# Theorem by Giné and Mason (2008)

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## 1 Introduction

Giné & Mason (2008) study the uniform in bandwidth behaviour of estimators of

$$T(F) = \int_R \phi\left(x, F(x), F^{(1)}(x), \cdots, F^{(r)}(x)\right) dF(x),$$

which have the form

$$T_n(h) = \frac{1}{n}\sum_{i=1}^{n} \phi\left(X_i, F_{n,i}(X_i), F^{(1)}_{n,h_1,i}(X_i), \cdots, F^{(r)}_{n,h_r,i}(X_i)\right),$$

where $h = (h_1, \cdots, h_r), h_i > 0$, is a vector of bandwidths and

$$F_{n,i}(X_i) = \frac{1}{n-1}\sum_{1\leq i\neq j\leq n} K_0^{(-1)}(X_i - X_j),$$

with $K_0^{(-1)}(x) = I\{x > 0\}$, and for $1 \leq m \leq r$,

$$F^{(m)}_{n,h_m,i}(X_i) = \frac{1}{n-1}\frac{1}{h^m}\sum_{1\leq i\neq j\leq n} K_m^{(m-1)}\left(\frac{X_i - X_j}{h}\right), \quad i = 1, \cdots, n,$$

where $K_m^{(0)} = K_m$ is an $L_1$ kernel, $m-1$ times differentiable with $K_m^{(m-1)}(x) = \frac{d^{m-1}K_m(x)}{dx^{m-1}}$ satisfying

$$\int_R K_m(u)du = 1.$$

Define the sequence of processes in $\vec{\lambda} = (\lambda_1, \cdots, \lambda_r), 0 < a \leq \lambda_m \leq b < \infty$, for $m = 1, \cdots r$, by

$$v_n\left(\vec{\lambda}\right) := \sqrt{n}\left\{T_n(\vec{\lambda}\otimes \mathbf{h_n}) - T(F)\right\},$$

where $\{\mathbf{h_n}\}$ is a sequence of vectors $\mathbf{h_n} = (h_{1,n}, \cdots h_{r,n})$ with positive coordinates converging to zero and

$$\vec{\lambda}\otimes \mathbf{h_n} = (\lambda_1 h_{1,n}, \cdots, \lambda_r h_{r,n}).$$

Giné and Mason construct i.i.d. random variables $Y_1, Y_2, \cdots Y_n$ with mean 0 and finite variance, such that

$$\sup_{\vec{\lambda}\in[a,b]^r}\left| v_n\left(\vec{\lambda}\right) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i\right| = o_p(1). \tag{1}$$

## 2 Notation

Towards defining the random variables $Y_1, Y_2, \cdots Y_n$, let

$$\varphi(x) := \varphi\left(x, F(x), F^{(1)}(x), \cdots, F^{(r)}(x)\right),$$

and

$$\varphi_m(x) := \frac{\partial}{\partial y_m}\varphi(x, y_0, y_1, \cdots, y_r)|_{(y_0,y_1,\cdots,y_r)=(F(x),F^{(1)}(x),\cdots,F^{(r)}(x))}.$$

Let

$$\zeta(X_i) = \varphi(X_i) - E\varphi(X) = \varphi(X_i) - T(F),$$

$$\zeta_0(X_i) = \int_{X_i}^{\infty} \varphi_0(X_i)f(y)dy - \int_R F(y)\varphi_0(y)f(y)dy,$$

and, for $m = 1, \cdots, r$, let

$$\chi_m(y) := \frac{d^{m-1}}{dy^{m-1}}\left(\varphi(y)f(y)\right) = -\frac{d^m}{dy^m}\int_y^{\infty} \varphi(x)f(x)dx,$$

where $\frac{d^0}{dy^0}g(y) = g(y)$, and set for $i \geq 1$,

$$\zeta_m(X_i) = (-1)^{m-1}\left\{\chi_m(X_i) - E\chi_m(X)\right\}.$$

Define

$$Y_i = \zeta(X_i) + \sum_{m=0}^{r} \zeta_m(X_i), \quad i \geq 1. \tag{2}$$

## 3 Conditions

For the main result one requires the following assumptions:

*I* For each $m = 1, \cdots, r$, $f^{(m-1)}$ is bounded.

*II* For some $0 < \alpha \leq 1$, and each $m = 1, \cdots, r$, $f^{(m-1)}$ is in $G_{2r+\alpha-1,K_m}(H)$, where $H$ is some non-negative measurable function satisfying $E\left[H^2(X)\right] < \infty$, which may depend on $f^{(m-1)}$. $G_{s,k}(H)$ is the class of all measurable functions $g$ on $\mathbb{R}$ for which there is a positive constant $M_K(g)$ such that for all $h$ sufficiently small

$$\left|\frac{1}{h}\int_R g(u)K\left(\frac{x-u}{h}\right)du - g(x)\right| \leq h^s M_k(g)H(x), \quad \forall x.$$

*III* Uniformly in $x \in \text{supp}(f)$, the function $\varphi(x, y_0, y_1, \cdots, y_r)$ and its partial derivatives $\partial\varphi/\partial y_i, i = 0, \cdots, r$, satisfy a global Lipschitz condition with respect to the variables $y_0, \cdots, y_r$ on a bounded open convex subset of $\mathbb{R}^{r+1}$ containing the closure of the range,

$$\left\{\left(F(x), f(x), f^{(1)}(x), \cdots, f^{(r-1)}(x)\right)\middle| x \in \mathbb{R}.\right\}.$$

*IV* $\varphi_m, m = 0, 1, \cdots, r$, are bounded on the support of $f$.

*V* For each $m = 1, \cdots, r$, the function $\chi_m$ is Lipschitz of order $0 < \beta \leq 1$ and

$$\int_R |u|^{\beta}|K_m(u)|du < \infty.$$

*VI* The kernels $K_m^{(m-1)}, m = 1, \cdots, r$, are in $L_2(\mathbb{R})$ and of bounded variation in $\mathbb{R}$.

*VII* The kernels $K_m$ is smooth and has bounded support.

*VIII* With $0 < \alpha \leq 1$ as in condition *II*, for $m = 1, \cdots, r$,

$$\sqrt{n}\frac{h_{m,n}^{2r-1}}{\log(1/h_{m,n})} \to \infty \text{ and } \sqrt{n}h_{m,n}^{2r+\alpha-1} \to 0.$$

# 4 Theorem

Under conditions *I-VIII*, for all $0 < a < b < \infty$, (1) holds. Moreover, if $0 < \mathrm{Var}(Y) < \infty$, where $Y$ is defined as in (2), then the processes $\left\{v_n\left(\vec{\lambda}\right) : \vec{\lambda} \in [a,b]^r\right\}$ converge in law in $l_\infty([a,b]^r)$ (in the sense of Hoffmann-Jorgensen) to the constant Gaussian process $G\left(\vec{\lambda}\right) = G$ where $G$ is $N(0, \mathrm{Var}(Y_1))$, that is,

$$\left\{T_n(\vec{\lambda} \otimes \mathbf{h_n}) - T(F)\right\} \longrightarrow_d N(0, \mathrm{Var}(Y_1)),$$

uniformly in $\vec{\lambda} \in [a,b]^r$.

# References

Giné, E. & Mason, D. (2008), 'Uniform in bandwidth estimation of integral functionals of the density function', *Scandinavian Journal of Statistics* **35**(4), 739–761.