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On the Simes inequality in elliptical models

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Abstract We provide some necessary and some sufficient conditions for the validity of the inequality of Simes in models with elliptical dependencies. Necessary conditions are presented in terms of sufficient conditions for the reverse Simes inequality. One application of our main results concerns the problem of model misspecification, in particular the case that the assumption of Gaussianity of test statistics is violated. Since our sufficient conditions require non-negativity of correlation coefficients between test statistics, we also develop two exact tests for vectors of correlation coefficients and compare their powers in computer simulations.

Keywords Covariance matrix · Distributional transform · Multiple testing · Multivariate normal distribution · p value · Student's t · Total positivity

1 Introduction

It is fair to say that one of the major foundations of modern multiple test theory is Simes' inequality. This inequality concerns the joint distribution of the order statistics

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of m marginally uniformly distributed random variables $U_1, \ldots, U_m$. In its original form, it was proven (as an equality) by Simes (1986) under joint independence of the $U_i : 1 \leq i \leq m$.

Proposition 1 (Simes 1986) *Let $U_1, \ldots, U_m$ denote stochastically independent, identically UNI$[0, 1]$-distributed random variables and $U_{1:m} \leq \cdots \leq U_{m:m}$ their order statistics. Define $\alpha_{i:m} = i\alpha/m$, $1 \leq i \leq m$, for $\alpha \in [0, 1]$. Then, it holds*

$$\mathbb{P}(U_{1:m} > \alpha_{1:m}, \ldots, U_{m:m} > \alpha_{m:m}) = 1 - \alpha.$$

The constants $(\alpha_{i:m})_{1 \leq i \leq m}$ are referred to as Simes' critical values in the multiple testing literature. Based on Proposition 1, they have been implemented into various stepwise rejective multiple tests for testing m null hypotheses $H_1, \ldots, H_m$ against alternatives $K_1, \ldots, K_m$. These stepwise rejective tests uniformly improve single-step procedures like the Bonferroni correction in terms of power. For example, the multiple test by Hommel (1988) is a powerful improvement of the Bonferroni test. It keeps the family-wise error rate (FWER) at level α when applied to marginal p values $P_1, \ldots, P_m$ which are under the corresponding null hypotheses distributed as the U_i in Proposition 1. Moreover, Simes' critical values also build the basis for the linear step-up test φ^{LSU} by Benjamini and Hochberg (1995), and the authors proved that φ^{LSU} controls the false discovery rate (FDR) under independence, again by making use of Proposition 1. Nowadays, φ^{LSU} is presumably the most widely applied multiple test procedure in practice, with more than 22,000 citations according to Google Scholar.

Already in his original article, Simes (1986) argued that the inequality

$$\mathbb{P}(P_{1:m} > \alpha_{1:m}, \ldots, P_{m:m} > \alpha_{m:m}) \geq 1 - \alpha \tag{1}$$

(which is actually sufficient for type I error control of multiple tests based on Simes' critical values) is not valid in general, but "... may well be true for a large family of multivariate distributions as suggested by [...] simulation studies". This assertion is known as the Simes conjecture. An important step towards the characterization of multivariate distributions for which the Simes conjecture is true was the paper by Sarkar (1998). He proved that multivariate total positivity of order 2 (MTP_2 for short) among $P_1, \ldots, P_m$ is sufficient for the validity of (1). This considerably extends the applicability of φ^{LSU} to models with dependency, see Benjamini and Yekutieli (2001) and Sarkar (2002).

Often, the p values $P_1, \ldots, P_m$ are constructed as distributional transforms (in the sense of Rüschendorf 2009) of real-valued test statistics $T_1, \ldots, T_m$, meaning that

$$P_i = F(T_i), \quad 1 \leq i \leq m, \tag{2}$$

where F denotes the (common) marginal cumulative distribution function (cdf) of each T_i under H_i. This construction is reasonable if each T_i tends to smaller values under the alternative K_i. A detailed discussion about the interrelation of test statistics and p values in multiple hypotheses testing is provided in Chapter 2 of Dickhaus (2014). Assuming F as known, one may equivalently analyze the dependency structure of the vector $\mathbf{T} = (T_1, \ldots, T_m)^\top$ of test statistics instead of that of $\mathbf{P} = (P_1, \ldots, P_m)^\top$, because the right-hand side of (2) is a deterministic transformation of T_i. If, moreover, F is continuous and strictly increasing, Simes' inequality can equivalently be stated in terms of $\mathbf{T}$ as

$$\mathbb{P}(T_{1:m} > a_1, \ldots, T_{m:m} > a_m) \geq 1 - \alpha = 1 - F(a_m), \tag{3}$$

where $a_i = F^{-1}(\alpha_{i:m})$, $1 \leq i \leq m$.

Recently, Block et al. (2013) extended the work by Sarkar (1998) by considering the multivariate Student's t distribution. This distribution is highly relevant for many applications in multiple testing (see, for instance, Hothorn et al. 2008), but unfortunately does not exhibit MTP_2 dependence. Block et al. (2013) derived sufficient conditions for the validity of (3) in the case that the random vector $\mathbf{T} = (T_1, \ldots, T_m)^\top$ follows a multivariate Student's t distribution; see Theorem 3.1(i) in their paper. Since the multivariate Student's t distribution belongs to the broad class of elliptical distributions (see the monograph by Gupta et al. 2013 for a comprehensive overview) and the dependence structure among the components of a random vector $\mathbf{T}$ which follows an elliptical distribution is entirely captured by the covariance matrix $\mathbf{\Sigma}$ of $\mathbf{T}$ and the distribution of the generating variate R of the elliptical distribution, the results by Block et al. (2013) provoke the question if sufficient conditions on $\mathbf{\Sigma}$, R, and $\mathbf{a} = (a_1, \ldots, a_m)^\top$ can be obtained such that (3) is generally valid for such $\mathbf{T}$. This issue is addressed in the present work.

Remark 1 If T_i tends to larger values under K_i, one typically considers $P_i = 1 - F(T_i)$. Then, the analog of (3) is given by

$$\mathbb{P}(T_{1:m} \leq b_1, \ldots, T_{m:m} \leq b_m) \geq 1 - \alpha = F(b_1), \tag{4}$$

where $b_i = F^{-1}(1 - \alpha_{m-i+1:m})$, $1 \leq i \leq m$. This case has been treated in part (ii) of Theorem 3.1 by Block et al. (2013). However, as argued by Block et al. (2013), (4) directly follows from (3) under, respectively, modified conditions on $\mathbf{b} = (b_1, \ldots, b_m)^\top$. Therefore, we will mainly consider (3) in the present work.

The rest of the paper is structured as follows. In Sect. 2, we formally define the class of elliptically contoured distributions and derive some sufficient and necessary conditions for the validity of Simes' inequality under such distributions of $\mathbf{T}$. One application of our results concerns the problem of model misspecification, i.e., the case that $F = \Phi$ is assumed, where Φ denotes the cumulative distribution function (cdf) of the standard normal law, but the actual distribution of T_1 is elliptical with $F \neq \Phi$. It will turn out that non-negativity of the entries of $\mathbf{\Sigma}$ is crucial for all of our main results. Thus, for practical purposes, we develop exact confidence regions for (vectors of) correlation coefficients in Sect. 3, and we compare the powers of the corresponding tests for non-negativity of correlation coefficients in Sect. 4 by means of a simulation study. We conclude with a discussion in Sect. 5.

2 Simes' inequality under elliptically contoured distributions

Throughout the work, we assume that the vector $\mathbf{T}$ of test statistics has an elliptically contoured distribution. Assuming that the density of $\mathbf{T}$ exists for all $m \geq 1$, $\mathbf{T}$ possesses the stochastic representation

$$\mathbf{T} \stackrel{d}{=} R\mathbf{Z}, \tag{5}$$

where R and $\mathbf{Z}$ are stochastically independent, $\mathbf{Z} \sim \mathcal{N}_m(\mathbf{0}, \mathbf{\Sigma})$, and R is a non-negative univariate random variable. We denote this class of distributions by $\mathbf{T} \sim E_m(\mathbf{0}, \mathbf{\Sigma}, R)$.

In the class of multivariate normal distributions of $\mathbf{T}$ ($R \equiv 1$), non-negativity of all entries of $\mathbf{\Sigma}$ is sufficient for the validity of Simes' inequality, because it entails the positive dependent through stochastic ordering (PDS) property; cf. Block et al. (1985). As mentioned in the introduction, Block et al. (2013) provided conditions for the validity of Simes' inequality for the class of multivariate t distributions of $\mathbf{T}$, where R follows an inverse gamma distribution. These conditions are stronger than the ones in case of the multivariate normal distribution (see also Sarkar 2008). Namely, it is required that all non-diagonal elements of $\mathbf{\Sigma}$ are non-negative (as for the normal distribution) and, in addition, certain restrictions on $\mathbf{a}$ are imposed.

The latter conditions have been derived by Block et al. (2013) by exploiting the identity

$$\mathbb{P}(T_{1:m} > a_1, \ldots, T_{m:m} > a_m) = 1 - F(a_m) + \sum_{i=1}^{m} \sum_{j=1}^{m-1} \Delta_{i,j}(\mathbf{T}, \mathbf{a}), \tag{6}$$

where

$$\Delta_{i,j}(\mathbf{T}, \mathbf{a}) = \mathbb{E}\Bigg[\left(\frac{\mathbb{1}(T_i \leq a_{j+1})}{j+1} - \frac{\mathbb{1}(T_i \leq a_j)}{j} \right) \times \mathbb{1}(T^{(-i)}_{j:m-1} > a_{j+1}, \ldots, T^{(-i)}_{m-1:m-1} > a_m) \Bigg], \tag{7}$$

which was presented in Lemma 2.1 of their paper. In (7), $\mathbb{1}(A)$ denotes the indicator function of set A, and $T^{(-i)}_{1:m-1} \leq T^{(-i)}_{2:m-1} \leq \cdots \leq T^{(-i)}_{m-1:m-1}$ are the order statistics obtained for the vector $\mathbf{T}$ after removing T_i. The derivations by Block et al. (2013) depend on (6) and on specific properties of the multivariate t distribution, hence, they do not generalize to the class $E_m(\mathbf{0}, \mathbf{\Sigma}, R)$.

In Theorem 1, we analyze (6) for broader classes of elliptically contoured distributions. This leads to sufficient conditions on $\mathbf{\Sigma}$, $\mathbf{a}$, and R which imply that Simes' inequality holds. Moreover, we also provide related conditions under which the reverse Simes inequality holds, meaning that the order relation in (3) is in the opposite direction. To this end, we define for any $1 \leq j \leq m-1$ the function $G_j : (0, \infty) \to \mathbb{R}$ by

$$G_j(r) = \frac{\mathbb{P}\left(Z_1 \leq \frac{a_{j+1}}{r}\right)}{j+1} - \frac{\mathbb{P}\left(Z_1 \leq \frac{a_j}{r}\right)}{j} = \frac{\Phi\left(\frac{a_{j+1}}{r}\right)}{j+1} - \frac{\Phi\left(\frac{a_j}{r}\right)}{j}. \tag{8}$$

Theorem 1 *Assume that* $\mathbf{T} \sim E_m(\mathbf{0}, \mathbf{\Sigma}, R)$ *and that* $a_1 \leq a_2 \leq \cdots \leq a_m \leq 0$. *Let*

$$A_j(R, \mathbf{\Sigma}, \mathbf{a}) = \int_0^\infty \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \leq \frac{a_j}{r}\right] G_j(r) f_R(r) \mathrm{d}r.$$

Then, the following two assertions hold true.

(a) (*Sufficient conditions for Simes' inequality*)
If $A_j(R, \mathbf{\Sigma}, \mathbf{a}) \geq 0$ *for all* $1 \leq j \leq m$ *and* $\mathbf{\Sigma}$ *is such that the PDS condition is satisfied for* $\mathbf{Z} \sim \mathcal{N}_m(\mathbf{0}, \mathbf{\Sigma})$, *then Simes' inequality holds for* $\mathbf{T}$.

(b) (*Sufficient conditions for the reverse Simes inequality*)
Assume that $\mathbf{\Sigma}$ *is a diagonal matrix. If* $A_j(R, \mathbf{\Sigma}, \mathbf{a}) \leq 0$ *for all* $1 \leq j \leq m$, *then the reverse Simes inequality holds for* $\mathbf{T}$. *If, furthermore, at least one of the m inequalities is strict, then the reverse Simes inequality for* $\mathbf{T}$ *is also strict.*

Proof To prove the statement of the theorem, it suffices to show that each $\Delta_{i,j}(\mathbf{T}, \mathbf{a})$ is non-negative (part (a)) or non-positive (part (b)), respectively. We note that

$$\begin{aligned}
\Delta_{i,j}(\mathbf{T}, \mathbf{a}) &= \frac{1}{j+1}\mathbb{P}\left[T^{(-i)}_{j:m-1} > a_{j+1}, \ldots, T^{(-i)}_{m-1:m-1} > a_m, T_i \leq a_{j+1}\right] \\
&- \frac{1}{j}\mathbb{P}\left[T^{(-i)}_{j:m-1} > a_{j+1}, \ldots, T^{(-i)}_{m-1:m-1} > a_m, T_i \leq a_j\right] \\
&= \int_0^\infty \left\{ \frac{\mathbb{P}\left(Z_i \leq \frac{a_{j+1}}{r}\right)}{j+1} \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \leq \frac{a_{j+1}}{r}\right] \right. \\
&\left. - \frac{\mathbb{P}\left(Z_i \leq \frac{a_j}{r}\right)}{j} \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \leq \frac{a_j}{r}\right] \right\} f_R(r) \mathrm{d}r.
\end{aligned} \tag{9}$$

To prove part (a), we use that the PDS property for $\mathbf{Z}$ implies that (cf. Section 5 in Block et al. 1985)

$$\begin{aligned}
&\mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \leq \frac{a_{j+1}}{r}\right] \\
&\geq \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \leq \frac{a_j}{r}\right].
\end{aligned}$$

Utilizing this relation in (9), we get that

$$\begin{aligned}
\Delta_{i,j}(\mathbf{T}, \mathbf{a}) &\geq \int_0^\infty \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \leq \frac{a_j}{r}\right] G_j(r) f_R(r) \mathrm{d}r \\
&= A_j(R, \mathbf{\Sigma}, \mathbf{a}),
\end{aligned}$$

and our assumption on $A_j(R, \mathbf{\Sigma}, \mathbf{a})$ yields the assertion.

If $\mathbf{\Sigma}$ is a diagonal matrix, then the Z_i are stochastically independent, leading to

$$\mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \le \frac{a_{j+1}}{r}\right]$$
$$= \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r}\right]$$
$$= \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r} | Z_i \le \frac{a_j}{r}\right].$$

Consequently,

$$\Delta_{i,j}(\mathbf{T}, \mathbf{a}) = \int_0^\infty \mathbb{P}\left[Z^{(-i)}_{j:m-1} > \frac{a_{j+1}}{r}, \ldots, Z^{(-i)}_{m-1:m-1} > \frac{a_m}{r}\right] G_j(r) f_R(r) \mathrm{d}r$$
$$= A_j(R, \mathbf{\Sigma}, \mathbf{a}),$$

and our assumption on $A_j(R, \mathbf{\Sigma}, \mathbf{a})$ entails the first assertion of part (b). The second assertion of part (b) follows immediately. □

Theorem 1 has several interesting applications. First, we recover the previously mentioned result by Sarkar (2008) and Block et al. (2013).

Corollary 1 (Sarkar 2008; Block et al. 2013) *Assume that* $\mathbf{T}$ *follows a centered multivariate t distribution with dispersion matrix* $\mathbf{\Sigma}$*. Let all non-diagonal elements of* $\mathbf{\Sigma}$ *be non-negative and let* $a_1 \le a_2 \le \cdots \le a_m \le 0$*. If* $j^{-1}F(a_j)$ *is non-decreasing in* $j = 1, \ldots, m$*, then Simes' inequality holds for* $\mathbf{T}$.

Proof The assertion follows from Theorem 1 by analyzing $A_j(R, \mathbf{\Sigma}, \mathbf{a})$ in an analogous manner as done by Sarkar (2008) in the proof of his Theorem 3.1. □

Next, we consider another class of elliptically contoured distributions for which Simes' inequality or the reverse Simes inequality, respectively, holds. In this class, the support of R is restricted. To this end, we need the following auxiliary result.

Lemma 1 *For each* $1 \le j \le m - 1$*, the equation* $G_j(r) = 0$ *has a unique solution on* $(0, \infty)$*, which we denote by* r_j.

Proof The first derivative of G_j defined in (8) is given by

$$\frac{\partial G_j(r)}{\partial r} = \frac{\phi\left(\frac{a_{j+1}}{r}\right)}{j+1}\left(-\frac{a_{j+1}}{r^2}\right) - \frac{\phi\left(\frac{a_j}{r}\right)}{j}\left(-\frac{a_j}{r^2}\right)$$
$$= -\frac{1}{r^2}\left(\phi\left(\frac{a_{j+1}}{r}\right)\frac{a_{j+1}}{j+1} - \phi\left(\frac{a_j}{r}\right)\frac{a_j}{j}\right),$$

where ϕ denotes the probability density function (pdf) of the standard normal distribution. Setting this derivative to zero and solving the equation, we get only one extremal point of G_j with abscissa

$$r_{j,\max} = \sqrt{\frac{a_j^2 - a_{j+1}^2}{2\log\left(\frac{-a_j/j}{-a_{j+1}/(j+1)}\right)}}.$$

Moreover, it holds that $\frac{\partial G_j(r)}{\partial r} > 0$ for $r \in (0, r_{j,\max})$ and $\frac{\partial G_j(r)}{\partial r} < 0$ for $r \in (r_{j,\max}, \infty)$, implying that the extremum is a maximum. Finally, we note that $G_j(r) \to 0$ as $r \to 0$ and

$$G_j(r) \to \frac{1}{2(j+1)} - \frac{1}{2j} < 0 \quad \text{as} \quad r \to \infty.$$

This completes the proof. □

Corollary 2 *Let* $\mathbf{T} \sim E_m(\mathbf{0}, \mathbf{\Sigma}, R)$ *and assume that* $a_1 \leq a_2 \leq \cdots \leq a_m \leq 0$. *Let* $(r_j)_{1 \leq j \leq m-1}$ *be as in Lemma 1. Define* $\bar{r} = \min_{1 \leq j \leq m-1}\{r_j\}$ *and* $\underline{r} = \max_{1 \leq j \leq m-1}\{r_j\}$.

(a) (*Sufficient conditions for Simes' inequality*)
If all non-diagonal elements of $\mathbf{\Sigma}$ *are non-negative and* $\mathbb{P}(0 \leq R \leq \bar{r}) = 1$, *then Simes' inequality holds for* $\mathbf{T}$.

(b) (*Sufficient conditions for the reverse Simes inequality*)
If $\mathbf{\Sigma}$ *is a diagonal matrix and* $\mathbb{P}(\underline{r} \leq R \leq \infty) = 1$, *then the reverse Simes inequality holds for* $\mathbf{T}$.

Proof To prove part (a), we notice that our assumptions and the curvature of the functions G_j which we have discussed in Lemma 1 imply that

$$\Delta_{i,j}(\mathbf{T}, \mathbf{a}) = \int_0^{\bar{r}} \mathbb{P}\left[T_{j:m-1}^{(-i)} > \frac{a_{j+1}}{r}, \ldots, T_{m-1:m-1}^{(-i)} > \frac{a_m}{r} | T_i \leq \frac{a_j}{r}\right] G_j(r) f_R(r) \mathrm{d}r \\ \geq 0.$$

Furthermore, recall from the proof of Theorem 1 that under the conditions of part (b) we have

$$\Delta_{i,j}(\mathbf{T}, \mathbf{a}) = \int_{\underline{r}}^{\infty} \mathbb{P}\left[Z_{j:m-1}^{(-i)} > \frac{a_{j+1}}{r}, \ldots, Z_{m-1:m-1}^{(-i)} > \frac{a_m}{r}\right] G_j(r) f_R(r) \mathrm{d}r.$$

Now, the curvature of G_j leads to $G_j(r) \leq 0$ for all $1 \leq j \leq m-1$ under the assumptions of part (b), completing the proof. □

Corollary 2 can be used to analyze the effect of model misspecification on the validity of Simes' inequality or the reverse Simes inequality, respectively. Namely, consider $a_j = \Phi^{-1}(j\alpha/m)$ for $1 \leq j \leq m$. In view of the discussion around (3), these constants correspond to the assumption that $F = \Phi$. Corollary 3 analyzes the effect of making this assumption, while the true distribution of $\mathbf{T}$ is elliptical, but non-Gaussian.

Corollary 3 *Assume that* $\mathbf{T} \sim E_m(\mathbf{0}, \boldsymbol{\Sigma}, R)$ *and let* $a_j = \Phi^{-1}(j\alpha/m)$, $j = 1, \ldots, m$.

(a) (*Sufficient conditions for Simes' inequality*)
If all non-diagonal elements of $\boldsymbol{\Sigma}$ *are non-negative and* $\mathbb{P}(0 \le R \le 1) = 1$, *then Simes' inequality holds for* $\mathbf{T}$.

(b) (*Sufficient conditions for the reverse Simes inequality*)
If $\boldsymbol{\Sigma}$ *is a diagonal matrix and* $\mathbb{P}(1 \le R \le \infty) = 1$, *then the reverse Simes inequality holds for* $\mathbf{T}$.

Proof For the vector $\mathbf{a} = (a_1, \ldots, a_m)^\top$, we have $r_j = 1$ for all $j \in \{1, 2, \ldots, m-1\}$. Hence, $\bar{r} = \underline{r} = 1$ and the assertion follows from Corollary 2. □

Remark 2 The reasoning of Theorem 1 and Corollaries 1–3 can also be applied to the analog of Simes' inequality considered in Remark 1. For example, we get that

$$\mathbb{P}(T_{1:m} < b_1, \ldots, T_{m:m} < b_m) \ge F(b_1)$$

if $0 \le b_1 \le b_2 \le \cdots \le b_m$, all elements of $\boldsymbol{\Sigma}$ are non-negative, and $\mathbb{P}(0 \le R \le \bar{r}_{\mathbf{b}}) = 1$, where $\bar{r}_{\mathbf{b}} = \min_{1 \le j \le m-1}\{r_{j,\mathbf{b}}\}$ and $r_{j,\mathbf{b}}$ is the unique solution of

$$\frac{1 - \Phi\left(\frac{b_{m-j}}{r}\right)}{j+1} - \frac{1 - \Phi\left(\frac{b_{m-j+1}}{r}\right)}{j} = 0.$$

3 Exact tests on vectors of correlation coefficients

In practical applications of multiple testing, the joint distribution of test statistics is often not known exactly, even under the global hypothesis. As mentioned in Sect. 1, we make the general assumption that the common marginal cdf F of each test statistic under the respective null hypothesis is specified. This implies that conditions imposed on the quantile function F^{-1} (as in the case of the multivariate t distribution; see Corollary 1) as well as conditions imposed on the support of the distribution of R (cf. Corollary 2) can be checked straightforwardly. However, the correlation (or covariance) matrix is often an unknown nuisance parameter. As a result, the non-negativity of its non-diagonal elements (a sufficient condition for Simes' inequality which appeared throughout Sect. 2) cannot be checked analytically and has to be tested. This is the motivation to deal with the latter problem in this section.

First, we derive a test under the assumption of normality. To this end, we assume that a data matrix $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n) \in \mathbb{R}^{m \times n}$ is available from which the vector $\mathbf{T}$ of test statistics is computed. Let $\mathbf{X} \sim \mathcal{N}_{m,n}(\boldsymbol{\mu}\mathbf{1}_n^\top, \boldsymbol{\Sigma} \otimes \mathbf{I}_n)$ ($m \times n$-dimensional matrix-variate normal distribution with mean matrix $\boldsymbol{\mu}\mathbf{1}_n^\top$ and covariance matrix $\boldsymbol{\Sigma} \otimes \mathbf{I}_n$), where $\mathbf{1}_n$ denotes the n-dimensional vector of ones and $\mathbf{I}_n$ is the $n \times n$-dimensional identity matrix. For clarity of exposition, we introduce the following assumption which connects $\boldsymbol{\Sigma}$ with the covariance matrix of $\mathbf{T}$.

Assumption 1 There exists a constant $\gamma \in (0, \infty)$ such that $\mathrm{Cov}(\mathbf{T}) = \gamma \boldsymbol{\Sigma}$.

Assumption 1 justifies our slight abuse of notation (the symbol $\boldsymbol{\Sigma}$ was used to denote the covariance matrix of $\mathbf{T}$ in Sect. 2).

Remark 3 Assumption 1 is for instance fulfilled if $\mathbf{T}$ is the vector of row-wise means of $\mathbf{X}$ (with $\gamma = 1/n$). However, in other practical applications, it is often violated. In asymptotic considerations ($n \to \infty$), one can relax Assumption 1 and only assume that $\mathrm{Cov}(\mathbf{T}) = h(\boldsymbol{\Sigma})$, where $h : \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m}$ is a known deterministic function. Application of the Delta method then leads to asymptotic analogs of our proposed tests.

The covariance matrix $\boldsymbol{\Sigma}$ is estimated by its empirical counterpart

$$\tilde{\mathbf{S}} = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})^\top \quad \text{with} \quad \bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i.$$

It holds that $\mathbf{S} = (n-1)\tilde{\mathbf{S}} \sim W_m(n-1, \boldsymbol{\Sigma})$ for $n \geq m+1$ (cf. Muirhead 1982), where the symbol $W_m(n-1, \boldsymbol{\Sigma})$ denotes the m-dimensional Wishart distribution with $n-1$ degrees of freedom and covariance matrix $\boldsymbol{\Sigma}$. Moreover, it holds that $\mathbf{S}$ and $\bar{\mathbf{X}}$ are stochastically independent; see, e.g., Theorem 3.1.2 in Muirhead (1982). Our proposed tests rely on $\mathbf{S}$.

Let us consider the problem of simultaneous testing for non-negativity of all non-diagonal elements of $\boldsymbol{\Sigma}$ in a given column i. This condition ensures positive regression dependency of $\mathbf{X}_1$ on the subset (PRDS) $I_0 = \{i\}$ in the sense of Benjamini and Yekutieli (2001). If $\mathbf{Z}$ in the stochastic representation of $\mathbf{T}$ in (5) is PRDS on $\{i\}$ for any $1 \leq i \leq m$, then $\mathbf{Z}$ fulfills the PDS property used in the proof of Theorem 1. This is indicated in the discussion on page 1173 in Benjamini and Yekutieli (2001), see also Condition 1.1 by Sarkar (2008). The advantage of column-wise testing is that exact tests can be derived which do not depend on unknown model parameters and can be applied to any matrix-variate elliptical distribution of $\mathbf{X}$.

For given index $1 \leq i \leq m$, we are thus interested in testing

$$H_i^{<} : \sigma_{ij} < 0 \text{ for at least one } 1 \leq j \leq m,\ j \neq i \text{ versus } K_i^{<} : \boldsymbol{\sigma}_i \geq \mathbf{0}, \tag{10}$$

where $\boldsymbol{\sigma}_i = (\sigma_{i1}, \ldots, \sigma_{i,i-1}, \sigma_{i,i+1}, \ldots, \sigma_{im})^\top$. The test problem in (10) can be solved by constructing a confidence region in $\mathbb{R}^{m-1}$ for the standardized version of $\boldsymbol{\sigma}_i$. Following Aitchison (1964), we exploit the duality of tests and confidence regions and consider the (auxiliary) family of point hypotheses

$$H_i^{(\boldsymbol{\delta})} : \sigma_{ii}^{-1}\boldsymbol{\sigma}_i = \boldsymbol{\delta} \text{ versus } K_i^{(\boldsymbol{\delta})} : \sigma_{ii}^{-1}\boldsymbol{\sigma}_i \neq \boldsymbol{\delta}, \quad \boldsymbol{\delta} \in \mathbb{R}^{m-1}. \tag{11}$$

Let s_{ii} be the ith diagonal element of $\mathbf{S}$, let $\mathbf{s}_i$ denote the ith column of $\mathbf{S}$ without s_{ii}, and let $\mathbf{S}_{(ii)}$ stand for $\mathbf{S}$ without its ith row and ith column. We denote $\mathbf{V}_i = \mathbf{S}_{(ii)} - \mathbf{s}_i\mathbf{s}_i^\top / s_{ii}$. For testing (11) we consider the test statistic

$$Q_i^{(\delta)} = Q_i^{(\delta)}(\mathbf{X}) = \frac{n-m}{m-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \mathbf{V}_i^{-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right) s_{ii}.$$

Let $\boldsymbol{\Sigma}_{(ii)}$ be obtained from $\boldsymbol{\Sigma}$ by deleting its ith row and ith column and let $\boldsymbol{\Omega}_i = \boldsymbol{\Sigma}_{(ii)} - \boldsymbol{\sigma}_i\boldsymbol{\sigma}_i^\top/\sigma_{ii}$. In Theorem 2, we derive the distribution of $Q_i^{(\delta)}$ both under $H_i^{(\delta)}$ and under $K_i^{(\delta)}$.

Theorem 2 *Let* $\mathbf{X} \sim \mathcal{N}_{m,n}(\boldsymbol{\mu}\mathbf{1}_n^\top, \boldsymbol{\Sigma} \otimes \mathbf{I}_n)$.

(a) *Under* $H_i^{(\delta)}$ *it holds that* $Q_i^{(\delta)} \sim F_{m-1,n-m}$.
(b) *Let* $\boldsymbol{\vartheta}_i \in K_i^{(\delta)}$. *Then, the pdf of* $Q_i^{(\delta)}$ *under* $\boldsymbol{\vartheta}_i$ *is given by*

$$f_{Q_i^{(\delta)}}(x) = \frac{f_{m-1,n-m}(x)}{(1+\lambda_i)^{(n-1)/2}}\, {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}; \frac{m-1}{2}; \frac{\lambda_i}{(1+\lambda_i)}\frac{\frac{m-1}{n-m}x}{1+\frac{m-1}{n-m}x}\right),$$

where

$$\lambda_i = \sigma_{ii}(\boldsymbol{\vartheta}_i - \boldsymbol{\delta})^\top \boldsymbol{\Omega}_i^{-1}(\boldsymbol{\vartheta}_i - \boldsymbol{\delta}). \tag{12}$$

Proof Applying Theorem 3.2.10 by Muirhead (1982), we get that $s_{ii} \sim W_1(n-1, \sigma_{ii})$ (i. e., $s_{ii}/\sigma_{ii} \sim \chi^2_{n-1}$), $\mathbf{V}_i \sim W_{m-1}(n-2, \boldsymbol{\Omega}_i)$,

$$\mathbf{s}_i | s_{ii} \sim \mathcal{N}_{m-1}\left(\boldsymbol{\sigma}_i \frac{s_{ii}}{\sigma_{ii}}, s_{ii}\boldsymbol{\Omega}_i\right),$$

and that $\mathbf{V}_i$ is stochastically independent of s_{ii} and $\mathbf{s}_i$.

Now, we consider the representation

$$Q_i^{(\delta)} = \frac{n-m}{m-1}\frac{\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \mathbf{V}_i^{-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)}{\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \boldsymbol{\Omega}_i^{-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \boldsymbol{\Omega}_i^{-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right) s_{ii}.$$

Because $\mathbf{V}_i$ is stochastically independent of s_{ii} and $\mathbf{s}_i$, application of Theorem 3.2.12 of Muirhead (1982) leads to

$$\frac{\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \boldsymbol{\Omega}_i^{-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)}{\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \mathbf{V}_i^{-1}\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)} \sim \chi^2_{n-m}, \tag{13}$$

where the latter statistic is stochastically independent of s_{ii} and $\mathbf{s}_i$.

Since $\sigma_{ii}^{-1}\boldsymbol{\sigma}_i = \boldsymbol{\delta}$ under $H_i^{(\boldsymbol{\delta})}$, we have

$$\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \boldsymbol{\Omega}_i^{-1} \left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right) s_{ii} \sim \chi^2_{m-1} \tag{14}$$

in this case. Noticing that the statistic in (14) depends only on s_{ii} and $\mathbf{s}_i$, the assertion of part (a) follows by combining (13) and (14).

For proving part (b), we use that

$$\left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top \boldsymbol{\Omega}_i^{-1} \left(\frac{\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right) s_{ii} | s_{ii} = y \sim \chi^2_{m-1}(y\tilde{\lambda}_i), \tag{15}$$

where $\tilde{\lambda}_i = \sigma_{ii}^{-1}\lambda_i$ with λ_i defined in (12). Hence, from (13) and (15), we get

$$Q_i^{(\boldsymbol{\delta})} | s_{ii} = y \sim F_{m-1,n-m}(y\tilde{\lambda}_i).$$

Making use of $s_{ii}/\sigma_{ii} \sim \chi^2_{n-1}$ yields that

$$\begin{aligned} f_{Q_i^{(\boldsymbol{\delta})}}(x) = & \frac{1}{2^{(n-1)/2}\sigma_{ii}^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)} \\ & \times \int_0^\infty y^{(n-1)/2-1} \exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_{ii}}\right)\right) f_{F_{m-1,n-m}(y\tilde{\lambda}_i)}(x)\mathrm{d}y. \end{aligned}$$

Let $f_{m-1,n-m}$ denote the pdf of the $F_{m-1,n-m}$ distribution. Application of Theorem 1.3.6 in Muirhead (1982) leads to

$$\begin{aligned} f_{Q_i^{(\boldsymbol{\delta})}}(x) = & \frac{f_{m-1,n-m}(x)}{2^{(n-1)/2}\sigma_{ii}^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty y^{(n-1)/2-1} \exp\left(-\frac{1}{2}(\sigma_{ii}^{-1} + \tilde{\lambda}_i)y\right) \\ & \times {}_1F_1\left(\frac{n-1}{2}; \frac{m-1}{2}; \frac{1}{2}\frac{\frac{m-1}{n-m}x}{1+\frac{m-1}{n-m}x}\tilde{\lambda}_i y\right)\mathrm{d}y. \end{aligned}$$

The last integral can be evaluated using Lemma 1.3.3 of Muirhead (1982), yielding

$$\begin{aligned} f_{Q_i^{(\boldsymbol{\delta})}}(x) = & \frac{f_{m-1,n-m}(x)}{(1+\sigma_{ii}\tilde{\lambda}_i)^{(n-1)/2}} \\ & \times {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}; \frac{m-1}{2}; \frac{\tilde{\lambda}_i}{(\sigma_{ii}^{-1}+\tilde{\lambda}_i)}\frac{\frac{m-1}{n-m}x}{1+\frac{m-1}{n-m}x}\right). \end{aligned}$$

Noting that $\lambda_i = \sigma_{ii}\tilde{\lambda}_i$ completes the proof of Theorem 2. □

Corollary 4 *For each $1 \le i \le m$, the following assertions hold true.*

(a) *The test $\varphi_i^{(\boldsymbol{\delta})} = \mathbb{1}\{Q_i^{(\boldsymbol{\delta})} > c_\alpha\}$ is a level α test for $H_i^{(\boldsymbol{\delta})}$ versus $K_i^{(\boldsymbol{\delta})}$, where $c_\alpha = F_{m-1,n-m;1-\alpha}$.*

(b) *The set* $\mathcal{C}_{\alpha,i} = \{\boldsymbol{\delta} \in \mathbb{R}^{m-1} : Q_i^{(\boldsymbol{\delta})} \leq c_\alpha\}$ *constitutes a* $(1-\alpha)$*-confidence region for* $\sigma_{ii}^{-1}\boldsymbol{\sigma}_i$.
(c) *The hypothesis* $H_i^<$ *can be rejected at significance level* α *if* $H_i^< \cap \mathcal{C}_{\alpha,i} = \emptyset$.

Proof Part (a) is an immediate consequence of part (a) in Theorem 2. Parts (b) and (c) follow from Section 1 by Aitchison (1964). □

Next, we extend the results obtained for the normal distribution of $\mathbf{X}$ to the family of elliptically contoured distributions.

Theorem 3 *Assume that* $\mathbf{X} \sim E_{m,n}(\boldsymbol{\mu}\mathbf{1}_n^\top, \boldsymbol{\Sigma}\otimes\mathbf{I}_n, R)$ *(matrix-variate elliptically contoured distributed with location matrix* $\boldsymbol{\mu}\mathbf{1}^\top$*, scale matrix* $\boldsymbol{\Sigma}\otimes\mathbf{I}_n$ *and generating variable* R*) with* $\mathbb{P}(\mathbf{X} = \boldsymbol{\mu}\mathbf{1}_n^\top) = 0$. *Let* $n > m$ *and* $\mathbf{Y} \sim \mathcal{N}_{m,n}(\boldsymbol{\mu}\mathbf{1}_n^\top, \boldsymbol{\Sigma}\otimes\mathbf{I}_n)$. *Then, for any* $1 \leq i \leq m$ *and any* $\boldsymbol{\delta} \in \mathbb{R}^{m-1}$*, the distribution of* $Q_i^{(\boldsymbol{\delta})}(\mathbf{X})$ *is the same as the distribution of* $Q_i^{(\boldsymbol{\delta})}(\mathbf{Y})$*, i.e., these distributions do not depend on* R.

Proof We only provide the proof of part (a) and note that the results of part (b) are obtained in the same way.

Let $\mathbf{A} = \mathbf{I}_n - \mathbf{1}_n\mathbf{1}_n^\top$. Then, the sample covariance matrix for a data matrix $\mathbf{Y}$ is calculated by $\tilde{\mathbf{S}}(\mathbf{Y}) = \frac{1}{n-1}\mathbf{Y}\mathbf{A}\mathbf{Y}^\top$. First, note that $\tilde{\mathbf{S}}(\mathbf{Y}) = \tilde{\mathbf{S}}(\mathbf{Y} - \boldsymbol{\mu}\mathbf{1}_n^\top)$ and, therefore, without loss of generality, we can assume $\boldsymbol{\mu} = \mathbf{0}$.

Clearly, if $\mathbf{Y} \in \mathbb{R}^{m\times n}$ and $a > 0$, then $a\mathbf{Y} \in \mathbb{R}^{m\times n}$ as well. Furthermore, if $\mathbf{Y} \in \mathbb{R}^{m\times n}$ and $a > 0$, then obviously, $Q_i^{(\boldsymbol{\delta})}(a\mathbf{Y}) = Q_i^{(\boldsymbol{\delta})}(\mathbf{Y})$. Now, applying Theorem 5.12 of Gupta et al. (2013) with $\mathbf{K} = Q_i^{(\boldsymbol{\delta})}$, we obtain the assertion of Theorem 3. □

Finally, consider the problem of testing all non-diagonal elements in $\boldsymbol{\Sigma}$ simultaneously for non-negativity. This condition entails the PDS property of $\mathbf{Z}$. The alternative hypothesis that all non-diagonal elements in $\boldsymbol{\Sigma}$ are simultaneously non-negative can be expressed as $K^< = \bigcap_{i=1}^{m-1} K_i^<$, with corresponding null hypothesis given by $H^< = \bigcup_{i=1}^{m-1} H_i^<$. Let for each $1 \leq i \leq m-1$ the test $\varphi_i^<$ be defined via the decision rule given in part (c) of Corollary 3.1. Then, a test for $H^<$ versus $K^<$ is defined by

$$\varphi^< = \prod_{i=1}^{m-1} \varphi_i^<, \tag{16}$$

meaning that we reject $H^<$ iff all $H_i^<$ are rejected for $1 \leq i \leq m-1$.

Theorem 4 *Let* $\mathbf{X}$ *be distributed as in Theorem 3. Then, the test* $\varphi^<$ *has level* α.

Proof Assume that $\boldsymbol{\Sigma}$ is such that $H^<$ holds true. This means that at least one $H_i^<$ must hold true. Let i^* denote one of the indices for which $H_{i^*}^<$ is true. Then, the rejection probability of $\varphi_{i^*}^<$ under this $\boldsymbol{\Sigma}$ is bounded by α due to Corollary 4. But, the rejection event of the test $\varphi^<$ is a subset of the rejection event of $\varphi_{i^*}^<$. Thus, its probability under the considered $\boldsymbol{\Sigma}$ is bounded by α. □

Remark 4 If one is only interested in testing $H^<$ versus $K^<$, one may consider the modified pairs of hypotheses

$$\begin{aligned}\tilde{H}_i^< &: \sigma_{ji} < 0 \quad \text{for at least one } i < j \le m \text{ versus} \\ \tilde{K}_i^< &: \sigma_{ji} \ge 0 \quad \text{for all } i < j \le m,\end{aligned} \tag{17}$$

where $1 \le i \le m-1$, because it still holds that $K^< = \bigcap_{i=1}^{m-1} \tilde{K}_i^<$.

Let $\mathbf{L} = [\mathbf{0}\ \mathbf{I}_{m-i}]$ be a $(m-i) \times (m-1)$ matrix of zeros and ones. Then, the (auxiliary) family of point hypotheses pertaining to (17) is given by

$$\tilde{H}_i^{(\boldsymbol{\delta})} : \sigma_{ii}^{-1}\mathbf{L}\boldsymbol{\sigma}_i = \boldsymbol{\delta} \text{ versus } \tilde{K}_i^{(\boldsymbol{\delta})} : \sigma_{ii}^{-1}\mathbf{L}\boldsymbol{\sigma}_i \ne \boldsymbol{\delta}, \quad \boldsymbol{\delta} \in \mathbb{R}^{m-i} \tag{18}$$

for $i = 1, \ldots, m-1$. For testing (18), we construct the test statistic

$$\tilde{Q}_i^{(\boldsymbol{\delta})} = \frac{n-m}{m-i}\left(\frac{\mathbf{L}\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)^\top (\mathbf{L}\mathbf{V}_i\mathbf{L})^{-1}\left(\frac{\mathbf{L}\mathbf{s}_i}{s_{ii}} - \boldsymbol{\delta}\right)s_{ii}. \tag{19}$$

In analogy to the proof of Theorem 2, the distribution of $\tilde{Q}_i^{(\boldsymbol{\delta})}$ can be derived both under $\tilde{H}_i^{(\boldsymbol{\delta})}$ and under $\tilde{K}_i^{(\boldsymbol{\delta})}$. In particular, under $\tilde{H}_i^{(\boldsymbol{\delta})}$, it holds that $\tilde{Q}_i^{(\boldsymbol{\delta})} \sim F_{m-i,n-m}$. Consequently, the test $\tilde{\varphi}_i^{(\boldsymbol{\delta})} = 1\{\tilde{Q}_i^{(\boldsymbol{\delta})} > F_{m-i,n-m;1-\alpha}\}$ is a level α test for $\tilde{H}_i^{(\boldsymbol{\delta})}$ versus $\tilde{K}_i^{(\boldsymbol{\delta})}$. Exploiting again the duality of tests and confidence regions, the test for $\tilde{H}_i^<$ versus $\tilde{K}_i^<$ is defined according to the decision rule in part (c) of Corollary 4. Finally, a test $\tilde{\varphi}^<$ for $H^<$ versus $K^<$ is obtained as in (16), i.e., $H^<$ is rejected iff all $\tilde{H}_i^<$ are rejected.

4 Simulation study

In this section, we study the power functions of the two tests on non-negativity of correlation coefficients suggested in Sect. 3. In Theorem 3, we proved that the distribution of the test statistic $Q_i^{(\boldsymbol{\delta})}$ does not depend on the type of elliptical distribution. For this reason and for convenience, we simulated samples of size $n \in \{150, 250\}$ from the m-variate normal distribution, $m \in \{10, 20\}$, with mean vector zero and covariance matrix $\boldsymbol{\Sigma} = (1-\rho)\mathbf{I}_p + \rho\mathbf{1}_p\mathbf{1}_p^\top$, where $\rho \in [0, 0.5]$. The results are based on $B = 10{,}000$ independent Monte Carlo repetitions.

In case of $\varphi^<$, the empirical power of the test is defined as the relative frequency of Monte Carlo simulation runs in which all the confidence sets $\mathcal{C}_{\alpha,i}$, $i = 1, \ldots, m$, defined in Corollary 4 are contained in $\mathbb{R}_+^{m-1}$, i.e.,

$$\widehat{\text{power}}(\varphi^<) = \frac{1}{B}\sum_{b=1}^{B}\prod_{i=1}^{m}\mathbb{1}(\mathcal{C}_{\alpha,i}^{(b)} \subseteq \mathbb{R}_+^{m-1}), \tag{20}$$

where $\mathcal{C}_{\alpha,i}^{(b)}$ is the confidence region in the bth simulation run. In the analogous way, we define the empirical power of the test $\tilde{\varphi}^<$.

The condition $\mathcal{C}_{\alpha,i} \subseteq \mathbb{R}_+^{m-1}$ appearing in (20) means that the ellipsoid $\{Q_i^{(\delta)} \leq c_\alpha\}$ with $c_\alpha = F_{m-1,n-m;1-\alpha}$ lies inside $\mathbb{R}_+^{m-1}$. This condition is checked in two steps:

– Check if the central point of $\{Q_i^{(\delta)} \leq c_\alpha\}$ lies in $\mathbb{R}_+^{m-1}$, i. e., if all components of $\mathbf{s}_i$ are non-negative.
– If the first condition is fulfilled, then check if the ellipsoid $\{Q_i^{(\delta)} \leq c_\alpha\}$ does not intersect the axes. The latter condition is equivalent to

$$\frac{\left(\mathbf{s}_i^{(-j)}\right)^\top \left(\mathbf{V}_i^{-j}\right)^{-1} \left(\mathbf{s}_i^{(-j)}\right)}{s_{ii}^2} - \frac{c_\alpha}{s_{ii}} \geq 0 \quad \text{for} \quad j = 1, \ldots, m-1,$$

where $\mathbf{s}_i^{(-j)}$ is obtained from $\mathbf{s}_i$ by deleting its jth element and, similarly, $\mathbf{V}_i^{-j}$ is obtained from $\mathbf{V}_i$ by deleting its jth column and its jth row.

In Fig. 1, we present the results of the simulation study. The solid lines correspond to $\varphi^<$, whereas the dashed lines refer to $\tilde{\varphi}^<$. We observe that already for $\rho = 0.3$ the empirical powers of both tests approach one, meaning that the decision in favor of the alternative hypothesis of non-negative correlations is taken with probability close to one. The multiple test $\tilde{\varphi}^<$ is slightly more powerful than $\varphi^<$. The difference between the empirical powers of the two tests becomes larger for larger values of m. If n equals 250, then the empirical powers of the tests approach one already for $\rho = 0.2$.

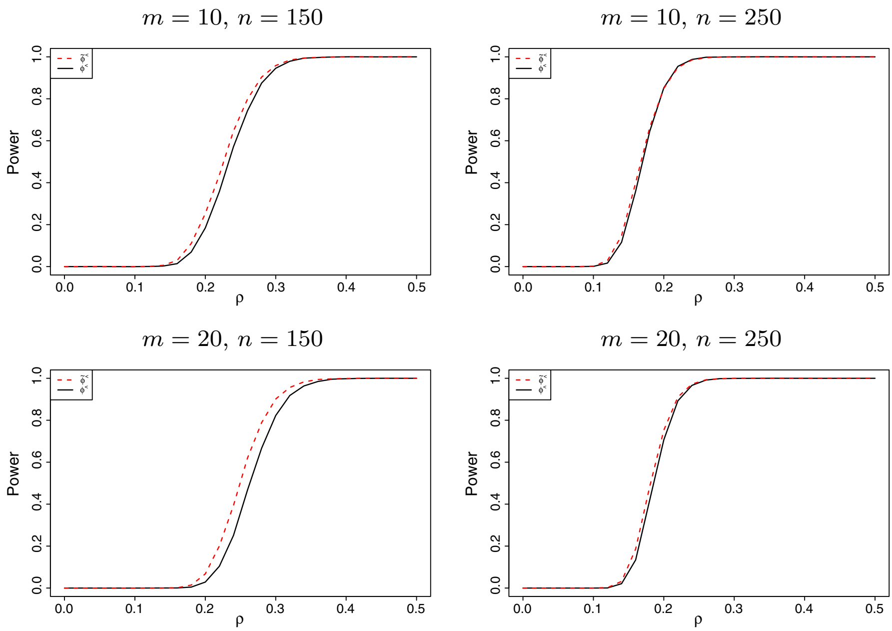

Fig. 1 Empirical powers of the tests $\varphi^<$ and $\tilde{\varphi}^<$ in case of $\mathbf{\Sigma} = (1-\rho)\mathbf{I}_p + \rho\mathbf{1}_p\mathbf{1}_p^\top$ as functions of $\rho \in [0, 0.5]$ for $m \in \{10, 20\}$ and $n \in \{150, 250\}$

5 Discussion

We have provided necessary and sufficient conditions for the validity of Simes' inequality in the broad class of elliptically contoured distributions. Our sufficient conditions can be checked in practice by means of the cdf F of T_1 under the null, together with the tests on $\mathbf{\Sigma}$ that we have derived in Sect. 3. Our necessary conditions (i.e., the sufficient conditions for the validity of the reverse Simes inequality) contribute to a characterization of classes of multivariate probability distributions for which the Simes conjecture is true. The latter problem is still an active area of multiple test theory, not least because of its practical relevance due to the popularity of φ^{LSU}. For example, Läuter (2013) conjectured, based on extensive computer simulations, that Simes' inequality is always valid in applications of the two-sided F or Beta test to normally distributed data with any covariance matrix $\mathbf{\Sigma}$. In contrast, part (b) of our Theorem 1 shows that conditions on $\mathbf{\Sigma}$ are necessary for the validity of Simes' inequality in the broader class $E_m(\mathbf{0}, \mathbf{\Sigma}, R)$. Further counterexamples (in non-elliptical models) have been presented by Finner and Strassburger (2014).

Furthermore, our tests on non-negativity of correlation coefficients are contributions to multivariate analysis of independent value. It is well known (see, e.g., Theorem 5.1.8 in Muirhead 1982) that a uniformly most powerful test for the one-sided hypothesis about a single population correlation coefficient ρ_{ij} (say) with corresponding pair of indices (i, j) in the vector $\mathbf{X}_1$ can be based on the test statistic

$$Q_{ij} = \sqrt{n-2}\frac{r_{ji}}{\sqrt{1-r_{ji}^2}},$$

where $r_{ji} = s_{ji}/\sqrt{s_{ii}s_{jj}}$ is the corresponding sample correlation coefficient. Under $\rho_{ij} = 0$, Q_{ij} follows a central univariate Student's t distribution. However, the joint distribution of several of the Q_{ij}, which is needed for multiple test problems regarding several ρ_{ij} simultaneously, is not pivotal, because the dependency structure among the Q_{ij} depends on unknown model parameters. Therefore, Westfall and Young (1993, pp. 194–199) have considered resampling-based approaches which reproduce this unknown dependency structure at least asymptotically as $n \to \infty$. In contrast, the exact tests developed in Sect. 3 are non-asymptotic and distribution free for any sample size n.

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