, \mathrm{d}\theta \\ &= \int_0^\infty \int_{\mathbb{R}^p} \frac{1}{(2\pi v)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2v}\right) \pi(\|\theta\|^2)\, \mathrm{d}\theta\, \mathrm{d}G(v) \\ &= E\big[E_{\|x\|^2/2V}[\pi(V\,W)]\big], \end{aligned} \tag{73}$$

by Fubini's Theorem for the second equality and noticing, for the third equality, that $W = \|\theta\|^2/V|V \sim \chi_p^2(\lambda)$ with $\lambda = \|x\|^2/2V$. Similarly, thanks to (6) and to (69) and by definition of F in (7), we have

$$
\begin{aligned}
M(\|x\|^2) &= \int_{\mathbb{R}^p} \frac{1}{2} \int_{\|x-\theta\|^2}^{\infty} \int_0^{\infty} \frac{1}{(2\pi v)^{p/2}} \exp\Big(-\frac{u}{2v}\Big) \, \mathrm{d}G(v) \, \mathrm{d}u \, \pi(\|\theta\|^2) \, \mathrm{d}\theta \\
&= \int_0^{\infty} \frac{1}{(2\pi v)^{p/2}} \int_{\mathbb{R}^p} \int_{\|x-\theta\|^2}^{\infty} \frac{1}{2} \exp\Big(-\frac{u}{2v}\Big) \, \mathrm{d}u \, \pi(\|\theta\|^2) \, \mathrm{d}\theta \, \mathrm{d}G(v) \\
&= \int_0^{\infty} \int_{\mathbb{R}^p} \frac{v}{(2\pi v)^{p/2}} \exp\Bigg(-\frac{\|x-\theta\|^2}{2v}\Bigg) \pi(\|\theta\|^2) \, \mathrm{d}\theta \, \mathrm{d}G(v) \\
&= E\big[V \, E_{\|x\|^2/2V}[\pi(V \, W)]\big].
\end{aligned} \tag{74}
$$

Therefore, setting $y = \|x\|^2$ and differentiating with respect to y, we have

$$
m'(y) = E\left[\frac{1}{2V} H'\Big(\frac{y}{2V}, V\Big)\right] \tag{75}
$$

and

$$
M'(y) = E\left[\frac{1}{2} H'\Big(\frac{y}{2V}, V\Big)\right] \tag{76}
$$

with $H'(\lambda, V)$ defined in (70). Note that, as $\pi'(\|\theta\|^2) \leq 0$, we have $H'(\lambda, V) \leq 0$, for any $\lambda > 0$, since $\chi_p^2(\lambda)$ has increasing monotone likelihood ratio in λ. Therefore, it follows from (72), (75) and (76) that

$$
\frac{\gamma(y)}{\Gamma(y)} = \frac{E[\frac{1}{V} H'(\frac{y}{2V}, V)]}{E\big[H'(\frac{y}{2V}, V)\big]} = \frac{E[\frac{1}{V} |H'(\frac{y}{2V}, V)|]}{E\big[|H'(\frac{y}{2V}, V)|\big]} = E_y^*[V^{-1}],
$$

which is the desired result. □

Lemma 9 *Assume that the sampling density in* (1) *and the prior density in* (3) *are variance mixtures of normals as in* (33) *and* (34), *respectively, that is,*

$$
f(t) = \int_0^{\infty} \frac{1}{(2\pi v)^{p/2}} \exp\Big(-\frac{t}{2v}\Big) \, \mathrm{d}G(v), \tag{77}
$$

where G is the distribution of a random variable V on $\mathbb{R}_+$, and

$$
\pi(\|\theta\|^2) = \int_0^{\infty} \frac{1}{(2\pi t)^{p/2}} \exp\Bigg(-\frac{\|\theta\|^2}{2t}\Bigg) h(t) \, \mathrm{d}t, \tag{78}
$$

for a certain (possibly improper) mixing density h. Then the superharmonicity of M^β can be expressed as

$$
Q + R \; \leq \; (1-\beta) \, \frac{p-2}{2}, \tag{79}
$$

where

$$
Q = -h(0) \left[\frac{\int_0^{\infty} v^{1-p/2} \exp\big(\frac{-s}{v}\big) \, \mathrm{d}G(v)}{\int_0^{\infty} J_{p/2+1}(v) \, v \, \mathrm{d}G(v)} - (1-\beta) \frac{\int_0^{\infty} v^{2-p/2} \exp\big(\frac{-s}{v}\big) \, \mathrm{d}G(v)}{\int_0^{\infty} J_{p/2}(v) \, v \, \mathrm{d}G(v)} \right] \tag{80}
$$

and

$$R = \frac{\int_0^\infty \int_0^\infty -\frac{h'(t)}{(v+t)^{p/2}} \exp\left(\frac{-s}{v+t}\right) \mathrm{d}t \, v \, \mathrm{d}G(v)}{\int_0^\infty J_{1+p/2}(v) \, v \, \mathrm{d}G(v)}$$
$$-(1-\beta) \frac{\int_0^\infty \int_0^\infty -\frac{h'(t)}{(v+t)^{p/2-1}} \exp\left(\frac{-s}{v+t}\right) \mathrm{d}t \, v \, \mathrm{d}G(v)}{\int_0^\infty J_{p/2}(v) \, v \, \mathrm{d}G(v)} \tag{81}$$

with

$$J_k(v) = \int_0^\infty (t+v)^{-k} \exp\left(\frac{-s}{t+v}\right) h(t) \, \mathrm{d}t. \tag{82}$$

Proof By the last equality in (31), $\Delta M^\beta(\|x\|^2) \leq 0$ is equivalent to

$$\frac{\Delta M(\|x\|^2)}{\left\|\nabla M(\|x\|^2)\right\|} - (1-\beta) \frac{\left\|\nabla M(\|x\|^2)\right\|}{M(\|x\|^2)} \leq 0, \tag{83}$$

Now, for a sampling density (77) and a prior (78), it easy to derive

$$M(\|x\|^2) = \frac{1}{(2\pi)^{p/2}} \int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2}} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v),$$

so that

$$\nabla M(\|x\|^2) = -\frac{1}{(2\pi)^{p/2}} \int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v) \; x,$$

and

$$\Delta M(\|x\|^2) = \frac{1}{(2\pi)^{p/2}} \int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \left(\frac{\|x\|^2}{t+v} - p\right)$$
$$\times \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v).$$

Hence, Inequality (83) becomes

$$\frac{\int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \left(\frac{\|x\|^2}{t+v} - p\right) \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v)}{\|x\| \int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v)}$$
$$-(1-\beta) \frac{\|x\| \int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v)}{\int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2}} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{t+v}\right) h(t) \, \mathrm{d}t \, \mathrm{d}G(v)} \leq 0,$$

which is equivalent to

$$\frac{\int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+2}} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{t+v}\right) h(t)\,\mathrm{d}t\,\mathrm{d}G(v)}{\int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{t+v}\right) h(t)\,\mathrm{d}t\,\mathrm{d}G(v)}$$

$$-(1-\beta)\,\frac{\int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2+1}} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{t+v}\right) h(t)\,\mathrm{d}t\,\mathrm{d}G(v)}{\int_0^\infty \int_0^\infty \frac{v}{(t+v)^{p/2}} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{t+v}\right) h(t)\,\mathrm{d}t\,\mathrm{d}G(v)} \leq \frac{p}{\|x\|^2}$$

and can be conveniently expressed as

$$\frac{\int_0^\infty J_{p/2+2}(v)\,v\,\mathrm{d}G(v)}{\int_0^\infty J_{p/2+1}(v)\,v\,\mathrm{d}G(v)} - (1-\beta)\,\frac{\int_0^\infty J_{p/2+1}(v)\,v\,\mathrm{d}G(v)}{\int_0^\infty J_{p/2}(v)\,v\,\mathrm{d}G(v)} \leq \frac{p}{2\,s} \tag{84}$$

where $J_k(v)$ is in (82) and $s = \|x\|^2/2$. Setting $\mathrm{d}w = (v+t)^{-2}\exp(-s/(t+v))\,\mathrm{d}t$ and $u = (v+t)^{-k+2}\,h(t)$, so that $w = 1/s\,\exp(-s/(t+v))$ and $\mathrm{d}u = [(v+t)^{1-k}\,(2-k)\,h(t) + (v+t)^{2-k}\,h^{'}(t)]\,\mathrm{d}t$, we have

$$\begin{aligned} J_k(v) = {} & \frac{-1}{s}\,v^{2-k}\,h(0)\,\exp\left(\frac{-s}{v}\right) + \frac{k-2}{s}\,J_{k-1}(v) \\ & -\frac{1}{s}\int_0^\infty (v+t)^{2-k} h^{'}(t)\,\exp\left(\frac{-s}{t+v}\right)\mathrm{d}t, \end{aligned}$$

since

$$\lim_{t\to\infty} (v+t)^{2-k}\,h(t) = 0.$$

Then using this representation of $J_k(v)$ in (84) gives rise to (79) with Q in (80) and R in (81). □

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