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# Efficient ANOVA for directional data

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**Abstract** In this paper, we tackle the ANOVA problem for directional data. We apply the invariance principle to construct locally and asymptotically most stringent rank-based tests. Our semi-parametric tests improve on the optimal parametric tests by being valid under the whole class of rotationally symmetric distributions. Moreover, they keep the optimality property of the latter under a given $m$-tuple of rotationally symmetric distributions. Asymptotic relative efficiencies are calculated and the finite-sample behavior of the proposed tests is investigated by means of a Monte Carlo simulation. We conclude by applying our findings to a real-data example involving geological data.

**Keywords** Directional statistics · Local asymptotic normality · Pseudo-FvML tests · Rank-based inference · ANOVA

## 1 Introduction

Spherical or directional data naturally arise in a broad range of earth sciences such as geology, astrophysics, meteorology, oceanography, in studies of animal behavior or

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even in neuroscience (see Mardia and Jupp 2000 and the references therein). Although primitive statistical analysis of directional data can already be traced back to early nineteenth century works by the likes of C. F. Gauss and D. Bernoulli, the methodical and systematic study of such non-linear data by means of tools tailored for their specificities only began in the 1950s under the impetus of Sir Ronald Fisher's pioneering work (see Fisher 1953). We refer the reader to the monographs (Fisher et al. 1987; Mardia and Jupp 2000) for a thorough introduction and a comprehensive overview of this discipline.

Spherical or directional data are modeled as realizations of random vectors $\mathbf{X}$ taking values on the surface of the unit hypersphere $\mathcal{S}^{k-1} := \{\mathbf{v} \in \mathbb{R}^k : \mathbf{v}'\mathbf{v} = 1\}$, the distribution of $\mathbf{X}$ depending only on its angular distance from a fixed point $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ which is to be viewed as a "north pole" for the problem under study. A natural, flexible and realistic family of probability distributions for these data is the class of *rotationally symmetric distributions* introduced by Saw (1978) (in the circular case $k = 2$, one rather speaks of *reflective symmetry*, see Jones and Pewsey 2005). Roughly speaking rotationally symmetric distributions allow to model all spherical data that are spread out uniformly around a central parameter $\boldsymbol{\theta}$ with the concentration of the data waning as the angular distance from the north pole increases. This class of distributions contains, for instance, the most used and best studied directional distribution: the *Fisher-von Mises-Langevin* (*FvML*) distribution; we refer the interested reader to Breitenberger (1963), Bingham and Mardia (1975) or Duerinckx and Ley (2012) for details and references on the FvML distribution. Precise definitions and notations will be provided in Sect. 2.

Within this setup, an important question goes as follows: "do two or more sets of spherical data spring from the same source?". Such questions are important, e.g., in the study of magnetism and palaeomagnetic data, as recognized in the seminal paper (Graham 1949) where the classical *fold test* for palaeomagnetic data was developed (see also McFadden and Jones 1981 for the two-sample and McFadden and Lowes 1981 for the multi-sample problem). In mathematical terms, the fold test can be described as follows. Suppose that there are $m$ different data sets spread around $i$ sources of magnetism $\boldsymbol{\theta}_i \in \mathcal{S}^{k-1}$, $i = 1, \ldots, m$. The question then becomes that of testing for the problem $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_m$ against $\mathcal{H}_1 : \exists\, 1 \leq i \neq j \leq m$ such that $\boldsymbol{\theta}_i \neq \boldsymbol{\theta}_j$, that is, an ANOVA problem for directional data.

This ANOVA problem is of course of interest for all disciplines dealing with directional observations and thus has been studied considerably in the statistical literature. The technical difficulty of the task, however, entails that most available methods are either of parametric nature (assuming, as in McFadden and Jones 1981; McFadden and Lowes 1981, that the data follow an FvML distribution; see also Sections 10.5 and 10.6 of Mardia and Jupp (2000), where several FvML-based procedures are discussed), suffer from computational difficulties/slowness (such as Wellner's 1979 permutation test or Fisher and Hall's 1990; Beran and Fisher's 1998 bootstrap test), lack desirable geometric properties or are restricted to the circular ($k = 2$) setting (e.g., Eplett 1979 or Eplett 1982) or are confined to the two-sample case (e.g., Jupp 1987 or Tsai 2009). To the best of the authors' knowledge, the only computationally simple, rotationally equivariant test for the general multi-sample null hypothesis $\mathcal{H}_0$ above is the score test $\varphi_{\mathrm{W}}^{(n)}$ given in Watson (1983).

In this paper, we use a spherical adaptation of Le Cam's methodology and propose two families of tests for the spherical ANOVA problem. First we take advantage of the aforementioned analogy between the Gaussian and the FvML to introduce *pseudo-FvML* tests $\varphi^{(n)}$ (see Sect. 3). These tests have very good properties (they are locally and asymptotically most stringent in the FvML case), but turn out to be asymptotically equivalent to Watson's $\varphi_{\mathrm{W}}^{(n)}$ although they perform better in a small sample setting, as shown in Sect. 5. The asymptotic optimality property of $\varphi_{\mathrm{W}}^{(n)}$ and $\varphi^{(n)}$ is, by construction, restricted to the case where the underlying $m$-tuple of distributions is FvML. To compensate for this limitation, we also use the invariance principle to introduce a family of rank-based tests based on estimated versions of the spherical signs and ranks (see Sect. 4) which

(i) remain valid (in the sense that they meet the nominal level constraint) under *any* $m$-tuple $(Q_1, \ldots, Q_m)$ of rotationally symmetric densities satisfying the general null hypothesis $\mathcal{H}_0$;
(ii) are asymptotically optimal under a given $m$-tuple of distributions $(P_1, \ldots, P_m)$ which *need not* be of FvML type;
(iii) do not assume that the $P_i$s are all equal (thus allowing for distinct concentrations of the data).

In particular, these tests are asymptotically distribution-free within the semi-parametric class of rotationally symmetric distributions.

The more detailed outline of the paper is as follows. In Sect. 2, we define the class of rotationally symmetric distributions, we collect the main assumptions required in the sequel and discuss the ULAN property of rotationally symmetric distributions and its consequences. We construct pseudo-FvML tests in Sect. 3 and rank-based tests in Sect. 4. A comparison between the two procedures on basis of asymptotic relative efficiencies as well as via a Monte Carlo simulation study is provided in Sect. 5. We apply our findings to a real-data application in Sect. 6. Finally an appendix collects the precise ULAN statements as well as the main technical proofs and details.

## 2 Rotational symmetry, main assumptions and notations

Throughout, the $m(\geq 2)$ independent samples of data points $\mathbf{X}_{i1}, \ldots, \mathbf{X}_{in_i}$, $i = 1, \ldots, m$, are assumed to belong to the unit sphere $\mathcal{S}^{k-1}$ of $\mathbb{R}^k$, $k \geq 2$, and to satisfy

**Assumption A** (*Rotational symmetry*) For all $i = 1, \ldots, m$, $\mathbf{X}_{i1}, \ldots, \mathbf{X}_{in_i}$ are i.i.d. with common distribution $P_{\boldsymbol{\theta}_i; f_i}$ characterized by a density (with respect to the usual surface area measure on spheres)

$$\mathbf{x} \mapsto c_{k,f_i}\; f_i(\mathbf{x}'\boldsymbol{\theta}_i), \quad \mathbf{x} \in \mathcal{S}^{k-1}, \tag{1}$$

where $\boldsymbol{\theta}_i \in \mathcal{S}^{k-1}$ is a location parameter, $f_i : [-1, 1] \to \mathbb{R}_0^+$ is absolutely continuous and strictly monotone increasing and $c_{k,f_i}$ is a normalizing constant. If $\mathbf{X}$ has density (1), then the density of $\mathbf{X}'\boldsymbol{\theta}_i$ is

$$t \mapsto \tilde{f}_i(t) := \frac{\omega_k \, c_{k,f_i}}{B\left(\frac{1}{2}, \frac{1}{2}(k-1)\right)} f_i(t)(1-t^2)^{(k-3)/2}, \quad -1 \le t \le 1,$$

where $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of $\mathcal{S}^{k-1}$ and $B(\cdot,\cdot)$ is the beta function. The corresponding cumulative distribution function (cdf) is denoted by $\tilde{F}_i(t)$, $i = 1, \ldots, m$.

The functions $f_i$ are called *angular functions* (because the distribution of each $\mathbf{X}_{ij}$ depends only on the angle between it and the location $\boldsymbol{\theta}_i \in \mathcal{S}^{k-1}$). Throughout the rest of this paper, we denote by $\mathcal{F}^m$ the collection of $m$-tuples of angular functions $\underline{f} := (f_1, f_2, \ldots, f_m)$. The assumption of rotational symmetry entails appealing stochastic properties. Indeed, as shown in Watson (1983), for a random vector $\mathbf{X}$ distributed according to some $P_{\boldsymbol{\theta}_i; f_i}$ as in Assumption A, not only is the multivariate sign vector

$$\mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}) = \frac{\mathbf{X} - (\mathbf{X}'\boldsymbol{\theta}_i)\boldsymbol{\theta}_i}{||\mathbf{X} - (\mathbf{X}'\boldsymbol{\theta}_i)\boldsymbol{\theta}_i||}$$

uniformly distributed on $\mathcal{S}^{\boldsymbol{\theta}_i^\perp} := \{\mathbf{v} \in \mathbb{R}^k \mid \|\mathbf{v}\| = 1, \mathbf{v}'\boldsymbol{\theta}_i = 0\}$ but also the angular distance $\mathbf{X}'\boldsymbol{\theta}_i$ and the sign vector $\mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X})$ are stochastically independent.

The class of rotationally symmetric distributions contains a wide variety of useful spherical distributions including the FvML, the spherical linear, the spherical logarithmic, the spherical logistic and the Purkayastha distributions (see Purkayastha 1991). Often used reflectively symmetric distributions (in the circular setting) are the cardioid, the wrapped Cauchy and the wrapped normal; see Jammalamadaka and SenGupta (2001), Jones and Pewsey (2005) or Abe et al. (2010). The most popular and most used rotationally symmetric distribution is the aforementioned FvML distribution (named, according to Watson 1983, after von Mises 1918; Fisher 1953; Langevin 1905) with density

$$\phi_\kappa(\mathbf{x}) := f_{\mathrm{FvML}(\kappa)}(\mathbf{x}; \boldsymbol{\theta}) = c_k(\kappa)\exp(\kappa \mathbf{x}'\boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{k-1},$$

where $\kappa > 0$ is a concentration or dispersion parameter, $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ is a location parameter and $c_k(\kappa)$ is the corresponding normalizing constant.

Throughout the paper, our asymptotic results require a certain amount of control on the respective sample sizes $n_i$, $i = 1, \ldots, m$. This we achieve via the following

**Assumption B** Letting $n = \sum_{i=1}^m n_i$, for all $i = 1, \ldots, m$ the ratio $r_i^{(n)} := n_i/n$ converges to a finite constant $r_i$ as $n \to \infty$.

In particular Assumption B entails that the specific sizes $n_i$ are, up to a point, irrelevant; hence, in what precedes and in what follows, we simply use the superscript $^{(n)}$ for the different quantities at play and do not specify whether they are associated with a given $n_i$. In the sequel we let $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ stand for the $m \times m$ block-diagonal matrix with blocks $\mathbf{A}_1, \ldots, \mathbf{A}_m$, and use the notations $\boldsymbol{\nu}^{(n)} := \mathrm{diag}((r_1^{(n)})^{-1/2}\mathbf{I}_k, \ldots, (r_m^{(n)})^{-1/2}\mathbf{I}_k)$, $\mathcal{M}(\mathbf{A})$ for the linear subspace spanned by the columns of the matrix $\mathbf{A}$, $\mathbf{1}_m := (1, \ldots, 1)' \in \mathbb{R}^m$ and $\mathbf{A} \otimes \mathbf{B}$ for the Kronecker product between $\mathbf{A}$ and $\mathbf{B}$.

Letting $\boldsymbol{\vartheta} := (\boldsymbol{\theta}'_1, \ldots, \boldsymbol{\theta}'_m)'$ and putting $P^{(n)}_{\boldsymbol{\vartheta};\underline{f}}$ for the joint distribution of the $\mathbf{X}_{ij}$s under Assumption A, we show in Appendix A that the parametric model $\left\{P^{(n)}_{\boldsymbol{\vartheta};\underline{f}} \mid \boldsymbol{\vartheta} \in (\mathcal{S}^{k-1})^m\right\}$ is ULAN under the following usual assumption.

**Assumption C** The Fisher information associated with the spherical location parameter is finite; this finiteness is ensured if, for $i = 1, \ldots, m$,

$$\mathcal{J}_k(f_i) := \int_{-1}^{1} \psi^2_{f_i}(t)(1-t^2)\tilde{f}_i(t)\mathrm{d}t < +\infty$$

with $\psi_{f_i} := \dot{f}_i/f_i$ ($\dot{f}_i$ is the a.e. derivative of the function $u \to f_i(u)$ (from $[-1, 1] \to \mathbb{R}^+_0$)).

The *central sequence* (a score-like quantity in the Le Cam methodology) associated with the model $\left\{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta};\underline{f}} \mid \boldsymbol{\vartheta} \in (\mathcal{S}^{k-1})^m\right\}$ is the sequence

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta};\underline{f}} := \left(\left(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_1;f_1}\right)', \ldots, \left(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_m;f_m}\right)'\right)',$$

where

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_i;f_i} := n_i^{-1/2} \sum_{j=1}^{n_i} \psi_{f_i}(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)(1-(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)^2)^{1/2}\mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}_{ij}), \quad i = 1, \ldots, m,$$

and the Fisher information matrix is given by

$$\boldsymbol{\Gamma}_{\boldsymbol{\vartheta};\underline{f}} := \mathrm{diag}(\boldsymbol{\Gamma}_{\boldsymbol{\theta}_1;f_1}, \ldots, \boldsymbol{\Gamma}_{\boldsymbol{\theta}_m;f_m}),$$

with

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta}_i;f_i} := \frac{\mathcal{J}_k(f_i)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}_i\boldsymbol{\theta}'_i), \quad i = 1, \ldots, m.$$

For the sake of readability, the technical details related with this ULAN property (along with an explicit statement of said property, see Proposition 4) are delayed to the same Appendix A. The central sequence $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta};\underline{f}}$ will play an important role in the construction of our tests.

## 3 FvML score tests

For a given $m$-tuple of FvML densities $(\phi_{\kappa_1}, \ldots, \phi_{\kappa_m})$ with respective concentration parameters $\kappa_1, \ldots, \kappa_m > 0$ (where we do not assume $\kappa_1 = \cdots = \kappa_m$), the score functions $\psi_{\phi_{\kappa_i}}$ reduce to the constants $\kappa_i$, $i = 1, \ldots, m$, and hence the central sequences for each sample take the simplified form

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_i;\phi_{\kappa_i}} := \kappa_i n_i^{-1/2} \sum_{j=1}^{n_i} (1 - (\mathbf{X}'_{ij}\boldsymbol{\theta}_i)^2)^{1/2} \mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}_{ij})$$
$$= \kappa_i (\mathbf{I}_k - \boldsymbol{\theta}_i \boldsymbol{\theta}'_i) n_i^{1/2} (\bar{\mathbf{X}}_i - \boldsymbol{\theta}_i), \quad i = 1, \ldots, m.$$

Asymptotic tests for the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_m$ can be obtained by studying the asymptotic distribution of

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta};\underline{\phi}} = \left( \left( \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_1;\phi_{\kappa_1}} \right)', \ldots, \left( \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_m;\phi_{\kappa_m}} \right)' \right)'$$

under the null (and any $m$-tuple of rotationally symmetric distributions). The asymptotic distribution of $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta};\underline{\phi}}$ (for known common location $\boldsymbol{\theta}$) easily follows from the multivariate central limit theorem and leads to consideration of the following construction for the $\boldsymbol{\theta}$-unspecified setting we are interested in.

Let $\hat{\boldsymbol{\theta}}$ be a root-$n$ consistent estimator of $\boldsymbol{\theta}$ under $\mathcal{H}_0$ and define, for $i = 1, \ldots, m$, the quantities

$$\hat{B}_i := 1 - n_i^{-1} \sum_{j=1}^{n_i} (\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}})^2 \text{ and } \hat{E}_i := n_i^{-1} \sum_{j=1}^{n_i} (\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}})$$

as well as

$$\hat{D}_i := \frac{\hat{E}_i}{\hat{B}_i} \text{ and } \hat{H} := \sum_{i=1}^{m} r_i^{(n)} \hat{D}_i^2 \hat{B}_i.$$

Following the rationale behind Hallin and Paindaveine (2008) and Hallin et al. (2013), one sees that a *pseudo-FvML* test for $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_m$ against $\mathcal{H}_1 : \exists\, 1 \leq i \neq j \leq m$ such that $\boldsymbol{\theta}_i \neq \boldsymbol{\theta}_j$ rejects the null when the statistic

$$Q^{(n)} = (k-1) \left( \sum_{i=1}^{m} \frac{n_i \hat{D}_i}{\hat{E}_i} \bar{\mathbf{X}}'_i (\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}') \bar{\mathbf{X}}_i \right.$$
$$\left. - \sum_{i,j}^{m} \frac{n_i n_j}{n} \frac{\hat{D}_i \hat{D}_j}{\hat{H}} \bar{\mathbf{X}}'_i (\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}') \bar{\mathbf{X}}_j \right) \tag{2}$$

exceeds the $\alpha$-upper quantile of the Chi-square distribution with $(m-1)(k-1)$ degrees of freedom.

The following proposition yields the asymptotic properties of $Q^{(n)}$ under the entire class of rotationally symmetric distributions. Before proceeding we need some more notation. Given $\underline{g} = (g_1, \ldots, g_m) \in \mathcal{F}^m$, define for $i = 1, \ldots, m$

$$B_{g_i} := 1 - \mathrm{E}_{g_i}[(\mathbf{X}'_{ij}\boldsymbol{\theta})^2], \quad E_{g_i} := \mathrm{E}_{g_i}[\mathbf{X}'_{ij}\boldsymbol{\theta}]$$

and

$$C_{g_i} := \mathrm{E}_{g_i}[(1 - (\mathbf{X}'_{ij}\boldsymbol{\theta})^2) \psi_{g_i}(\mathbf{X}'_{ij}\boldsymbol{\theta})]$$

($E_{g_i}$ denotes an expectation taken under $g_i$, i.i.d.ness ensures that these quantities do not depend on $j$) as well as

$$D_{g_i} := E_{g_i}/B_{g_i} \text{ and } H_{\underline{\phi},\underline{g}} := \sum_{i=1}^{m} r_i^{(n)} D_{g_i}^2 B_{g_i}.$$

Also let

$$l_{\mathbf{t};\underline{\phi},\underline{g}} := \frac{1}{(k-1)} \left( \sum_{i=1}^{m} \frac{C_{g_i}^2}{B_{g_i}} \mathbf{t}_i' \mathbf{t}_i - \left( \sum_{i=1}^{m} \frac{r_i E_{g_i}^2}{B_{g_i}} \right)^{-1} \right. \\ \left. \times \sum_{i,j=1}^{m} \frac{\sqrt{r_i}\sqrt{r_j} C_{g_i} E_{g_i} C_{g_j} E_{g_j}}{B_{g_i} B_{g_j}} \mathbf{t}_i' \mathbf{t}_j \right) \tag{3}$$

for $\mathbf{t} := (\mathbf{t}_1', \ldots, \mathbf{t}_m')'$. Finally, putting $\boldsymbol{\vartheta}_0 := \mathbf{1}_m \otimes \boldsymbol{\theta}$ ($\boldsymbol{\theta}$ is the common value of $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ under the null) we assume the existence of an estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ such that the following assumption holds.

**Assumption D** The estimator $\hat{\boldsymbol{\vartheta}} = \mathbf{1}_m \otimes \hat{\boldsymbol{\theta}}$, with $\hat{\boldsymbol{\theta}} \in \mathcal{S}^{k-1}$, is $n^{1/2}\big(\boldsymbol{\nu}^{(n)}\big)^{-1}$*-consistent:* for all $\boldsymbol{\vartheta}_0 = \mathbf{1}_m \otimes \boldsymbol{\theta} \in \mathcal{H}_0$, $n^{1/2}\big(\boldsymbol{\nu}^{(n)}\big)^{-1}(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0) = O_P(1)$, as $n \to \infty$ under $P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$ for any $\underline{g} \in \mathcal{F}^m$.

Typical examples of estimators satisfying Assumption D belong to the class of $M$-estimators (see Chang 2004) or $R$-estimators (see Ley et al. 2013).

**Proposition 1** *Let Assumptions A, B and C hold and let $\hat{\boldsymbol{\vartheta}}$ be an estimator of $\boldsymbol{\vartheta}_0$ such that Assumption D holds. Then*

(i) $Q^{(n)}$ *is asymptotically Chi-square with* $(m-1)(k-1)$ *degrees of freedom under* $\bigcup_{\boldsymbol{\vartheta}_0 \in \mathcal{H}_0} \bigcup_{\underline{g} \in \mathcal{F}^m} P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$;

(ii) $Q^{(n)}$ *is asymptotically non-central Chi-square with* $k-1$ *degrees of freedom and non-centrality parameter* $l_{\mathbf{t};\underline{\phi},\underline{g}}$ *(defined in 3) under* $P^{(n)}_{\boldsymbol{\vartheta}_0 + n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$, *where* $\mathbf{t}^{(n)}$ *is as in (10) and* $\mathbf{t} := \lim_{n\to\infty} \mathbf{t}^{(n)}$;

(iii) *the test* $\varphi^{(n)}$ *which rejects the null hypothesis as soon as* $Q^{(n)}$ *exceeds the* $\alpha$*-upper quantile of the Chi-square distribution with* $(m-1)(k-1)$ *degrees of freedom has asymptotic level* $\alpha$ *under* $\bigcup_{\boldsymbol{\vartheta}_0 \in \mathcal{H}_0} \bigcup_{\underline{g} \in \mathcal{F}^m} \{P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}\}$;

(iv) $\varphi^{(n)}$ *is locally and asymptotically most stringent, at asymptotic level* $\alpha$, *for* $\bigcup_{\boldsymbol{\vartheta}_0 \in \mathcal{H}_0} \bigcup_{\underline{g} \in \mathcal{F}^m} \{P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}\}$ *against alternatives of the form* $\bigcup_{\boldsymbol{\vartheta} \notin \mathcal{H}_0} \{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta};\underline{\phi}}\}$.

The proof follows along the same lines as that of the forthcoming Proposition 3 and is, therefore, omitted. Moreover, this proof is not enlightening because it is easy to see that our construction is, asymptotically, equivalent to that proposed for the same problem in Watson (1983). More precisely, using Watson's Watson (1983) arguments, one easily shows that the difference between our $Q^{(n)}$ and Watson's test statistic is an $o_P(1)$ quantity under the null (and, therefore, also under contiguous alternatives). See Watson (1983, p. 145) for details. However, our construction is worth noting because of its very good behavior in fixed sample situations: the simulation results presented in Sect. 5.2 clearly show that our test $\varphi^{(n)}$ outperforms the Watson test when small samples are considered.

As mentioned in the Introduction, the asymptotic optimality property of $\varphi_{\mathrm{W}}^{(n)}$ and $\varphi^{(n)}$ is, by construction, restricted to the case where the underlying $m$-tuple of distributions is FvML. This is why we propose rank-based tests in the next section.

## 4 Rank-based tests

Our aim in the present section is to provide tests which are asymptotically optimal under any fixed (possibly non-FvML) $m$-tuple of rotationally symmetric distributions. Starting from any given $m$-tuple $\underline{f} \in \mathcal{F}^m$, our objective is to provide tests which are asymptotically valid under any $m$-tuple of (non-necessarily equal) rotationally symmetric distributions and which remain optimal under $\underline{f}$. To obtain such a test, we have recourse here to the invariance principle. This principle advocates that if the sub-model identified by the null hypothesis is invariant under the action of a group of transformations $\mathcal{G}_T$, one should exclusively use procedures whose outcome does not change along the orbits of that group $\mathcal{G}_T$. This is the case if and only if these procedures are measurable with respect to the maximal invariant associated with $\mathcal{G}_T$. The invariance principle is accompanied by an appealing corollary for our purposes here: provided that the group $\mathcal{G}_T$ is a generating group for $\mathcal{H}_0$, the invariant procedures are distribution-free under the null.

Invariance with respect to "common rotations" is crucial in this context. More precisely, letting $\mathbf{O} \in \mathcal{SO}_k := \{\mathbf{A} \in \mathbb{R}^{k\times k}, \mathbf{A}'\mathbf{A} = \mathbf{I}_k,\ \det(\mathbf{A}) = 1\}$, the null hypothesis is unquestionably invariant with respect to a transformation of the form

$$g_{\mathbf{O}} : \mathbf{X}_{11}, \ldots, \mathbf{X}_{mn_m} \mapsto \mathbf{O}\mathbf{X}_{11}, \ldots, \mathbf{O}\mathbf{X}_{mn_m}.$$

However, this group is not a generating group for $\mathcal{H}_0$ as it does not take into account the underlying angular functions $\underline{f}$, which are an infinite-dimensional nuisance under $\mathcal{H}_0$. This group is actually rather generating for $\bigcup_{\boldsymbol{\vartheta}_0\in\mathcal{H}_0} \mathrm{P}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{f}}$ with fixed $\underline{f}$. Now, denote as in the previous section the common value of $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ under the null as $\boldsymbol{\theta}$. Note also that, by definition,

$$\mathbf{X}_{ij} = (\mathbf{X}'_{ij}\boldsymbol{\theta})\boldsymbol{\theta} + \sqrt{1-(\mathbf{X}'_{ij}\boldsymbol{\theta})^2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})$$

for all $j = 1, \ldots, n_i$ and $i = 1, \ldots, m$. Let $\mathcal{G}_{\underline{h}}$ ($\underline{h} := (h_1, \ldots, h_m)$) be the group of transformations of the form

$$g_{h_i} : \mathbf{X}_{ij} \mapsto g_{h_i}(\mathbf{X}_{ij}) = h_i(\mathbf{X}'_{ij}\boldsymbol{\theta})\boldsymbol{\theta} + \sqrt{1-(h_i(\mathbf{X}'_{ij}\boldsymbol{\theta}))^2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}),$$

for $i = 1, \ldots, m$, where the $h_i : [-1, 1] \to [-1, 1]$ are monotone continuous nondecreasing functions such that $h_i(1) = 1$ and $h_i(-1) = -1$. For any $m$-tuple of (possibly different) transformations $(g_{h_1}, \ldots, g_{h_m}) \in \mathcal{G}_{\underline{h}}$, it is easy to verify that $\|g_{h_i}(\mathbf{X}_{ij})\| = 1$; thus, $g_{h_i}$ is a monotone transformation from $\mathcal{S}^{k-1}$ to $\mathcal{S}^{k-1}$, $i = 1, \ldots, m$. Note furthermore that $g_{h_i}$ does not modify the signs $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij})$. Hence, the group of transformations $\mathcal{G}_{\underline{h}}$ is a generating group for $\bigcup_{\underline{f} \in \mathcal{F}^m} \mathrm{P}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{f}}$ and the null is invariant under the action of $\mathcal{G}_{\underline{h}}$. Letting $R_{ij}$ denote the rank of $\mathbf{X}'_{ij}\boldsymbol{\theta}$ among $\mathbf{X}'_{i1}\boldsymbol{\theta}, \ldots, \mathbf{X}'_{in_i}\boldsymbol{\theta}, i = 1, \ldots, m$, it is easy to show that the maximal invariant associated with $\mathcal{G}_{\underline{h}}$ is the vector of signs $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{11}), \ldots, \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{1n_1}), \ldots, \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{m1}), \ldots, \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{mn_m})$ and ranks $R_{11}, \ldots, R_{1n_1}, \ldots, R_{m1}, \ldots, R_{mn_m}$. As a consequence, we choose to base our tests in this section on a rank-based version of the central sequence $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{f}}$, namely on

$$\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}} := \left( \left( \underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\theta};K_1} \right)', \ldots, \left( \underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\theta};K_m} \right)' \right)'$$

with

$$\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\theta};K_i} = n_i^{-1/2} \sum_{j=1}^{n_i} K_i \left( \frac{R_{ij}}{n_i + 1} \right) \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}), \quad i = 1, \ldots, m,$$

where $\underline{K} := (K_1, \ldots, K_m)$ is a $m$-tuple of *score (generating) functions* satisfying the following assumption.

**Assumption E** The score functions $K_i, i = 1, \ldots, m$, are continuous functions from $[0, 1]$ to $\mathbb{R}$.

The following result, which is a direct corollary (using again the inner sample independence and the mutual independence between the $m$ samples) of Proposition 3.1 in Ley et al. (2013), characterizes the asymptotic behavior of $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}}$ under any $m$-tuple of densities with respective angular functions $g_1, \ldots, g_m$.

**Proposition 2** *Let Assumptions A, B, C and E hold and fix* $\underline{g} = (g_1, \ldots, g_m) \in \mathcal{F}^m$. *Then the rank-based central sequence* $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}}$

(i) *is such that* $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}} - \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K};\underline{g}} = o_P(1)$ *under* $P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$ *as* $n \to \infty$, *where* ($\tilde{G}_i$ *standing for the common cdf of the* $\mathbf{X}'_{ij}\boldsymbol{\theta}$*s under* $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$, $i = 1, \ldots, m$)

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K};\underline{g}} = \left( \left( \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta};K_1;g_1} \right)', \ldots, \left( \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta};K_m;g_m} \right)' \right)'$$

*with*

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta};K_i;g_i} := n_i^{-1/2} \sum_{j=1}^{n_i} K_i(\tilde{G}_i(\mathbf{X}'_{ij}\boldsymbol{\theta}))\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}), \quad i = 1, \ldots, m.$$

*In particular, for* $\underline{K} = \underline{K}_{\underline{f}} := (K_{f_1}, \ldots, K_{f_m})$ *with* $K_{f_i}(u) := \psi_{f_i}(\tilde{F}_i^{-1}(u))(1 - (\tilde{F}_i^{-1}(u))^2)^{1/2}$, $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}_{\underline{f}}}$ *is asymptotically equivalent to the efficient central sequence* $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{f}}$ *under* $P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{f}}$.

(ii) *is asymptotically normal under* $P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$ *with mean zero and covariance matrix*

$$\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K}} := \mathrm{diag}\left(\frac{\mathcal{J}_k(K_1)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \ldots, \frac{\mathcal{J}_k(K_m)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')\right),$$

*where* $\mathcal{J}_k(K_i) := \int_0^1 K_i^2(u)du$.

(iii) *is asymptotically normal under* $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$ ($\mathbf{t}^{(n)}$ *as in* (10)) *with mean* $\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\mathbf{t}$ ($\mathbf{t} := \lim_{n\to\infty}\mathbf{t}^{(n)}$) *and covariance matrix*

$$\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}} := \mathrm{diag}\left(\frac{\mathcal{J}_k(K_1,g_1)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}'), \ldots, \frac{\mathcal{J}_k(K_m,g_m)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')\right),$$

*where*

$$\mathcal{J}_k(K_i, g_i) := \int_0^1 K_i(u)K_{g_i}(u)du$$

*for* $i = 1, \ldots, m$.

(iv) *satisfies, under* $P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$ *as* $n \to \infty$, *the asymptotic linearity property*

$$\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{K}} - \underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}} = -\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\mathbf{t}^{(n)} + o_P(1),$$

*for* $\mathbf{t}^{(n)} = (\mathbf{t}_1^{(n)\prime}, \ldots, \mathbf{t}_m^{(n)\prime})'$ *as in* (10).

Now the common value $\boldsymbol{\theta}$ of $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ under $\mathcal{H}_0$ will have to be estimated to provide our tests. To this end we will assume the existence of an estimator $\hat{\boldsymbol{\vartheta}}$ satisfying the following strengthened version of Assumption D:

**Assumption D$'$** Besides $n^{1/2}(\boldsymbol{\nu}^{(n)})^{-1}$-consistency under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$ for any $\underline{g} \in \mathcal{F}^m$, the estimator $\hat{\boldsymbol{\vartheta}} \in (\mathcal{S}^{k-1})^m$ is further *locally and asymptotically discrete*, meaning that it only takes a bounded number of distinct values in $\boldsymbol{\vartheta}_0$-centered balls of the form $\{\mathbf{t} \in \mathbb{R}^{mk} : n^{1/2}\|\left(\boldsymbol{\nu}^{(n)}\right)^{-1}(\mathbf{t} - \boldsymbol{\vartheta}_0)\| \leq c\}$.

Estimators satisfying the above assumption are easy to construct. Indeed the consistency is not a problem and the discretization condition is a purely technical requirement (needed to deal with these rank-based test statistics, see pages 125 and 188 of Le Cam and Yang 2000 for a discussion) with little practical implications (in fixed-$n$ practice, such discretizations are irrelevant as the radius can be taken arbitrarily large). We will, therefore, tacitly assume that $\hat{\boldsymbol{\theta}} \in \mathcal{S}^{k-1}$ (and, therefore, $\hat{\boldsymbol{\vartheta}} = \mathbf{1}_m \otimes \hat{\boldsymbol{\theta}}$) is locally and asymptotically discrete throughout this section. Following Lemma 4.4 in Kreiss (1987), the local discreteness allows to replace in Part (iv) of Proposition 2 non-random perturbations of the form $\boldsymbol{\vartheta} + n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)}$ with $\mathbf{t}^{(n)}$ such that $\boldsymbol{\vartheta} + n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)}$ still belongs to $\mathcal{H}_0$ by a $n^{1/2}(\boldsymbol{\nu}^{(n)})^{-1}$-consistent estimator $\hat{\boldsymbol{\vartheta}} := \mathbf{1}_m \otimes \hat{\boldsymbol{\theta}}$. Based on the asymptotic result of Proposition 2, the tests we propose below require the consistent estimation of the cross-information quantities

$$\mathcal{J}_k(K_{f_1}, g_1), \ldots, \mathcal{J}_k(K_{f_m}, g_m). \tag{4}$$

To this end define, for any $\rho \geq 0$,

$$\tilde{\boldsymbol{\theta}}_i(\rho) := \hat{\boldsymbol{\theta}} + n_i^{-1/2}\rho\,(k-1)(\mathbf{I}_k - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}')\,\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\hat{\boldsymbol{\theta}};K_i}, \quad i = 1, \ldots, m.$$

Then, letting $\hat{\boldsymbol{\theta}}_i(\rho) := \tilde{\boldsymbol{\theta}}_i(\rho)/\|\tilde{\boldsymbol{\theta}}_i(\rho)\|$, we consider the piecewise continuous quadratic form

$$\rho \mapsto h_i^{(n)}(\rho) := \frac{k-1}{\mathcal{J}(K_i)}\left(\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\hat{\boldsymbol{\theta}};K_i}\right)'\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{\hat{\boldsymbol{\theta}}_i(\rho);K_i}.$$

Consistent estimators of the quantities $\mathcal{J}_k^{-1}(K_1, g_1), \ldots, \mathcal{J}_k^{-1}(K_m, g_m)$ (and, therefore, readily of (4)) can be obtained by taking

$$\hat{\rho}_i := \inf\{\rho > 0 : h_i^{(n)}(\rho) < 0\}$$

for $i = 1, \ldots, m$ (see Ley et al. 2013; Hallin et al. 2013 or Hallin et al. 2014. Denoting by $\hat{\mathcal{J}}_k(K_i, g_i)$, for $i = 1, \ldots, m$, the resulting estimators, setting

$$\hat{H}_{\underline{K},\underline{g}} := \sum_{i=1}^{m} r_i^{(n)}\hat{\mathcal{J}}_k^2(K_i, g_i)/\mathcal{J}_k(K_i)$$

and letting

$$\bar{\mathbf{U}}_i = n_i^{-1}\sum_{j=1}^{n_i} K_i(\hat{R}_{ij}/(n_i+1))\mathbf{S}_{\hat{\boldsymbol{\theta}}}(\mathbf{X}_{ij})$$

($\hat{R}_{ij}$ naturally stands for the rank of $\mathbf{X}'_{ij}\hat{\boldsymbol{\theta}}$ among $\mathbf{X}'_{i1}\hat{\boldsymbol{\theta}}, \ldots, \mathbf{X}'_{in_i}\hat{\boldsymbol{\theta}}$), the proposed rank test $\underset{\sim}{\varphi}^{(n)}_{\underline{K}}$ rejects the null hypothesis of homogeneity of the locations when

$$\underset{\approx}{Q}^{(n)}_{\underline{K}} := (k-1)\left(\sum_{i=1}^{m}\frac{n_i}{\mathcal{J}_k(K_i)}\bar{\mathbf{U}}_i'\bar{\mathbf{U}}_i \right. \\ \left. -\hat{H}^{-1}_{\underline{K},\underline{g}}\sum_{i,j=1}^{m}\frac{n_i n_j}{n}\frac{\hat{\mathcal{J}}_k(K_i, g_i)}{\mathcal{J}_k(K_i)}\frac{\hat{\mathcal{J}}_k(K_j, g_j)}{\mathcal{J}_k(K_j)}\bar{\mathbf{U}}_i'\bar{\mathbf{U}}_j\right) \tag{5}$$

exceeds the $\alpha$-upper quantile of the Chi-square distribution with $(m-1)(k-1)$ degrees of freedom. This asymptotic behavior under the null as well as the asymptotic distribution of $\underset{\approx}{Q}^{(n)}_{\underline{K}}$ under a sequence of contiguous alternatives are summarized in the following proposition. Defining

$$l_{\mathbf{t};\underline{K},\underline{g}} := \frac{1}{(k-1)} \left( \sum_{i=1}^{m} \frac{\mathcal{J}_k^2(K_i, g_i)}{\mathcal{J}_k(K_i)} \mathbf{t}_i'\mathbf{t}_i - \left( \sum_{i=1}^{m} \frac{r_i \mathcal{J}_k^2(K_i, g_i)}{\mathcal{J}_k(K_i)} \right)^{-1} \right.$$
$$\left. \times \sum_{i,j=1}^{m} \frac{\sqrt{r_i}\sqrt{r_j}\mathcal{J}_k^2(K_i, g_i)\mathcal{J}_k^2(K_j, g_j)}{\mathcal{J}_k(K_i)\mathcal{J}_k(K_j)} \mathbf{t}_i'\mathbf{t}_j \right), \tag{6}$$

we have the following result.

**Proposition 3** *Let Assumptions A, B, C and E hold and let $\hat{\boldsymbol{\vartheta}}$ be an estimator such that Assumption D' holds. Then*

(i) $\underset{\approx}{Q}^{(n)}_{\underline{K}}$ *is asymptotically Chi-square with $(m-1)(k-1)$ degrees of freedom under* $\bigcup_{\boldsymbol{\vartheta}_0 \in \mathcal{H}_0} \bigcup_{\underline{g} \in \mathcal{F}^m} \{P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}\}$;

(ii) $\underset{\approx}{Q}^{(n)}_{\underline{K}}$ *is asymptotically non-central Chi-square, still with $(m-1)(k-1)$ degrees of freedom, but with non-centrality parameter $l_{\mathbf{t};\underline{K},\underline{g}}$ defined in (6) under $P^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$, where $\mathbf{t}^{(n)}$ is as in (10) and $\mathbf{t} := \lim_{n\to\infty} \mathbf{t}^{(n)}$;*

(iii) *the test $\underset{\sim}{\varphi}^{(n)}_{\underline{K}}$ which rejects the null hypothesis as soon as $\underset{\approx}{Q}^{(n)}_{\underline{K}}$ exceeds the $\alpha$-upper quantile of the Chi-square distribution with $(m-1)(k-1)$ degrees of freedom has asymptotic level $\alpha$ under $\bigcup_{\boldsymbol{\vartheta}_0 \in \mathcal{H}_0} \bigcup_{\underline{g} \in \mathcal{F}^m} \{P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}\}$;*

(iv) *in particular, for $\underline{K} = \underline{K}_{\underline{f}} := (K_{f_1}, \ldots, K_{f_m})$ with $K_{f_i}(u) := \varphi_{f_i}(\tilde{F}_i^{-1}(u))(1-(\tilde{F}_i^{-1}(u))^2)^{1/2}$, $\underset{\sim}{\varphi}^{(n)}_{\underline{K}_{\underline{f}}}$ is locally and asymptotically most stringent, at asymptotic level $\alpha$, for $\bigcup_{\boldsymbol{\vartheta}_0 \in \mathcal{H}_0} \bigcup_{\underline{g} \in \mathcal{F}^m} \{P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}\}$ against alternatives of the form $\bigcup_{\boldsymbol{\vartheta} \notin \mathcal{H}_0} \{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta};\underline{f}}\}$.*

See the appendix for the proof. The asymptotic distributions under local alternatives of the pseudo-FvML tests and the signed-rank-based tests obtained in Propositions 1 and 3, respectively, allow to compare them using asymptotic relative efficiencies.

## 5 Comparison of the proposed test procedures

In what follows, we will compare the optimal pseudo-FvML test (hence, the Watson test) $\varphi^{(n)}$ to optimal rank-based tests $\underset{\sim}{\varphi}^{(n)}_{\underline{K}_{\underline{f}}}$ for several choices of $\underline{f} \in \mathcal{F}^m$, both at asymptotic level via the calculation of Pitman's asymptotic relative efficiencies (AREs, Sect. 5.1) and at finite-sample level via a Monte Carlo simulation study (Sect. 5.2).

### 5.1 Asymptotic relative efficiencies

The Pitman asymptotic relative efficiency $\mathrm{ARE}_{\boldsymbol{\vartheta}_0;\underline{g}}(\varphi_1^{(n)}, \varphi_2^{(n)})$ under $P^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$ of $\varphi_1^{(n)}$ with respect to $\varphi_2^{(n)}$ is defined as the limit, when it exists, as $n \to \infty$, of the ratio $N(n)/n$ of the number $N(n)$ of observations it takes for the test $\varphi_2^{(n)}$, under $P^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$, to match the local performance of $\varphi_1^{(n)}$ based on $n$ observations. Since our procedures converge to non-centered Chi-square limit distributions under our local alternatives, this comparison will be obtained as the ratio of the respective non-central quantities, see Hallin (2012) for details.

Let $\mathrm{ARE}_{\boldsymbol{\vartheta}_0;\underline{g}}(\varphi_1^{(n)}, \varphi_2^{(n)})$ denote the ARE of a test $\varphi_1^{(n)}$ with respect to another test $\varphi_2^{(n)}$ under $P^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$. Thanks to Propositions 1 and 3, we find that

$$\mathrm{ARE}_{\boldsymbol{\vartheta}_0;\underline{g}}(\underset{\sim}{\varphi}^{(n)}_{\underline{K}_f}, \varphi^{(n)}) = l_{\mathbf{t};\underline{K}_f,\underline{g}}/l_{\mathbf{t};\underline{\phi},\underline{g}}. \tag{7}$$

In the homogeneous case $\underline{g} = (g_1, \ldots, g_1)$ (the angular density is the same for the $m$ samples) and if the same score function—namely, $K_{f_1}$—is used for the $m$ rankings (the test is, therefore, denoted by $\underset{\sim}{\varphi}^{(n)}_{K_{f_1}}$), the ratio in (7) simplifies into

$$\mathrm{ARE}_{\boldsymbol{\vartheta}_0;\underline{g}}(\underset{\sim}{\varphi}^{(n)}_{K_{f_1}}, \varphi^{(n)}) = \frac{\mathcal{J}_k^2(K_{f_1}, g_1)B_{g_1}}{\mathcal{J}_k(K_{f_1})C^2_{g_1}}. \tag{8}$$

Numerical values of the AREs in (8) are reported in Table 1 for the three-dimensional setup under various angular densities and various choices of the score function $K_{f_1}$. More precisely, besides the FvML we consider the spherical linear, logarithmic, logistic and squared distributions (see Ley et al. 2013) with respective angular functions

$$f_{\mathrm{lin(a)}}(t) := t + a, \quad f_{\mathrm{log(a)}}(t) := \log(t + a)$$
$$f_{\mathrm{logis}(a,b)}(t) := \frac{a\ \exp(-b\ \arccos(t))}{(1 + a\ \exp(-b\ \arccos(t)))^2} \quad \text{and} \quad f_{\mathrm{Sq}(a)}(t) = \sqrt{t + a}.$$

The constants $a$ and $b$ are chosen so that all the above functions are true angular functions satisfying Assumption A. The score functions associated with all these angular functions are denoted by $K_{\mathrm{lin(a)}}$ for $f_{\mathrm{lin(a)}}$, $K_{\mathrm{log(a)}}$ for $f_{\mathrm{log(a)}}$, $K_{\mathrm{logis(a,b)}}$ for $f_{\mathrm{logis(a,b)}}$ and $K_{\mathrm{Sq(a)}}$ for $f_{\mathrm{Sq(a)}}$. For the FvML distribution with concentration $\kappa$, the score function will be denoted by $K_{\phi_\kappa}$.

Inspection of Table 1 confirms the theoretical results. As expected, the pseudo-FvML test $\varphi^{(n)}$ dominates the rank-based tests under FvML densities, whereas rank-based tests mostly outperform the pseudo-FvML test under other densities, especially so when they are based on the score function associated with the underlying density (in which case the rank-based tests are optimal).

**Table 1** Asymptotic relative efficiencies of (homogeneous) rank-based tests $\underset{\sim}{\varphi}^{(n)}_{K_{f_1}}$ with respect to the pseudo-FvML test $\varphi^{(n)}$ under various three-dimensional rotationally symmetric densities

| Underlying density | ARE($\underset{\sim}{\varphi}^{(n)}_{K_{f_1}}, \varphi^{(n)}$) | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\underset{\sim}{\varphi}^{(n)}_{K_{\phi_2}}$ | $\underset{\sim}{\varphi}^{(n)}_{K_{\phi_6}}$ | $\underset{\sim}{\varphi}^{(n)}_{K_{\mathrm{lin}(2)}}$ | $\underset{\sim}{\varphi}^{(n)}_{K_{\mathrm{lin}(4)}}$ | $\underset{\sim}{\varphi}^{(n)}_{K_{\log(2.5)}}$ | $\underset{\sim}{\varphi}^{(n)}_{K_{\mathrm{logis}(1,1)}}$ | $\underset{\sim}{\varphi}^{(n)}_{K_{\mathrm{Sq}(2)}}$ |
| FvML(1) | 0.9744 | 0.8787 | 0.9813 | 0.9979 | 0.9027 | 0.9321 | 0.9992 |
| FvML(2) | 1 | 0.9556 | 0.9978 | 0.9586 | 0.9749 | 0.9823 | 0.9919 |
| FvML(6) | 0.9555 | 1 | 0.9381 | 0.8517 | 0.9768 | 0.9911 | 0.9154 |
| Lin(2) | 1.0539 | 0.9909 | 1.0562 | 1.0215 | 1.0212 | 1.0247 | 1.0531 |
| Lin(4) | 0.9709 | 0.8627 | 0.9795 | 1.0128 | 0.8856 | 0.9231 | 0.9957 |
| Log(2.5) | 1.1610 | 1.1633 | 1.1514 | 1.0413 | 1.1908 | 1.1625 | 1.1252 |
| Log(4) | 1.0182 | 0.9216 | 1.0261 | 1.0347 | 0.9503 | 0.9741 | 1.0359 |
| Logis(1,1) | 1.0768 | 1.0865 | 1.0635 | 0.9991 | 1.0701 | 1.0962 | 1.0485 |
| Logis(2,1) | 1.3182 | 1.4426 | 1.2946 | 1.0893 | 1.4294 | 1.3865 | 1.2411 |
| Sq(1.1) | 1.2303 | 1.3460 | 1.1964 | 1.0264 | 1.3158 | 1.3004 | 1.1478 |
| Sq(2) | 1.0502 | 0.9692 | 1.0556 | 1.0408 | 1.0003 | 1.0127 | 1.0587 |

### 5.2 Monte Carlo simulation results

To study the finite-sample behavior of the Watson test $\varphi^{(n)}_{\mathrm{W}}$, the pseudo-FvML test $\varphi^{(n)}$ and various rank-based tests $\underset{\sim}{\varphi}^{(n)}_{\underline{K}_f}$, we have conducted a Monte Carlo simulation study on R for moderate and small sample sizes for the two-sample spherical location problem, that is, for an ANOVA with $m = 2$. We generated $M = 2{,}500$ replications of four pairs of mutually independent samples of ($k$ =)3-dimensional rotationally symmetric random vectors

$$\boldsymbol{\varepsilon}_{\ell;ij_i}, \quad \ell = 1, 2, 3, 4, \quad j_i = 1, \ldots, n_i, \quad i = 1, 2,$$

with FvML densities and linear densities: the $\boldsymbol{\varepsilon}_{1;1j_1}$s have an FvML(5) distribution and the $\boldsymbol{\varepsilon}_{1;2j_2}$s have a FvML(2) distribution; the $\boldsymbol{\varepsilon}_{2;1j_1}$s have a Lin(5) distribution and the $\boldsymbol{\varepsilon}_{2;2j_2}$s have an Lin(2) distribution; the $\boldsymbol{\varepsilon}_{3;1j_1}$s have an FvML(5) distribution and the $\boldsymbol{\varepsilon}_{3;2j_2}$s have a Lin(2) distribution and finally the $\boldsymbol{\varepsilon}_{4;1j_1}$s have a Lin(5) distribution and the $\boldsymbol{\varepsilon}_{4;2j_2}$s have an FvML(2) distribution.

The rotationally symmetric vectors $\boldsymbol{\varepsilon}_{\ell;ij_i}$s have all been generated with a common spherical location $\boldsymbol{\theta}_0 = (1, 0, 0)'$. Then, each replication of the $\boldsymbol{\varepsilon}_{\ell;ij_i}$s was transformed into

$$\begin{cases} \mathbf{X}_{\ell;1j_1} = \boldsymbol{\varepsilon}_{\ell;1j_1}, & \ell = 1, 2, 3, 4 \quad j_1 = 1, \ldots, n_1 \\ \mathbf{X}_{\ell;2j_2;\xi} = \mathbf{O}_\xi \boldsymbol{\varepsilon}_{\ell;2j_2}, & \ell = 1, 2, 3, 4 \quad j_2 = 1, \ldots, n_2, \quad \xi = 0, 1, 2, 3, \end{cases}$$

**Table 2** Rejection frequencies (out of $M = 2,500$ replications), under the null and under increasingly distant alternatives, of the Watson test $\varphi_{\mathrm{W}}^{(n)}$, the pseudo-FvML test $\varphi^{(n)}$ and various rank-based tests $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\phi_5})}$ (based on FvML(2) and FvML(5) scores), $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}$ (based on Lin(2) and Lin(5) scores), $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}$ (based on Lin(2) and FvML(5) scores), $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_5},K_{\mathrm{Lin}(2)})}$ (based on FvML(5) and Lin(2) scores) and $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ (the sign test based on constant scores)

| Test | True densities | $\xi$ | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0320 | 0.0384 | 0.0532 | 0.0824 |
| $\varphi^{(n)}$ | | 0.0444 | 0.0528 | 0.0668 | 0.1008 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\phi_5})}$ | | 0.0524 | 0.0592 | 0.0872 | 0.1264 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}$ | $(\phi_2, \phi_5)$ | 0.0468 | 0.0552 | 0.0720 | 0.1120 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}$ | | 0.0460 | 0.0536 | 0.0808 | 0.1148 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}$ | | 0.0544 | 0.0588 | 0.0800 | 0.1212 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0392 | 0.0464 | 0.0612 | 0.0880 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0184 | 0.0168 | 0.0196 | 0.0152 |
| $\varphi^{(n)}$ | | 0.0360 | 0.0352 | 0.0428 | 0.0432 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\phi_5})}$ | | 0.0424 | 0.0452 | 0.0496 | 0.0504 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}$ | (Lin(2), Lin(5)) | 0.0388 | 0.0432 | 0.0436 | 0.0468 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}$ | | 0.0360 | 0.0396 | 0.0432 | 0.0428 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}$ | | 0.0444 | 0.0516 | 0.0496 | 0.0492 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0364 | 0.0360 | 0.0400 | 0.0420 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0252 | 0.0332 | 0.0308 | 0.0412 |
| $\varphi^{(n)}$ | | 0.0456 | 0.0528 | 0.0528 | 0.0744 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\phi_5})}$ | | 0.0544 | 0.0720 | 0.0728 | 0.0948 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}$ | $(\phi_2$, Lin(5)) | 0.0472 | 0.0572 | 0.0592 | 0.0816 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}$ | | 0.0476 | 0.0592 | 0.0656 | 0.0848 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}$ | | 0.0540 | 0.0680 | 0.0688 | 0.0928 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0428 | 0.0516 | 0.0512 | 0.0648 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0328 | 0.0276 | 0.0312 | 0.0380 |
| $\varphi^{(n)}$ | | 0.0480 | 0.0488 | 0.0560 | 0.0640 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\phi_5})}$ | | 0.0612 | 0.0676 | 0.0768 | 0.0940 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}$ | (Lin(2), $\phi_5$) | 0.0532 | 0.0552 | 0.0652 | 0.0756 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}$ | | 0.0524 | 0.0548 | 0.0664 | 0.0828 |

**Table 2** continued

| Test | True densities | $\xi$ 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}$ | | 0.0632 | 0.0640 | 0.0776 | 0.0852 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0508 | 0.0520 | 0.0632 | 0.0696 |

Sample sizes are $n_1 = 30$ and $n_2 = 30$

where

$$\mathbf{O}_\xi = \begin{pmatrix} \cos(\pi\xi/25) & -\sin(\pi\xi/25) & 0 \\ \sin(\pi\xi/25) & \cos(\pi\xi/25) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the spherical locations of the $\mathbf{X}_{\ell;1j_1}$s and the $\mathbf{X}_{\ell;2j_2;0}$s coincide while the spherical location of the $\mathbf{X}_{\ell;2j_2;\xi}$s, $\xi = 1, 2, 3$, is different from the spherical location of the $\mathbf{X}_{\ell;1j_1}$s, characterizing alternatives to the null hypothesis of common spherical locations. Rejection frequencies based on the asymptotic Chi-square critical values at nominal level 5 % are reported in Table 2 for $n_1 = n_2 = 30$, in Table 3 for $n_1 = 100$ and $n_2 = 500$ and in Table 4 for $n_1 = n_2 = 500$ below. The pseudo-FvML test and the various rank-based tests have been performed using

$$\hat{\boldsymbol{\theta}} := \frac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}\mathbf{X}_{ij}}{\left\|\sum_{i=1}^{2}\sum_{j=1}^{n_i}\mathbf{X}_{ij}\right\|} \tag{9}$$

as a root-$n$ consistent estimator of the common value $\boldsymbol{\theta}$ of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ under the null. Inspection of the various tables reveals the following results:

(i) The pseudo-FvML test and all the rank-based tests are valid under heterogeneous densities. They reach the 5 % nominal level constraint under any considered pair of densities. The Watson test clearly requires large sample sizes to be valid.
(ii) The comparison of the empirical powers reveals that when based on scores associated with the underlying distributions, the rank-based tests are quite powerful.
(iii) The proposed procedures (even the rank-based tests) perform better than the Watson test under small sample sizes.

## 6 Real-data example

In this section, we apply our new tests to a real-data example. The data consist of measurements of remanent magnetization in red slits and claystones made at two different locations in Eastern New South Wales, Australia, the first location yielding $n_1 = 39$, the second $n_2 = 36$ observations; see Embleton and Mc Donnell (1980) for details. As can be seen from Fig. 1, the rotational symmetry assumption in the two

**Table 3** Rejection frequencies (out of $M = 2,500$ replications), under the null and under increasingly distant alternatives, of the Watson test $\varphi_{\mathrm{W}}^{(n)}$, the pseudo-FvML test $\varphi^{(n)}$ and various rank-based tests $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\phi_5})}^{(n)}$ (based on FvML(2) and FvML(5) scores), $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}^{(n)}$ (based on Lin(2) and Lin(5) scores), $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}^{(n)}$ (based on Lin(2) and FvML(5) scores), $\underset{\sim}{\varphi}{}_{(K_{\phi_5},K_{\mathrm{Lin}(2)})}^{(n)}$ (based on FvML(5) and Lin(2) scores) and $\underset{\sim}{\varphi}{}_{\mathrm{S}}^{(n)}$ (the sign test based on constant scores)

| Test | True densities | $\xi$ | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0424 | 0.0748 | 0.1788 | 0.3636 |
| $\varphi^{(n)}$ | | 0.0476 | 0.0808 | 0.1916 | 0.3816 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\phi_5})}^{(n)}$ | | 0.0644 | 0.1020 | 0.2476 | 0.4748 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}^{(n)}$ | $(\phi_2, \phi_5)$ | 0.0508 | 0.0820 | 0.2008 | 0.4024 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}^{(n)}$ | | 0.0508 | 0.0876 | 0.2220 | 0.4300 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}^{(n)}$ | | 0.0588 | 0.0952 | 0.2300 | 0.4480 |
| $\underset{\sim}{\varphi}{}_{\mathrm{S}}^{(n)}$ | | 0.0448 | 0.0772 | 0.1792 | 0.3544 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0304 | 0.0404 | 0.0564 | 0.0872 |
| $\varphi^{(n)}$ | | 0.0492 | 0.0596 | 0.0868 | 0.1212 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\phi_5})}^{(n)}$ | | 0.0620 | 0.0692 | 0.0940 | 0.1348 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}^{(n)}$ | (Lin(2), Lin(5)) | 0.0500 | 0.0600 | 0.0868 | 0.1216 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}^{(n)}$ | | 0.0528 | 0.0596 | 0.0780 | 0.1192 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}^{(n)}$ | | 0.0600 | 0.0716 | 0.0956 | 0.1380 |
| $\underset{\sim}{\varphi}{}_{\mathrm{S}}^{(n)}$ | | 0.0456 | 0.0552 | 0.0808 | 0.1092 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0376 | 0.0356 | 0.0584 | 0.0928 |
| $\varphi^{(n)}$ | | 0.0504 | 0.0572 | 0.0836 | 0.1432 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\phi_5})}^{(n)}$ | | 0.0584 | 0.0684 | 0.1100 | 0.1644 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}^{(n)}$ | $(\phi_2$, Lin(5)) | 0.0480 | 0.0524 | 0.0900 | 0.1396 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}^{(n)}$ | | 0.0480 | 0.0564 | 0.0916 | 0.1444 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\mathrm{Lin}(5)})}^{(n)}$ | | 0.0584 | 0.0644 | 0.1020 | 0.1592 |
| $\underset{\sim}{\varphi}{}_{\mathrm{S}}^{(n)}$ | | 0.0504 | 0.0552 | 0.0784 | 0.1344 |
| $\varphi_{\mathrm{W}}^{(n)}$ | | 0.0372 | 0.0340 | 0.0668 | 0.0920 |
| $\varphi^{(n)}$ | | 0.0528 | 0.0588 | 0.0992 | 0.1364 |
| $\underset{\sim}{\varphi}{}_{(K_{\phi_2},K_{\phi_5})}^{(n)}$ | | 0.0716 | 0.0752 | 0.1200 | 0.1648 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\mathrm{Lin}(5)})}^{(n)}$ | (Lin(2), $\phi_5$) | 0.0612 | 0.0604 | 0.1044 | 0.1412 |
| $\underset{\sim}{\varphi}{}_{(K_{\mathrm{Lin}(2)},K_{\phi_5})}^{(n)}$ | | 0.0588 | 0.0608 | 0.1096 | 0.1428 |

**Table 3** continued

| Test | True densities | $\xi$ | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\mathrm{Lin}(5)})}$ | | 0.0716 | 0.0728 | 0.1224 | 0.1616 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0560 | 0.0564 | 0.0948 | 0.1348 |

Sample sizes are $n_1 = 100$ and $n_2 = 500$

**Table 4** Rejection frequencies (out of $M = 2,500$ replications), under the null and under increasingly distant alternatives, of the Watson test $\varphi^{(n)}_{\mathrm{W}}$, the pseudo-FvML test $\varphi^{(n)}$ and various rank-based tests $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\phi_5})}$ (based on FvML(2) and FvML(5) scores), $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\mathrm{Lin}(5)})}$ (based on Lin(2) and Lin(5) scores), $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\phi_5})}$ (based on Lin(2) and FvML(5) scores), $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_5}, K_{\mathrm{Lin}(2)})}$ (based on FvML(5) and Lin(2) scores) and $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ (the sign test based on constant scores)

| Test | True densities | $\xi$ | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| $\varphi^{(n)}_{\mathrm{W}}$ | | 0.0444 | 0.1648 | 0.5136 | 0.8748 |
| $\varphi^{(n)}$ | | 0.0456 | 0.1664 | 0.5148 | 0.8772 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\phi_5})}$ | | 0.0512 | 0.2024 | 0.6212 | 0.9364 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\mathrm{Lin}(5)})}$ | $(\phi_2, \phi_5)$ | 0.0456 | 0.1544 | 0.5004 | 0.8616 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\phi_5})}$ | | 0.0444 | 0.1856 | 0.5924 | 0.9232 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\mathrm{Lin}(5)})}$ | | 0.0500 | 0.1656 | 0.5228 | 0.8736 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0416 | 0.1588 | 0.4940 | 0.8444 |
| $\varphi^{(n)}_{\mathrm{W}}$ | | 0.0440 | 0.0752 | 0.1560 | 0.2940 |
| $\varphi^{(n)}$ | | 0.0452 | 0.0796 | 0.1616 | 0.3016 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\phi_5})}$ | | 0.0500 | 0.0752 | 0.1680 | 0.3172 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\mathrm{Lin}(5)})}$ | (Lin(2), Lin(5)) | 0.0464 | 0.0776 | 0.1612 | 0.3072 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\phi_5})}$ | | 0.0456 | 0.0712 | 0.1560 | 0.3016 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\mathrm{Lin}(5)})}$ | | 0.0528 | 0.0848 | 0.1704 | 0.3228 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0460 | 0.0712 | 0.1468 | 0.2668 |
| $\varphi^{(n)}_{\mathrm{W}}$ | | 0.0588 | 0.0884 | 0.2244 | 0.4488 |
| $\varphi^{(n)}$ | | 0.0608 | 0.0904 | 0.2300 | 0.4568 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\phi_5})}$ | | 0.0712 | 0.1108 | 0.2836 | 0.5184 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\mathrm{Lin}(5)})}$ | $(\phi_2, \mathrm{Lin}(5))$ | 0.0580 | 0.0916 | 0.2372 | 0.4580 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\phi_5})}$ | | 0.0616 | 0.0968 | 0.2596 | 0.4928 |

**Table 4** continued

| Test | True densities | $\xi$ | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\mathrm{Lin}(5)})}$ | | 0.0648 | 0.1048 | 0.2616 | 0.4856 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0636 | 0.0940 | 0.2200 | 0.4172 |
| $\varphi^{(n)}_{\mathrm{W}}$ | | 0.0564 | 0.0888 | 0.2220 | 0.4480 |
| $\varphi^{(n)}$ | | 0.0576 | 0.0916 | 0.2280 | 0.4568 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\phi_5})}$ | | 0.0628 | 0.1104 | 0.2740 | 0.5240 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\mathrm{Lin}(5)})}$ | $(\mathrm{Lin}(2), \phi_5)$ | 0.0556 | 0.0900 | 0.2224 | 0.4668 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(2)}, K_{\phi_5})}$ | | 0.0532 | 0.0944 | 0.2456 | 0.4960 |
| $\underset{\sim}{\varphi}^{(n)}_{(K_{\phi_2}, K_{\mathrm{Lin}(5)})}$ | | 0.0624 | 0.1016 | 0.2468 | 0.4932 |
| $\underset{\sim}{\varphi}^{(n)}_{\mathrm{S}}$ | | 0.0544 | 0.0896 | 0.2160 | 0.4272 |

Sample sizes are $n_1 = 500$ and $n_2 = 500$

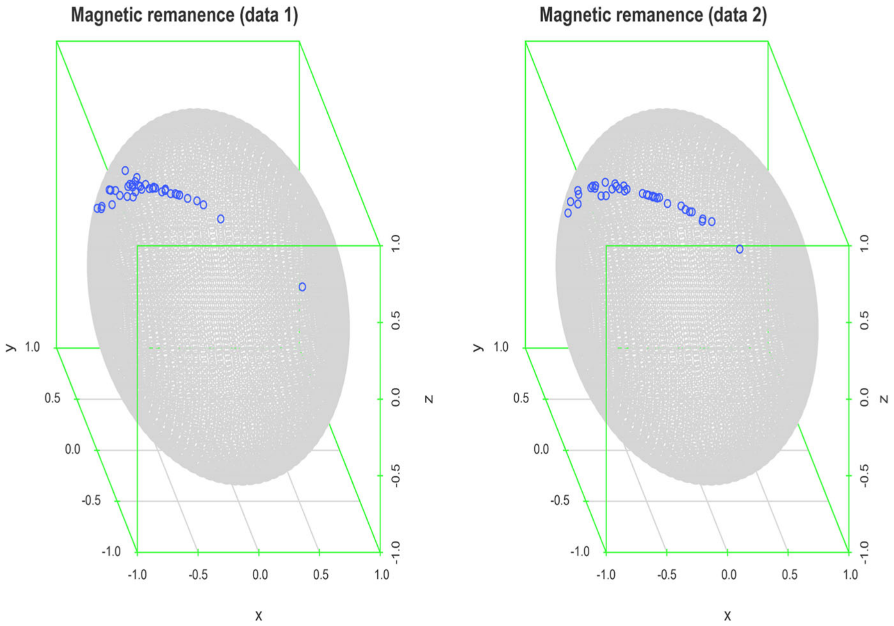


**Fig. 1** Measurements of remanent magnetization in red slits and claystones made at two different locations in Australia

samples seems to be appropriate since data are clearly concentrated. However, the specification of the angular functions is not reasonable, whence our semi-parametric procedures are quite useful in this setting.

As explained in the Introduction, the main task for the practitioner consists in solving the fold problem, that is, to test whether the remanent magnetization obtained in those samples comes from a single source of magnetism or not. Therefore, we test here the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$. For this purpose, we used the Watson test $\phi_{\mathrm{W}}^{(n)}$, the pseudo-FVML test $\varphi^{(n)}$ and rank-based tests $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}}(1.1), K_{\phi_{10}})}$ and $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(1.1)}, K_{\phi_{100}})}$ based, respectively, on the couples of linear and FvML scores (Lin(1.1), FvML(10)) and (Lin(1.1), FvML(100)). All the tests were performed using (9). The corresponding test statistics are given by $Q_{\mathrm{W}} = 5.946859$ ($Q_{\mathrm{W}}$ is the Watson 1983 test statistic; see p. 144 of Watson 1983), $Q^{(n)} = 5.96652$, $\underset{\sim}{Q}^{(n)}_{(K_{\mathrm{Lin}(1.1)}, K_{\phi_{10}})} = 5.477525$ and $\underset{\sim}{Q}^{(n)}_{(K_{\mathrm{Lin}(1.1)}, K_{\phi_{100}})} = 5.525854$. At the asymptotic nominal level 5 %, the tests $\varphi_{\mathrm{W}}^{(n)}$, $\varphi^{(n)}$, $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(1.1)}, K_{\phi_{10}})}$ and $\underset{\sim}{\varphi}^{(n)}_{(K_{\mathrm{Lin}(1.1)}, K_{\phi_{100}})}$ do not reject the null hypothesis of equality of the modal directions since the 5 % upper quantile of the Chi-square distribution with 2 degrees of freedom is equal to 5.991465.

## Appendix A: ULAN property and optimal parametric tests

In this appendix, we give the technical details leading to the ULAN property used to derive the different testing procedures in the paper. Informally, a sequence of rotationally symmetric models $\{P^{(n)}_{\boldsymbol{\vartheta};\underline{f}} \mid \boldsymbol{\vartheta} \in (\mathcal{S}^{k-1})^m\}$ is ULAN if, uniformly in $\boldsymbol{\vartheta}^{(n)} = (\boldsymbol{\theta}_1^{(n)\prime}, \ldots, \boldsymbol{\theta}_m^{(n)\prime})' \in (\mathcal{S}^{k-1})^m$ such that $\boldsymbol{\vartheta}^{(n)} - \boldsymbol{\vartheta} = O(n^{-1/2})$, the log-likelihood ratio

$$\log\left(P^{(n)}_{\boldsymbol{\vartheta}^{(n)}+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{f}} \Big/ P^{(n)}_{\boldsymbol{\vartheta}^{(n)};\underline{f}}\right)$$

allows a specific form of (probabilistic) Taylor expansion (see Eq. (11) below) as a function of $\mathbf{t}^{(n)} := (\mathbf{t}_1^{(n)\prime}, \ldots, \mathbf{t}_m^{(n)\prime})' \in \mathbb{R}^{mk}$. Of course the local perturbations $\mathbf{t}^{(n)}$ must be chosen so that $\boldsymbol{\vartheta}^{(n)} + n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)}$ remains on $(\mathcal{S}^{k-1})^m$ and thus, in particular, the $\mathbf{t}_i^{(n)}$ need to satisfy

$$\begin{aligned} 0 &= (\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)})'(\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)}) - 1 \\ &= 2n_i^{-1/2}(\boldsymbol{\theta}_i^{(n)})'\mathbf{t}_i^{(n)} + n_i^{-1}(\mathbf{t}_i^{(n)})'\mathbf{t}_i^{(n)} \end{aligned} \tag{10}$$

for all $i = 1, \ldots, m$. Consequently, $\mathbf{t}_i^{(n)}$ must be such that $2n_i^{-1/2}(\boldsymbol{\theta}_i^{(n)})'\mathbf{t}_i^{(n)} + o(n_i^{-1/2}) = 0$ and thus, for $\boldsymbol{\theta}_i^{(n)} + n_i^{-1/2}\mathbf{t}_i^{(n)}$ to remain in $\mathcal{S}^{k-1}$, the perturbation $\mathbf{t}_i^{(n)}$ must belong, up to a $o(n_i^{-1/2})$ quantity, to the tangent space to the sphere $\mathcal{S}^{k-1}$ at $\boldsymbol{\theta}_i^{(n)}$.

The domain of the parameter being the non-linear manifold $\left(\mathcal{S}^{k-1}\right)^m$ it is all but easy to establish the ULAN property of a sequence of rotationally symmetric models. A natural way to handle this difficulty consists, as in Ley et al. (2013), in resorting to a re-parameterization of the problem in terms of spherical coordinates $\boldsymbol{\eta}$, say, for which it is possible to prove ULAN. After obtaining the ULAN property for the $\boldsymbol{\eta}$-

parameterization, one can use a lemma from Hallin et al. (2010) to transpose the ULAN property in the spherical $\boldsymbol{\eta}$-coordinates back in terms of the original $\boldsymbol{\theta}$-coordinates. Finally the inner sample independence and the mutual independence between the $m$ samples entail that we can deduce the required ULAN property which is relevant for our purposes (this we state without proof because it follows directly from Proposition 2.2 of Ley et al. 2013).

**Proposition 4** *Let Assumptions A, B and C hold. Then the model* $\{P^{(n)}_{\boldsymbol{\vartheta};\underline{f}} \mid \boldsymbol{\vartheta} \in (\mathcal{S}^{k-1})^m\}$ *is ULAN with central sequence*

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta};\underline{f}} := \left(\left(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_1;f_1}\right)', \ldots, \left(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_m;f_m}\right)'\right)',$$

*where*

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_i;f_i} := n_i^{-1/2}\sum_{j=1}^{n_i}\psi_{f_i}(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)(1-(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)^2)^{1/2}\mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}_{ij}), \quad i=1,\ldots,m,$$

*and Fisher information matrix* $\boldsymbol{\Gamma}_{\boldsymbol{\vartheta};\underline{f}} := \mathrm{diag}(\boldsymbol{\Gamma}_{\boldsymbol{\theta}_1;f_1}, \ldots, \boldsymbol{\Gamma}_{\boldsymbol{\theta}_m;f_m})$ *where*

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta}_i;f_i} := \frac{\mathcal{J}_k(f_i)}{k-1}(\mathbf{I}_k - \boldsymbol{\theta}_i\boldsymbol{\theta}'_i), \quad i=1,\ldots,m.$$

*More precisely, for any* $\boldsymbol{\vartheta}^{(n)} \in (\mathcal{S}^{k-1})^m$ *such that* $\boldsymbol{\vartheta}^{(n)} - \boldsymbol{\vartheta} = O(n^{-1/2})$ *and any bounded sequences* $\mathbf{t}^{(n)} = (\mathbf{t}_1^{(n)\prime}, \ldots, \mathbf{t}_m^{(n)\prime})'$ *satisfying (10), we have*

$$\log\left(\frac{P^{(n)}_{\boldsymbol{\vartheta}^{(n)}+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{f}}}{P^{(n)}_{\boldsymbol{\vartheta}^{(n)};\underline{f}}}\right) = (\mathbf{t}^{(n)})'\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}^{(n)};\underline{f}} - \frac{1}{2}(\mathbf{t}^{(n)})'\boldsymbol{\Gamma}_{\boldsymbol{\vartheta};\underline{f}}\mathbf{t}^{(n)} + o_{\mathrm{P}}(1), \qquad (11)$$

*where* $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}^{(n)};\underline{f}} \xrightarrow{\mathcal{L}} \mathcal{N}_{mk}(\mathbf{0}, \boldsymbol{\Gamma}_{\boldsymbol{\vartheta};\underline{f}})$*, both under* $P^{(n)}_{\boldsymbol{\vartheta};\underline{f}}$*, as* $n \to \infty$.

Proposition 4 provides us with all the necessary tools for building optimal $\underline{f}$-parametric procedures (i.e., under any $m$-tuple of densities with respective specified angular functions $f_1, \ldots, f_m$) for testing $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_m$ against $\mathcal{H}_1 : \exists\, 1 \le i \ne j \le m$ such that $\boldsymbol{\theta}_i \ne \boldsymbol{\theta}_j$.

The null hypothesis $\mathcal{H}_0$ consists in the intersection between $(\mathcal{S}^{k-1})^m$ and the linear subspace (of $\mathbb{R}^{mk}$)

$$\mathcal{C} := \{\mathbf{v} = (\mathbf{v}'_1, \ldots, \mathbf{v}'_m)' \mid \mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^k \text{ and } \mathbf{v}_1 = \cdots = \mathbf{v}_m\} =: \mathcal{M}(\mathbf{1}_m \otimes \mathbf{I}_k).$$

Such a restriction, namely an intersection between a linear subspace and a non-linear manifold, has already been considered in Hallin et al. (2010) in the context of Principal Component Analysis (in that paper, the authors obtained very general results related to hypothesis testing in ULAN families with curved experiments). In particular from their results we can deduce that to obtain a locally and asymptotically most stringent test in the present context, one has to consider the locally and asymptotically most stringent test for the (linear) null hypothesis defined by the intersection between $\mathcal{C}$ and the tangent to $(\mathcal{S}^{k-1})^m$. Let $\boldsymbol{\theta}$ denote the common value of $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ under the null. In the vicinity of $\mathbf{1}_m \otimes \boldsymbol{\theta}$, the intersection between $\mathcal{C}$ and the tangent to $(\mathcal{S}^{k-1})^m$ is given by

$$\Big\{(\boldsymbol{\theta}' + n^{-1/2}(r_1^{(n)})^{-1/2}\mathbf{t}_1^{(n)\prime}, \ldots, \boldsymbol{\theta}' + n^{-1/2}(r_m^{(n)})^{-1/2}\mathbf{t}_m^{(n)\prime})', \tag{12}$$
$$\boldsymbol{\theta}'\mathbf{t}_1^{(n)} = \cdots = \boldsymbol{\theta}'\mathbf{t}_m^{(n)} = 0, (r_1^{(n)})^{-1/2}\mathbf{t}_1^{(n)} = \cdots = (r_m^{(n)})^{-1/2}\mathbf{t}_m^{(n)}\Big\}.$$

Solving the system (12) yields

$$\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)} = \left((r_1^{(n)})^{-1/2}\mathbf{t}_1^{(n)\prime}, \ldots, (r_m^{(n)})^{-1/2}\mathbf{t}_m^{(n)\prime}\right)' \in \mathcal{M}(\mathbf{1}_m \otimes (\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')). \tag{13}$$

Loosely speaking we have "transcripted" the initial null hypothesis $\mathcal{H}_0$ into a linear restriction of the form (13) in terms of local perturbations $\mathbf{t}^{(n)}$, for which Le Cam's asymptotic theory then provides a locally and asymptotically optimal parametric test under fixed $\underline{f}$. Using Proposition 4, an asymptotically most stringent test $\phi_{\underline{f}}$ is then obtained by rejecting $\mathcal{H}_0$ as soon as ($\mathbf{A}^-$ stands for the Moore–Penrose pseudo-inverse of $\mathbf{A}$)

$$Q_{\underline{f}}^{(n)} := \boldsymbol{\Delta}'_{\boldsymbol{\vartheta};\underline{f}}\left(\boldsymbol{\Gamma}^-_{\boldsymbol{\vartheta};\underline{f}} - \boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta};\boldsymbol{\nu}}\left((\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta};\boldsymbol{\nu}})'\boldsymbol{\Gamma}_{\boldsymbol{\vartheta};\underline{f}}\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta};\boldsymbol{\nu}}\right)^-(\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta};\boldsymbol{\nu}})'\right)\boldsymbol{\Delta}_{\boldsymbol{\vartheta};\underline{f}} \tag{14}$$

exceeds the $\alpha$-upper quantile of a Chi-square distribution with $(m-1)(k-1)$ degrees of freedom, where $\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta};\boldsymbol{\nu}} := (\boldsymbol{\nu}^{(n)})^{-1}\mathbf{1}_m \otimes (\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')$. Hence, the optimal parametric tests are now known.

## Appendix B: Proof of Proposition 3

Using the definition of $\underset{\approx}{Q}{}^{(n)}_{\underline{K}}$, the consistency of the cross-information quantities $\hat{\mathcal{J}}_k(K_1, g_1), \ldots, \hat{\mathcal{J}}_k(K_m, g_m)$ together with Proposition 2, it is easy to show that the $n^{1/2}(\boldsymbol{\nu}^{(n)})^{-1}$-consistency and the local discreteness of $\hat{\boldsymbol{\vartheta}}$ entail that

$$\underset{\approx}{Q}{}^{(n)}_{\underline{K}} = \left(\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}}\right)'\boldsymbol{\Gamma}^{\perp}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\underset{\sim}{\boldsymbol{\Delta}}{}^{(n)}_{\boldsymbol{\vartheta}_0;\underline{K}} + o_P(1)$$

where

$$\boldsymbol{\Gamma}^{\perp}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}} := \boldsymbol{\Gamma}^-_{\boldsymbol{\vartheta}_0;\underline{K}} - \boldsymbol{\Gamma}^-_{\boldsymbol{\vartheta}_0;\underline{K}}\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta}_0;\boldsymbol{\nu}}[(\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta}_0;\boldsymbol{\nu}})'\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\boldsymbol{\Gamma}^-_{\boldsymbol{\vartheta}_0;\underline{K}}\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta}_0;\boldsymbol{\nu}}]^-$$
$$(\boldsymbol{\Upsilon}^{(n)}_{\boldsymbol{\vartheta}_0;\boldsymbol{\nu}})'\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\boldsymbol{\Gamma}^-_{\boldsymbol{\vartheta}_0;\underline{K}}. \tag{15}$$

Then Proposition 2 entails that since $\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K}}\boldsymbol{\Gamma}^{\perp}_{\boldsymbol{\vartheta}_0;\underline{\phi},\underline{g}}$ is idempotent with trace $(m-1)(k-1)$, $\underset{\approx}{Q}{}^{(n)}_{\underline{K}}$ is asymptotically Chi-square with $(m-1)(k-1)$ degrees of freedom under $P^{(n)}_{\boldsymbol{\vartheta}_0;\underline{g}}$, and asymptotically non-central Chi-square, still with $(m-1)(k-1)$ degrees of freedom, and with non-centrality parameter $\mathbf{t}'\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\boldsymbol{\Gamma}^{\perp}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}_0;\underline{K},\underline{g}}\mathbf{t}$ under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}_0+n^{-1/2}\boldsymbol{\nu}^{(n)}\mathbf{t}^{(n)};\underline{g}}$. Parts (i) and (ii) follow. Point (iii) is a direct consequence of point (i). Finally, point (iv) directly follows from the fact that for $\underline{K} = \underline{K}_f := (K_{f_1}, \ldots, K_{f_m})$ with $K_{f_i}(u) := \psi_{f_i}(\tilde{F}_i^{-1}(u))(1-(\tilde{F}_i^{-1}(u))^2)^{1/2}$, $\underset{\approx}{Q}{}^{(n)}_{\underline{K}}$ is asymptotically equivalent to $Q^{(n)}_{\underline{f}}$ in (14). $\square$

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## References

Abe, T., Shimizu, K., Pewsey, A. (2010). Symmetric unimodal models for directional data motivated by inverse stereographic projection. *Journal of the Japan Statistical Society*, *40*, 45–61.

Beran, R., Fisher, N. I. (1998). Nonparametric comparison of mean directions or mean axes. *Annals of Statistics*, *26*, 472–493.

Bingham, M. S., Mardia, K. V. (1975). Characterizations and applications. In S. Kotz, G. P. Patil, J. K. Ord (Eds.), *Statistical Distributions for Scientific Work* (Vol. 3, pp. 387–398). Dordrecht and Boston: Reidel.

Breitenberger, E. (1963). Analogues of the normal distribution on the circle and the sphere. *Biometrika*, *50*, 81–88.

Chang, T. (2004). Spatial statistics. *Statistical Science*, *19*, 624–635.

Duerinckx, M., Ley, C. (2012). Maximum likelihood characterization of rotationally symmetric distributions. *Sankhyā Series A*, *74*, 249–262.

Embleton, B. J. J., Mc Donnell, K. L. (1980). Magnetostratigraphy in the Sidney Basin, Southern Australia. Journal of Geomagnetism and Geoelectricity *32*(Suppl. III), 304.

Eplett, W. J. R. (1979). The small sample distribution of a Mann–Whitney type statistic for circular data. *Annals of Statistics*, *7*, 446–453.

Eplett, W. J. R. (1982). Two Mann–Whitney type rank tests. *Journal of the Royal Statistical Society, Series B*, *44*, 270–286.

Fisher, R. A. (1953). Dispersion on a sphere. *Proceedings of the Royal Society of London, Series A*, *217*, 295–305.

Fisher, N. I., Hall, P. (1990). New statistical methods for directional data I. Bootstrap comparison of mean directions and the fold test in palaeomagnetism. *Geophysical Journal International*, *101*, 305–313.

Fisher, N. I., Lewis, T., Embleton, B. J. J. (1987). *Statistical analysis of spherical data*. UK: Cambridge University Press.

Graham, J. W. (1949). The stability and significance of magnetism in sedimentary rocks. *Journal of Geophysical Research*, *54*, 131–167.

Hallin, M. (2012). Asymptotic relative efficiency. In: W. Piegorsch & A. El Shaarawi (Eds.), Encyclopedia of environmetrics (pp. 106–110, 2nd ed.). UK: Wiley.

Hallin, M., Paindaveine, D. (2008). A general method for constructing pseudo-Gaussian tests. *Journal of the Japan Statistical Society*, *38*, 27–39.

Hallin, M., Paindaveine, D., Verdebout, T. (2010). Optimal rank-based testing for principal components. *Annals of Statistics*, *38*, 3245–3299.

Hallin, M., Paindaveine, D., Verdebout, T. (2013). Optimal rank-based tests for common principal components. *Bernoulli*, *19*, 2524–2556.

Hallin, M., Paindaveine, D., Verdebout, T. (2014). Efficient R-estimation of principal and common principal components. *Journal of the American Statistical Association*, *109*, 1071–1083.

Jammalamadaka, S. R., SenGupta, A. (2001). *Topics in circular statistics*. Singapore: World Scientific.

Jones, M. C., Pewsey, A. (2005). A family of symmetric distributions on the circle. *Journal of the American Statistical Association*, *100*, 1422–1428.

Jupp, P. E. (1987). A non-parametric correlation coefficient and a two-sample test for random vectors or directions. *Biometrika*, *74*, 887–890.

Kreiss, J. P. (1987). On adaptive estimation in stationary ARMA processes. *Annals of Statistics*, *15*, 112–133.

Langevin, P. (1905). Sur la théorie du magnétisme. *Journal de Physique*, *4*, 678–693.

Le Cam, L., Yang, G. L. (2000). *Asymptotics in statistics* (2nd ed.). New York: Springer.

Ley, C., Swan, Y., Thiam, B., Verdebout, T. (2013). Optimal $R$-estimation of a spherical location. *Statistica Sinica*, *23*, 305–332.

Mardia, K. V., Jupp, P. E. (2000). *Directional statistics*. New York: Wiley.

McFadden, P. L., Jones, D. L. (1981). The fold test in palaeomagnetism. *Geophysical Journal of the Royal Astronomical Society*, *67*, 53–58.

McFadden, P. L., Lowes, F. J. (1981). The discrimination of mean directions drawn from Fisher distributions. *Geophysical Journal of the Royal Astronomical Society*, *67*, 19–33.

Purkayastha, S. (1991). A rotationally symmetric directional distribution: Obtained through maximum likelihood characterization. *Sankhyā Series A*, *53*, 70–83.

Saw, J. G. (1978). A family of distributions on the $m$-sphere and some hypothesis tests. *Biometrika*, *65*, 69–73.

Tsai, M. T. (2009). Asymptotically efficient two-sample rank tests for modal directions on spheres. *Journal of Multivariate Analysis*, *100*, 445–458.

von Mises, R. (1918). Über die Ganzzahligkeit der Atomgewichte und verwandte Fragen. *Physikalische Zeitschrift*, *19*, 490–500.

Watson, G. S. (1983). *Statistics on spheres*. New York: Wiley.

Wellner, J. A. (1979). Permutation tests for directional data. *Annals of Statistics*, *7*, 929–943.