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Testing the constancy of Spearman's rho in multivariate time series

Ivan Kojadinovic[1] · **Jean-François Quessy**[2] · **Tom Rohmer**[3]

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Abstract A class of tests for change-point detection designed to be particularly sensitive to changes in the cross-sectional rank correlation of multivariate time series is proposed. The derived procedures are based on several multivariate extensions of Spearman's rho. Two approaches to carry out the tests are studied: the first one is based on resampling and the second one consists of estimating the asymptotic null distribution. The asymptotic validity of both techniques is proved under the null for strongly mixing observations. A procedure for estimating a key bandwidth parameter involved in both approaches is proposed, making the derived tests parameter-free. Their finite-sample behavior is investigated through Monte Carlo experiments. Practical recommendations are made and an illustration on trivariate financial data is finally presented.

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✉ Ivan Kojadinovic
ivan.kojadinovic@univ-pau.fr

Jean-François Quessy
jean-francois.quessy@uqtr.ca

Tom Rohmer
tom.rohmer@univ-nantes.fr

1 Laboratoire de mathématiques et applications, UMR CNRS 5142, Université de Pau et des Pays de l'Adour, B.P. 1155, 64013 Pau Cedex, France

2 Département de mathématiques et d'informatique, Université du Québec à Trois-Rivières, C.P. 500, Trois-Rivières, QC G9A 5H7, Canada

3 Laboratoire de mathématiques Jean Leray, Université de Nantes, B.P. 92208, 44322 Nantes Cedex 3, France

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1 Introduction

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a multivariate times series of d-dimensional observations and, for any $i \in \{1, \ldots, n\}$, let $F^{(i)}$ denote the cumulative distribution function (c.d.f.) of $\boldsymbol{X}_i$. We are interested in procedures for testing $H_0 : F^{(1)} = \cdots = F^{(n)}$ against $\neg H_0$. Notice that the aforementioned null hypothesis can be simply rewritten as

$$H_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have c.d.f. } F. \tag{1}$$

Such statistical procedures are commonly referred to as *tests for change-point detection* (see, e.g., Csörgő and Horváth 1997, for an overview of possible approaches). The majority of tests for H_0 developed in the literature deal with the case $d = 1$. We aim at developing *nonparametric* tests for *multivariate* time series that are particularly sensitive to changes in the *dependence* among the components of the d-dimensional observations. The availability of such tests seems to be of great practical importance for the analysis of economic data, among others. In particular, assessing whether the dependence among financial assets can be considered constant or not over a given time period appears crucial for risk management, portfolio optimization and related statistical modeling (see, e.g., Wied et al. 2014; Dehling et al. 2014, and the references therein for a more detailed discussion about the motivation for such statistical procedures).

The above context, rather naturally, suggests to address the informal notion of *dependence* through that of *copula* (see, e.g., Nelsen 2006). Assume that H_0 in (1) holds and that, additionally, the common marginal c.d.f.s $F_1, \ldots, F_d$ of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are continuous. Then, from the work of Sklar (1959), the common multivariate c.d.f. F of the observations can be written as

$$F(\boldsymbol{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

where the function $C : [0, 1]^d \to [0, 1]$ is the unique *copula* associated with F. It follows that H_0 can be rewritten as $H_{0,m} \cap H_{0,c}$, where

$$H_{0,m} : \exists\, F_1, \ldots, F_d \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_1, \ldots, F_d, \tag{2}$$

$$H_{0,c} : \exists\, C \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have copula } C. \tag{3}$$

Several nonparametric tests designed to be particularly sensitive to certain alternatives under $H_{0,m} \cap \neg H_{0,c}$ were proposed in the literature. Tests for the constancy of Kendall's tau (which is a functional of C) were investigated by Gombay and Horváth (1999) (see also, Gombay and Horváth 2002) and Quessy et al. (2013) in the case of serially independent observations. A version of the previous tests adapted to a very general class of bivariate time series was proposed by Dehling et al. (2014). Recent multivariate alternatives are the tests studied in (Bücher et al. 2014, see also the references therein) based on Cramér–von Mises functionals of the *sequential empirical copula process*.

The aim of this work was to derive tests for the constancy of several multivariate extensions of Spearman's rho (which are also functionals of C) in multivariate strongly mixing time series. A similar problem was recently tackled by Wied et al. (2014). However, as the functional they considered does not exactly correspond to a multivariate extension of Spearman's rho (because of the way ranks are calculated), the corresponding test turns out to have a rather low power. We remedy that situation by computing ranks with respect to the relevant subsamples. From a theoretical perspective, as in Wied et al. (2014), no assumptions on the first order partial derivatives of the copula are made. The latter is actually an advantage of the studied tests over that investigated in Bücher et al. (2014). An inconvenience with respect to the aforementioned approach is however that, as all tests based on moments of copulas (such as Spearman's rho or Kendall's tau), the derived tests will have no power, by construction, against alternatives involving changes in the copula at a constant value of Spearman's rho.

To carry out the tests, we propose two approaches for computing approximate p values: the first one is based on resampling while the second one consists of estimating the asymptotic null distribution. In addition, a procedure for estimating a key bandwidth parameter involved in both approaches is proposed, making the derived tests fully data-driven. The versions of the studied tests based on the estimation of the asymptotic null distribution can be seen as alternatives to the test based on Kendall's tau recently proposed by Dehling et al. (2014).

The paper is organized as follows: The test statistics are defined in Sect. 2 and their limiting null distribution is established under strong mixing. Section 3 presents two approaches for computing approximate p values based, respectively, on bootstrapping and on the estimation of an asymptotic variance. The fourth section partially reports the results of Monte Carlo experiments involving bivariate and fourvariate time series generated from autoregressive and GARCH-like models. The fifth section contains practical recommendations and an illustration on trivariate financial data, while the last section concludes.

In the rest of the paper, the arrow '$\rightsquigarrow$' denotes weak convergence in the sense of Definition 1.3.3 in van der Vaart and Wellner (2000). Also, given a set T, $\ell^\infty(T; \mathbb{R})$ denotes the space of all bounded real-valued functions on T equipped with the uniform metric. The proofs of the stated theoretical results are available in the online supplementary material and the studied tests for change-point detection are implemented in the package `npcp` (Kojadinovic 2014) for the R statistical system (R Development Core Team 2014).

2 Test statistics

2.1 Multivariate extensions of Spearman's rho and their estimation

Spearman's rho is a very well-known measure of bivariate dependence (see, e.g., Nelsen 2006, Section 5.1 and the references therein). For a bivariate random vector with continuous margins and copula C, it can be expressed as

$$\rho(C) = 12 \int_{[0,1]^2} C(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} - 3 = 12 \int_{[0,1]^2} u_1 u_2 \mathrm{d}C(\boldsymbol{u}) - 3.$$

When the random vector of interest is d-dimensional with $d > 2$, the following three possible extensions were proposed by Schmid and Schmidt (2007):

$$\rho_1(C) = \frac{d+1}{2^d - d - 1}\left\{2^d \int_{[0,1]^d} C(\boldsymbol{u})\mathrm{d}\boldsymbol{u} - 1\right\},$$
$$\rho_2(C) = \rho_1(\bar{C}),$$
$$\rho_3(C) = \binom{d}{2}^{-1} \sum_{1\le i<j\le d} \rho\left(C^{(i,j)}\right),$$

where $C^{(i,j)}$ is the bivariate margin obtained from C by keeping dimensions i and j, and $\bar{C}$ is the survival function corresponding to C. It is well known that the latter can be expressed in terms of C. To see this, let $D = \{1, \dots, d\}$ and, for any $\boldsymbol{u} \in [0,1]^d$ and $A \subseteq D$, let $\boldsymbol{u}^A$ be the vector of $[0,1]^d$ such that $u_i^A = u_i$ if $i \in A$ and $u_i^A = 1$ otherwise. Then, for any $\boldsymbol{u} \in [0,1]^d$, $\bar{C}(\boldsymbol{u}) = \sum_{A\subseteq D}(-1)^{|A|}C(\boldsymbol{u}^A)$. Other related d-dimensional coefficients are considered in Quessy (2009).

Let us now discuss the estimation of the above theoretical quantities. Specifically, we assume that we have at hand n copies $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ of a d-dimensional random vector $\boldsymbol{X}$ with copula C and continuous margins. Given an estimator of C, natural estimators of $\rho_1(C)$, $\rho_2(C)$ and $\rho_3(C)$ can be obtained using the plug-in principle. Restricting attention to a sample $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$, $1 \le k \le l \le n$, for reasons that will become clear in the next subsection, a natural estimator of C is given by

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}\left(\hat{\boldsymbol{U}}_i^{k:l} \le \boldsymbol{u}\right), \quad \boldsymbol{u} \in [0,1]^d, \tag{4}$$

where

$$\hat{\boldsymbol{U}}_i^{k:l} = \frac{1}{l-k+1}\left(R_{i1}^{k:l}, \dots, R_{id}^{k:l}\right), \quad i \in \{k, \dots, l\}, \tag{5}$$

with $R_{ij}^{k:l} = \sum_{t=k}^{l} \mathbf{1}(X_{tj} \le X_{ij})$ the maximal rank of X_{ij} among $X_{kj}, \dots, X_{lj}$. The quantity given by (4) is commonly referred to as the *empirical copula* of $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$ (see, e.g., Rüschendorf 1976; Deheuvels 1981). Corresponding natural estimators of the three aforementioned multivariate versions of Spearman's rho are, therefore, $\rho_1(C_{k:l})$, $\rho_2(C_{k:l})$ and $\rho_3(C_{k:l})$, respectively.

It is important to notice that we do not necessarily assume the observations to be serially independent. Serial independence *and* continuity of the marginal distributions together guarantee the absence of ties in the d component series. However, continuity of the marginal distributions alone is *not* sufficient to guarantee the absence of ties when the observations are serially dependent (see,e.g., Bücher and Segers 2014, Example 4.2). This is the reason why maximal ranks are used in (5). The possible presence of ties in the component series makes the study of the tests under consideration substantially more complicated.

2.2 Change-point statistics

To derive tests for change-point detection particularly sensitive to changes in the strength of the cross-sectional dependence, one natural possibility is to base these tests on differences of Spearman's rhos. By analogy with the classical approach to change-point analysis (see, e.g., Csörgő and Horváth 1997), one could for instance consider the following three test statistics:

$$S_{n,i} = \max_{1 \le k \le n-1} \frac{k(n-k)}{n^{3/2}} \left| \rho_i(C_{1:k}) - \rho_i(C_{k+1:n}) \right|, \qquad i \in \{1, 2, 3\}, \tag{6}$$

where $C_{1:k}$ and $C_{k+1:n}$ are the empirical copulas of the subsamples $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ and $\boldsymbol{X}_{k+1}, \ldots, \boldsymbol{X}_n$, respectively, defined analogously to (4). All three statistics above turn out to be particular cases of a generic statistic which is the primary focus of this work. Before we can define it, some additional notation is necessary.

For any $A \subseteq D = \{1, \ldots, d\}$, let ϕ_A be the map from $\ell^\infty([0,1]^d; \mathbb{R})$ to $\mathbb{R}$ defined by

$$\phi_A(g) = \int_{[0,1]^d} g\left(\boldsymbol{u}^A\right) \mathrm{d}\boldsymbol{u}, \quad g \in \ell^\infty\left([0,1]^d; \mathbb{R}\right). \tag{7}$$

Then, define the empirical process

$$\mathbb{T}_{n,A}(s) = \sqrt{n}\, \lambda_n(0,s)\, \lambda_n(s,1) \{\phi_A(C_{1:\lfloor ns \rfloor}) - \phi_A(C_{\lfloor ns \rfloor+1:n})\}, \qquad s \in [0,1],$$

where $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ for $(s,t) \in \Delta = \{(s,t) \in [0,1]^2 : s \le t\}$, and with the additional convention that $C_{k:l} = 0$ whenever $k > l$. Simple calculations reveal that $\mathbb{T}_{n,\emptyset} = 0$. Next, consider the $\mathbb{R}^{2^d-1}$-valued empirical process

$$\mathbb{T}_n(s) = (\mathbb{T}_{n,\{1\}}(s), \mathbb{T}_{n,\{2\}}(s), \ldots, \mathbb{T}_{n,D}(s)), \qquad s \in [0,1]. \tag{8}$$

Finally, given a function $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$, define the generic change-point statistic

$$S_{n,f} = \sup_{s \in [0,1]} |f\{\mathbb{T}_n(s)\}| = \max_{1 \le k \le n-1} |f\{\mathbb{T}_n(k/n)\}|. \tag{9}$$

We shall now verify that the statistics $S_{n,i}$, $i \in \{1,2,3\}$, given by (6) are particular cases of $S_{n,f}$ when f is linear, that is, when there exists a vector $\boldsymbol{a} \in \mathbb{R}^{2^d-1}$ such that, for any $\boldsymbol{x} \in \mathbb{R}^{2^d-1}$, $f(\boldsymbol{x}) = \boldsymbol{a}^\top \boldsymbol{x}$. As we continue, with some abuse of notation, we index the components of vectors of $\mathbb{R}^{2^d-1}$ by subsets of D of cardinality greater than 1, i.e., for any $\boldsymbol{x} \in \mathbb{R}^{2^d-1}$, we write $\boldsymbol{x} = (x_{\{1\}}, x_{\{2\}}, \ldots, x_D)$. Then, we have $S_{n,i} = S_{n,f_i}$, $i \in \{1,2,3\}$, where, for any $\boldsymbol{x} \in \mathbb{R}^{2^d-1}$,

$$f_1(\boldsymbol{x}) = \frac{(d+1)2^d}{2^d - d - 1} x_D, \quad f_2(\boldsymbol{x}) = \frac{(d+1)2^d}{2^d - d - 1} \sum_{\substack{A \subseteq D \\ |A| \ge 1}} (-1)^{|A|} x_A,$$

$$f_3(\boldsymbol{x}) = \frac{24}{d(d-1)} \sum_{\substack{A \subseteq D \\ |A| \geq 2}} x_A.$$

Similar relationships hold for the statistics constructed from the additional coefficients mentioned in Quessy (2009), though the corresponding functions f are not necessarily linear anymore but only continuous.

Let us make a brief remark concerning the statistic $S_{n,2}$. Expressing it as S_{n,f_2} above is clearly not the most efficient way to compute it. To see this, for any $1 \leq k \leq l \leq n$, define

$$\bar{C}_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}\left(\hat{\boldsymbol{U}}_i^{k:l} > \boldsymbol{u}\right), \qquad \boldsymbol{u} \in [0,1]^d,$$

where the $\hat{\boldsymbol{U}}_i^{k:l}$ are defined in (5), and notice that, for any $\boldsymbol{u} \in [0,1]^d$, $\bar{C}_{k:l}(\boldsymbol{u}) = \sum_{A \subseteq D} (-1)^{|A|} C_{k:l}(\boldsymbol{u}^A)$, where $C_{k:l}$ is defined in (4). Then, by definition of ρ_2,

$$S_{n,2} = \max_{1 \leq k \leq n-1} \frac{k(n-k)}{n^{3/2}} \left| \rho_1(\bar{C}_{1:k}) - \rho_1(\bar{C}_{k+1:n}) \right|.$$

Under the assumption of no ties in the d component series, some additional simple calculations reveal that the latter is actually nothing else than $S_{n,1}$ computed from the sample $-\boldsymbol{X}_1, \ldots, -\boldsymbol{X}_n$.

We end this section by a discussion of the differences between $S_{n,1}$ and the similar statistic considered in Wied et al. (2014). Instead of basing their approach on the empirical copula, these authors considered the alternative estimator of C defined, for any $1 \leq k \leq l \leq n$, as

$$C_{k:l,n}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}\left(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}\right), \qquad \boldsymbol{u} \in [0,1]^d, \tag{10}$$

with the convention that $C_{k:l,n} = 0$ if $k > l$. The apparently subtle yet crucial difference between $C_{k:l}$ in (4) and $C_{k:l,n}$ above is that the scaled ranks are computed relative to the complete sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ for $C_{k:l,n}$, while, for $C_{k:l}$, they are computed relative to the subsample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$. As a consequence, the analogue of the statistic $S_{n,1}$ considered in Wied et al. (2014) is not really a maximally selected absolute difference of sample Spearman's rhos. From a practical perspective, as illustrated empirically in Bücher et al. (2014), the use of $C_{k:l}$ instead of $C_{k:l,n}$ in a change-point detection framework results in tests that are more powerful when the change in distribution in only due to a change in the copula. We provide similar empirical evidence in Sect. 4: tests based on $S_{n,1}$ appear substantially more powerful than their analogues based on (10) for alternatives involving a change of $\rho_1(C)$ at constant margins. Reasons that explain this improved efficiency are discussed in Bücher et al. (2014, Section 2).

2.3 Limiting null distribution under strong mixing

Let us first recall the notion of *strongly mixing sequence*. For a sequence of d-dimensional random vectors $(\boldsymbol{Y}_i)_{i \in \mathbb{Z}}$, the σ-field generated by $(\boldsymbol{Y}_i)_{a \leq i \leq b}$, $a, b \in$

$\mathbb{Z} \cup \{-\infty, +\infty\}$, is denoted by $\mathcal{F}_a^b$. The strong mixing coefficients corresponding to the sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ are defined by

$$\alpha_r = \sup_{p\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^p, B\in\mathcal{F}_{p+r}^{+\infty}} |P(A \cap B) - P(A)P(B)|$$

for strictly positive integer r. The sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ is said to be *strongly mixing* if $\alpha_r \to 0$ as $r \to \infty$.

The limiting null distribution of the vector-valued empirical process $\mathbb{T}_n$ defined in (8) can be obtained by rewriting its components in terms of the processes

$$\mathbb{S}_{n,A}(s,t) = \sqrt{n}\lambda_n(s,t)\{\phi_A(C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}) - \phi_A(C)\}, \qquad (s,t) \in \Delta, \tag{11}$$

for $A \subseteq D$, $|A| \geq 1$. Indeed, it is easy to verify that, under H_0 defined in (1),

$$\mathbb{T}_{n,A}(s) = \lambda_n(s,1)\mathbb{S}_{n,A}(0,s) - \lambda_n(0,s)\mathbb{S}_{n,A}(s,1), \qquad s \in [0,1]. \tag{12}$$

As we shall see below, the limiting null distribution of $\mathbb{T}_n$ is then a mere consequence of the fact that the empirical processes $\mathbb{S}_{n,A}$, $A \subseteq D$, $|A| \geq 1$, are asymptotically equivalent to continuous functionals of the sequential empirical process

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{13}$$

where $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ is the unobservable sample obtained from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ by the probability integral transforms $U_{ij} = F_j(X_{ij})$, $i \in \{1, \ldots, n\}$, $j \in D$.

If $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ with $a > 1$, we have from Bücher (2014) that $\mathbb{B}_n(0,\cdot,\cdot)$ converges weakly in $\ell^\infty([0,1]^{d+1};\mathbb{R})$ to a tight centered Gaussian process $\mathbb{B}_C^\circ$ with covariance function $\mathrm{cov}\{\mathbb{B}_C^\circ(s,\boldsymbol{u}), \mathbb{B}_C^\circ(t,\boldsymbol{v})\} = (s \wedge t)\kappa_C(\boldsymbol{u},\boldsymbol{v})$, $(s,\boldsymbol{u}), (t,\boldsymbol{v}) \in [0,1]^{d+1}$, where

$$\kappa_C(\boldsymbol{u},\boldsymbol{v}) = \mathrm{cov}\left\{\mathbb{B}_C^\circ(1,\boldsymbol{u}), \mathbb{B}_C^\circ(1,\boldsymbol{v})\right\} = \sum_{k\in\mathbb{Z}} \mathrm{cov}\{\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \leq \boldsymbol{v})\}. \tag{14}$$

As a consequence of the continuous mapping theorem, $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta \times [0,1]^d;\mathbb{R})$, where

$$\mathbb{B}_C(s,t,\boldsymbol{u}) = \mathbb{B}_C^\circ(t,\boldsymbol{u}) - \mathbb{B}_C^\circ(s,\boldsymbol{u}), \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d. \tag{15}$$

The following proposition, proved in Section A of the supplementary material, is the key step for obtaining the limiting null distribution of the vector-valued process $\mathbb{T}_n$ defined in (8).

Proposition 1 *Assume that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins and whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 1$. Then, for any $A \subseteq D$, $|A| \geq 1$,*

$$\sup_{(s,t)\in\Delta} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| = o_{\mathrm{P}}(1), \tag{16}$$

where $\psi_{C,A}$ *is a linear map from* $\ell^\infty([0,1]^d;\mathbb{R})$ *to* $\mathbb{R}$ *defined by*

$$\psi_{C,A}(g) = \phi_A(g) - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l) g(\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}), \qquad g \in \ell^\infty([0,1]^d;\mathbb{R}), \tag{17}$$

with ϕ_A *given in* (7).

From the work of Mokkadem (1988), we know that the strong mixing conditions stated in the previous proposition (as well as those stated in the forthcoming propositions and corollaries) are for instance satisfied (with much to spare) when $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ is drawn from a stationary vector ARMA process with absolutely continuous innovations. A similar conclusion holds for a large class of GARCH processes (see Lindner 2009, Section 5, and the references therein).

The next result, proved in Section B of the supplementary material, is a consequence of the previous proposition and establishes the limiting null distribution of the generic statistic $S_{n,f}$ defined in (9) under strong mixing.

Corollary 2 *Under the conditions of Proposition 1,*

$$\mathbb{T}_n \rightsquigarrow s \mapsto \mathbb{T}_C(s) = \big(\mathbb{T}_{C,\{1\}}(s), \mathbb{T}_{C,\{2\}}(s), \dots, \mathbb{T}_{C,D}(s)\big) \tag{18}$$

in $\ell^\infty([0,1];\mathbb{R}^{2^d-1})$*, where*

$$\mathbb{T}_C(s) = \psi_C\{\mathbb{B}_C(0,s,\cdot) - s\mathbb{B}_C(0,1,\cdot)\}, \qquad s\in[0,1], \tag{19}$$

with $\mathbb{B}_C$ *defined in* (15) *and* ψ_C *a map from* $\ell^\infty([0,1]^d;\mathbb{R})$ *to* $\mathbb{R}^{2^d-1}$ *defined by*

$$\psi_C(g) = \big(\psi_{C,\{1\}}(g), \psi_{C,\{2\}}(g), \dots, \psi_{C,D}(g)\big), \qquad g\in \ell^\infty([0,1]^d;\mathbb{R}). \tag{20}$$

As a consequence, for any $f:\mathbb{R}^{2^d-1}\to\mathbb{R}$ *continuous,*

$$S_{n,f} = \sup_{s\in[0,1]} |f\{\mathbb{T}_n(s)\}| \rightsquigarrow S_{C,f} = \sup_{s\in[0,1]} |f\{\mathbb{T}_C(s)\}|,$$

and, if f *is additionally linear and* $\sigma^2_{C,f} = \mathrm{var}[f\circ\psi_C\{\mathbb{B}_C(0,1,\cdot)\}] > 0$*, the weak limit of* $\sigma^{-1}_{C,f} S_{n,f}$ *is equal in distribution to* $\sup_{s\in[0,1]} |\mathbb{U}(s)|$*, where* $\mathbb{U}$ *is a standard Brownian bridge on* $[0,1]$.

3 Computation of approximate *p* values

Corollary 2 suggests two related ways to compute p values for the generic test statistic $S_{n,f}$ defined in (9). The first approach, based on resampling, consists of exploiting the fact that, under H_0, $\mathbb{T}_n$ defined in (8) is asymptotically equivalent to a continuous functional of the sequential empirical process $\mathbb{B}_n$ defined in (13) and can be applied as soon as $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$ is continuous. The second approach, restricted to the situation when f is linear, is motivated by the last claim of Corollary 2. It consists of estimating $\sigma^2_{C,f}$ and thus the asymptotic null distribution of $S_{n,f}$.

3.1 Approximate *p* values by bootstrapping

The first approach that we consider consists of bootstrapping the vector-valued empirical process $\mathbb{T}_n$ defined in (8) using a bootstrap for the sequential empirical process $\mathbb{B}_n$. This way of proceeding actually allows us to consider not only linear but also *continuous* functions f in (9). More specifically, we consider a *multiplier bootstrap* for $\mathbb{B}_n$ in the spirit of van der Vaart and Wellner (2000, Chapter 2.9) when observations are serially independent, or Bühlmann (1993, Section 3.3) when they are serially dependent. In the latter case, we rely on the recent work of Bücher and Kojadinovic (2014).

The notion of *multiplier sequence* is central to this resampling technique. We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is an *i.i.d. multiplier sequence* if:

(M0) $(\xi_{i,n})_{i\in\mathbb{Z}}$ is i.i.d., independent of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$, with distribution not changing with n, having mean 0, variance 1, and being such that $\int_0^\infty \{\mathrm{P}(|\xi_{0,n}| > x)\}^{1/2} \mathrm{d}x < \infty$.

We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a *dependent multiplier sequence* if

(M1) the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is strictly stationary with $\mathrm{E}(\xi_{0,n}) = 0$, $\mathrm{E}(\xi^2_{0,n}) = 1$ and $\sup_{n\geq 1} \mathrm{E}(|\xi_{0,n}|^\nu) < \infty$ for all $\nu \geq 1$, and is independent of the available sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$.

(M2) There exists a sequence $\ell_n \to \infty$ of strictly positive constants such that $\ell_n = o(n)$ and the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is ℓ_n-dependent, i.e., $\xi_{i,n}$ is independent of $\xi_{i+h,n}$ for all $h > \ell_n$ and $i \in \mathbb{N}$.

(M3) There exists a function $\varphi : \mathbb{R} \to [0, 1]$, symmetric around 0, continuous at 0, satisfying $\varphi(0) = 1$ and $\varphi(x) = 0$ for all $|x| > 1$ such that $\mathrm{E}(\xi_{0,n}\xi_{h,n}) = \varphi(h/\ell_n)$ for all $h \in \mathbb{Z}$.

The choice of the function φ and an approach to generate dependent multiplier sequences is briefly discussed in Sect. 4. More details can be found in (Bücher and Kojadinovic 2014, Section 5.2).

Let M be a large integer and let $(\xi^{(1)}_{i,n})_{i\in\mathbb{Z}}, \dots, (\xi^{(M)}_{i,n})_{i\in\mathbb{Z}}$ be M independent copies of the same multiplier sequence. Then, following Bücher and Kojadinovic (2014) and Bücher et al. (2014), for any $m \in \{1, \dots, M\}$ and $(s, t, \boldsymbol{u}) \in \Delta \times [0, 1]^d$, let

$$\hat{\mathbb{B}}^{(m)}_n(s, t, \boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi^{(m)}_{i,n} \left\{ \mathbf{1}\left(\hat{\boldsymbol{U}}^{1:n}_i \leq \boldsymbol{u}\right) - C_{1:n}(\boldsymbol{u}) \right\},$$

$$\check{\mathbb{B}}^{(m)}_n(s, t, \boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\xi^{(m)}_{i,n} - \bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\right) \mathbf{1}\left(\hat{\boldsymbol{U}}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}_i \leq \boldsymbol{u}\right), \qquad (21)$$

where $\bar{\xi}^{(m)}_{k:l}$ is the arithmetic mean of $\xi^{(m)}_{i,n}$ for $i \in \{k, \dots, l\}$.

The following proposition is a consequence of Theorem 1 in Holmes et al. (2013), Theorem 2.1 and the proof of Proposition 4.2 in Bücher and Kojadinovic (2014), as well as the proof of Proposition 4.3 in Bücher et al. (2014). It suggests interpreting the multiplier replicates $\hat{\mathbb{B}}_n^{(1)}, \dots, \hat{\mathbb{B}}_n^{(M)}$ (resp. $\check{\mathbb{B}}_n^{(1)}, \dots, \check{\mathbb{B}}_n^{(M)}$) as "almost" independent copies of $\mathbb{B}_n$ as n increases.

Proposition 3 *Assume that either*

(i) *the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are i.i.d. with continuous margins and the sequences $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \dots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ are independent copies of a multiplier sequence satisfying (M0),*
(ii) *or the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ for some $a > 3 + 3d/2$, and $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}$, ..., $(\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ are independent copies of a dependent multiplier sequence satisfying (M1)–(M3) with $\ell_n = O(n^{1/2-\varepsilon})$ for some $0 < \varepsilon < 1/2$.*

Then,

$$\left(\mathbb{B}_n, \hat{\mathbb{B}}_n^{(1)}, \dots, \hat{\mathbb{B}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}\right),$$
$$\left(\mathbb{B}_n, \check{\mathbb{B}}_n^{(1)}, \dots, \check{\mathbb{B}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}\right)$$

in $\{\ell^\infty(\Delta \times [0,1]^d; \mathbb{R})\}^{M+1}$, where $\mathbb{B}_C$ is given in (15) and $\mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}$ are independent copies of $\mathbb{B}_C$.

Starting from the quantities defined above, we shall now define appropriate multiplier replicates under H_0 of $\mathbb{T}_n$ defined in (8). From (12), we see that to do so, we first need to define multiplier replicates of the processes $\mathbb{S}_{n,A}$, $A \subseteq D$, $|A| \geq 1$, defined in (11). From (16) and Proposition 3, natural candidates would be the processes $(s,t) \mapsto \psi_{C,A}\{\hat{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}$ or the processes $(s,t) \mapsto \psi_{C,A}\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}$, $m \in \{1,\dots,M\}$, where the map $\psi_{C,A}$ is defined in (17). These, however, still depend on the unknown copula C. The latter could be estimated either by $C_{1:n}$ or by $C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}$, which led us to consider the following two computable versions instead:

$$\hat{\mathbb{S}}_{n,A}^{(m)}(s,t) = \psi_{C_{1:n},A}\left\{\hat{\mathbb{B}}_n^{(m)}(s,t,\cdot)\right\}, \quad \check{\mathbb{S}}_{n,A}^{(m)}(s,t) = \psi_{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor},A}\left\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\right\},$$

for $(s,t) \in \Delta$. The processes $\check{\mathbb{S}}_{n,A}^{(m)}$ were found to lead to better behaved tests than the $\hat{\mathbb{S}}_{n,A}^{(m)}$ in our Monte Carlo experiments, which is why, from now on, we focus solely on the former. It is easy to verify that the $\check{\mathbb{S}}_{n,A}^{(m)}$ can be rewritten as

$$\check{\mathbb{S}}_{n,A}^{(m)}(s,t) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}\right) \mathcal{I}_{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor},A}\left(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\right),$$

where, for any $\boldsymbol{u} \in [0, 1]^d$,

$$\mathcal{I}_{C,A}(\boldsymbol{u}) = \psi_{C,A}\{\mathbf{1}(\boldsymbol{u} \leq \cdot)\} \\ = \prod_{l \in A}(1 - u_l) - \int_{[0,1]^d} \sum_{j \in A} \prod_{l \in A \setminus \{j\}} (1 - v_l)\mathbf{1}(u_j \leq v_j)\mathrm{d}C(\boldsymbol{v}). \tag{22}$$

Next, by analogy with (12), for any $m \in \{1, \ldots, M\}$, $A \subseteq D$, $|A| \geq 1$, let

$$\check{\mathbb{T}}_{n,A}^{(m)}(s) = \lambda_n(s, 1)\check{\mathbb{S}}_{n,A}^{(m)}(0, s) - \lambda_n(0, s)\check{\mathbb{S}}_{n,A}^{(m)}(s, 1), \qquad s \in [0, 1],$$

and let $\check{\mathbb{T}}_n^{(m)}$ be the corresponding version of $\mathbb{T}_n$ in (8). Finally, for some continuous function $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$, let $\check{S}_{n,f}^{(m)} = \sup_{s \in [0,1]} |f\{\check{\mathbb{T}}_n^{(m)}(s)\}|$ by analogy with (9). Interpreting the $\check{S}_{n,f}^{(m)}$ as multiplier replicates of $S_{n,f}$ under H_0, it is natural to compute an approximate p value for the test as

$$\frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\check{S}_{n,f}^{(m)} \geq S_{n,f}\right). \tag{23}$$

The null hypothesis is rejected if the estimated p value is smaller than the desired significance level.

The following result, proved in Section C of the supplementary material, can be combined with Proposition F.1 in Bücher and Kojadinovic (2014) to show that a test based on $S_{n,f}$ whose p value is computed as in (23) will hold its level asymptotically as $n \to \infty$ followed by $M \to \infty$.

Proposition 4 *Under the conditions of Proposition 3, for any $A \subseteq D$, $|A| \geq 1$,*

$$\left(\mathbb{S}_{n,A}, \check{\mathbb{S}}_{n,A}^{(1)}, \ldots, \check{\mathbb{S}}_{n,A}^{(M)}\right) \rightsquigarrow \left(\mathbb{S}_{C,A}, \mathbb{S}_{C,A}^{(1)}, \ldots, \mathbb{S}_{C,A}^{(M)}\right)$$

in $\{\ell^\infty(\Delta; \mathbb{R})\}^{M+1}$, where, for any $(s, t) \in \Delta$, $\mathbb{S}_{C,A}(s, t) = \psi_{C,A}\{\mathbb{B}_C(s, t, \cdot)\}$ and $\mathbb{S}_{C,A}^{(1)}, \ldots, \mathbb{S}_{C,A}^{(M)}$ are independent copies of $S_{C,A}$. As a consequence,

$$\left(\mathbb{T}_n, \check{\mathbb{T}}_n^{(1)}, \ldots, \check{\mathbb{T}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{T}_C, \mathbb{T}_C^{(1)}, \ldots, \mathbb{T}_C^{(M)}\right)$$

in $\{\ell^\infty([0, 1]; \mathbb{R}^{2^d-1})\}^{M+1}$, where $\mathbb{T}_C$ is given in (19) and $\mathbb{T}_C^{(1)}, \ldots, \mathbb{T}_C^{(M)}$ are independent copies of T_C, and, for any continuous function $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$,

$$\left(S_{n,f}, \check{S}_{n,f}^{(1)}, \ldots, \check{S}_{n,f}^{(M)}\right) \rightsquigarrow \left(S_{C,f}, S_{C,f}^{(1)}, \ldots, S_{C,f}^{(M)}\right)$$

in $\mathbb{R}^{M+1}$, where $S_{C,f} = \sup_{s \in [0,1]} |f\{\mathbb{T}_C(s)\}|$ and $S_{C,f}^{(1)}, \ldots, S_{C,f}^{(M)}$ are independent copies of $S_{C,f}$.

The finite-sample behavior of the tests under consideration based on the processes $\check{\mathbb{S}}_{n,A}^{(m)}$ is not, however, completely satisfactory: the tests appear too liberal for multivariate time series with strong cross-sectional dependence. This prompted us to try other asymptotically equivalent versions of the $\check{\mathbb{S}}_{n,A}^{(m)}$. Under an additional assumption on the partial derivatives of the copula, the generic test statistic $S_{n,f}$ defined in (9) can be written under H_0 as a functional of the *two-sided sequential empirical copula process* studied in Bücher and Kojadinovic (2014), and could, therefore, be bootstrapped via the multiplier processes defined in (4.4) of Bücher et al. (2014). Without imposing any condition on the partial derivatives of the copula, the latter remark led us to consider, instead of the processes

$$\check{\mathbb{S}}_{n,A}^{(m)}(s,t) = \phi_A\left\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\right\} - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l)\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}), \quad (24)$$

the processes

$$\tilde{\mathbb{S}}_{n,b_n,A}^{(m)}(s,t) = \phi_A\left\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\right\} - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l)\tilde{\mathbb{B}}_{n,b_n,j}^{(m)}(s,t,v_j)\mathrm{d}C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}), \quad (25)$$

where, for any $j \in D$, $\tilde{\mathbb{B}}_{n,b_n,j}^{(m)}$ is a linearly smoothed version of $(s,t,u) \mapsto \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}_j)$ with $\boldsymbol{u}_j$ the vector of $[0,1]^d$ whose components are all equal to 1 except the jth one which is equal to u, and b_n a strictly positive sequence of constants converging to 0. Specifically, for $(s,t,v) \in \Delta \times [0,1]$,

$$\tilde{\mathbb{B}}_{n,b_n,j}^{(m)}(s,t,v) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}\right) \mathcal{L}_{b_n}\left(\hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}, v\right),$$

where

$$\mathcal{L}_{b_n}(u,v) = \frac{u_+ \wedge v - u_- \wedge v}{u_+ - u-}, \qquad u, v \in [0,1],$$

with $u_+ = (u+b_n) \wedge 1$ and $u_- = (u-b_n) \vee 0$. It is easy to verify that, for any $u \in [0,1]$, $\mathcal{L}_{b_n}(u,\cdot)$ differs from $\mathbf{1}(u \leq \cdot)$ only on the interval (u_-, u_+) on which it linearly increases from 0 to 1.

Notice that (25) can be rewritten as

$$\tilde{\mathbb{S}}_{n,b_n,A}^{(m)}(s,t) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}\right) \mathcal{I}_{b_n,C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor},A}\left(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\right),$$

where, for any $\boldsymbol{u} \in [0,1]^d$,

$$\mathcal{I}_{b_n,C,A}(\boldsymbol{u}) = \prod_{l\in A}(1-u_l) - \int_{[0,1]^d} \sum_{j\in A}\prod_{l\in A\setminus\{j\}} (1-v_l)\mathcal{L}_{b_n}(u_j, v_j)\mathrm{d}C(\boldsymbol{v}). \quad (26)$$

For any $m \in \{1,\ldots,M\}$, let $\tilde{\mathbb{T}}_{n,b_n}^{(m)}$ and $\tilde{S}_{n,b_n,f}^{(m)}$ be the analogues of $\check{\mathbb{T}}_n^{(m)}$ and $\check{S}_{n,f}^{(m)}$, respectively, defined from the processes $\tilde{\mathbb{S}}_{n,b_n,A}^{(m)}$ in (25). The following result, proved in Section C of the supplementary material, is then the analogue of Proposition 4 above.

Proposition 5 *If $b_n = o(n^{-1/2})$, Proposition 4 holds with $\check{\mathbb{S}}_{n,A}^{(m)}$ replaced by $\tilde{\mathbb{S}}_{n,b_n,A}^{(m)}$, $\check{\mathbb{T}}_n^{(m)}$ replaced by $\tilde{\mathbb{T}}_{n,b_n}^{(m)}$ and $\check{S}_{n,f}^{(m)}$ replaced by $\tilde{S}_{n,b_n,f}^{(m)}$.*

Finally, notice that it is possible to consider a version of the above construction in which the smoothing sequence is $b_{\lfloor nt\rfloor-\lfloor ns\rfloor}$ instead of b_n. We focused above only on the latter approach as it led to better behaved tests in our Monte Carlo experiments.

3.2 Estimating the asymptotic null distribution

When the function f used in the definition of $S_{n,f}$ in (9) is linear, Corollary 2 gives conditions under which, provided $\sigma_{C,f}^2 = \mathrm{var}[f\circ\psi_C\{\mathbb{B}_C(0,1,\cdot)\}] > 0$, the weak limit of $\sigma_{C,f}^{-1}S_{n,f}$ under H_0 is equal in distribution to $\sup_{s\in[0,1]}|\mathbb{U}(s)|$. The distribution of the latter random variable can be approximated very well (this aspect is discussed in more detail in Sect. 4). To be able to estimate an asymptotic p value for $S_{n,f}$, it thus remains to estimate the unknown variance $\sigma_{C,f}^2$.

Let E_ξ and var_ξ denote the expectation and variance, respectively, conditional on the data. By analogy with the classical way of proceeding when estimating variances using resampling procedures (see,e.g., Künsch 1989; Shao 2010), in our context, a first natural estimator of the unknown variance under H_0 is of the form

$$\check{\sigma}_{n,C,f}^2 = \mathrm{var}_\xi\left[f\circ\psi_C\left\{\check{\mathbb{B}}_n^{(m)}(0,1,\cdot)\right\}\right], \quad (27)$$

where $\check{\mathbb{B}}_n^{(m)}$ is defined in (21). To simplify the notation, we shall drop the superscript (m) in the rest of this section. The previous estimator is not computable as C is unknown, which is why we will eventually consider the estimator $\check{\sigma}_{n,C_{1:n},f}^2$ instead.

To obtain a more explicit expression of $\check{\sigma}_{n,C,f}^2$, first, let

$$\mathcal{I}_C(\boldsymbol{u}) = \left(\mathcal{I}_{C,\{1\}}(\boldsymbol{u}), \mathcal{I}_{C,\{2\}}(\boldsymbol{u}),\ldots,\mathcal{I}_{C,D}(\boldsymbol{u})\right), \qquad \boldsymbol{u}\in[0,1]^d, \quad (28)$$

where $\mathcal{I}_{C,A}$, $A\subseteq D$, $|A|\geq 1$, is defined in (22). From the linearity of $f\circ\psi_C$, we then obtain that

$$\begin{aligned}\check{\sigma}_{n,C,f}^2 &= \mathrm{var}_\xi\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_{i,n}-\bar{\xi}_{1:n})f\circ\mathcal{I}_C\left(\hat{\boldsymbol{U}}_i^{1:n}\right)\right\}\\ &= \mathrm{var}_\xi\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n\xi_{i,n}\left\{f\circ\mathcal{I}_C\left(\hat{\boldsymbol{U}}_i^{1:n}\right)-\frac{1}{n}\sum_{j=1}^n f\circ\mathcal{I}_C\left(\hat{\boldsymbol{U}}_j^{1:n}\right)\right\}\right].\end{aligned}$$

Using the fact that, from (22) and (28),

$$\frac{1}{n}\sum_{i=1}^{n} f \circ \mathcal{I}_C\left(\hat{\boldsymbol{U}}_i^{1:n}\right) = \frac{1}{n}\sum_{i=1}^{n} f \circ \psi_C\left\{\mathbf{1}\left(\hat{\boldsymbol{U}}_i^{1:n} \leq \cdot\right)\right\} = f \circ \psi_C(C_{1:n}),$$

we obtain that

$$\begin{aligned}\check{\sigma}^2_{n,C,f} = \frac{1}{n}\sum_{i,j=1}^{n} \mathrm{E}_\xi(\xi_{i,n}\xi_{j,n}) f\left\{\mathcal{I}_C\left(\hat{\boldsymbol{U}}_i^{1:n}\right) - \psi_C(C_{1:n})\right\} \\ \times f\left\{\mathcal{I}_C\left(\hat{\boldsymbol{U}}_j^{1:n}\right) - \psi_C(C_{1:n})\right\}.\end{aligned}$$

On the one hand, should the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ be an i.i.d. multiplier sequence, that is, should it satisfy (M0), unsurprisingly, the above estimator simplifies to

$$\check{\sigma}^2_{n,C,f} = \frac{1}{n}\sum_{i=1}^{n}\left[f\left\{\mathcal{I}_C\left(\hat{\boldsymbol{U}}_i^{1:n}\right) - \psi_C(C_{1:n})\right\}\right]^2. \tag{29}$$

On the other hand, if the multiplier sequence satisfies (M1)–(M3), one obtains

$$\begin{aligned}\check{\sigma}^2_{n,C,f} = \frac{1}{n}\sum_{i,j=1}^{n} \varphi\left(\frac{i-j}{\ell_n}\right) f\left\{\mathcal{I}_C\left(\hat{\boldsymbol{U}}_i^{1:n}\right) - \psi_C(C_{1:n})\right\} \\ \times f\left\{\mathcal{I}_C\left(\hat{\boldsymbol{U}}_j^{1:n}\right) - \psi_C(C_{1:n})\right\},\end{aligned} \tag{30}$$

which has the form of the HAC kernel estimator of de Jong and Davidson (2000).

Very naturally, once C has been replaced by $C_{1:n}$, we use the form in (29) (resp. (30)) for serially independent (resp. weakly dependent) observations. The following result, proved in Section D of the supplementary material, establishes the consistency of $\check{\sigma}^2_{n,C_{1:n},f}$ under H_0.

Proposition 6 *Assume that $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$ in the definition of (9) is linear and that either*

(i) *the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d. with continuous margins,*
(ii) *or the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ for some $a > 6$, and $\ell_n = O(n^{1/2-\varepsilon})$ for some $0 < \varepsilon < 1/2$ such that, additionally, φ defined in (M3) is twice continuously differentiable on $[-1, 1]$ with $\varphi''(0) \neq 0$ and is Lipschitz continuous on $\mathbb{R}$.*

Then, $\check{\sigma}^2_{n,C_{1:n},f} \xrightarrow{\mathrm{P}} \sigma^2_{C,f}$. As a consequence, the weak limit of $\check{\sigma}^{-1}_{n,C_{1:n},f} S_{n,f}$ is equal in distribution to $\sup_{s\in[0,1]} |\mathbb{U}(s)|$.

As in the previous subsection, better behaved tests are obtained if (26) is used instead of (22) in the above developments. Let

$$\mathcal{I}_{b_n,C}(\boldsymbol{u}) = \left(\mathcal{I}_{b_n,C,\{1\}}(\boldsymbol{u}), \mathcal{I}_{b_n,C,\{2\}}(\boldsymbol{u}), \ldots, \mathcal{I}_{b_n,C,D}(\boldsymbol{u})\right), \qquad \boldsymbol{u} \in [0,1]^d,$$

and let $\tilde{\sigma}^2_{n,b_n,C_{1:n},f}$ be the corresponding estimator of $\sigma^2_{C,f}$. Proceeding as above, for serially independent data, the appropriate form of $\tilde{\sigma}^2_{n,b_n,C_{1:n},f}$ is

$$\tilde{\sigma}^2_{n,b_n,C_{1:n},f} = \frac{1}{n}\sum_{i=1}^{n}\left[f\left\{\mathcal{I}_{b_n,C_{1:n}}\left(\hat{\boldsymbol{U}}_i^{1:n}\right) - \bar{\mathcal{I}}_{b_n,C_{1:n}}\right\}\right]^2, \tag{31}$$

where $\bar{\mathcal{I}}_{b_n,C_{1:n}} = n^{-1}\sum_{i=1}^{n}\mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_i^{1:n})$, while, for weakly dependent observations,

$$\begin{aligned}\tilde{\sigma}^2_{n,b_n,C_{1:n},f} = \frac{1}{n}\sum_{i,j=1}^{n}\varphi\left(\frac{i-j}{\ell_n}\right) f\left\{\mathcal{I}_{b_n,C_{1:n}}\left(\hat{\boldsymbol{U}}_i^{1:n}\right) - \bar{\mathcal{I}}_{b_n,C_{1:n}}\right\}\\ \times f\left\{\mathcal{I}_{b_n,C_{1:n}}\left(\hat{\boldsymbol{U}}_j^{1:n}\right) - \bar{\mathcal{I}}_{b_n,C_{1:n}}\right\}.\end{aligned} \tag{32}$$

The following analogue of Proposition 6 is proved in Section D of the supplementary material.

Proposition 7 *If $b_n = o(n^{-1/2})$, Proposition 6 holds with $\check{\sigma}^2_{n,C_{1:n},f}$ replaced with $\tilde{\sigma}^2_{n,b_n,C_{1:n},f}$.*

3.3 Estimation of the bandwidth parameter ℓ_n

When the available observations are weakly dependent, both the approaches based on resampling presented in Sect. 3.1 and the one based on the estimation of the asymptotic null distribution discussed in Sect. 3.2 require the choice of the bandwidth parameter ℓ_n. The latter quantity appears in the definition of the dependent multiplier sequences and, as mentioned in Bücher and Kojadinovic (2014), plays a role somehow analogous to that of the block length in the block bootstrap. The value of ℓ_n is, therefore, expected to have a crucial influence on the finite-sample performance of the two versions of the test based on $S_{n,f}$ described previously.

The aim of this subsection was to propose an estimator of ℓ_n in the spirit of that investigated in Paparoditis and Politis (2001), Politis and White (2004) and Patton et al. (2009), among others, for other resampling schemes. By analogy with (27), we start from the non computable estimator of $\sigma^2_{C,f}$ defined by

$$\sigma^2_{n,C,f} = \mathrm{var}_\xi[f \circ \psi_C\{\bar{\mathbb{B}}_n(0,1,\cdot)\}], \tag{33}$$

where

$$\bar{\mathbb{B}}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_{i,n}\{\mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d,$$

and $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a dependent multiplier sequence. Proceeding as for (27), it is easy to verify that

$$\sigma^2_{n,C,f} = \frac{1}{n}\sum_{i,j=1}^{n} \varphi\left(\frac{i-j}{\ell_n}\right) f\left\{\mathcal{I}_C(\boldsymbol{U}_i) - \psi_C(C)\right\} f\left\{\mathcal{I}_C(\boldsymbol{U}_j) - \psi_C(C)\right\}. \tag{34}$$

Under the conditions of Proposition 6 (ii) and from the fact that the random variables $|f \circ \mathcal{I}_C(\boldsymbol{U}_i)|$ are bounded by $\sup_{\boldsymbol{x}\in[-1,1]^{2^d-1}} |f(\boldsymbol{x})| < \infty$ (since $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}(\boldsymbol{u})| \leq 1$ for all $A \subseteq D$ $|A| \geq 1$), we can proceed as in the proofs of Propositions 5.1 and 5.2 in Bücher and Kojadinovic (2014) (see also Lemmas 3.12 and 3.13 in Bühlmann 1993 and Proposition 2.1 in Shao 2010) to obtain that

$$\mathrm{E}\left(\sigma^2_{n,C,f}\right) - \sigma^2_{C,f} = \frac{\Gamma}{\ell_n^2} + o\left(\ell_n^{-2}\right) \quad \text{and} \quad \mathrm{var}\left(\sigma^2_{n,C,f}\right) = \frac{\ell_n}{n}\Delta + o(\ell_n/n),$$

where $\Gamma = \varphi''(0)/2\sum_{k=-\infty}^{\infty} k^2\tau(k)$ with $\tau(k) = \mathrm{cov}\{f \circ \mathcal{I}_C(\boldsymbol{U}_0), f \circ \mathcal{I}_C(\boldsymbol{U}_k)\}$, and $\Delta = 2\sigma^4_{C,f}\int_{-1}^{1}\varphi(x)^2\mathrm{d}x$. As a consequence, the mean squared error of $\sigma^2_{n,C,f}$ is

$$\mathrm{MSE}\left(\sigma^2_{n,C,f}\right) = \frac{\Gamma^2}{\ell_n^4} + \Delta\frac{\ell_n}{n} + o\left(\ell_n^{-4}\right) + o(\ell_n/n). \tag{35}$$

Differentiating the function $x \mapsto \Gamma^2/x^4 + \Delta x/n$ and equating the derivative to zero, we obtain that the value of ℓ_n that minimizes the mean square error of $\sigma^2_{n,C,f}$ is, asymptotically,

$$\ell_n^{\mathrm{opt}} = \left(\frac{4\Gamma^2}{\Delta}\right)^{1/5} n^{1/5}.$$

To estimate ℓ_n^{opt}, it is necessary to estimate the infinite sum $\sum_{k\in\mathbb{Z}} k^2\tau(k)$ as well as $\sigma^2_{C,f} = \sum_{k\in\mathbb{Z}}\tau(k)$ through a *pilot* estimate. To do so, we adapt the approach described in (Paparoditis and Politis 2001, page 1111) and (Politis and White 2004, Section 3) to the current context (see also Patton et al. 2009). Let $\hat\tau_n(k)$ be the sample autocovariance at lag k computed from the sequence $f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_1^{1:n}), \dots, f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_n^{1:n})$. Then, we estimate Γ and Δ by

$$\hat\Gamma_n = \varphi''(0)/2\sum_{k=-L}^{L}\lambda(k/L)k^2\hat\tau_n(k)$$

and

$$\hat\Delta_n = 2\left\{\sum_{k=-L}^{L}\lambda(k/L)\hat\tau_n(k)\right\}^2\left\{\int_{-1}^{1}\varphi(x)^2\mathrm{d}x\right\},$$

respectively, where $\lambda(x) = [\{2(1-|x|)\} \vee 0] \wedge 1$, $x \in \mathbb{R}$, is the “flat top” (trapezoidal) kernel of Politis and Romano (1995) and L is an integer estimated by adapting the procedure described in (Politis and White 2004, Section 3.2). Let $\hat\varrho_n(k)$ be the sample autocorrelation at lag k estimated from $f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_1^{1:n}), \ldots, f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_n^{1:n})$. The parameter L is then taken as the smallest integer k after which $\hat{\varrho}_n(k)$ appears negligible. The latter is determined automatically by means of the algorithm described in detail in (Politis and White 2004, Section 3.2) . Our implementation is based on Matlab code by A.J. Patton (available on his web page) and its R version by J. Racine and C. Parmeter.

4 Monte Carlo experiments

In the previous section, two ways to compute approximate p values for generic change-point tests based on (9) were studied under the null. These asymptotic results do not, however, guarantee that such tests will behave satisfactorily in finite-samples, which is why additional numerical simulations are needed. In our experiments, we restricted attention to the three statistics given in (6). For each statistic $S_{n,i}$, $i \in \{1, 2, 3\}$, an approximate p value was computed using either the resampling approach based on the processes in (25), or the estimated asymptotic null distribution based on variance estimators of the form (31) or (32). To distinguish between these two situations, we shall talk about *the test* $\tilde{S}_{n,i}$ and *the test* $S^a_{n,i}$, respectively, in the rest of the paper.

The experiments were carried out in the R statistical system using the `copula` package (Hofert et al. 2013). The sequence b_n involved in both classes of tests was taken equal to $n^{-0.51}$. The only (asymptotically negligible) difference with the theoretical developments presented in the previous sections is that the rescaled maximal ranks in (5) were computed by dividing the ranks by $l - k + 2$ instead of $l - k + 1$.

Data generating procedure Two multivariate time series models were used to generate d-dimensional samples of size n in our Monte Carlo experiments: a simple autoregressive model of order one and a GARCH(1,1)-like model. Apart from d, n and the parameters of the models, the other inputs of the procedure are a real $t \in (0, 1)$ determining the location of the possible change-point in the innovations, and two d-dimensional copulas C_1 and C_2. The procedure used to generate a d-dimensional sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ then consists of

1. generating independent random vectors $\boldsymbol{U}_i$, $i \in \{-100, \ldots, 0, \ldots, n\}$ such that $\boldsymbol{U}_i$, $i \in \{-100, \ldots, 0, \ldots, \lfloor nt \rfloor\}$ are i.i.d. from copula C_1 and $\boldsymbol{U}_i$, $i \in \{\lfloor nt \rfloor + 1, \ldots, n\}$ are i.i.d. from copula C_2,
2. computing $\boldsymbol{\epsilon}_i = (\Phi^{-1}(U_{i1}), \ldots, \Phi^{-1}(U_{id}))$, where Φ is the c.d.f. of the standard normal distribution,
3. setting $\boldsymbol{X}_{-100} = \boldsymbol{\epsilon}_{-100}$ and, for any $j \in D$, computing recursively either

$$X_{ij} = \gamma X_{i-1,j} + \epsilon_{ij}, \tag{AR1}$$

or

$$\sigma_{ij}^2 = \omega_j + \beta_j \sigma_{i-1,j}^2 + \alpha_j \epsilon_{i-1,j}^2 \quad \text{and} \quad X_{ij} = \sigma_{ij}\epsilon_{ij}, \tag{GARCH}$$

for $i = -99, \ldots, 0, \ldots, n$.

If the copulas C_1 and C_2 are chosen equal, the above procedure generates samples under H_0 defined in (1). Three possible values were considered for the parameter γ controlling the strength of the serial dependence in (AR1): 0 (serial independence), 0.25 (mild serial dependence), and 0.5 (strong serial dependence). Model (GARCH) was only considered in the bivariate case, and following Bücher and Ruppert (2013), with $(\omega_1, \beta_1, \alpha_1) = (0.012, 0.919, 0.072)$ and $(\omega_2, \beta_2, \alpha_2) = (0.037, 0.868, 0.115)$. The latter values were estimated by Jondeau et al. (2007) from SP500 and DAX daily logreturns, respectively.

Samples under $H_{0,m} \cap (\neg H_{0,c})$, where $H_{0,m}$ and $H_{0,c}$ are defined in (2) and (3), respectively, were obtained by taking $C_1 \neq C_2$ and $t \in \{0.1, 0.25, 0.5\}$. Notice that when $\gamma = 0$ in (AR1), the latter are samples under $H_{0,m} \cap H_{1,c}$, where

$$\begin{aligned} H_{1,c} : &\exists \text{ distinct } C_1 \text{ and } C_2, \text{ and } t \in (0, 1) \text{ such that} \\ &\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{\lfloor nt \rfloor} \text{ have copula } C_1 \text{ and } \boldsymbol{X}_{\lfloor nt \rfloor + 1}, \ldots, \boldsymbol{X}_n \text{ have copula } C_2. \end{aligned}$$

This is not the case anymore when $\gamma > 0$ as the change in cross-sectional dependence is then gradual by (AR1).

Other factors of the experiments Five copula families were considered (the Clayton, the Gumbel–Hougaard, the Normal, the Frank, and the Student), the cross-sectional dimensional d was taken in {2, 4}, and the values 50, 100, 200, 400, and 500 were used for n. To estimate the power of the tests, 1000 samples were generated under each combination of factors and all the tests were carried out at the 5 % significance level.

Computation of the test statistics and of the corresponding p values The data-generating procedure above generates multivariate time series whose component series do not contain ties with probability one. Consequently, as explained in Sect. 2.2, $S_{n,2}$ is merely $S_{n,1}$ computed from the sample $-\boldsymbol{X}_1, \ldots, -\boldsymbol{X}_n$. Furthermore, if $d = 2$, it is easy to see that $S_{n,1} = S_{n,2} = S_{n,3}$. However, it can be verified that only the approximate p values for the tests $\tilde{S}_{n,1}$ and $\tilde{S}_{n,3}$ (resp. $S^a_{n,1}$ and $S^a_{n,3}$) will be equal. Indeed, the multiplier replicates based on the processes in (25) (resp. the variance estimators of the form (31) or (32)) computed from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ do not coincide in general with those computed from $-\boldsymbol{X}_1, \ldots, -\boldsymbol{X}_n$, even in dimension two.

From Proposition 7, we see that, to compute an asymptotic p value for the tests $S^a_{n,i}$, it is necessary to be able to compute the c.d.f. of the random variable $\sup_{s \in [0,1]} |\mathbb{U}(s)|$. The distribution of the latter random variable is known as the Kolmogorov distribution. As classically done in other contexts, we approach this distribution by that of the statistic of the classical Kolmogorov–Smirnov goodness-of-fit test for a simple hypothesis. Specifically, we use the function `pkolmogorov1x` given in the code of the R function `ks.test`.

Empirical levels and power of the tests based on i.i.d. multipliers/a variance estimator of the form (31) Table 1 gives the empirical levels of the tests when the observations are serially independent. For the sake of brevity, the results are reported only for two copula families. Overall, we find that the tests $\tilde{S}_{n,i}$ with multiplier sequences satisfying (M0) (here standard normal sequences) hold there level rather well both for $d = 2$ and $d = 4$, and all the considered degrees of cross-sectional dependence. This is not the case for the tests $S^a_{n,i}$ which frequently appear way too liberal when the cross-sectional dependence is high.

Table 1 Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200, 400\}$ generated with $\gamma = 0$ in (AR1) and when $C_1 = C_2 = C$ is either the d-dimensional Clayton (Cl) or Gumbel–Hougaard (GH) copula the bivariate margins of which have a Kendall's tau of τ

C	n	τ	$d = 2$				$d = 4$					
			$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	$S^a_{n,1}$	$S^a_{n,2}$	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	$\tilde{S}_{n,3}$	$S^a_{n,1}$	$S^a_{n,2}$	$S^a_{n,3}$
Cl	50	0.1	6.8	7.4	2.6	3.0	4.6	5.1	4.0	1.2	2.1	0.7
		0.3	4.1	5.2	1.7	4.2	4.9	5.4	3.7	0.5	2.6	0.7
		0.5	3.1	2.7	2.5	8.6	7.1	3.9	4.9	2.8	2.8	1.2
		0.7	3.0	0.5	8.3	23.8	7.4	4.1	3.3	5.4	10.3	3.1
	100	0.1	3.5	4.3	2.3	2.7	4.1	5.3	4.4	1.6	3.4	2.5
		0.3	4.0	4.4	2.3	3.6	5.7	4.7	4.4	2.0	2.8	1.4
		0.5	4.2	4.0	4.9	8.3	4.3	4.0	3.5	2.2	3.7	1.9
		0.7	5.7	1.6	12.6	23.1	9.1	3.9	7.6	11.3	9.5	7.4
	200	0.1	4.9	4.7	2.8	3.1	6.1	5.1	5.2	3.1	3.4	3.3
		0.3	4.9	5.3	3.7	4.9	4.1	5.6	4.2	2.3	3.6	1.9
		0.5	4.6	4.3	4.8	6.9	4.6	5.5	4.2	4.1	4.8	3.2
		0.7	5.6	3.1	11.2	15.1	10.5	5.3	11.1	14.1	8.3	9.9
	400	0.1	4.6	4.9	3.7	3.8	6.3	6.7	6.5	4.5	5.5	4.8
		0.3	4.3	4.6	4.0	4.4	5.8	5.3	5.5	4.1	4.2	3.8
		0.5	4.8	4.6	4.2	4.8	5.8	4.5	5.5	5.5	4.0	4.7
		0.7	5.9	4.0	9.3	10.8	8.5	6.6	8.7	13.5	8.1	8.2
GH	50	0.1	6.7	6.3	3.4	2.3	5.8	5.3	4.7	2.4	0.8	2.5
		0.3	4.1	3.9	3.5	2.1	5.9	6.0	5.3	1.8	0.7	3.1
		0.5	3.1	3.4	6.9	3.4	4.6	4.9	4.0	3.0	2.5	6.5
		0.7	2.0	1.8	15.5	10.7	3.4	6.2	2.0	6.2	4.2	10.3
	100	0.1	5.2	5.1	2.7	2.5	4.3	4.8	4.1	2.5	1.5	2.1
		0.3	5.9	5.3	5.2	3.9	6.1	6.7	6.7	3.1	1.9	4.5
		0.5	3.7	3.7	6.6	5.1	5.3	4.8	5.3	3.6	3.4	6.4
		0.7	1.3	2.3	16.9	13.8	4.5	7.0	2.7	8.6	9.0	14.2
	200	0.1	5.2	5.2	3.8	3.5	4.8	4.3	4.5	3.3	2.6	3.1
		0.3	5.2	5.1	4.7	3.9	6.0	6.5	5.3	4.7	3.3	4.3
		0.5	4.5	4.5	5.2	4.7	4.2	3.9	4.0	3.2	3.6	3.9
		0.7	2.2	3.7	12.8	10.8	4.6	7.0	4.9	6.6	9.0	10.9
	400	0.1	6.4	6.1	4.8	4.7	5.1	5.7	4.3	4.0	3.1	3.1
		0.3	4.7	4.6	4.1	3.8	4.6	5.3	5.6	3.7	3.6	4.4
		0.5	3.3	3.3	3.5	3.0	4.3	5.1	4.5	3.9	4.5	4.7
		0.7	4.6	5.8	10.1	9.9	5.3	7.1	5.9	6.3	9.5	10.4

The tests $\tilde{S}_{n,i}$ are carried out with i.i.d. multiplier sequences, while the tests $S^a_{n,i}$ use variance estimators of the form (31)

Table 2 partially reports the percentages of rejection of the i.i.d. multiplier tests for serially independent observations generated under $H_{0,m} \cap H_{1,c}$ resulting from a change of the copula parameter within a copula family. The columns CvM give the results of

Table 2 Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200\}$ generated with $\gamma = 0$ in (AR1), $t \in \{0.1, 0.25, 0.5\}$ and when C_1 and C_2 are both d-dimensional normal (N) or Frank (F) copulas such that the bivariate margins of C_1 have a Kendall's tau of 0.2 and those of C_2 a Kendall's tau of τ

C	n	τ	t	$d = 2$			$d = 4$			
				CvM	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	CvM	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	$\tilde{S}_{n,3}$
N	50	0.4	0.10	5.6	6.0	5.6	5.9	7.9	7.9	8.3
			0.25	9.1	8.7	8.9	12.2	17.3	18.9	19.5
			0.50	13.4	12.6	12.6	24.3	25.1	27.6	28.2
		0.6	0.10	9.0	8.7	8.9	7.1	20.7	21.7	22.4
			0.25	32.3	34.7	32.6	45.6	66.3	67.0	69.9
			0.50	46.7	42.7	41.6	76.1	78.0	77.5	80.8
	100	0.4	0.10	5.7	7.8	7.6	7.6	11.2	12.2	12.3
			0.25	14.9	19.7	19.1	27.0	35.3	37.2	43.0
			0.50	25.9	28.9	29.2	54.5	54.6	53.5	59.6
		0.6	0.10	14.6	22.7	23.4	26.1	47.5	51.1	58.8
			0.25	60.0	68.6	69.0	90.3	94.9	94.8	97.6
			0.50	81.9	84.8	84.2	98.8	98.4	99.0	99.5
	200	0.4	0.10	9.1	11.7	12.3	13.2	18.2	17.9	23.3
			0.25	26.5	36.7	36.9	58.9	64.9	67.1	75.5
			0.50	47.7	54.2	53.7	83.4	83.5	83.3	88.9
		0.6	0.10	34.5	57.7	58.0	63.1	87.3	87.8	93.8
			0.25	92.6	96.5	96.7	100.0	100.0	100.0	100.0
			0.50	99.1	99.5	99.5	100.0	100.0	100.0	100.0
F	50	0.4	0.10	6.9	5.7	6.2	4.5	7.8	9.0	8.4
			0.25	10.8	9.7	10.0	12.9	17.9	19.7	19.9
			0.50	15.1	13.6	13.6	24.7	30.2	31.1	29.1
		0.6	0.10	11.1	10.6	11.3	7.3	23.3	29.7	24.8
			0.25	33.1	32.7	31.9	42.3	67.2	70.2	69.5
			0.50	50.9	46.1	46.2	78.3	81.9	82.3	85.5
	100	0.4	0.10	6.1	7.0	7.4	6.5	9.2	13.6	11.9
			0.25	16.5	18.2	18.7	26.5	38.8	46.8	49.6
			0.50	26.4	28.6	28.3	48.9	52.7	58.3	61.6
		0.6	0.10	17.7	27.3	27.2	22.7	55.3	63.9	68.6
			0.25	66.5	73.6	74.0	91.9	97.7	98.2	99.5
			0.50	86.2	87.3	87.5	99.3	98.8	99.4	99.8
	200	0.4	0.10	10.2	15.7	15.6	12.5	19.7	25.3	27.1
			0.25	34.3	41.3	41.5	53.6	64.4	76.2	78.8
			0.50	50.7	54.3	54.4	83.2	83.9	90.4	93.2
		0.6	0.10	39.0	64.7	65.6	60.3	88.0	92.2	96.4
			0.25	95.4	98.3	98.3	99.9	100.0	100.0	100.0
			0.50	99.5	99.8	99.8	100.0	100.0	100.0	100.0

The columns CvM give the results for the test studied in Bücher et al. (2014). All the tests were carried out with i.i.d. multiplier sequences the i.i.d. multiplier test based on the maximally selected Cramér–von Mises statistic studied in Bücher et al. (2014) (with multiplier replicates of the form (4.6) in the latter reference) and implemented in the R package `npcp`. Overall, we find that the tests $\tilde{S}_{n,i}$ are more powerful than that studied in Bücher et al. (2014) for such scenarios, especially when the change in the copula occurs early or late. Among the tests $\tilde{S}_{n,i}$, we observed that the test $\tilde{S}_{n,3}$ (which coincides with the test $\tilde{S}_{n,1}$ in dimension two) led frequently to slightly higher rejection rates, although this conclusion is based on a limited number of simulation scenarios. The rejection rates of the tests $S^a_{n,i}$ with a variance estimator of the form (31) are not reported for the sake of brevity. They were found to be slightly less powerful than the tests $\tilde{S}_{n,i}$ when $\tau = 0.4$. For $\tau = 0.6$, a comparison of the two classes of tests is not necessarily meaningful as the tests $S^a_{n,i}$ were often found to be way too liberal under strong cross-sectional dependence.

Empirical levels and power of the tests based on dependent multipliers/a variance estimator of the form (32) Part of Table 3 reports the empirical levels of the test $\tilde{S}_{n,1}$ when dependent multiplier sequences satisfying (M1)–(M3) are used. These sequences were generated using the "moving average approach" proposed initially in Bühlmann (1993, Section 6.2) and revisited in Bücher and Kojadinovic (2014, Section 5.2). A standard normal sequence was used for the required initial i.i.d. sequence. The kernel function κ in that approach was chosen to be the Parzen kernel defined by $\kappa_P(x) = (1-6x^2+6|x|^3)\mathbf{1}(|x| \leq 1/2)+2(1-|x|)^3\mathbf{1}(1/2 < |x| \leq 1), x \in \mathbb{R}$, which amounts to choosing the function φ in (M3) as $x \mapsto (\kappa_P \star \kappa_P)(2x)/(\kappa_P \star \kappa_P)(0)$, where '$\star$' denotes the convolution operator. The value of the bandwidth parameter ℓ_n defined in (M2) was estimated using the data-driven procedure described in Sect. 3.3. The same value of ℓ_n was used to carry out the test $S^a_{n,1}$ relying on a variance estimator of the form (32).

From the first three vertical blocks of Table 3, we see that an increase in the degree of serial dependence in (AR1) (controlled by γ) appears to result in a small inflation of the empirical levels of the test $\tilde{S}_{n,1}$. As expected, the situation improves as n increases from 100 to 400. For sequences generated using (GARCH), the empirical levels of the test $\tilde{S}_{n,1}$ appear always reasonably close to the 5 % nominal level. The test $S^a_{n,1}$ remains overall way too liberal when the cross-sectional dependence is high.

The last vertical block of Table 3 reports, for strongly serially dependent observations generated using (AR1), the empirical levels of the test $\tilde{S}_{n,1}$ based on i.i.d. multipliers, as well as those of the test $S^a_{n,1}$ based on an inappropriate variance estimator of the form (31). As expected, both tests strongly fail to hold their level.

Table 4 partially reports the rejection percentages of the tests based on dependent multipliers / a variance estimator of the form (32) for observations generated under $H_{0,m} \cap (\neg H_{0,c})$ resulting from a change of the copula parameter within a copula family. The rejection rates of the test $S^a_{n,1}$ should be considered with care when $\tau = 0.6$ as that test was found to be way too liberal under strong cross-sectional dependence. Despite that issue, the test $\tilde{S}_{n,1}$ appears almost always more powerful than the test $S^a_{n,1}$. Also, as it could have been expected, the presence of strong serial dependence ($\gamma = 0.5$) leads to lower rejection percentages when compared with serial independence ($\gamma = 0$). Finally, comparing the results for the test $\tilde{S}_{n,1}$ when $\gamma = 0$ with the analogue results

Table 3 Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200, 400\}$ when $C_1 = C_2 = C$ is either the bivariate Clayton (Cl), Gumbel–Hougaard (GH) or Frank (F) copula with a Kendall's tau of τ

C	n	τ	$\gamma = 0$		$\gamma = 0.25$		$\gamma = 0.5$		GARCH		$\gamma = 0.5$/ind	
			$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$
Cl	100	0.10	5.2	2.3	6.6	3.5	8.2	3.3	6.2	2.5	14.5	10.2
		0.30	3.5	1.8	6.7	3.1	7.1	4.7	5.2	3.3	15.0	11.6
		0.50	4.0	3.4	5.0	4.5	5.2	4.7	4.6	4.5	12.0	13.5
		0.70	8.3	12.0	7.5	11.8	7.2	11.2	7.2	13.2	8.9	20.0
	200	0.10	4.2	2.3	5.1	2.8	6.9	3.6	5.0	3.1	17.2	13.5
		0.30	5.1	2.6	6.2	3.4	7.2	4.4	5.3	3.8	15.7	13.0
		0.50	4.4	4.1	5.0	5.1	4.6	5.1	4.5	4.5	14.1	14.2
		0.70	6.5	12.2	6.6	9.8	7.4	11.2	6.5	10.8	12.4	20.0
	400	0.10	4.7	3.3	5.6	4.3	6.0	3.5	5.3	3.8	19.4	16.9
		0.30	4.4	3.4	6.3	4.3	6.0	4.2	4.0	3.5	17.3	15.2
		0.50	4.7	4.7	5.9	5.7	5.6	5.0	6.1	5.7	14.6	14.2
		0.70	6.4	8.7	5.7	7.9	5.1	6.8	6.6	9.5	15.7	19.0
GH	100	0.10	4.8	2.5	5.1	2.0	7.7	2.7	5.6	2.8	15.3	11.2
		0.30	5.0	3.7	5.9	4.4	7.5	4.5	4.9	2.9	15.0	14.2
		0.50	4.5	6.7	4.3	7.1	6.3	7.9	4.9	6.9	10.7	15.7
		0.70	3.5	16.0	4.3	18.9	5.1	18.9	3.7	16.2	4.5	25.4
	200	0.10	6.4	3.9	5.6	3.7	7.3	3.9	5.8	3.8	18.2	14.1
		0.30	6.0	5.1	6.4	4.6	6.7	4.6	5.4	4.5	19.1	16.4
		0.50	5.1	4.9	6.0	6.4	6.9	8.0	3.7	4.9	15.6	17.2
		0.70	3.8	14.4	2.8	13.0	4.4	12.4	3.5	12.2	10.0	25.4
	400	0.10	5.0	4.0	5.8	4.8	6.3	5.1	5.2	3.9	18.5	16.3
		0.30	4.1	3.0	5.1	4.3	6.3	4.6	4.9	4.1	18.5	17.2
		0.50	3.2	3.6	5.0	6.3	7.9	7.5	4.9	4.7	16.7	17.2
		0.70	5.2	9.8	3.8	8.7	5.4	10.6	3.8	8.2	14.5	22.4
F	100	0.10	5.5	2.1	5.3	2.3	10.6	4.2	5.0	2.4	15.2	10.2
		0.30	4.4	2.2	5.9	3.9	7.7	4.1	6.4	4.7	13.3	10.3
		0.50	4.0	7.6	4.0	6.0	5.4	7.1	4.2	6.7	12.8	18.0
		0.70	5.2	29.3	4.8	26.5	5.4	18.1	5.4	23.9	5.9	28.5
	200	0.10	4.0	2.1	6.0	3.9	8.3	4.5	5.1	2.9	17.5	13.4
		0.30	5.0	3.9	5.7	4.1	7.1	3.9	5.3	3.4	17.0	14.5
		0.50	4.8	6.2	4.5	5.7	6.9	7.1	4.4	5.6	15.0	17.3
		0.70	3.2	19.9	4.0	17.5	4.6	13.4	4.9	20.1	8.9	25.1
	400	0.10	4.1	3.1	6.0	4.4	6.0	4.0	4.5	3.0	18.0	14.8
		0.30	5.5	4.6	6.7	5.6	5.9	4.2	5.2	4.3	14.7	12.5
		0.50	4.6	4.7	4.7	5.0	4.0	3.8	4.8	5.5	15.7	16.5
		0.70	5.3	13.2	4.5	12.3	6.2	9.9	5.7	13.2	14.2	21.7

In the first four vertical blocks of the table, the test $\tilde{S}_{n,1}$ (resp. $S^a_{n,1}$) is carried out using dependent multiplier sequences (resp. a variance estimator of the form (32)). In the last vertical block, i.i.d. multipliers and a variance estimator of the form (31) are used instead

Table 4 Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated with $t \in \{0.1, 0.25, 0.5\}$ and when C_1 and C_2 are both bivariate Clayton (Cl), Gumbel–Hougaard (GH) or normal (N) copulas with a Kendall's tau of 0.2 for C_1 and a Kendall's tau of τ for C_2

C	n	τ	t	$\gamma = 0$			$\gamma = 0.5$			GARCH		
				CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$	CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$	CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$
Cl	100	0.4	0.10	6.5	6.5	4.3	6.5	8.0	5.0	6.6	6.7	3.8
			0.25	17.9	20.4	13.4	14.0	19.7	10.6	17.2	18.1	11.2
			0.50	23.5	23.2	15.0	18.3	22.4	9.7	28.6	27.6	17.1
		0.6	0.10	12.6	20.6	19.7	9.4	17.1	17.0	13.9	20.1	19.4
			0.25	61.3	65.7	52.7	44.2	53.6	36.4	61.1	64.8	50.7
			0.50	80.0	78.8	61.1	58.4	61.8	34.9	80.3	78.3	59.3
	200	0.4	0.10	8.2	9.6	7.5	6.9	10.4	7.0	8.3	11.1	8.9
			0.25	26.5	31.8	25.2	19.9	27.7	20.2	27.8	32.0	26.2
			0.50	45.3	47.0	37.0	34.2	40.0	27.9	47.1	48.8	40.1
		0.6	0.10	30.4	42.1	42.3	12.6	28.8	28.6	29.7	43.9	43.4
			0.25	93.2	94.2	87.4	71.1	79.2	65.9	91.1	92.2	83.5
			0.50	98.5	98.3	94.1	89.5	90.5	80.1	98.7	98.2	94.1
GH	100	0.4	0.10	5.3	8.0	7.1	5.0	8.2	7.1	6.3	7.6	6.9
			0.25	12.4	17.1	12.1	11.6	18.6	11.1	14.9	18.6	14.9
			0.50	22.5	25.2	16.9	18.2	24.2	14.0	26.0	27.7	19.9
		0.6	0.10	10.4	18.5	26.1	7.7	19.4	25.7	10.2	19.9	26.6
			0.25	53.3	63.1	54.7	41.2	58.0	43.7	55.0	63.8	52.4
			0.50	78.1	80.4	67.4	62.7	69.5	46.1	76.0	76.3	63.1
	200	0.4	0.10	7.0	10.5	10.0	7.1	11.4	9.9	6.9	10.2	9.0
			0.25	25.2	31.9	27.7	19.1	30.9	22.8	24.6	32.3	26.7
			0.50	43.0	48.3	42.1	31.4	39.3	30.0	43.2	49.1	41.3
		0.6	0.10	25.9	42.7	47.2	13.0	30.1	34.0	23.5	43.4	46.3
			0.25	89.0	92.9	86.3	72.1	83.5	70.0	88.9	94.5	85.0
			0.50	98.3	98.5	95.9	89.6	92.0	83.4	98.4	98.7	93.6
N	100	0.4	0.10	6.1	7.8	6.2	6.9	10.2	7.8	6.1	7.0	5.5
			0.25	14.4	19.3	14.7	13.7	19.2	13.2	14.7	17.8	13.3
			0.50	25.6	27.7	19.4	17.5	24.1	12.5	25.2	28.7	19.2
		0.6	0.10	10.6	27.1	32.0	8.2	19.7	23.7	10.2	19.3	24.7
			0.25	61.5	70.1	61.3	46.0	62.3	44.8	58.4	69.2	59.3
			0.50	82.6	85.1	72.3	64.9	71.3	44.9	79.0	82.0	65.7
	200	0.4	0.10	8.0	10.8	9.2	5.9	12.6	9.2	7.0	9.3	8.9
			0.25	27.7	37.4	33.2	20.4	31.0	24.7	26.8	35.1	30.7
			0.50	47.0	51.5	43.6	33.2	41.7	30.7	43.0	49.5	41.3
		0.6	0.10	27.1	47.3	49.6	14.5	35.6	39.2	28.8	48.3	51.8
			0.25	91.5	96.5	88.4	72.3	85.2	71.0	90.7	96.1	85.7
			0.50	98.8	99.7	96.3	91.7	95.5	83.6	99.1	99.3	94.8

The columns CvM give the results for the test studied in Bücher et al. (2014). The latter test and the test $\tilde{S}_{n,1}$ (resp. the test $S^a_{n,1}$) are (resp. is) carried out using dependent multiplier sequences (resp. a variance estimator of the form (32))

Table 5 Percentage of rejection of H_0 computed from 1000 samples of size $n = 500$ generated with $\gamma = 0$ in (AR1) and when C_1 and C_2 are both either bivariate Student copulas with 1 d.f. (t_1), with 3 d.f. (t_3) or with 5 d.f. (t_5) with a Spearman's rho of 0.4 for C_1 and a Spearman's rho of ρ for C_2

ρ	t_1			t_3			t_5		
	W	$\tilde{S}_{n,1}$	$S^a_{n,1}$	W	$\tilde{S}_{n,1}$	$S^a_{n,1}$	W	$\tilde{S}_{n,1}$	$S^a_{n,1}$
0.4	4.5	3.9	2.8	4.5	5.2	4.0	4.7	6.3	4.4
0.6	8.1	43.3	38.7	8.5	57.9	54.3	8.5	66.5	63.8
0.8	20.5	99.4	98.6	21.7	100.0	99.9	21.5	100.0	100.0
0.2	7.9	33.7	29.2	8.8	51.0	46.6	8.9	52.9	48.4
0.0	19.9	87.7	84.7	23.0	95.7	94.9	24.0	97.2	96.3
−0.2	41.8	99.7	99.6	49.5	100.0	100.0	51.5	100.0	100.0
−0.4	70.2	100.0	100.0	78.6	100.0	100.0	80.4	100.0	99.9
−0.6	91.7	100.0	99.9	95.8	100.0	100.0	96.6	100.0	100.0

The test $\tilde{S}_{n,1}$ was carried out with dependent multiplier sequences, while the test $S^a_{n,1}$ used a variance estimator of the form (32). The columns W contain the rejection rates of the similar test studied in Wied et al. (2014). The results are taken from Table 1 in the latter reference

reported in Table 2 reveals that, rather naturally, the use of dependent multipliers in the case of serially independent observations results in a small loss of power.

We end this section by a comparison of the tests $\tilde{S}_{n,1}$ and $S^a_{n,1}$ with the similar test studied in Wied et al. (2014). To do so, we reproduced one of the experiments carried out in the latter reference. The results are reported in Table 5 and confirm that tests for change-point detection based on (4) are potentially substantially more powerful than tests based on (10).

5 Practical recommendations and illustration

Based on the experiments partially reported in the previous section, we recommend, among the tests $\tilde{S}_{n,i}$ and $S^a_{n,i}$, the tests $\tilde{S}_{n,i}$. Indeed, the tests $S^a_{n,i}$ did not hold their level well in the case of strong cross-sectional dependence. Furthermore, because of their form, the tests $S^a_{n,i}$ might suffer from some of the practical issues described in Shao and Zhang (2010), and, in future research, it might be of interest to study a *self-normalization* version of these as advocated in the latter reference.

The pros and cons of the tests $\tilde{S}_{n,i}$ compared with the test studied in Bücher et al. (2014) are as follows: the tests $\tilde{S}_{n,i}$ seem more powerful for alternatives involving a change in Spearman's rho at constant margins; they are also substantially faster to compute. Their main weakness is that, by construction, they have no power against alternatives involving a change in the copula at a constant value of Spearman's rho and constant margins.

Among the tests $\tilde{S}_{n,i}$, we recommend the test $\tilde{S}_{n,3}$ merely because of its slightly better finite-sample behavior in our simulations.

We end this section by a brief illustration of the studied tests on real financial observations. Specifically, we consider a trivariate version of the data analyzed in Dehling et al. (2014, Section 7). The observations consist of $n = 990$ daily logreturns computed from the DAX, the CAC 40 and the Standard and Poor 500 indices for the years 2006–2009. An approximate p value of 0.045 was obtained for the test $\tilde{S}_{n,3}$ with dependent multipliers, providing some evidence against H_0. It is, however, important to bear in mind that it is only under the assumption that $H_{0,m}$ in (2) holds that it would be fully justified to decide to reject $H_{0,c}$ in (3).

6 Conclusion

Tests for change-point detection based on the generic statistic $S_{n,f}$ defined in (9) were first studied theoretically. These tests, designed to be particularly sensitive to changes in the cross-sectional dependence of multivariate time series, can be carried out using either resampling based on multipliers, or by estimating the asymptotic null distribution of $S_{n,f}$. Both approaches were shown to be asymptotically valid under strong mixing and suitable conditions on the underlying function f. In addition, a procedure for estimating a key bandwidth parameter involved in both techniques for computing p values was suggested, making the tests fully data-driven. Next, their finite-sample behavior was investigated by means of extensive simulations for three particular choices of the function f resulting in the test statistics defined in (6) measuring changes in the cross-sectional dependence in terms of multivariate extensions of Spearman's rho. Practical recommendations and an illustration were finally given.

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