

Is the Brownian bridge a good noise model on the boundary of a circle?

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Abstract In this paper, we study periodical stochastic processes, and we define the conditions that are needed by a model to be a good noise model on the circumference. The classes of processes that fit the required conditions are studied together with their expansion in random Fourier series to provide results about their path regularity. Finally, we discuss a simple and flexible parametric model with prescribed regularity that is used in applications, and we prove the asymptotic properties of the maximum likelihood estimates of model parameters.

Keywords Fourier transform · Karhunen–Loève’s theorem · Gaussian processes · Periodic processes · Stationary processes · Maximum likelihood

1 Introduction

1.1 Literature review

Modeling the random boundaries of star-shaped planar objects is a topic that is receiving an increasing interest in recent times. Some examples can be found in neurology (see [Hobolth 2003](#) and the references therein), geography ([Burrough and Frank 1996](#)), stereology (see [Hobolth et al. 2003](#); [Hobolth and Vedel Jensen 2002](#) and the refer-

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ences therein), fractal geometry analysis (see [Dioguardi et al. 2003](#) and the references therein).

A common way to model such a phenomena is to model the radius-vector function as a periodic stochastic process from an interval to \mathbb{R} ; a detailed geometric description of this model is provided in [van Lieshout \(2013\)](#). In such a framework, the radius of the star-shaped planar object is a periodic and continuous function, as a function of the independent variable, representing the angle.

A very standard and well-known model that apparently suits these needs is the Brownian bridge. The Brownian bridge is a universally known model, used in several areas of applied mathematical science. As only an example, the recent publication of [Kroese et al \(2011\)](#) and [Manganaro \(2011\)](#) and the reference therein provide a huge relevant literature, while [Bass \(2011\)](#) provides a theoretical analysis of such a process. The main aspect of the Brownian bridge is its periodicity that makes this process a good model for a noise on the finite domain $[0, 1]$. On the other hand, a deficiency of this model is its non-stationarity, which is almost a must when one models pure noise. This is due to the fact that Brownian bridge is assumed to be 0 at $t = 0$.

A second approach that is being obtaining success in recent time is to exploit the asymptotic results of the random Fourier series to provide general models for the boundaries of star-shaped objects. In [Hobolth et al. \(2003\)](#), the authors propose a parametric random Fourier series model (called generalized p -order model) to describe the border of random planar star-shaped objects in terms of normalized radius-vector function. Again, in [Hobolth et al. \(2003\)](#), the authors also provided results about sample path regularity, and an expression for the maximum likelihood function for the model parameters, even if there are not asymptotic results about these estimators.

1.2 Overview and insights of the paper

In this paper, we deal with the second approach, using random Fourier series as a flexible modeling tool, finding interesting properties of the studied processes thanks to the standard representation they provide.

First, we define the theoretical conditions that are needed by a process to be a good noise model on the boundary of a circle, admitting models with a fixed zero value in the origin only as the conditioning of such a process, as the result of a selective sampling. More precisely, two classes of processes are considered:

- \mathcal{H} , the set of Gaussian, stationary, $[0, 1]$ -periodic processes;
- \mathcal{H}_0 , the set of processes generated by a process in \mathcal{H} conditioned to be 0 when $t = 0$.

Then, we remark that the Brownian bridge is not contained in \mathcal{H}_0 . Furthermore, we find a standard Fourier decomposition for a process $\{x_t\}_{t \in [0,1]}$ in \mathcal{H} , by expressing its covariance function $C(s, t)$ as $C(s, t) = c_0^2 + 2 \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(s - t))$. Thanks to Karhunen–Loève's theorem, the process $\{x_t\}_{t \in [0,1]}$ may be represented as:

$$x_t = c_0 Y'_0 + \sum_{k=1}^{\infty} c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t))$$

where $\{Y_k\}_{k \geq 1}$ and $\{Y'_k\}_{k \geq 0}$ are two independent sequences of independent standard Gaussian variables.

As a consequence, we prove that:

- the random Fourier series expansion of a process in \mathcal{H}_0 shares the same asymptotic behavior for the spectrum with its generator in \mathcal{H} ;
- the path regularity of a process in \mathcal{H}_0 depends on path regularity of its generator in \mathcal{H} ; in particular, we show that the regularity properties of the trajectories of a process in \mathcal{H} and of its generated process in \mathcal{H}_0 have the same lower bound in terms of Hölder regularity;
- the path regularity of a process in \mathcal{H} (and of its generated process in \mathcal{H}_0) can be deduced by the Fourier coefficients of the generator process covariance function, looking at their decrease rate. In particular, we show that, for any $0 < \alpha \leq 1$,

$$c_k^2 = O(1/k^{1+2m+\alpha}) \implies \{x_t\}_{t \in [0,1]} \in C^{m,\beta}([0, 1]), \quad \text{with } \beta < \alpha/2,$$

where $C^{m,\beta}([0, 1])$ is the Hölder space of the functions on $[0, 1]$ having continuous derivatives up to order m and such that the m th-derivative is Hölder continuous with exponent β .

Finally, as in [Hobolth et al. \(2003\)](#), we discuss a general and flexible parametric model in \mathcal{H} :

$$x_t = \sum_{k=1}^N \frac{a}{k^p} (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)),$$

together with the generated model in \mathcal{H}_0 . We underline that

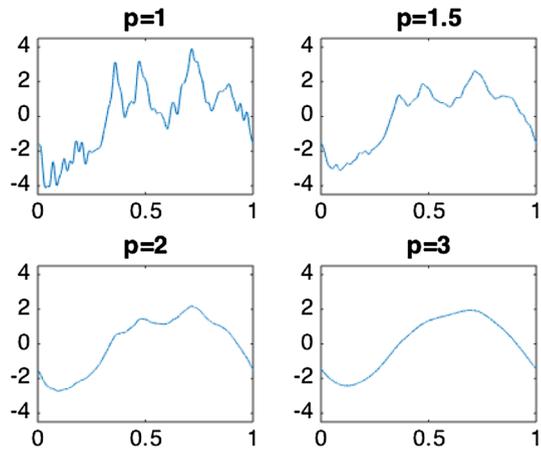
- these models provide a very easy way to represent stochastic processes in computer memory, where only a finite number of coefficient may be stored. In addition, the representation of the first one is built on a finite dimensional subspace made by only trigonometric functions;
- the path regularity of $\{x_t\}_{t \in [0,1]}$ is determined by its parameters, see [Figure 1](#);
- we provide maximum likelihood estimates for the first model, together with asymptotic properties of the estimators.

Summing up, these models can be very useful in the applications: on one hand, they might shape particular characteristics of the observed phenomena, allowing on the other hand properties similar to the Brownian bridge when these are needed, but with a stronger theoretical support.

1.3 Structure of the paper

In [Sect. 2](#), we define the two classes of interest, \mathcal{H} and the space of conditioned processes \mathcal{H}_0 , studying their properties and analyzing their random Fourier series expansion. In [Sect. 3](#), we show the connection between the spectrum of the processes in \mathcal{H}_0 with respect to their generators in \mathcal{H} . In [Sect. 4](#), it is proven that also the

Fig. 1 Parametric model in \mathcal{H} , where path regularity is determined by the parameter p



path regularity is maintained for such couples, as a consequence of Kolmogorov’s continuity conditions and a result of Boas. In Sect. 5, simple parametric models in \mathcal{H} and \mathcal{H}_0 are presented, together with the properties of the maximum likelihood estimators for the parameters.

Summary of notations

The variables s, t, \dots relate to time variables, and will often belong to $[0, 1]$. We denote by $\{x_t\}_{t \in [0,1]}, \{y_t\}_{t \in [0,1]}, \dots$ stochastic adapted process defined on a given filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, \mathbb{P})$, while $\{X_n\}_{n \geq 1}, \{Y_n\}_{n \geq 1}, \{Z_n\}_{n \geq 1}, \dots$ are sequences of random variables. $C(s, t)$ is a positive semidefinite function (it will be the correlation function of a stochastic process). When a process has stationary increments, its covariance function will often be replaced by the associated covariogram function $\tilde{C}(t - s) = C(s, t)$. The sequence $\{e_k(t)\}_{k \geq 0}$ denotes a sequence of orthogonal function on $L^2([0, 1])$. Finally, we denote by $\|t\|_1$ the fractional part of the real number t that is the sawtooth wave defined by the formula $\|t\|_1 = t - \text{floor}(t)$.

2 Preliminaries and Karhunen–Loève’s decomposition theorem

In this section, we recall some basic results from Gaussian processes theory. The first theorem we need is the Karhunen–Loève’s decomposition theorem (see Karhunen 1947) that states what follows.

Theorem 1 (Karhunen–Loève) *Let $\{x_t\}_{t \in [0,1]}$, such that $E[x_t] \equiv 0$, and $Cov(x_t, x_s) = C(t, s)$, continuous in both variables. Then, $x_t = \sum_{k=1}^{\infty} Z_k e_k(t)$, where*

- the functions $\{e_k(\cdot)\}_{k \geq 1}$ are the eigenfunctions of the following integral operator from $L^2[0, 1]$ in itself

$$f \in L^2[0, 1] \longrightarrow g(t) = \int_0^1 C(t, \tau) f(\tau) d\tau, \tag{1}$$

- and $\{e_k(\cdot)\}_{k \geq 1}$ form an orthonormal basis for the space spanned by the eigenfunctions corresponding to nonzero eigenvalues;
- the random variables Z_1, Z_2, \dots are given by $Z_k = \int_0^1 x_t e_k(t) dt$ and form a zero-mean orthogonal system (i.e., $E(Z_k Z_j) = 0$ for $k \neq j$) with variance λ_k^2 , where λ_k is the eigenvalue corresponding to the eigenfunction $e_k(\cdot)$.

The series $\sum_{k=1}^\infty Z_k e_k(t)$ converges in mean square to x_t , uniformly in t :

$$\sup_{t \in [0,1]} E \left(\left[x_t - \sum_{k=1}^\infty Z_k e_k(t) \right]^2 \right) \xrightarrow{n \rightarrow \infty} 0.$$

Finally, x_t is a Gaussian process if and only if $\{Z_n\}_{n \geq 1}$ is a sequence of independent Gaussian random variables.

2.1 Representation of the set \mathcal{H} with respect to the Fourier basis $\{\mathbf{s}_k(t), \mathbf{c}_k(t)\}_{k \geq 0}$

We deal in this paper with the following class \mathcal{H} of processes, thought of as the set of ‘pure Gaussian noises’ on the circumference.

Definition 1 Let $\{x_t\}_{t \in [0,1]}$ be a stochastic process with covariance function $C(s, t) = Cov(x_t, x_s)$. \mathcal{H} is the set of real Gaussian stochastic processes $\{x_t\}_{t \in [0,1]}$ such that

- zero-mean: $E(x_t) = 0, \forall t \in [0, 1]$;
- continuously stationary: there exists a continuous real function \tilde{C} such that $C(s, t) = \tilde{C}(s - t), \forall s, t \in [0, 1]$;
- periodic: $\{x_t\}_{t \in [0,1]}$ admits a periodic extension to \mathbb{R} (i.e., $x_0 = x_1, a.s.$).

Remark 1 A necessary and sufficient condition for a continuously stationary process to be periodic is that $\tilde{C}(1) = \tilde{C}(0)$. This allows a continuous version of the process with $Var(x_{t+1} - x_t) = 0$ for any $t \in \mathbb{R}$. We remark that if $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$ and if $\tilde{C}(s - t) = C(s, t)$ is its covariogram function, then $\tilde{C}(t) = \tilde{C}(t + 1)$.

The set \mathcal{H} is a Hilbert space, when it is equipped with the inner product given by $\langle x(\cdot), y(\cdot) \rangle = \int_0^1 E(x_t y_t) dt$. Karhunen–Loève’s decomposition theorem can be specialized to \mathcal{H} , in order to show that a process is in \mathcal{H} if and only if it can be written as limit of a canonical trigonometric random series, namely the constant function equal to 1 together with the sequence $\{\mathbf{s}_k(t), \mathbf{c}_k(t)\}_{k \geq 1}$, where $\mathbf{s}_k(t) = \sqrt{2} \sin(2k\pi t)$ and $\mathbf{c}_k(t) = \sqrt{2} \cos(2k\pi t)$.

Theorem 2 Let $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$ with covariance $C(s, t) = \tilde{C}(t - s)$; then in mean square, uniformly in t ,

$$x_t = c_0 Y'_0 + \sum_{k=1}^\infty c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t))$$

where $\{Y_k\}_{k \geq 1}$ and $\{Y'_k\}_{k \geq 0}$ are two independent sequences of independent standard Gaussian variables, and $\{c_k\}_{k \geq 0} \in \ell^2$ is such that

$$c_n^2 = \int_0^1 \tilde{C}(s) \cos(2n\pi s) ds, \quad n = 0, 1, 2, \dots$$

Proof See Appendix 1. □

Theorem 3 Let $\{Y_k\}_{k \geq 1}$ and $\{Y'_k\}_{k \geq 0}$ be two independent sequences of independent standard Gaussian variables, and $\{c_k\}_{k \geq 0} \in \ell^2$. Then, the sequence

$$y_t^{(n)} = c_0 Y'_0 + \sum_{k=1}^n c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t))$$

converges in mean square, uniformly in t to $\{y_t\}_{t \in [0,1]} \in \mathcal{H}$. Moreover, if $C(s, t)$ is the covariance function of y_t , then uniformly, absolutely and in $L^2[0, 1] \times [0, 1]$,

$$\begin{aligned} C(s, t) &= c_0^2 + \sum_{k=1}^{\infty} c_k^2 \mathbf{c}_k(s) \mathbf{c}_k(t) + \sum_{k=1}^{\infty} c_k^2 \mathbf{s}_k(s) \mathbf{s}_k(t) \\ &= c_0^2 + 2 \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi s) \cos(2k\pi t) + 2 \sum_{k=1}^{\infty} c_k^2 \sin(2k\pi s) \sin(2k\pi t) \\ &= c_0^2 + 2 \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(s - t)). \end{aligned} \tag{2}$$

Proof See Appendix 1. □

Remark 2 As a consequence of Theorems 2 and 3, we can observe that periodic processes with period $\frac{1}{m}$ have only terms of form mk in their expansion:

$$x_{t+\frac{1}{m}} = c_0 Y'_0 + \sum_{k=1}^{\infty} c_{mk} (Y_{mk} \mathbf{s}_{mk}(t + \frac{1}{m}) + Y'_{mk} \mathbf{c}_{mk}(t + \frac{1}{m})) = x_t.$$

More fancy processes having only odd terms are antiperiodic with period $\frac{1}{2}$, i.e.,

$$x_{t+\frac{1}{2}} = \sum_{k=0}^{\infty} c_{2k+1} (Y_{2k+1} \mathbf{s}_{2k+1}(t + \frac{1}{2}) + Y'_{2k+1} \mathbf{c}_{2k+1}(t + \frac{1}{2})) = -x_t.$$

An immediate consequence of this remark is that when one needs to model a pure noise on the boundary of a circle, then he must choose processes whose expansion has both odd and even terms.

2.2 The quotient set \mathcal{H}_Z

It is easy to see that \mathcal{H} can be seen as a Hilbert space, isometrically equivalent to the space of the coefficients ℓ^2 ; let us consider two independent sequences $\{\bar{Y}_n\}_{n \geq 1}$ and $\{\bar{Y}'_n\}_{n \geq 0}$ of independent standard Gaussian variables. For each $\{z_t\}_{t \in [0,1]} \in \mathcal{H}$, there exists an $\{x_t\}_{t \in [0,1]} \in \mathcal{H}_Z$ having the same law, where

$$\mathcal{H}_Z = \left\{ \{x_t\}_{t \in [0,1]} \in \mathcal{H} : x_t = a_0 \bar{Y}'_0 + \sum_{k=1}^{\infty} a_k (\bar{Y}_k \mathbf{s}_k(t) + \bar{Y}'_k \mathbf{c}_k(t)), \{a_n\}_{n \geq 0} \in \ell^2 \right\}$$

and the limit is in mean square and uniformly in t . From Theorems 2 and 3, it is naturally defined an isometry between the representative space \mathcal{H}_Z and ℓ^2 :

$$x_t = a_0 \bar{Y}'_0 + \sum_{k=1}^{\infty} a_k (\bar{Y}_k \mathbf{s}_k(t) + \bar{Y}'_k \mathbf{c}_k(t)) \longleftrightarrow \{a_0, \sqrt{2}a_1, \sqrt{2}a_2, \sqrt{2}a_3, \dots\} \in \ell^2,$$

where $\|x_t\|_{\mathcal{H}_Z} = \sqrt{a_0^2 + 2 \sum_n a_n^2}$.

2.3 The space \mathcal{H}_0 and its relation with \mathcal{H}_Z

By Theorem 3 given $\{c_i\}_{i \geq 0} \in \ell^2$, there exists a unique $\{x_t\}_{t \in [0,1]} \in \mathcal{H}_Z$, with covariance function given by

$$C(s, t) = c_0^2 + \sum_{k=1}^{\infty} c_k^2 \mathbf{c}_k(s) \mathbf{c}_k(t).$$

Let us define the set \mathcal{H}_0 of the process generated by those in \mathcal{H} conditioned to be 0 at $t = 0$.

Definition 2 Let \mathcal{H}_0 be the following set:

$$\mathcal{H}_0 = \{ \{y_t\}_{t \in [0,1]} : \exists \{x_t\}_{t \in [0,1]} \in \mathcal{H} \text{ such that } \mathcal{L}((y_{t_1}, \dots, y_{t_n})) = \mathcal{L}((x_{t_1}, \dots, x_{t_n}) | x_0 = 0), \forall \underline{t} \in [0, 1]^n, n \in \mathbb{N} \}.$$

We call:

- Generator process: the process $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$;
- Generated process: the process $\{y_t\}_{t \in [0,1]} \in \mathcal{H}_0$.

In other words, the process $\{x_t\}_{t \in [0,1]} \in \mathcal{H}_Z$, conditioned to be 0 at $t = 0$, is the periodic zero-mean Gaussian process $\{y_t\}_{t \in [0,1]} \in \mathcal{H}_0$ with covariance function

$$R(s, t) = C(s, t) - \frac{C(s, 0)C(0, t)}{C(0, 0)}. \tag{3}$$

It is easy to show that $\{y_t\}_{t \in [0,1]} \notin \mathcal{H}$ because it is not stationary. However, the function $R(s, t)$ is symmetric, and hence it is the L^2 -limit of its 2-D Fourier series. With the notation given above, with $\mathbf{c}_0(t) = 1$, we get the series expansion:

$$R(s, t) = \sum_{k,j=0}^{\infty} r_{kj}^{cc} \mathbf{c}_k(s) \mathbf{c}_j(t) + \sum_{k,j=1}^{\infty} r_{kj}^{ss} \mathbf{s}_k(s) \mathbf{s}_j(t) + \sum_{k=1,j=0}^{\infty} r_{kj}^{sc} \mathbf{s}_k(s) \mathbf{c}_j(t) + \sum_{k=0,j=1}^{\infty} r_{kj}^{cs} \mathbf{c}_k(s) \mathbf{s}_j(t). \tag{4}$$

The following theorem gives a necessary and sufficient condition for a process $\{y_t\}_{t \in [0,1]}$ with covariance function $R(s, t)$ to have a unique process $\{x_t\}_{t \in [0,1]} \in \mathcal{H}_{\mathbb{Z}}$ which generates it. The trivial case when $R(s, t) = 0$ (generated by a constant process) is omitted since it is the sole case when the solution is not unique. The proof may be found in Appendix 1.

Theorem 4 *For any Gaussian process $\{y_t\}_{t \in [0,1]}$ such that $y_0 = 0$, $E(y_t) = 0$ and continuous covariance function $R(s, t) \neq 0$, there exists a unique (in law) stationary process $\{x_t\}_{t \in [0,1]} \in \mathcal{H}_{\mathbb{Z}}$ which generates $\{y_t\}_{t \in [0,1]}$ if and only if the Fourier coefficients of $R(s, t)$ satisfy:*

- the mixed matrices cos–sin and sin–cos are null:

$$\{r_{jk}^{cs}\}_{j \geq 0, k \geq 1} = \{r_{jk}^{sc}\}_{j \geq 1, k \geq 0} = 0;$$

- the sin–sin matrix is a nonnegative diagonal in ℓ^1 :

$$\{r_{jk}^{ss}\}_{j,k \geq 1} = \begin{pmatrix} r_{11}^{ss} & 0 & 0 & 0 & \dots \\ 0 & r_{22}^{ss} & 0 & 0 & \dots \\ 0 & 0 & r_{33}^{ss} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

with $r_{kk}^{ss} \geq 0$ and $r_{00}^{cc} < \bar{r} = \sum_k r_{kk}^{ss} < \infty$;

- defined $r_{00}^{ss} = \frac{r_{00}^{cc} \bar{r}}{\bar{r} - r_{00}^{cc}}$, the cos–cos matrix is built from the sin–sin matrix and r_{00}^{cc} :

$$\{r_{jk}^{cc}\}_{j,k \geq 0} = \begin{pmatrix} r_{00}^{ss} & 0 & 0 & 0 & \dots \\ 0 & r_{11}^{ss} & 0 & 0 & \dots \\ 0 & 0 & r_{22}^{ss} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} - \frac{\bar{r} - r_{00}^{cc}}{\bar{r}^2} \begin{pmatrix} r_{00}^{ss} r_{00}^{ss} & r_{00}^{ss} r_{11}^{ss} & r_{00}^{ss} r_{22}^{ss} & \dots \\ r_{11}^{ss} r_{00}^{ss} & r_{11}^{ss} r_{11}^{ss} & r_{11}^{ss} r_{22}^{ss} & \dots \\ r_{22}^{ss} r_{00}^{ss} & r_{22}^{ss} r_{11}^{ss} & r_{22}^{ss} r_{22}^{ss} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Remark 3 One of the models mainly used for periodic noise is the Brownian bridge, i.e., the process $\{B_t\}_{t \in [0,1]}$ such that $B_t = W_t - tW_1$, where $\{W_t\}_{t \in [0,1]}$ is a Brownian motion. This process is Gaussian, periodic and has the following standard represen-

tation in Fourier random series:

$$B_t = \sum_{k=1}^{\infty} Z_k \frac{\sqrt{2} \sin(k\pi t)}{k\pi},$$

where Z_1, Z_2, \dots are independent identically distributed standard normal random variables. Starting from the results explained in this section, it is straightforward to prove that $\{B_t\}_{t \in [0,1]} \notin \mathcal{H}_0$; so we cannot consider it a “good” noise model on the boundary of a circle.

3 A process in \mathcal{H}_0 shares the same asymptotic behavior for the spectrum with its generator

We want to get information about Fourier coefficients of Karhunen–Loève expansion for processes in \mathcal{H}_0 with respect to the coefficients of their generators in \mathcal{H} . To do this, as described in the Theorem 1, it is sufficient to study the spectrum of the integral operator induced by the covariance function of the process $\{y_t\}_{t \in [0,1]} \in \mathcal{H}_0$ generated by $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$.

Theorem 5 Denote by $\{y_t\}_{t \in [0,1]}$ a process in \mathcal{H}_0 and by $\{x_t\}_{t \in [0,1]}$ its generator in \mathcal{H} . Let $\{c_n\}_{n \geq 0} \in \ell^2$ be the sequence of Fourier coefficients of Karhunen–Loève expansion of the process $\{x_t\}_{t \in [0,1]}$, such that, as in Theorems 2 and 3,

$$x_t = c_0 Y'_0 + \sum_{k=1}^{\infty} c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t)).$$

Then, the Karhunen–Loève expansion of the process $\{y_t\}_{t \in [0,1]}$ has the following form:

$$y_t = \sum_{k=0}^{\infty} (c_k Y_k \mathbf{s}_k(t) + \tilde{c}_k Y'_k f_k(t))$$

where $f_k(t)$ is the eigenfunction related to the eigenvalue $\tilde{a}_n = \tilde{c}_n^2$, and, for all $n \in \mathbb{N}$,

$$\begin{aligned} a_{k_n} &= \tilde{a}_{k_n} = a_{k_{n+1}} && \text{if } a_{k_n} = a_{k_{n+1}} \\ a_{k_n} &> \tilde{a}_{k_n} > a_{k_{n+1}} && \text{if } a_{k_n} > a_{k_{n+1}} \end{aligned}$$

where $\{a_{k_n}\}_{n \geq 0}$ is a decreasing reordering of the sequence $\{a_n\}_{n \geq 0}$.

Proof See Appendix 2. □

Remark 4 Theorems 4 and 5 give a theoretical approach to build the Karhunen–Loève expansion of processes in \mathcal{H}_0 . A numerical example of such a procedure may be found in the example of the Sect. 5.

4 A process in \mathcal{H}_0 shares the same path regularity properties with its generator

We showed in Theorem 5 that a process in \mathcal{H}_0 and its generator in \mathcal{H} share the same asymptotic behavior for the spectrum. In this section, we show that the regularity of the paths is also maintained.

4.1 Hölder regularity of the paths of processes in \mathcal{H} and in \mathcal{H}_0

We first remind that the Hölder space $C^{m,\alpha}([0, 1])$, where $m \geq 0$ is an integer and $0 < \alpha \leq 1$, consists of those functions on $[0, 1]$ having continuous derivatives up to order m and such that the m th-derivative is Hölder continuous with exponent α . We recall a classic regularity theorem.

Theorem 6 (Kolmogorov–Centsov continuity criterion, Revuz and Yor 1999) *Let $\{x_t\}_{t \in [0,1]}$ be a real stochastic process such that there exist three positive constants γ , c and ϵ so that*

$$E(|x_t - x_s|^\gamma) \leq c|t - s|^{1+\epsilon};$$

then there exists a modification $\{\tilde{x}_t\}_{t \in [0,1]}$ of $\{x_t\}_{t \in [0,1]}$, such that

$$E \left(\left(\sup_{s \neq t} \frac{|\tilde{x}_t - \tilde{x}_s|}{|t - s|^\alpha} \right)^\gamma \right) < \infty$$

for all $\alpha \in [0, \frac{\epsilon}{\gamma})$; in particular, the trajectories of $\{\tilde{x}_t\}_{t \in [0,1]}$ belong to $C^{0,\alpha}([0, 1])$.

The following results are an immediate consequence of this last theorem (proofs may be found in Appendix 3), where the processes $\{\tilde{x}_t\}_{t \in [0,1]}$ and $\{\tilde{y}_t\}_{t \in [0,1]}$ are thought modified as in the Theorem 6.

Theorem 7 *Let $\{x_t\}_{t \in [0,1]}$, a stationary stochastic process with null expectation, and let $R(s, t)$ be its covariance function; if $R \in C^{0,\alpha}([0, 1] \times [0, 1])$, with $0 < \alpha \leq 1$, then almost all trajectories of $\{x_t\}_{t \in [0,1]}$ belong to $C^{0,\beta}([0, 1])$ with $\beta < \frac{\alpha}{2}$.*

It is simple to apply this last theorem to processes laying in \mathcal{H} and in \mathcal{H}_0 : assume that $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$ and let $C(s, t) = \tilde{C}(s - t)$ be its covariance function. If $\tilde{C} \in C^{0,\alpha}([0, 1])$, then almost all trajectories of $\{\tilde{x}_t\}_{t \in [0,1]}$ belong to $C^{0,\beta}([0, 1])$, for any $\beta < \frac{\alpha}{2}$. The same argument can be applied to \mathcal{H}_0 processes.

In fact we can say something more.

Theorem 8 *Let $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$ and let $C(s, t) = \tilde{C}(s - t)$ be its covariance function. Consider its generated process $\{y_t\}_{t \in [0,1]} \in \mathcal{H}_0$, and let $R(s, t)$ be its covariance function. Then we have, for any $\beta < \frac{\alpha}{2}$,*

$$\tilde{C} \in C^{0,\alpha}([0, 1]) \Rightarrow R \in C^{0,\alpha}([0, 1] \times [0, 1]) \Rightarrow \begin{cases} \{\tilde{y}_t\}_{t \in [0,1]} \in C^{0,\beta}([0, 1]) \\ \{\tilde{x}_t\}_{t \in [0,1]} \in C^{0,\beta}([0, 1]) \end{cases}.$$

This last result implies that regularity properties of almost all trajectories of $\{\tilde{x}_t\}_{t \in [0,1]}$ and of its generated process $\{\tilde{y}_t\}_{t \in [0,1]}$ have the same lower bound, obtained by studying regularity of their covariance function.

4.2 Upper order regularity

In Sect. 2.2, a sequence in ℓ^2 is uniquely associated with each stochastic process in \mathcal{H} . We are now showing how the decrease rate of such sequence is associated with the regularity of the process trajectory path.

A very useful result for our analysis will be the following one, whose proof may be found in Lorentz (1948).

Theorem 9 (Boas' Theorem) *Let $f \in L^1[0, 1]$ be a function whose Fourier expansion has only nonnegative cosine terms, and let $\{a_n\}_{n \geq 0}$ be the sequence of its cosine coefficient. Then*

$$f \in C^{0,\alpha}([0, 1]) \iff a_k = O\left(\frac{1}{k^{\alpha+1}}\right).$$

Boas' Theorem may be used in connection with Theorems 2 and 3 to deduce more regularity properties of the processes in \mathcal{H} , since \tilde{C} is a function whose Fourier expansion has only nonnegative cosine terms. In fact, take $\{c_n\}_{n \geq 0}$ as in Theorem 2 and Theorem 3. From Boas' Theorem, we have that if $k^2 c_k^2 = O\left(\frac{1}{k^{1+\alpha}}\right)$ for $0 < \alpha \leq 1$, then $\tilde{C} \in C^{2,\alpha}([0, 1])$. This link between the regularity of \tilde{C} and the paths of $\{x_t\}_{t \in [0,1]}$ is underlined in the following theorem. The proof is in the Appendix 3.

Theorem 10 *With the notations of Theorem 3, if $c_k^2 = O\left(\frac{1}{k^{3+\alpha}}\right)$, then there exists a version of $\{x_t\}_{t \in [0,1]}$ whose trajectories belong to $C^{1,\beta}([0, 1])$, with $\beta < \frac{\alpha}{2}$.*

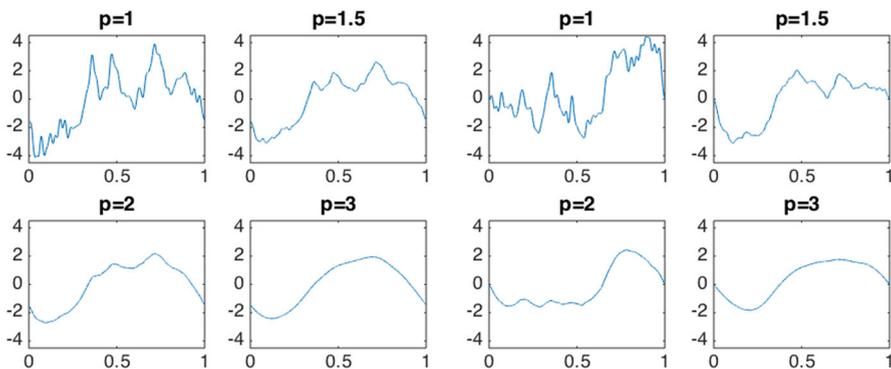


Fig. 2 Change of path regularity: a comparison between trajectories of processes for fixed i.i.d. gaussian $\{Y_k, Y'_k\}_{k \geq 0}$ and varying the magnitude of $\{c_k\}_{k \geq 0}$. *Left* parametric model in \mathcal{H} given in (5), where $a = 1.5$ (dilation coefficient, fixed) and varying p . *Right* parametric model in \mathcal{H}_0 given in (6), obtained by conditioning the model (5) to be 0 at $t = 0$. The coefficients $\{\tilde{c}_k\}_{k \geq 0}$ and the eigenbase $\{f_k\}_{k \geq 0}$ of the model are obtained as explained in Sect.5. As a consequence of the Corollary 1, each trajectory belongs to $C^{m,\beta}([0, 1])$, where $m + \beta < 2p - 1$. The series are truncated at $N = 40$

A natural generalization of this result is the following corollary, exemplified in Fig. 2.

Corollary 1 *With the notations of Theorem 3, if there exists an $m \in \mathbb{N}$ such that $c_k^2 = O(1/k^{1+2m+\alpha})$, then there exists a version of $\{x_t\}_{t \in [0,1]}$ whose trajectories belong to $C^{m,\beta}([0, 1])$, with $\beta < \frac{\alpha}{2}$.*

5 A parametric model in \mathcal{H} and in \mathcal{H}_0

Results provided in this paper suggest to create a Gaussian parametric family of stationary and periodic processes of arbitrary regularity. In fact, let us consider the following family of processes in \mathcal{H} :

$$x_t = \sum_{k=1}^N \frac{a}{k^p} (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)). \tag{5}$$

This family is the discrete approximation of the model given in Hobolth et al. (2003), obtained when N goes to infinite. We note that for this limiting process, Theorem 7 states that the paths become more regular as p increases. This property is shown in Fig. 2 (left), which suggests how to smooth a process by changing p .

By Theorem 4, it is possible to build a parametric model in \mathcal{H}_0 of the form given in the Theorem 5

$$x_t = \sum_{k=1}^N \frac{a}{k^p} Y_k \sin(2k\pi t) + \sum_{k=0}^N \tilde{c}_k Y'_k f_k(t). \tag{6}$$

The functions $\{f_k(t)\}_{k \geq 0}$ are the eigenfunctions of the cos–cos part of the covariance function $R(s, t)$ given in (4). To find an approximation of these first eigenfunctions, given

$$R^{cc}(s, t) = \sum_{k,j=0}^N R_{kj}^{cc} \mathbf{c}_k(s) \mathbf{c}_j(t),$$

we may find the spectral representation of the cos–cos matrix $R^{cc} = UDU^T$, with D diagonal and U unitary. Then, $\tilde{c}_k = \sqrt{D_{kk}}$ and $f_k(t) = \sum_{j=0}^N U_{jk} \mathbf{c}_j(t)$.

Model (5) gives a family of Gaussian processes. In application, maximum likelihood estimates of a and p are a straightforward consequence of a fast Fourier transform of the observed discretized process $\{x_t\}_{t \in [0,i/n]}$, $i = 0, \dots, n$. The properties of these estimators are studied in the following section.

5.1 Maximum likelihood estimators of (5)

Given $(x_{t_0}, x_{t_1}, \dots, x_{t_n})$ sampled from (5), we want to find the property of the maximum likelihood estimator (\hat{a}, \hat{p}) of the parameters (a, p) .

More precisely, with a equispaced or nonequispaced Fourier transform (see, e.g., Brigham 1982; Dutt and Rokhlin 1993), we first transform $(x_{t_0}, x_{t_1}, \dots, x_{t_n})$ into $(y_1^{(1)}, y_2^{(1)}, \dots, y_n^{(1)})$ and $(y_1^{(2)}, y_2^{(2)}, \dots, y_n^{(2)})$ (real and imaginary part). As a consequence of Theorem 2 applied to (5), there exist two sequences $\{Y_k\}_{k \geq 1}$ and $\{Y'_k\}_{k \geq 1}$ of independent Gaussian standard random variables such that

$$y_1^{(1)} = aY_1, \quad y_2^{(1)} = \frac{a}{2^p}Y_2, \dots, y_n^{(1)} = \frac{a}{n^p}Y_n,$$

$$y_1^{(2)} = aY'_1, \quad y_2^{(2)} = \frac{a}{2^p}Y'_2, \dots, y_n^{(2)} = \frac{a}{n^p}Y'_n.$$

The log-likelihood function then reads

$$\begin{aligned} \ell_n(a, p) &= \sum_{k=1}^n \log \left(\frac{1}{\sqrt{2\pi} \frac{a^2}{k^{2p}}} \exp \left(-\frac{1}{2} \frac{(y_k^{(1)})^2}{\frac{a^2}{k^{2p}}} \right) \right) \\ &\quad + \sum_{k=1}^n \log \left(\frac{1}{\sqrt{2\pi} \frac{a^2}{k^{2p}}} \exp \left(-\frac{1}{2} \frac{(y_k^{(2)})^2}{\frac{a^2}{k^{2p}}} \right) \right) \\ &= -n \log(2\pi) - 2n \log(a) + 2p \sum_{k=1}^n \log(k) \\ &\quad - \frac{1}{2a^2} \sum_{k=1}^n k^{2p} ((y_k^{(1)})^2 + (y_k^{(2)})^2) \end{aligned}$$

and hence, if $o_k = (y_k^{(1)})^2 + (y_k^{(2)})^2, k = 1, \dots, n$, we get

$$\frac{\partial \ell_n}{\partial a} = -\frac{2n}{a} + \frac{1}{a^3} \sum_{k=1}^n k^{2p} o_k$$

$$\frac{\partial \ell_n}{\partial p} = 2 \sum_{k=1}^n \log(k) - \frac{1}{a^2} \sum_{k=1}^n \log(k) k^{2p} o_k = \sum_{k=1}^n \log(k) \left(2 - \frac{k^{2p} o_k}{a^2} \right). \tag{7}$$

As expected, when p_0 is a known parameter,

$$\hat{a}^2 = \frac{1}{2n} \sum_{k=1}^n k^{2p_0} o_k, \quad 2n \frac{\hat{a}^2}{a_0^2} \sim \chi_{2n}^2,$$

where χ_{2n}^2 is a chi-square distribution with $2n$ degree of freedom, while nothing is known about the distribution of \hat{p} , for small n , and for the distribution of the couple (\hat{a}, \hat{p}) . We have the following asymptotic results, whose proof may be found in Appendix 4.

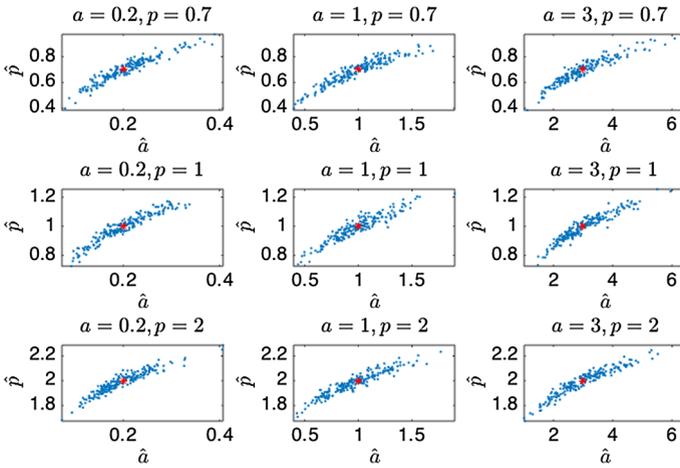


Fig. 3 Plot of 200 (\hat{a}, \hat{p}) -joint simulations (blue point) of data coming from (5) for different values of a and p (red stars). In these pictures, $n = 40$ (see Sect. 5.1 for notations)

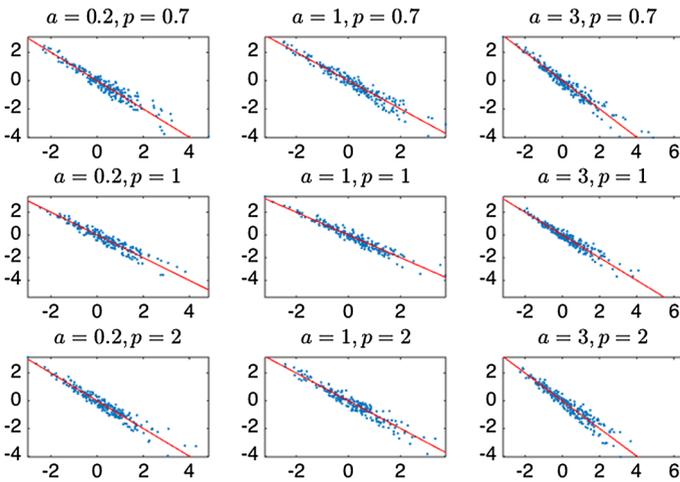


Fig. 4 Theorem 11 predicts that the left-hand part of (8) transforms data from Fig. 3 into i.i.d. vectors with a gaussian distribution concentrated on $y = -x$ (red line)

Theorem 11 *There exists an ML estimator $\{\hat{p}_n\}_{n \geq 1}$, zero of the Eq. (7), such that*

$$\hat{p}_n \xrightarrow[n \rightarrow \infty]{a.s.} p_0, \quad \frac{\hat{p}_n - p_0}{2\sqrt{\sum_1^n \log^2(k)}} \xrightarrow[n \rightarrow \infty]{L} N(0, 1).$$

Moreover,

$$2 \left(\begin{array}{cc} \frac{\sqrt{n}}{a_0} & -\frac{\sum_{k=1}^n \log(k)}{\sqrt{n}} \\ -\frac{\sum_{k=1}^n \log(k)}{a_0 \sqrt{\sum_{k=1}^n \log^2(k)}} & \sqrt{\sum_{k=1}^n \log^2(k)} \end{array} \right) \begin{pmatrix} \hat{a}_n - a_0 \\ \hat{p}_n - p_0 \end{pmatrix} \xrightarrow[n \rightarrow \infty]{L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} Z, \tag{8}$$

where Z is a standard Gaussian variable.

As a corollary of Theorem 11, the joint perfect correlation between \hat{a}_n and \hat{p}_n is asymptotically predicted. In Fig. 3, we show this fact by plotting the maximum likelihood estimates of 200 simulated processes from model (5), where the correlation coefficient $\rho > 0.94$ for $n = 40$ and different values of a_0 and p_0 . In Fig. 4, we plot the corresponding computed left-hand part of (8).

6 Conclusions and perspectives

The results we presented in this paper, together with the parametric models (5) and (6), are useful to represent the border of circular objects where random noise is present. The statistical results might help the practitioners to estimate the model parameters; confidence intervals may be found when dealing with large populations of objects.

A practical advantage of this model is its computer usability: often stochastic processes are represented in computers as discrete values, and only a subset of their Fourier coefficients are available; these models overcome this issue, allowing to find the parameters that best fit the represented process.

In conclusion, the parametric models might be considered in the applications more appropriate alternative than the Brownian bridge. Both the models have a more solid theoretical background, and they present more flexible in terms of path regularity.

A natural perspective would be to extend the presented results to model the shape of the border of a generic d -dimensional star-shaped object, starting from the three dimensional case, where the Fourier basis is more treatable, maintaining the usability of the models.

Appendix 1: Proofs of results of Sect. 2

Proof of the Theorem 2 By Mercer Theorem (see, e.g., Ash 1990), we know that if $\{e_n\}_{n \geq 0}$ is an orthonormal basis for the space spanned by the eigenfunctions corresponding to nonzero eigenvalues of the integral operator (1) then, uniformly, absolutely and in $L^2[0, 1] \times [0, 1]$,

$$C(s, t) = \sum_{k=0}^{\infty} e_k(t)e_k(s)\lambda_k, \tag{9}$$

where λ_k is the eigenvalue corresponding to e_k . By hypothesis, since $C(s, t) = \tilde{C}(|t - s|) = \tilde{C}(|t - s| + 1)$ by Remark 1, we get

$$\int_0^1 \tilde{C}(s) \cos(2n\pi s) ds = a_n, \quad \int_0^1 \tilde{C}(s) \sin(2n\pi s) ds = 0, \tag{10}$$

and hence

$$\tilde{C}(\tau) = a_0 + 2 \sum_{n=0}^{\infty} a_n \cos(2n\pi \tau). \tag{11}$$

It is simple to prove that the sequence $\{\mathbf{s}_n(t), \mathbf{c}_n(t)\}_{n \geq 0}$ contains all the eigenfunctions of the operator (1). In fact,

$$\begin{aligned} \int_0^1 C(t, \tau) \mathbf{c}_n(\tau) d\tau &= \sqrt{2} \int_0^1 \tilde{C}(s) \cos(2n\pi(t+s)) ds \\ &= \mathbf{c}_n(t) \int_0^1 \cos(2n\pi s) \tilde{C}(s) ds - \mathbf{s}_n(t) \int_{-1/2}^{1/2} \sin(2n\pi s) \tilde{C}(s) ds \\ &= a_n \mathbf{c}_n(t), \end{aligned} \tag{12}$$

the same relation holding when $\mathbf{c}_n(t)$ is replaced by $\mathbf{s}_n(t)$. By (11), we get

$$\begin{aligned} C(s, t) &= \tilde{C}(s-t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2k\pi(s-t)) \\ &= a_0 + 2 \sum_{k=1}^{\infty} a_k \cos(2k\pi s) \cos(2k\pi t) + 2 \sum_{k=1}^{\infty} a_k \sin(2k\pi s) \sin(2k\pi t) \\ &= a_0 + \sum_{k=1}^{\infty} a_k \mathbf{c}_k(s) \mathbf{c}_k(t) + \sum_{k=1}^{\infty} a_k \mathbf{s}_k(s) \mathbf{s}_k(t) \end{aligned}$$

where this equality holds uniformly, absolutely and in $L^2[0, 1] \times [0, 1]$ by Mercer Theorem (cfr. (9)).

Now, since $C(s, t)$ is a covariance function, it is positively definite, and hence $a_n \geq 0, \forall n$. Moreover, since $\{a_n\}_{n \geq 0} \in \ell^1$, if we define $c_n = \sqrt{a_n}$, then $\{c_n\}_{n \geq 0} \in \ell^2$. From Theorem 1, we deduce the existence of two independent sequences of independent standard Gaussian variables $\{Y_k\}_{k \geq 1}$ and $\{Y'_k\}_{k \geq 0}$ such that in mean square, uniformly in t

$$x_t = c_0 Y'_0 + \sum_{k=1}^{\infty} c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t)).$$

□

Proof of the Theorem 3 The sequence of Gaussian processes $y_t^{(n)}$ converges to a periodical $\{y_t\}_{t \in [0,1]}$ in mean square uniformly in t , since it is a Cauchy sequence:

$$\sup_{t \in [0,1]} E[|y_t^{(n)} - y_t^{(m)}|^2] = 2 \sum_{k=n}^m c_k^2 \xrightarrow{m,n \rightarrow \infty} 0.$$

Hence, $E[y_t] \equiv 0$, and

$$Cov(y_t, y_s) = c_0^2 + \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(s - t))$$

is a continuous function. Finally, $\{y_t\}_{t \in [0,1]}$ is a Gaussian process, since the two sequences $\{Y_k\}_{k \geq 1}$ and $\{Y'_k\}_{k \geq 0}$ are formed by independent Gaussian variables. \square

Proof of the Theorem 4 Necessity. Assume there exists a process $\{x_t\}_{t \in [0,1]} \in \mathcal{H}_Z$ which generates $\{y_t\}_{t \in [0,1]} \in \mathcal{H}_0$. The covariance function $C(s, t)$ of $\{x_t\}_{t \in [0,1]}$ is given as in (2):

$$C(s, t) = c_0^2 + \sum_{k=1}^{\infty} c_k^2 \mathbf{c}_k(s) \mathbf{c}_k(t) + \sum_{k=1}^{\infty} c_k^2 \mathbf{s}_k(s) \mathbf{s}_k(t).$$

If we define $x = C(0, 0) = \sum_0^{\infty} c_k^2$, $p_i = c_i^2/x$, and

$$D(s, t) = \frac{C(s, t)}{x} = p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s) \mathbf{c}_k(t) + \sum_{k=1}^{\infty} p_k \mathbf{s}_k(s) \mathbf{s}_k(t),$$

then, $x > 0$ and, by (3), we obtain

$$\begin{aligned} xR(s, t) &= D(s, t) - D(0, t)D(s, 0) \\ &= p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s) \mathbf{c}_k(t) + \sum_{k=1}^{\infty} p_k \mathbf{s}_k(s) \mathbf{s}_k(t) \\ &\quad - \left(p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s) \right) \left(p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(t) \right). \end{aligned} \tag{13}$$

(13) and (4) give

$$xr_{kj}^{ss} = \begin{cases} p_k & \text{if } k = j > 0 \\ 0 & \text{if } k \neq j \end{cases} \tag{14}$$

$$xr_{kj}^{cc} = \begin{cases} p_k - p_k^2 & \text{if } k = j \geq 0 \\ -p_k p_j & \text{if } k \neq j \end{cases} \tag{15}$$

$$r_{kj}^{sc} = r_{kj}^{cs} = 0.$$

Since $\sum_0^{\infty} p_k = 1$, if $\bar{r} = \sum_{k=1}^{\infty} r_{kk}^{ss}$, we obtain by (14)

$$x\bar{r} = 1 - p_0.$$

Assume $\bar{r} = 0$, then $p_0 = 1$, which is absurd since $R(s, t) \neq 0$. Hence $\bar{r} > 0$, and we define

$$x = \frac{\bar{r} - r_{00}^{cc}}{\bar{r}^2}. \tag{16}$$

This follows by combining (16) and (15).

Sufficiency. Given the matrices of the 2-D Fourier series as in the theorem assumption, set $x > 0$ as in (16). Define

$$p_k = r_{kk}^{ss} \frac{\bar{r} - r_{00}^{cc}}{\bar{r}^2}, \quad p_0 = \frac{r_{00}^{cc}}{\bar{r}}.$$

Then, $\{p_k\}_{k \geq 0}$ is a nonnegative sequence such that $\sum_k p_k = 1$. Define

$$x_t = \sqrt{x p_0} Y'_0 + \sum_{k=1}^n \sqrt{x p_k} (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t)).$$

By Theorem 2, we have

$$C(s, t) = x \left(p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s) \mathbf{c}_k(t) + \sum_{k=1}^{\infty} p_k \mathbf{s}_k(s) \mathbf{s}_k(t) \right).$$

It is straightforward to check that (14) and (15) hold. The fact that the solution is unique follows immediately from the necessary condition. \square

Appendix 2: Proof of the Theorem 5

The case $x_t \equiv k$ is obvious. Let $C(t, s) = \tilde{C}(t - s)$ be the covariogram function of $\{x_t\}_{t \in [0,1]}$ [see (2) for its expansion]. Since $x_t \equiv k \iff \tilde{C}(0) = 0$, we assume, without loss of generalities, that $\tilde{C}(0) = 1$.

A straightforward computation gives that, if $\{y_t\}_{t \in [0,1]} \in \mathcal{H}_0$ is generated by $\{x_t\}_{t \in [0,1]} \in \mathcal{H}$, then $\{y_t\}_{t \in [0,1]}$ is a Gaussian process with null expectation and continuous covariance function

$$R(t, s) = \tilde{C}(t - s) - \frac{\tilde{C}(t) \tilde{C}(s)}{\tilde{C}(0)} = \tilde{C}(t - s) - \tilde{C}(t) \tilde{C}(s). \tag{17}$$

Hence, given the covariogram function $C(s, t) = \tilde{C}(t - s)$ of the generating process $\{x_t\}_{t \in [0,1]}$, we need to study the spectrum of the operator (1), where C is replaced by R given in (17).

As in (11) and (2), we write $\tilde{C}(t) = a_0 + 2 \sum_{n=1}^{\infty} a_n \cos(2n\pi t)$ with $1 = a_0 + 2 \sum_{n=1}^{\infty} a_n$ since $\tilde{C}(0) = 1$.

Let $f(t)$ be an eigenfunction of (17); from the expansion theorem (see Ash 1990), we have in $L^2[0, 1]$,

$$f(t) = f_0 + \sum_{n=1}^{\infty} f_n^c \mathbf{c}_n(t) + f_n^s \mathbf{s}_n(t). \tag{18}$$

where $f_0 = \int_0^1 f(\tau) d\tau$, $f_n^c = \int_0^1 \mathbf{c}_n(\tau) f(\tau) d\tau$ and $f_n^s = \int_0^1 \mathbf{s}_n(\tau) f(\tau) d\tau$. Let's look for the eigenvalue related to f :

$$\int_0^1 R(s, t) f(t) dt = \int_0^1 \tilde{C}(t - s) f(t) dt - \tilde{C}(s) \int_0^1 \tilde{C}(t) f(t) dt = \tilde{a} f(s). \tag{19}$$

Substituting (18) into (19), and integrating with the results in (10) and (12), yields

$$a_0 f_0 + \sum_{n=1}^{\infty} a_n (f_n^c \mathbf{c}_n(s) + f_n^s \mathbf{s}_n(s)) - \tilde{C}(s) \left(a_0 f_0 + \sqrt{2} \sum_{n=1}^{\infty} a_n f_n^c \right) = \tilde{a} f(s). \tag{20}$$

$\mathbf{s}_n(s)$ eigenfunctions

For any $a_n \neq 0$, it is straightforward to see that $f(s) = \mathbf{s}_n(s)$ is an eigenfunction, by a direct substitution in (20), and that $\tilde{a} = a_n$. Moreover, we are going to state more: the only eigenfunctions which contain some $f_k^s \neq 0$ are indeed $\mathbf{s}_n(s)$ (when $a_n \neq 0$).

Assume that $\exists k: f_k^s \neq 0$ and, by contradiction, $f(t) \neq \mathbf{s}_k(t)$.

By multiplying both members of (20) by $\mathbf{s}_k(s)$ and integrating, we obtain $a_k f_k^s = \tilde{a} f_k^s$, i.e., $a_k = \tilde{a}$. Since $a_k \neq 0$, then $\mathbf{s}_k(t)$ is an eigenfunction. This eigenfunction is orthogonal to $f(s)$ by Mercer Theorem, and hence

$$0 = \int_0^1 \mathbf{s}_k(s) f(s) ds = f_k^s.$$

Summing up, for any $a_n \neq 0$, $\mathbf{s}_n(t)$ is an eigenfunction associated with $\tilde{a} = a_n$, and the other eigenfunctions do not contain the terms in $\{\mathbf{s}_n(t)\}_{n \geq 1}$ (they are even function).

The other eigenfunctions of (20)

To conclude the proof, we should find another sequence of eigenfunctions with eigenvalues $\{\tilde{a}_n\}_{n \geq 1} \asymp \{a_n\}_{n \geq 1}$. We will first obtain a simple result on the coefficients of the eigenfunctions. Then, we will introduce the multiplicity of the eigenvectors $\{a_n\}_{n \geq 1}$ to conclude the proof accordingly.

The other eigenfunction takes the form $f(t) = f_0 + \sum_{k=1}^{\infty} f_k^c \mathbf{c}_k(t)$. By multiplying both members of (20) by $\mathbf{c}_n(s)$ and integrating, we obtain

$$\begin{cases} a_0 f_0 - a_0(a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c) = \tilde{a} f_0, & n = 0; \\ a_n f_n^c - \sqrt{2} a_n(a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c) = \tilde{a} f_n^c, & n > 0. \end{cases} \tag{21}$$

As an immediate consequence, $(a_n = 0) \Rightarrow (f_n^c = 0)$.

Lemma 1 $\{f_n^c\}_{n \geq 0} \in \ell^1$, and $f_0 + \sqrt{2} \sum_{n=1}^\infty f_n^c = 0$.

Proof Recall that $a_n \geq 0$, and that $a_0 + 2 \sum_{n=1}^\infty a_n = \tilde{C}(0) = 1$. For $n > 0$, by (21), we have

$$|f_n^c| \leq \frac{a_n |f_n^c| - \sqrt{2} a_n (a_0 |f_0| + \sqrt{2} \sum_{k=1}^\infty a_k |f_k^c|)}{\tilde{a}},$$

and since $\{a_k |f_k^c|\}_{k \geq 0} \in \ell^1$ (as a product of two ℓ^2 sequences), and $\{a_n\}_{n \geq 1} \in \ell^1$, we obtain the first part of the thesis. By (21) and $a_0 + 2 \sum_{n=1}^\infty a_n = \tilde{C}(0) = 1$, we get

$$\begin{aligned} f_0 + \sqrt{2} \sum_{n=1}^\infty f_n^c &= \frac{a_0 f_0 - a_0 (a_0 f_0 + \sqrt{2} \sum_{k=1}^\infty a_k f_k^c)}{\tilde{a}} + \sqrt{2} \sum_{n=1}^\infty \frac{a_n f_n^c - \sqrt{2} a_n (a_0 f_0 + \sqrt{2} \sum_{k=1}^\infty a_k f_k^c)}{\tilde{a}} \\ &= \frac{a_0 f_0 + \sqrt{2} \sum_{n=1}^\infty a_n f_n^c}{\tilde{a}} - \frac{a_0 f_0 + \sqrt{2} \sum_{k=1}^\infty a_k f_k^c}{\tilde{a}} (a_0 + 2 \sum_{n=1}^\infty a_n) = 0. \end{aligned}$$

□

Definition 3 (*Multiplicity and support*) Given $\{a_n\}_{n \geq 1}$, we define the *support* $S_{\tilde{a}}$ of \tilde{a} :

$$S_{\tilde{a}} = \{k : a_k = \tilde{a}\}.$$

The *multiplicity* $m_{\tilde{a}}$ of a number $\tilde{a} > 0$ is the cardinality of $S_{\tilde{a}}$:

$$m_{\tilde{a}} = \#\{k : a_k = \tilde{a}\}.$$

It is clear that $m_{\tilde{a}} < \infty$ because $\{a_n\}_{n \geq 1} \in \ell^1$.

Lemma 2 *If $m_{\tilde{a}} = k > 0$, then there are exactly $k - 1$ orthogonal eigenfunctions of R related to \tilde{a} . Moreover for anyone of these $k - 1$ eigenfunctions,*

$$n \notin S_{\tilde{a}} \implies f_n^c = 0.$$

Proof Let $\tilde{a} > 0$ be such that $m_{\tilde{a}} > 1$.

It is simple to prove that there always exist $m_{\tilde{a}} - 1$ orthogonal eigenfunctions related to \tilde{a} with $f_n^c = 0$ if $a_n \notin S_{\tilde{a}}$. We have two possibilities:

- $0 \in S_{\tilde{a}}$ or, equivalently, $a_0 = \tilde{a}$. In this case, (21) is equivalent to the following system:

$$\begin{cases} f_n^c = 0, & n \notin S_{\tilde{a}} \\ \tilde{a} (f_0 + \sqrt{2} \sum_{n \in S_{\tilde{a}} \setminus \{0\}} f_n^c) = 0. \end{cases}$$

– $0 \notin S_{\tilde{a}}$. In this case, (21) is equivalent to the following system:

$$\begin{cases} f_n^c = 0, & n \notin S_{\tilde{a}} \\ \tilde{a} \left(\sum_{n \in S_{\tilde{a}}} f_n^c \right) = 0. \end{cases}$$

In both cases, there exist a $k - 1$ -dimensional orthogonal basis for the solution system.

We now need to prove that there are not other eigenfunctions related to \tilde{a} . Assume that $f_{\tilde{n}}^c \neq 0$. We recall that this fact implies $a_{\tilde{n}} \neq 0$. If $\tilde{n} = 0$, from (21) we have that

$$\begin{cases} \frac{(a_0 - \tilde{a})f_0}{a_0} = a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c, \\ \frac{(a_n - \tilde{a})f_n^c}{\sqrt{2}a_n} = a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c, \quad n \in S_{\tilde{a}}. \end{cases}$$

The second equation shows that $a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c = 0$, since $a_n = \tilde{a}$, and hence $a_0 = \tilde{a}$, which means that $\tilde{n} \in S_{\tilde{a}}$. Analogously, if $\tilde{n} \neq 0$, from (21) we can prove that $\tilde{n} \in S_{\tilde{a}}$ that completes the proof. □

Let $\{a_{(n)}\}_{n \geq 1}$ be the decreasing reordering of the sequence $\{a_n\}_{n \geq 1}$, positive and without repetition: $a_{(1)} > a_{(2)} > \dots > a_{(n)} > \dots$ and $\forall a_n > 0$, exists k such that $a_n = a_{(k)}$. To conclude the proof, we must find a sequence of eigenvalues $\{\tilde{a}_n\}_{n \geq 1}$ such that $a_{(n)} > \tilde{a}_n > a_{(n+1)}$.

Lemma 3 *For each $n \in \mathbb{N}$, there exists a unique eigenvalue \tilde{a}_n such that $a_{(n)} > \tilde{a}_n > a_{(n+1)}$. Moreover, $m_{\tilde{a}_n} = 1$.*

Proof We have already observed that $a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c = 0$ implies, for any n , $a_n = \tilde{a}$ or $f_n^c = 0$. Hence, without loss of generalities, we assume $a_0 f_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k f_k^c = c \neq 0$ and we continue the proof. From (21), we obtain

$$f_0 = c \frac{a_0}{a_0 - \tilde{a}}, \quad f_n^c = c \frac{\sqrt{2}a_n}{a_n - \tilde{a}}. \tag{22}$$

These relations with, again, $a_0 f_0 + \sqrt{2} \sum_{n=1}^{\infty} a_n f_n^c = c$, imply

$$\frac{a_0^2}{a_0 - \tilde{a}} + 2 \sum_{n=1}^{\infty} \frac{a_n^2}{a_n - \tilde{a}} = 1. \tag{23}$$

We are going to show that there exists a unique solution \tilde{a}_n of (23) such that $a_{(n)} > \tilde{a}_n > a_{(n+1)}$. This solution is the searched eigenvalue, whose corresponding eigenfunction' expansion is given in (22).

Let us consider the series

$$S(x) = \frac{a_0^2}{a_0 - x} + 2 \sum_{n=1}^{\infty} \frac{a_n^2}{a_n - x}$$

and the derivative series

$$S'(x) = \frac{a_0^2}{(a_0 - x)^2} + 2 \sum_{n=1}^{\infty} \frac{a_n^2}{(a_n - x)^2},$$

then they converge absolutely in each compact set not containing $\{a_n\}_{n \geq 1}$. We have that

$$\text{dom}(S) = \text{dom}(s) = \cup_n (a_{(n+1)}, a_{(n)}), \quad S'(x) = s(x), \forall x \in \text{dom}(S).$$

Moreover for each n ,

$$\lim_{x \rightarrow a_{(n+1)}^+} S(x) = -\infty, \quad \lim_{x \rightarrow a_{(n)}^-} S(x) = +\infty, \quad S'(x) > 0, \forall x \in (a_{(n+1)}, a_{(n)}).$$

Hence, there exists a unique $\tilde{a}_n \in (a_{(n+1)}, a_{(n)})$ such that $S(\tilde{a}_n) = 1$, i.e., for which (23) holds. The unique corresponding eigenfunction is given by (22) that implies also $m_{\tilde{a}_n} = 1$:

$$f(t) = \frac{a_0}{a_0 - \tilde{a}_n} + \sqrt{2} \sum_{n=1}^{\infty} \frac{a_n}{a_n - \tilde{a}_n} \mathbf{c}_n(t).$$

To complete the proof, we show that there are not eigenvalues greater than $a_{(1)} = \max_n a_n$ or smaller than any $a_n > 0$.

In fact, if we assume that there exists an eigenvalue $\hat{a} > \max a_n$, then (22) shows that the sequence $\{f_k^c\}_{k \geq 0}$ is made of either nonnegative or nonpositive numbers that together with Lemma 1 implies $f_k^c = 0$, for any f .

In the same way, it can be shown that there are no eigenvalues smaller than any $a_n > 0$. □

Appendix 3: Proofs of results of Sect. 4

We simply deduce the results based on the fact that if $Y \approx N(0, \sigma^2)$, then $E(|Y|^p) = \sigma^p \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}}$, (see, e.g., [Patel and Read 1982](#)).

Proof of the Theorem 7 Observe that

$$\begin{aligned} E(|x_{t+h} - x_t|^2) &= E(x_t^2 + x_{t+h}^2 - 2x_{t+h}x_t) = R(t+h, t+h) \\ &\quad + R(t, t) - 2R(t+h, t). \end{aligned}$$

Since there exists an M such that $|R(s + \delta_1, t + \delta_2) - R(s, t)| \leq M \|(\delta_1, \delta_2)\|^\alpha$, then there exists a D such that

$$E(|x_{t+h} - x_t|^2) \leq D|h|^\alpha.$$

The thesis follows. □

Proof of the Theorem 8 The first part of the theorem is a simple calculation. The second holds is a consequence of Theorem 6, since

$$E(|x_{t+h} - x_t|^2) = E(E(|x_{t+h} - x_t|^2|x_0)) = E(E((x_{t+h} - x_0) - (x_t - x_0))^2|x_0)) = R(t + h, t + h) + R(t, t) - 2R(t + h, t) \leq D|h|^\alpha.$$

□

Proof of the Theorem 10 It is clear that

$$\partial^2 \tilde{C}(\delta) = 2\partial^2 \sum_{k=1}^\infty c_k^2 \cos(2k\pi(\delta)) = -2 \sum_{k=1}^\infty (2\pi)^2 k^2 c_k^2 \cos(2k\pi(\delta))$$

and that $\partial^2 \tilde{C} \in C^{0,\alpha}([0, 1])$, for some $0 < \alpha \leq 1$. Moreover, we have that uniformly in t and in mean square

$$x_t = c_0 Y'_0 + \sum_{k=1}^\infty c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t)).$$

and, from Theorem 2, there also exists a stochastic process in \mathcal{H} such that uniformly in t and in mean square

$$\tilde{x}_t = 2\pi \sum_{k=1}^\infty k c_k (Y_k \mathbf{c}_k(t) - Y'_k \mathbf{s}_k(t)),$$

which has covariogram function belonging to $C^{0,\alpha}([0, 1])$ given by

$$\tilde{C}(\delta) = 2 \sum_{k=1}^\infty (2\pi)^2 k^2 c_k^2 \cos(2k\pi(\delta)).$$

If we define

$$y_t^{(n)} := c_0 Y'_0 + \sum_{k=1}^n c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t))$$

$$\tilde{y}_t^{(n)} := 2\pi \sum_{k=1}^n k c_k (Y_k \mathbf{c}_k(t) - Y'_k \mathbf{s}_k(t)),$$

than $y_t^{(n)} = y_0^{(n)} + \int_0^t \tilde{y}_\tau^{(n)} d\tau$, a.s. for any n , while for each fixed t , in mean square we have $\int_0^t \tilde{y}_\tau^{(n)} d\tau \rightarrow \int_0^t \tilde{x}_\tau d\tau$. Since

$$\begin{aligned} \sqrt{E\left([x_t - x_0 - \int_0^t \tilde{x}_\tau d\tau]^2\right)} &\leq \sqrt{E\left([x_t - y_t^{(n)}]^2\right)} \\ &+ \sqrt{E\left([y_0^{(n)} + \int_0^t \tilde{y}_\tau^{(n)} d\tau - x_0 - \int_0^t \tilde{x}_\tau d\tau]^2\right)} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

it follows that a.s. $x_t = x_0 + \int_0^t \tilde{x}_\tau d\tau$. By Theorem 7, we know that almost all trajectory path of \tilde{x}_t belongs to $C^{0,\beta}([0, 1])$, with $\beta < \frac{\alpha}{2}$, and thesis follows. \square

Appendix 4: Proofs of results of Sect. 5

Proof of Theorem 11 Assume p_0 be the true parameter, we may define a sequence of i.i.d. random variables $\{Z_k\}_{k \geq 0}$ in the following way:

$$Z_k \sim \frac{k^{2p_0} o_k}{a_0^2} \sim \chi_2^2 \sim \exp\left(\frac{1}{2}\right). \tag{24}$$

Equation (7), as a function of (p, p_0) and $\{Z_k\}_{k \geq 1}$, becomes

$$\frac{\partial \ell_n}{\partial p} = \sum_{k=1}^n \log(k) \left(2 - k^{2(p-p_0)} Z_k\right).$$

With the notation of (Hall and Heyde 1980, pp. 155–161), we have

$$\begin{aligned} I_n(p) &= \sum_1^n \log^2(k) E\left((2 - k^{2(p-p_0)} Z_k)^2 | Z_1, \dots, Z_{k-1}\right) \\ &= \sum_1^n \log^2(k) 2(1 + (1 - 2k^{2(p-p_0)})^2), \\ J_n(p) &= -\frac{2}{a_0^2} \sum_{k=1}^n \log^2(k) k^{2p} o_k = -2 \sum_{k=1}^n \log^2(k) k^{2(p-p_0)} Z_k \end{aligned}$$

and, in particular,

$$I_n(p_0) = 4 \sum_1^n \log^2(k), \tag{25}$$

$$J_n(p_0) = -2 \sum_1^n \log^2(k) Z_k. \tag{26}$$

The thesis is a consequence of (Hall and Heyde 1980, pp. 155–161), where the Assumption 1 and Assumption 2 on page 160 guarantee the existence of an ML estimator $\{\hat{p}_n\}_{n \geq 1}$ such that

$$\hat{p}_n \xrightarrow[n \rightarrow \infty]{a.s.} p_0, \quad \frac{\hat{p}_n - p_0}{\sqrt{I_n(p_0)}} \xrightarrow[n \rightarrow \infty]{L} N(0, 1).$$

Check of (Hall and Heyde 1980, Assumption 1, p. 160). The fact that $I_n(p_0) \xrightarrow[n \rightarrow \infty]{a.s.} \infty$ is a consequence of (25). As $I_n(p_0) = E(I_n(p_0))$, then $I_n(p_0)/E(I_n(p_0)) \rightarrow 1$ uniformly on compacts. By (25) and (26), we have

$$\frac{J_n(p_0)}{I_n(p_0)} = \frac{-2 \sum_1^n \log^2(k) Z_k}{4 \sum_1^n \log^2(k)},$$

and hence, by (24), we have

$$E\left(\frac{J_n(p_0)}{I_n(p_0)}\right) = 1, \quad \text{Var}\left(\frac{J_n(p_0)}{I_n(p_0)}\right) = \frac{\sum_1^n \log^4(k)}{(\sum_1^n \log^2(k))^2}.$$

Since, for $n \geq 4$,

$$\frac{\log^2(n)}{\sum_1^n \log^2(m)} \leq \frac{1}{\sum_{n/2}^n \left(\frac{\log(n/2)}{\log(n)}\right)^2} \leq \frac{1}{\sum_{n/2}^n \left(\frac{1}{2}\right)^2} \leq \frac{8}{n} \tag{27}$$

then $\sum_{n=1}^\infty \left(\frac{\log^2(n)}{\sum_1^n \log^2(m)}\right)^2 < \infty$, and hence $\text{Var}\left(\frac{J_n(p_0)}{I_n(p_0)}\right) \rightarrow 0$ by Kronecker’s Lemma, which ensures that $I_n(p_0)/E(I_n(p_0)) \rightarrow -1$ in probability uniformly on compacts. Check of (Hall and Heyde 1980, Assumption 2, p. 160). Since, for any p , $E_p(I_n(p))$ does not change, then Assumption 2.i) is automatically satisfied.

Now, if $|p_n - p_0| \leq \delta/\sqrt{I_n(p_0)}$, we get

$$|J_n(p_n) - J_n(p_0)| \leq 2 \sum_{k=1}^n \log^2(k) \left(k \frac{\delta}{\sqrt{\sum_1^n \log^2(m)}} - 1\right) Z_k \tag{28}$$

$$|I_n(p_n) - I_n(p_0)| \leq \sum_{k=1}^n \log^2(k) 8k^2 \frac{\delta}{\sqrt{I_n(p_0)}} \left(k^2 \frac{\delta}{\sqrt{I_n(p_0)}} - 1\right).$$

Note that, since $k \leq n$, we have

$$1 \leq k^2 \frac{\delta}{\sqrt{I_n(p_0)}} \leq e^{2 \frac{\delta}{\sqrt{I_n(p_0)}} \log(n)} \leq \exp(2\delta)$$

and hence, for sufficient large n and $k \leq n$, since

$$k^2 \frac{\delta}{\sqrt{I_n(p_0)}} - 1 \leq C_0 2 \frac{\delta}{\sqrt{I_n(p_0)}} \log(k), \tag{29}$$

we obtain

$$\left| \frac{I_n(p_n) - I_n(p_0)}{I_n(p_0)} \right| \leq \frac{C_1 \frac{\sum_{k=1}^n \log^3(k)}{\sqrt{\sum_1^n \log^2(k)}}}{4 \sum_1^n \log^2(k)} = C_2 \sum_{k=1}^n \left(\frac{\log^2(k)}{\sum_1^n \log^2(m)} \right)^{\frac{3}{2}}.$$

By (27), then $\sum_{n=1}^\infty \left(\frac{\log^2(n)}{\sum_1^n \log^2(m)} \right)^{\frac{3}{2}} < \infty$, and hence, by Kronecker’s Lemma, we get Assumption (2.ii), namely

$$\left| \frac{I_n(p_n) - I_n(p_0)}{I_n(p_0)} \right| \rightarrow 0.$$

The last Assumption (2.iii) requires that

$$\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)} \rightarrow 0, \quad \text{a.s.}$$

To check this, we first note that

$$\frac{\sum_{k=1}^n \log^2(k) \left(k \sqrt{\frac{\delta}{\sum_1^n \log^2(m)}} - 1 \right)}{\sum_1^n \log^2(k)} \rightarrow 0,$$

as a consequence of Kronecker’s Lemma, (29) and (27). Then,

$$\left| E \left(\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)} \right) \right| \leq \frac{E(|J_n(p_n) - J_n(p_0)|)}{I_n(p_0)} \rightarrow 0,$$

and hence, a sufficient condition for $\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)} \rightarrow 0$ to hold, is that

$$\text{Var} \left(\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)} \right) \rightarrow 0. \tag{30}$$

By (28), since $\text{Var}(X_k) = 4$, we obtain

$$\text{Var}(J_n(p_n) - J_n(p_0)) \leq 8 \sum_{k=1}^n \log^4(k) \left(k \sqrt{\frac{\delta}{\sum_1^n \log^2(m)}} - 1 \right)^2.$$

Again, by (29), we obtain

$$\text{Var} \left(\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)} \right) \leq \frac{C_1 \frac{\sum_{k=1}^n \log^6(k)}{\sum_1^n \log^2(k)}}{(4 \sum_1^n \log^2(k))^2} = C_2 \sum_{k=1}^n \left(\frac{\log^2(k)}{\sum_1^n \log^2(m)} \right)^3.$$

As above, by (27) and Kronecker’s Lemma, we obtain (30).

We sketch the second part of the proof, with the notation of (Heyde 1997, pag. 191). If we define

$$\mathbf{G}_n(\boldsymbol{\theta}) = \begin{cases} G_n^{(1)}(a, p) = \frac{1}{a} \sum_{k=1}^n \left(\frac{k^{2p}}{a^2} o_k - 2 \right) = \frac{1}{a} \sum_{k=1}^n \left(\frac{k^{2(p-p_0)} a_0^2}{a^2} Z_k - 2 \right) \\ G_n^{(2)}(a, p) = \sum_{k=1}^n \log(k) \left(2 - \frac{k^{2p}}{a^2} o_k \right) = \sum_{k=1}^n \log(k) \left(2 - \frac{k^{2(p-p_0)} a_0^2}{a^2} Z_k \right) \end{cases}$$

and

$$\mathbf{H}_n^{-1}(a_0, p_0) = \begin{pmatrix} \frac{a_0}{2\sqrt{n}} & 0 \\ 0 & \frac{1}{2\sqrt{\sum_{k=1}^n \log^2(k)}} \end{pmatrix}$$

it is simple to state that

$$\mathbf{H}_n^{-1}(a_0, p_0) \cdot \mathbf{G}_n(a_0, p_0) \xrightarrow[n \rightarrow \infty]{L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} Z. \tag{31}$$

In fact, since $\{Z_k\}_{k \geq 0}$ is a i.i.d. sequence of random variables with mean 2 and variance 4 (see (24)), we get

$$E(G_n^{(1)}(a_0, p_0)G_n^{(2)}(a_0, p_0)) = -E\left(\frac{1}{a_0} \sum_{k=1}^n \log(k)(2 - Z_k)^2\right) = -\frac{4}{a_0} \sum_{k=1}^n \log(k)$$

and hence

$$\text{Corr} \left(\frac{a_0}{2\sqrt{n}} G_n^{(1)}(a_0, p_0), \frac{G_n^{(2)}(a_0, p_0)}{2\sqrt{\sum_{k=1}^n \log^2(k)}} \right) = \frac{-\sum_{k=1}^n \log(k)}{\sqrt{n} \sqrt{\sum_{k=1}^n \log^2(k)}} \xrightarrow[n \rightarrow \infty]{} -1.$$

Now, since

$$\begin{aligned} \dot{\mathbf{G}}(a_0, p_0) &= \begin{pmatrix} -\frac{4n}{a_0^2} \left(1 + \frac{3}{4} \frac{\sum_{k=1}^n (Z_k - 2)}{n} \right) & \frac{4 \sum_{k=1}^n \log(k)}{a_0} \left(1 + \frac{\sum_{k=1}^n \log(k)(Z_k - 2)}{2 \sum_{k=1}^n \log(k)} \right) \\ \frac{4 \sum_{k=1}^n \log(k)}{a_0} \left(1 + \frac{\sum_{k=1}^n \log(k)(Z_k - 2)}{2 \sum_{k=1}^n \log(k)} \right) & -4 \sum_{k=1}^n \log^2(k) \left(1 + \frac{\sum_{k=1}^n \log^2(k)(Z_k - 2)}{2 \sum_{k=1}^n \log^2(k)} \right) \end{pmatrix} \\ &\asymp \begin{pmatrix} -\frac{4n}{a_0^2} & \frac{4 \sum_{k=1}^n \log(k)}{a_0} \\ \frac{4 \sum_{k=1}^n \log(k)}{a_0} & -4 \sum_{k=1}^n \log^2(k) \end{pmatrix} \end{aligned}$$

then, by (31) (see Heyde 1997, pag.191), we get

$$\mathbf{H}_n^{-1}(a_0, p_0) \cdot (-\dot{\mathbf{G}}(a_0, p_0)) \cdot \begin{pmatrix} \hat{a}_n - a_0 \\ \hat{p}_n - p_0 \end{pmatrix} \xrightarrow[n \rightarrow \infty]{L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} Z,$$

which is the thesis, once the conditions of uniformly boundedness are checked as for the previous case. \square

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