

Summary statistics for inhomogeneous marked point processes

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Abstract We propose new summary statistics for intensity-reweighted moment stationary marked point processes with particular emphasis on discrete marks. The new statistics are based on the n -point correlation functions and reduce to cross J - and D -functions when stationarity holds. We explore the relationships between the various functions and discuss their explicit forms under specific model assumptions. We derive ratio-unbiased minus sampling estimators for our statistics and illustrate their use on a data set of wildfires.

Keywords Generating functional · Intensity-reweighted moment stationarity · J -function · Marked point process · Multivariate point process · Nearest neighbour distance distribution function · n -point correlation function · Reduced Palm measure

1 Introduction

The analysis of a marked point pattern typically begins with computing some summary statistics which may be used to find specific structures in the data and suggest suitable models ([Chiu et al. 2013](#); [Daley and Vere-Jones 2003, 2008](#); [Gelfand et al. 2010](#); [Illian](#)

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et al. 2008; van Lieshout 2000). The choice of summary characteristic depends both on the pattern at hand and on the feature or hypothesis of interest. Indeed, under the working assumption of stationarity, for discrete marks, cross versions of the K - or nearest neighbour distance distribution function may be appropriate (Diggle 2014); for real-valued marks, the mark correlation functions of Penttinen and Stoyan (1989) are widely used. Various types of J -functions (van Lieshout 2006; van Lieshout and Baddeley 1999) offer useful alternatives.

Often, however, the assumption of homogeneity cannot be justified. In the unmarked case, Baddeley et al. (2000) proposed an inhomogeneous extension of the K -function for so-called second-order intensity-reweighted stationary point processes. Their ideas were extended to spatio-temporal point processes in Gabriel and Diggle (2009), Møller and Ghorbani (2012), whereas Cronie and van Lieshout (2015), van Lieshout (2011) extended the J -function under the somewhat stronger assumption of intensity-reweighted moment stationarity in space and time.

For non-stationary multivariate point processes, Møller and Waagepetersen (2004) proposed an extension of the K -function under the assumption of second-order intensity-reweighted stationarity. As we will indicate in this paper, this structure may be extended to K -functions for general marked point processes.

Regarding J -functions, in van Lieshout (2011), the author noted that the ideas in that paper could be combined with those in van Lieshout (2006) to define inhomogeneous J -functions with respect to mark sets. In this paper we do so, and, as a by-product, obtain a generalisation of the cross nearest neighbour distance distribution function.

The paper is structured as follows. In Sect. 2, we define marked point processes with locations in Euclidean spaces and give the necessary preliminaries. In Sects. 3.1 and 3.2, we define, respectively, cross D - and J -functions for inhomogeneous multivariate point processes and propose generalisations to point processes with real-valued marks. We show that D and J can be expressed in terms of the generating functional and discuss the relationships between these statistics and the cross K -function. In Sect. 4, we investigate the form of our statistics under various independence and marking assumptions. We derive minus sampling estimators and discuss tests in Sect. 5, which are applied to a data set on wildfires in New Brunswick, Canada, in Sect. 6. We finish the paper with a summary.

2 Definitions and notations

Throughout this paper, we consider marked point processes Y in the sense of Definition 6.4.1 in Daley and Vere-Jones (2003), with points in \mathbb{R}^d equipped with the Euclidean metric and Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$. We write ℓ for the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$. By definition, the ground process Z obtained from Y by ignoring the marks is a well-defined point process on \mathbb{R}^d in its own right. We shall assume that Z is simple, that is, almost surely does not contain multiple points.

We assume that the mark space \mathcal{M} is Polish and equipped with a finite reference measure ν on the Borel σ -algebra $\mathcal{B}(\mathcal{M})$. We denote by $\mathcal{B}(\mathbb{R}^d \times \mathcal{M})$ the Borel σ -algebra on the product space $\mathbb{R}^d \times \mathcal{M}$. In the special case that \mathcal{M} is finite, Y can be seen as a multivariate point process (Y_1, \dots, Y_k) , where Y_i contains the points marked $i \in \mathcal{M} = \{1, \dots, k\}$.

2.1 Product densities

Recall that the intensity measure of a marked point process is defined on product sets $A = B \times C \in \mathcal{B}(\mathbb{R}^d \times \mathcal{M})$ by

$$\Lambda(A) = \mathbb{E}Y(A) = \mathbb{E}Y(B \times C),$$

the expected number of points in B with marks in C . If Λ is locally finite as a set-function, it can be extended to a measure on $\mathcal{B}(\mathbb{R}^d \times \mathcal{M})$ (see e.g. Theorem A, p. 54, in [Halmos \(1974\)](#)). In this paper, additionally, we assume that Λ admits a density λ with respect to $\ell \times \nu$, which is referred to as the intensity function. In particular, for a finite mark space, $\lambda(z, i)\nu(i) = \lambda_i(z)$ is the intensity function of Y_i .

Since for fixed C , the measure $\Lambda(\cdot \times C)$ is absolutely continuous with respect to the intensity measure Λ_g of the ground process,

$$\Lambda(B \times C) = \int_B M^z(C) \Lambda_g(dz). \tag{1}$$

Here $M^z(C)$ is the probability that the mark of a point at location z falls in C . The members of the family $\{M^z : z \in \mathbb{R}^d\}$ of probability distributions on the Borel sets of \mathcal{M} are called *mark distributions*.

If Y is stationary, that is, if its distribution is invariant under translations of the locations, $\Lambda(B \times C) = \lambda\nu_M(C)\ell(B)$ for some probability distribution ν_M on \mathcal{M} , which is known as the *mark distribution*. In this case, we may take $\nu = \nu_M$ for the reference measure on \mathcal{M} so that Λ has constant intensity function λ with respect to $\ell \times \nu_M$, and, moreover, λ is the intensity of the ground process.

Higher order ‘intensity functions’ or *product densities* can be defined as densities $\rho^{(n)}$ of the factorial moment measures provided these exist, in which case they satisfy the following *n*th order *Campbell formula*. For any measurable function $f \geq 0$, the sum of f over n -tuples of different points of Y is a random variable with expectation

$$\begin{aligned} & \mathbb{E} \left[\sum_{(z_1, m_1), \dots, (z_n, m_n) \in Y}^{\neq} f((z_1, m_1), \dots, (z_n, m_n)) \right] \\ &= \int \cdots \int f((z_1, m_1), \dots, (z_n, m_n)) \rho^{(n)}((z_1, m_1), \dots, (z_n, m_n)) \prod_{i=1}^n dz_i d\nu(m_i) \end{aligned} \tag{2}$$

(with the left hand side being infinite if and only if the right hand side is infinite). Note that $\rho^{(1)} = \lambda$, the intensity function. Also, n -point mark distributions $M^{z_1, \dots, z_n}(C_1 \times \cdots \times C_n)$ can be defined analogously to the case $n = 1$. For further details, see for example the textbook [Chiu et al. \(2013\)](#). Note that, by the absolute continuity underlying the existence of $\rho^{(n)}$, there exist product densities $\rho_g^{(n)}(z_1, \dots, z_n)$ for the ground process and densities f_{z_1, \dots, z_n} of M^{z_1, \dots, z_n} with respect to the n -fold product of ν with itself such that

$$M^{z_1, \dots, z_n}(C_1 \times \dots \times C_n) = \int_{C_1} \dots \int_{C_n} f_{z_1, \dots, z_n}(m_1, \dots, m_n) \prod_{i=1}^n d\nu(m_i).$$

In particular, the intensity function of the ground process is given by $\lambda_g(z) = \rho_g^{(1)}(z)$ and $\lambda(z, m) = f_z(m)\lambda_g(z)$.

We will also need the related concept of *n-point correlation functions* ξ_n , $n \geq 1$, the intensity-reweighted densities of the factorial cumulant measures discussed in Section 9.5 of Daley and Vere-Jones (2008). These permutation invariant measurable functions are defined by the following recursive relation (see e.g. van Lieshout (2006), White (1979)). Set $\xi_1 \equiv 1$ and, for $n \geq 2$,

$$\sum_{k=1}^n \sum_{E_1, \dots, E_k} \prod_{j=1}^k \xi_{|E_j|}(\{(z_i, m_i) : i \in E_j\}) = \frac{\rho^{(n)}((z_1, m_1), \dots, (z_n, m_n))}{\lambda(z_1, m_1) \dots \lambda(z_n, m_n)}, \tag{3}$$

where \sum_{E_1, \dots, E_k} is a sum over all possible k -sized partitions $\{E_1, \dots, E_k\}$, $E_j \neq \emptyset$, of the set $\{1, \dots, n\}$ and $|E_j|$ denotes the cardinality of E_j . Note that for a Poisson process, $\xi_n \equiv 0$ for all $n \geq 2$.

2.2 Palm measures

Let Y be a simple marked point process whose intensity function exists. The summary statistics in this paper are defined in terms of *reduced Palm measures* satisfying the *reduced Campbell–Mecke* formula which states that, for any measurable function $f \geq 0$,

$$\mathbb{E} \left[\sum_{(z,m) \in Y} f((z, m), Y \setminus \{(z, m)\}) \right] = \int_{\mathbb{R}^d} \int_{\mathcal{M}} \mathbb{E}^{!(z,m)} [f((z, m), Y)] \lambda(z, m) dz d\nu(m) \tag{4}$$

(with the left hand side being infinite if and only if the right hand side is infinite). The probability measure $P^{!(z,m)}$ corresponding to $\mathbb{E}^{!(z,m)}$ can be interpreted as the conditional probability of $Y \setminus \{(z, m)\}$ given that $Y(\{(z, m)\}) = 1$. For further details, see Daley and Vere-Jones (2008).

A few remarks are in order. First, consider the special case that Y is stationary and the reference measure on \mathcal{M} is the mark distribution ν_M . In this case, it is possible to define reduced Palm measures with respect to arbitrary mark sets. Specifically, for $C \in \mathcal{B}(\mathcal{M})$ such that $\nu(C) = \nu_M(C) > 0$, set

$$P_C^{!z}(R) = \frac{1}{\nu(C)} \int_C P^{!(z,m)}(R) d\nu(m). \tag{5}$$

Then, $P_C^{!z}$ does not depend on the choice of $z \in \mathbb{R}^d$ and is a probability measure (Section 4.4.8 in Chiu et al. (2013)). It can be interpreted as the conditional distribution of Y on the complement of $\{z\} \times \mathcal{M}$, given that Y places a point at z with mark in C .

As a second example, consider multivariate point processes (Y_1, \dots, Y_k) and let ν be any finite measure on $\mathcal{M} = \{1, \dots, k\}$. Now, we have a family of reduced Palm measures $P^{!(z,i)}$ for $i = 1, \dots, k$ and may restrict ourselves to sets of the form $C = \{i\}$. Then, (5) reads

$$P_C^{!z}(R) = \frac{1}{\nu(i)} \nu(i) P^{!(z,i)}(R) = P^{!(z,i)}(R)$$

and does not depend on the specific choice of ν .

For non-finite mark spaces, the reference measure ν on \mathcal{M} may not correspond to a well-defined mark distribution. One pragmatic approach is to take a finite partition of the mark space, $\mathcal{M} = \cup_{i=1}^k \mathcal{M}_i$, and proceed as in the multivariate case. An alternative is to use (5) as definition for a ν -averaged reduced Palm distribution with respect to C , bearing in mind that the definition does depend on the choice of ν .

2.3 Generating functionals

When product densities of all orders exist, the *generating functional* $G(\cdot)$, which uniquely determines the distribution of Y (see e.g. Theorem 9.4.V, Daley and Vere-Jones 2008), is defined as follows. For all mappings $\nu = 1 - u$ such that $u : \mathbb{R}^d \times \mathcal{M} \rightarrow [0, 1]$ is measurable with bounded support, set

$$\begin{aligned} G(\nu) &= G(1 - u) = \mathbb{E} \left[\prod_{(z,m) \in Y} \nu(z, m) \right] \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d \times \mathcal{M}} \dots \int_{\mathbb{R}^d \times \mathcal{M}} \rho^{(n)}((z_1, m_1), \dots, (z_n, m_n)) \\ &\quad \times \prod_{i=1}^n u(z_i, m_i) \, dz_i \, d\nu(m_i) \\ &= \exp \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d \times \mathcal{M}} \dots \int_{\mathbb{R}^d \times \mathcal{M}} \xi_n((z_1, m_1), \dots, (z_n, m_n)) \right. \\ &\quad \left. \times \prod_{i=1}^n u(z_i, m_i) \lambda(z_i, m_i) \, dz_i \, d\nu(m_i) \right]. \end{aligned} \tag{6}$$

By convention, $\log 0 = -\infty$ and an empty product equals 1. The last equalities hold provided that the right hand sides converge (see e.g. p. 126, Chiu et al. 2013). Similarly, for $a \in \mathbb{R}^d$ and $C \in \mathcal{B}(\mathcal{M})$, we may define the generating functional $G_C^{!a}$ with respect to $P_C^{!a}$ (see the discussion around (5)) by

$$G_C^{!a}(\nu) = \frac{1}{\nu(C)} \int_C \mathbb{E}^{!(a,b)} \left[\prod_{(z,m) \in Y} \nu(z, m) \right] \, d\nu(b). \tag{7}$$

3 Definition of summary statistics

3.1 Inhomogeneous cross D -function

In this section, we define cross D -functions for marked point processes in analogy with the inhomogeneous nearest neighbour distance distribution function of van Lieshout (2011).

Write

$$\bar{\lambda}_E = \inf_{z \in \mathbb{R}^d, m \in E} \lambda(z, m).$$

Throughout we assume that Y is a simple marked point process whose product densities of all orders exist and for which the $\xi_n, n \geq 2$, are translation invariant in the sense that

$$\xi_n((z_1 + a, m_1), \dots, (z_n + a, m_n)) = \xi_n((z_1, m_1), \dots, (z_n, m_n))$$

for all $a \in \mathbb{R}^d$ and $\ell \otimes \nu$ -almost all $(z_i, m_i) \in \mathbb{R}^d \times \mathcal{M}$. If, moreover, $\bar{\lambda} = \bar{\lambda}_{\mathcal{M}} > 0$, then Y is said to be *intensity-reweighted moment stationary* (IRMS). This definition is due to van Lieshout (2011). In the context of multivariate point processes, a variation in which only ξ_2 is required to be translation invariant can be found in Møller and Waagepetersen (2004).

Clearly, if Y is stationary, it is also IRMS. Assuming that the intensities are bounded away from zero, further examples include Poisson processes, multivariate point processes with independent IRMS components (cf. Proposition 4.4 in Møller and Waagepetersen (2004)) or intensity-reweighted moment stationary point processes that are independently marked (cf. (16)). Since the ξ_n are invariant under independent thinning, further examples include such location or mark-dependent thinnings of stationary point processes. The special case of spatio-temporal point processes is treated in detail in Cronie and van Lieshout (2015).

Definition 1 Let Y be IRMS and let C and E be Borel sets in \mathcal{M} with $\nu(C)$ and $\nu(E)$ strictly positive. Write $B(a, r)$ for the closed ball centred at a with radius r . Set

$$u_r^{a,E}(z, m) = \frac{\bar{\lambda}_E \mathbf{1}\{(z, m) \in B(a, r) \times E\}}{\lambda(z, m)}, \quad a \in \mathbb{R}^d, \quad E \in \mathcal{B}(\mathcal{M}),$$

and define, for $r \geq 0$, the *inhomogeneous cross nearest neighbour distance distribution function* by

$$\begin{aligned} D_{\text{inhom}}^{CE}(r) &= 1 - G_C^{10}(1 - u_r^{0,E}) \\ &= 1 - \frac{1}{\nu(C)} \int_C \mathbb{E}^{!(0,b)} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda}_E \mathbf{1}\{(z, m) \in B(0, r) \times E\}}{\lambda(z, m)} \right) \right] \nu(b). \end{aligned} \tag{8}$$

We shall show in Theorem 1 below that the specific choice $a = 0$ in (8) is merely a matter of convenience. Moreover, $\bar{\lambda}_E$ may be replaced by smaller strictly positive scalars.

When Y is stationary and $\nu = \nu_M$, the mark distribution,

$$D_{\text{inhom}}^{CE}(r) = 1 - \frac{1}{\nu_M(C)} \int_C \mathbb{E}^{!(0,b)} \left[\prod_{(z,m) \in Y} \mathbf{1}\{(z,m) \notin B(0,r) \times E\} \right] d\nu_M(b) \\ = P_C^{!0}(Y \cap B(0,r) \times E \neq \emptyset),$$

so that (8) reduces to the C -to- E nearest neighbour distance distribution function of van Lieshout (2006) for marked point processes.

3.1.1 Multivariate point process

Consider a multivariate point process $Y = (Y_1, \dots, Y_k)$ that is intensity-reweighted moment stationary. Let $C = \{i\}$ and $E = \{j\}$ for $i \neq j \in \{1, \dots, k\}$. Write $\bar{\lambda}_j = \inf_{z \in \mathbb{R}^d} \lambda_j(z)$ and note that $\bar{\lambda}_E/\lambda(z, j)$ is equal to $\bar{\lambda}_j/\lambda_j(z)$. Therefore, (8) reduces to

$$D_{\text{inhom}}^{ij}(r) = 1 - \mathbb{E}^{!(0,i)} \left[\prod_{z \in Y_j} \left(1 - \frac{\bar{\lambda}_j}{\lambda_j(z)} \mathbf{1}\{z \in B(0,r)\} \right) \right] \tag{9}$$

which under the further assumption that Y is stationary is equal to

$$P^{!(0,i)}(Y_j \cap B(0,r) \neq \emptyset),$$

the classical cross nearest neighbour distance distribution function, see e.g. Chapter 21 in Gelfand et al. (2010). If Y is a Poisson process,

$$D_{\text{inhom}}^{ij}(r) = 1 - \exp[-\bar{\lambda}_j \ell(B(0,r))].$$

Smaller values of $D_{\text{inhom}}^{ij}(r)$ suggest that there are fewer points of type j in the r -neighbourhood, that is, inhibition; larger values indicate that points of type j are attracted by those of type i at range r . In the case $i = j$, we obtain the inhomogeneous D -function of Y_i .

With $C = \{i\}$ for some $i \in \{1, \dots, k\}$ and $E = \mathcal{M} = \{1, \dots, k\}$, (8) is equal to

$$D_{\text{inhom}}^{i\bullet}(r) = 1 - \mathbb{E}^{!(0,i)} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda} \mathbf{1}\{z \in B(0,r)\}}{\lambda(z,m)} \right) \right] \tag{10}$$

for $r \geq 0$. Note that the function $u_r^{0,\mathcal{M}}$ may depend on ν through $\lambda(z, m)$. If we give equal weight to each member of \mathcal{M} , however, $\bar{\lambda}/\lambda(z, m) = \bar{\lambda}/\lambda_m(z)$ is uniquely

defined in terms of the intensity functions of the components of Y and the minimal marginal intensity $\tilde{\lambda} = \inf\{\lambda_i(z) : z \in \mathbb{R}^d, i \in \{1, \dots, k\}\}$. If Y is stationary, $D_{\text{inhom}}^{i\bullet}$ is the classic i -to-any nearest neighbour distance distribution function.

3.2 Inhomogeneous cross J -functions

In this section, we define cross J -functions for marked point processes in analogy with the inhomogeneous J -function of van Lieshout (2011). Throughout we assume that Y is a simple intensity-reweighted moment stationary point process.

Definition 2 Let Y be IRMS and let C and E be Borel sets in \mathcal{M} with $\nu(C)$ and $\nu(E)$ strictly positive. For $r \geq 0$ and $n \geq 1$, set

$$J_n^{CE}(r) = \int_C \int \dots \int_{(B(0,r) \times E)^n} \xi_{n+1}((a, b), (z_1 + a, m_1), \dots, (z_n + a, m_n)) \times d\nu(b) \prod_{i=1}^n dz_i d\nu(m_i)$$

and define the *inhomogeneous cross J -function* by

$$J_{\text{inhom}}^{CE}(r) = \frac{1}{\nu(C)} \left(\nu(C) + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} J_n^{CE}(r) \right) \tag{11}$$

for all ranges $r \geq 0$ for which the series is absolutely convergent.

Note that there is an implicit dependence on $a \in \mathbb{R}^d$ in $J_n^{CE}(r)$ and consequently in $J_{\text{inhom}}^{CE}(r)$. However, the IRMS assumption implies that all $J_n^{CE}(r)$ (and therefore $J_{\text{inhom}}^{CE}(r)$) are ℓ -almost everywhere constant. Furthermore, Cauchy’s root test implies that whenever $\limsup_{n \rightarrow \infty} (\bar{\lambda}_E^n |J_n^{CE}(r)|/n!)^{1/n} < 1$, (11) is absolutely convergent.

When Y is stationary and $\nu = \nu_M$, the mark distribution, (11) reduces to the cross inhomogeneous J -function for marked point processes introduced in van Lieshout (2006) since in that case $\bar{\lambda}_E = \bar{\lambda}_{\mathcal{M}}$ regardless of the choice of E . Finally, note that for a Poisson process, $\xi_n \equiv 0$ for $n \geq 2$, so $J_{CE}(r) \equiv 1$. In general, the inhomogeneous J -function is not commutative with respect to the mark sets C and E , $C \neq E$.

Looking closer at Definition 2, we see that there is some resemblance between $J_{\text{inhom}}^{CE}(r)$ and the cross inhomogeneous K -function of Definition 4.8 in Møller and Waagepetersen (2004). Indeed, truncation of the series in (11) at $n = 1$ gives

$$J_{\text{inhom}}^{CE}(r) - 1 \approx -\frac{\bar{\lambda}_E}{\nu(C)} \int_C \int_{B(0,r) \times E} [g((0, b), (z, m)) - 1] d\nu(b) dz d\nu(m) = -\bar{\lambda}_E \nu(E) \left(K_{\text{inhom}}^{CE}(r) - \ell(B(0, r)) \right),$$

where

$$K_{\text{inhom}}^{CE}(r) = \frac{1}{\nu(C)\nu(E)} \int_C \int_{B(0,r) \times E} g((0, m_1), (z, m_2)) d\nu(m_1) dz d\nu(m_2) \tag{12}$$

is the generalisation of the cross inhomogeneous K -function to our set-up. Note that the inhomogeneous K -function defined by (12) requires translation invariance of the two-point correlation function only, in which case Y is said to be second-order intensity-reweighted stationary (SOIRS). Heuristically, $J_{CE}(r) < 1$ suggests that points with marks in E tend to cluster around points with marks in C at range $r \geq 0$; $J_{CE}(r) > 1$ indicates that points with marks in E avoid those with marks in C at range $r \geq 0$. This interpretation is confirmed by Theorem 1 below.

Definition 2 is hard to work with. A more natural representation can be given in terms of the generating functional. In order to do so, define the *inhomogeneous empty space function* of Y_E , the marked point process Y restricted to $\mathbb{R}^d \times E$, by

$$\begin{aligned}
 1 - F_{\text{inhom}}^E(r) &= G(1 - u_r^{a,E}) \\
 &= \mathbb{E} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda}_E \mathbf{1}\{(z,m) \in B(a,r) \times E\}}{\lambda(z,m)} \right) \right] \tag{13}
 \end{aligned}$$

with $u_r^{a,E}$ as in Definition 1. As for D_{inhom}^{CE} , the definition does not depend on the choice of origin $a = 0$ and $\bar{\lambda}_E$ may be replaced by smaller strictly positive scalars. At this point, it is important to stress that for $E = \mathcal{M}$, F_{inhom}^E is not necessarily equal to F_{inhom} , the empty space function of the ground process Z , since $u_r^{0,\mathcal{M}}$ depends on the marks both through the intensity function λ and the bound $\bar{\lambda}_{\mathcal{M}}$.

Theorem 1 *Let Y be as in Definition 2. Then, as a function of $a \in \mathbb{R}^d$, each $J_n^{CE}(r)$ is ℓ -almost everywhere constant. Moreover, if the \limsup for $n \rightarrow \infty$ of*

$$\left(\frac{(\bar{\lambda}_E)^n}{n!} \int \cdots \int_{(B(0,r) \times E)^n} \frac{\rho^{(n)}((z_1, m_1), \dots, (z_n, m_n))}{\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)} \prod_{i=1}^n dz_i dv(m_i) \right)^{1/n}$$

is strictly less than 1, then, for almost all $a \in \mathbb{R}^d$, the C -to- E inhomogeneous J -function of Definition 2 satisfies

$$J_{\text{inhom}}^{CE}(r) = \frac{1 - D_{\text{inhom}}^{CE}(r)}{1 - F_{\text{inhom}}^E(r)}$$

for all $r \geq 0$ for which $F_{\text{inhom}}^E(r) \neq 1$.

The proof is technical and relegated to an Appendix.

3.2.1 Multivariate point process

Consider a multivariate point process $Y = (Y_1, \dots, Y_k)$ that is intensity-reweighted moment stationary. By a suitable choice of mark set E , we obtain different types of inhomogeneous J -functions.

First, take $C = \{i\}$ and $E = \{j\}$ for $i \neq j \in \{1, \dots, k\}$. Then, writing F_{inhom}^j for the inhomogeneous empty space function of Y_j and recalling (9), the statistic (11) is equal to

$$J_{\text{inhom}}^{ij}(r) = \frac{1 - D_{\text{inhom}}^{ij}(r)}{1 - F_{\text{inhom}}^j(r)} \tag{14}$$

and compares the distribution of intensity-reweighted distances from a point of type i to the nearest one of type j to those from an arbitrarily chosen origin to Y_j . Therefore, it generalises the i -to- j cross J -function of van Lieshout and Baddeley (1999) for stationary multivariate point processes.

Set $C = \{i\}$ for some $i \in \{1, \dots, k\}$ and $E = \mathcal{M} = \{1, \dots, k\}$. Then, recalling (10), the statistic (11) can be written as:

$$J_{\text{inhom}}^{i\bullet}(r) = \frac{1 - D_{\text{inhom}}^{i\bullet}(r)}{1 - F_{\text{inhom}}^{\mathcal{M}}(r)} \tag{15}$$

and compares tails of the i -to-any nearest neighbour distance distribution and the empty space function of Y . Note that if ν is proportional to the counting measure, $F_{\text{inhom}}^{\mathcal{M}}(r)$ can be expressed in terms of the intensity functions of the components and the minimal marginal intensity (see the discussion following formula (10)). Hence, $J_{\text{inhom}}^{i\bullet}$ generalises the i -to-any J -function for stationary multivariate point processes (van Lieshout and Baddeley 1999).

4 Independence and random labelling

In this section, we investigate the effect of various independence assumptions and marking schemes on our summary statistics.

4.1 Independent marking mechanisms

Specific forms of marking are summarised in Definition 3 below, cf. Definition 6.4III in Daley and Vere-Jones (2003).

Definition 3 A marked point process Y is called *independently marked* if, given the ground process Z , the marks are independent random variables with a distribution that depends only on the corresponding location. If, additionally, M^z does not depend on the location, we say that Y has the *random labelling property*.

Proposition 1 Let C and E be Borel sets in \mathcal{M} with $\nu(C), \nu(E) > 0$ and assume that Y is independently marked.

- (a) If Y is SOIRS, the ground process Z is also SOIRS and $K_{\text{inhom}}^{CE}(r) = K_{\text{inhom}}^Z(r)$, the inhomogeneous K -function of Z .

Let $\mathbb{E}_Z^{!0}$ denote the expectation under the Palm distribution of the ground process Z and write $u_r^0(z) = \bar{\lambda}_g \mathbf{1}\{z \in B(0, r)\} / \lambda_g(z)$, $z \in \mathbb{R}^d$, $c_E = \bar{\lambda}_E \nu(E) / \bar{\lambda}_g$. Under the assumptions of Theorem 1, when Y is independently marked, Z is IRMS and

- (b) $F_{\text{inhom}}^E(r) = 1 - G_Z(1 - c_E u_r^0)$,
- (c) $D_{\text{inhom}}^{CE}(r) = 1 - G_Z^{I0}(1 - c_E u_r^0)$,
- (d) $J_{\text{inhom}}^{CE}(r) = G_Z^{I0}(1 - c_E u_r^0) / G_Z(1 - c_E u_r^0)$ for all $r \geq 0$ for which the denominator is non-zero.

If Y is randomly labelled with $\nu = \nu_M$, then $c_E = \nu_M(E)$ and

$$F_{\text{inhom}}^{\mathcal{M}}(r) = F_{\text{inhom}}^Z(r); \quad D_{\text{inhom}}^{C\mathcal{M}}(r) = D_{\text{inhom}}^Z(r); \quad J_{\text{inhom}}^{C\mathcal{M}}(r) = J_{\text{inhom}}^Z(r).$$

Proof Recall that

$$\rho^{(n)}((z_1, m_1), \dots, (z_n, m_n)) = f_{z_1, \dots, z_n}(m_1, \dots, m_n) \rho_g^{(n)}(z_1, \dots, z_n).$$

Under the independent marking assumption,

$$f_{z_1, \dots, z_n}(m_1, \dots, m_n) = \prod_{i=1}^n f_{z_i}(m_i). \tag{16}$$

Therefore,

$$\xi_n((z_1, m_1), \dots, (z_n, m_n)) = \xi_n^g(z_1, \dots, z_n),$$

the n -point correlation function of the ground process, so that Z is (second order) intensity-reweighted moment stationary whenever Y is so. Plugging (16) into (12) yields $K_{\text{inhom}}^{CE}(r) = \int_{B(0,r)} g_g(0, z) dz$, the inhomogeneous K -function of Z . Furthermore, under the assumption that the series expansion is absolutely convergent, by (6), (13) reduces to

$$\begin{aligned} 1 - F_{\text{inhom}}^E(r) &= 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \int_{(B(0,r) \times E)^n} \frac{\rho_g^{(n)}(z_1, \dots, z_n)}{\lambda_g(z_1) \cdots \lambda_g(z_n)} \prod_{i=1}^n dz_i d\nu(m_i) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E \nu(E))^n}{n!} \int_{B(0,r)^n} \frac{\rho_g^{(n)}(z_1, \dots, z_n)}{\lambda_g(z_1) \cdots \lambda_g(z_n)} \prod dz_i. \end{aligned}$$

Similarly,

$$1 - D_{\text{inhom}}^{CE}(r) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E \nu(E))^n}{n!} \int_{B(0,r)^n} \frac{\rho_g^{(n+1)}(0, z_1, \dots, z_n)}{\lambda_g(0) \lambda_g(z_1) \cdots \lambda_g(z_n)} \prod dz_i.$$

We conclude that $1 - D_{\text{inhom}}^{CE}(r) = G_Z^{I0}(1 - \bar{\lambda}_E \nu(E) \mathbf{1}_{B(0,r)}(\cdot) / \lambda_g(\cdot))$ and $1 - F_{\text{inhom}}^E(r) = G_Z(1 - \bar{\lambda}_E \nu(E) \mathbf{1}_{B(0,r)}(\cdot) / \lambda_g(\cdot))$.

Under random labelling, the right hand side of (16) is further simplified to $\prod_{i=1}^n f(m_i)$ for some probability density f that does not depend on location. If furthermore $\nu = M^z = \nu_M$, the mark distribution, the density is one, i.e. $f(m_i) \equiv 1$. Hence $\bar{\lambda}_E = \bar{\lambda}_g$ and $c_E = \nu_M(E)$. In particular, $c_{\mathcal{M}} = 1$. □

Proposition 1 generalises known results for stationary multivariate and marked point processes, cf. respectively, Lemma 4 in van Lieshout and Baddeley (1999) and Proposition 3.1 in van Lieshout (2006). Heuristically, since conditionally on the locations, the marks are assigned independently of one another, any interaction in Y is due to interactions in the ground process Z . Note that the summary statistics do not depend on the choice of C , but may depend on E through c_E .

4.2 Cross summary statistics for independent components

Recall that we use the notation $Y_C, C \in \mathcal{B}(\mathcal{M})$, for the restriction of Y to $\mathbb{R}^d \times C$. If Y_C and Y_E are independent, then the C -to- E cross J -function is identically 1. More precisely, the following result holds.

Proposition 2 Consider two disjoint Borel sets $C, E \subseteq \mathcal{M}$ with $v(C)$ and $v(E)$ strictly positive and assume that Y_C and Y_E are independent. Under the assumptions of Theorem 1,

$$D_{\text{inhom}}^{CE}(r) = F_{\text{inhom}}^E(r)$$

so that $J_{\text{inhom}}^{CE}(r) \equiv 1$ whenever well defined.

If Y is SOIRS, $K_{\text{inhom}}^{CE}(r) = \omega_d r^d = \ell(B(0, r))$ whenever Y_C and Y_E are independent by Proposition 4.4 in Møller and Waagepetersen (2004).

Proof By the Campbell formula (2), if Y_C and Y_E are independent, the product densities factorise with respect to C and E , i.e.

$$\begin{aligned} &\rho^{(n_C+n_E)}((z_1, m_1), \dots, (z_{n_C}, m_{n_C}), (\tilde{z}_1, \tilde{m}_1), \dots, (\tilde{z}_{n_E}, \tilde{m}_{n_E})) \\ &= \rho^{(n_C)}((z_1, m_1), \dots, (z_{n_C}, m_{n_C})) \rho^{(n_E)}((\tilde{z}_1, \tilde{m}_1), \dots, (\tilde{z}_{n_E}, \tilde{m}_{n_E})) \end{aligned}$$

for almost all $(z_i, m_i) \in \mathbb{R}^d \times C$ and $(\tilde{z}_i, \tilde{m}_i) \in \mathbb{R}^d \times E$. Then, by the proof of Theorem 1,

$$\begin{aligned} G_C^{!0}(1 - u_r^{0,E}) &= 1 + \frac{1}{v(C)} \\ &\times \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \int_C \int_{(B(0,r) \times E)^n} \frac{\rho^{(n+1)}((0, b), (z_1, m_1), \dots, (z_n, m_n))}{\lambda(0, b)\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)} \\ &\times \prod_{i=1}^n dz_i dv(m_i) dv(b). \end{aligned}$$

The integrand factorises as:

$$\frac{\lambda(0, b)\rho^{(n)}((z_1, m_1), \dots, (z_n, m_n))}{\lambda(0, b)\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)}$$

so that $G_C^{!0}(1 - u_r^{0,E}) = G(1 - u_r^{0,E})$. We conclude that $D_{\text{inhom}}^{CE}(r) = F_{\text{inhom}}^E(r)$ and $J_{\text{inhom}}^{CE}(r) \equiv 1$. \square

Propositions 2 and 3 (below) generalise well-known results for stationary multivariate point processes, cf. Diggle (2014), van Lieshout and Baddeley (1999). The next result collects mixture formulae.

Proposition 3 *Let $C \subseteq \mathcal{M}$ be a Borel set with $0 < \nu(C) < \nu(\mathcal{M})$. Set $E = \mathcal{M} \setminus C$ and assume that Y_C and Y_E are independent.*

(a) *If Y is SOIRS, $K_{\text{inhom}}^{CM}(r) = \frac{\nu(E)}{\nu(\mathcal{M})}\omega_d r^d + \frac{\nu(C)}{\nu(\mathcal{M})}K_{\text{inhom}}^{Y_C}(r)$, where $K_{\text{inhom}}^{Y_C}$ is the inhomogeneous K -function of Y_C .*

Write $c_A = \bar{\lambda}/\bar{\lambda}_A$ for $A \in \mathcal{B}(\mathcal{M})$. Under the assumptions of Theorem 1,

(b) $1 - F_{\text{inhom}}^M(r) = G(1 - c_C u_r^{0,C})G(1 - c_E u_r^{0,E})$,

(c) $1 - D_{\text{inhom}}^{CM}(r) = G_C^{!0}(1 - c_C u_r^{0,C})G(1 - c_E u_r^{0,E})$,

(d) $J_{\text{inhom}}^{CM}(r) = G_C^{!0}(1 - c_C u_r^{0,C})/G(1 - c_C u_r^{0,C})$ for all $r \geq 0$ for which the denominator is non-zero.

Note that if we would have used the global infimum $\bar{\lambda}$ in (8), (13) and Definition 2, the constants c_C and c_E would vanish and, e.g. $J_{\text{inhom}}^{CM}(r) \equiv J_{\text{inhom}}^{Y_C}(r)$ whenever defined.

Proof As in the proof of Proposition 2, $g((0, m_1), (z, m_2)) = 1$ if $m_1 \in C$ and $m_2 \in \mathcal{M} \setminus C$, so that

$$\begin{aligned} K_{\text{inhom}}^{CM}(r) &= \frac{1}{\nu(C)\nu(\mathcal{M})} \int_C \int_{B(0,r) \times \mathcal{M}} g((0, m_1), (z, m_2)) \, d\nu(m_1) dz d\nu(m_2) \\ &= \frac{\nu(\mathcal{M} \setminus C)}{\nu(\mathcal{M})} \omega_d r^d + \frac{\nu(C)}{\nu(\mathcal{M})} K_{\text{inhom}}^{CC}(r). \end{aligned}$$

Since $Y = Y_C \cup Y_{\mathcal{M} \setminus C}$ is the superposition of independent point processes,

$$\begin{aligned} 1 - F_{\text{inhom}}^M(r) &= \mathbb{E} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda} \mathbf{1}\{(z, m) \in B(0, r) \times \mathcal{M}\}}{\lambda(z, m)} \right) \right] \\ &= G \left(1 - \frac{\bar{\lambda}}{\bar{\lambda}_{\mathcal{M} \setminus C}} u_r^{0, \mathcal{M} \setminus C} \right) G \left(1 - \frac{\bar{\lambda}}{\bar{\lambda}_C} u_r^{0, C} \right). \end{aligned}$$

Furthermore, under each (conditional) distribution $P^{!(0,b)}$, $b \in C$, the points of $Y_{\mathcal{M} \setminus C}$ follow the distribution P , hence

$$D_{\text{inhom}}^{CM}(r) = 1 - \int_C \mathbb{E}^{!(0,b)} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda} \mathbf{1}\{(z, m) \in B(0, r) \times \mathcal{M}\}}{\lambda(z, m)} \right) \right] \frac{d\nu(b)}{\nu(C)}$$

$$\begin{aligned}
 &= 1 - \mathbb{E} \left[\prod_{(z,m) \in Y \cap (B(0,r) \times \mathcal{M} \setminus C)} \left(1 - \frac{\bar{\lambda}}{\lambda(z,m)} \right) \right] \\
 &\quad \times \frac{1}{\nu(C)} \int_C \mathbb{E}^{!(0,b)} \left[\prod_{(z,m) \in Y \cap (B(0,r) \times C)} \left(1 - \frac{\bar{\lambda}}{\lambda(z,m)} \right) \right] \mathrm{d}\nu(b) \\
 &= 1 - G \left(1 - \frac{\bar{\lambda}}{\bar{\lambda}_{\mathcal{M} \setminus C}} u_r^{0, \mathcal{M} \setminus C} \right) G_C^{!0} \left(1 - \frac{\bar{\lambda}}{\bar{\lambda}_C} u_r^{0,C} \right).
 \end{aligned}$$

□

The propositions in this section help to choose an appropriate summary statistic. Since a random labelling hypothesis is plausible in situations in which labels are allocated conditionally on the point positions, Proposition 1 suggests to consider $D_{\text{inhom}}^{C\mathcal{M}} - D_{\text{inhom}}^Z$ or $J_{\text{inhom}}^{C\mathcal{M}} - J_{\text{inhom}}^Z$. In situations in which points and marks arise jointly so that an unconditional analysis is called for, we prefer cross statistics J_{inhom}^{CE} to assess the independence or otherwise of Y_C and Y_E . An alternative in the bivariate case $E = \mathcal{M} \setminus C$ is the statistic $J_{\text{inhom}}^{C\mathcal{M}} - J_{\text{inhom}}^{Y_C}$ (cf. Proposition 3).

5 Statistical inference

Although defined on all of \mathbb{R}^d , in practice, the ground process Z is observed only in some compact spatial region $W \subseteq \mathbb{R}^d$ with boundary ∂W . To deal with edge effects, we apply a minus sampling scheme (see e.g. Chiu et al. (2013) for further details). The underlying idea is that when one is interested in the interactions up to range r , only observations in the eroded set

$$W_{\ominus r} = \{z \in W : d_{\mathbb{R}^d}(z, \partial W) \geq r\}$$

are taken into account. For clarity of exposition, we assume that the intensity function $\lambda(z, m)$ is known. If this is not the case, an estimator $\hat{\lambda}$ may be plugged in. We return to this point in Sect. 7.

In Sect. 5.1, we derive estimators for our inhomogeneous D - and J -functions; estimation of the cross K -function is discussed in Møller and Waagepetersen (2004). Finally, Sect. 5.2 is devoted to non-parametric test statistics for the independence and random labelling assumptions (cf. Sect. 4).

5.1 Estimation

Let Y be an intensity-reweighted moment stationary marked point process and consider the statistics

$$\sum_{(a,b) \in Y \cap (W_{\ominus r} \times C)} \frac{1}{\lambda(a,b)} \left[\prod_{(z,m) \in (Y \setminus \{(a,b)\}) \cap (B(a,r) \times E)} \left(1 - \frac{\bar{\lambda}_E}{\lambda(z,m)} \right) \right] \quad (17)$$

and

$$\frac{1}{|L \cap W_{\Theta r}|} \sum_{l \in L \cap W_{\Theta r}} \left[\prod_{(z,m) \in Y \cap (B(l,r) \times E)} \left(1 - \frac{\bar{\lambda}_E}{\lambda(z,m)} \right) \right], \tag{18}$$

where $L \subseteq W$ is some finite point grid. The following unbiasedness result holds.

Lemma 1 *Suppose $C, E \in \mathcal{B}(\mathcal{M})$ have strictly positive ν -content. Then, under the assumptions of Theorem 1, provided $\ell(W_{\Theta r}) > 0$, (17) and (18) are unbiased estimators of, respectively, $\ell(W_{\Theta r})\nu(C)G_C^{l_0}(1 - u_r^{0,E})$ and $G(1 - u_r^{0,E})$.*

Proof By the Campbell–Mecke formula and Fubini’s theorem, the expectation of (17) is equal to

$$\begin{aligned} & \int_{W_{\Theta r}} \int_C \mathbb{E}^{l(a,b)} \left[\frac{1}{\lambda(a,b)} \prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda}_E \mathbf{1}\{(z,m) \in B(a,r) \times E\}}{\lambda(z,m)} \right) \right] \\ & \lambda(a,b) \, \text{d}a \, \text{d}\nu(b) \\ & = \ell(W_{\Theta r}) \int_C \mathbb{E}^{l(0,b)} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda}_E \mathbf{1}\{(z,m) \in B(0,r) \times E\}}{\lambda(z,m)} \right) \right] \, \text{d}\nu(b), \end{aligned}$$

which can be written in generating functional terms as $\ell(W_{\Theta r})\nu(C)G_C^{l_0}(1 - u_r^{0,E})$. The expectation of (18) is

$$\frac{1}{|L \cap W_{\Theta r}|} \sum_{l \in L \cap W_{\Theta r}} G(1 - u_r^{l,E}).$$

Since the summands do not depend on l , the required unbiasedness follows. □

Lemma 1 implies that an estimator of $G_C^{l_0}(1 - u_r^{0,E})$ can be obtained from (17) upon division by $\ell(W_{\Theta r})\nu(C)$. For irregular windows, however, the volume of $W_{\Theta r}$ may be difficult to compute. To overcome this problem, we use the Hamilton principle advocated in [Stoyan and Stoyan \(2000\)](#) and estimate $\ell(W_{\Theta r})\nu(C)$ by

$$\sum_{(a,b) \in Y \cap (W_{\Theta r} \times C)} \frac{1}{\lambda(a,b)}.$$

The result is a ratio-unbiased estimator with the desirable property that it takes the value one at $r = 0$. Simulations suggest that the Hamilton principle is also helpful in reducing the sensitivity of the estimator with respect to misspecification of the intensity function λ . To define a non-parametric estimator for $J_{\text{inhom}}^{CE}(r)$, we use Theorem 1 and plug in the estimators for the numerator and denominator discussed above, i.e.

$$\begin{aligned}
 \widehat{J_{\text{inhom}}^{CE}}(r) &= \frac{1 - \widehat{D_{\text{inhom}}^{CE}}(r)}{1 - \widehat{F_{\text{inhom}}^E}(r)} = \left(\sum_{(a,b) \in Y \cap (W_{\Theta r} \times C)} \frac{1}{\lambda(a,b)} \right)^{-1} \\
 &\times \left(\sum_{(a,b) \in Y \cap (W_{\Theta r} \times C)} \frac{1}{\lambda(a,b)} \left[\prod_{(z,m) \in (Y \setminus \{(a,b)\}) \cap (B(a,r) \times E)} \left(1 - \frac{\bar{\lambda}_E}{\lambda(z,m)} \right) \right] \right) \\
 &/ \left(\frac{1}{|L \cap W_{\Theta r}|} \sum_{l \in L \cap W_{\Theta r}} \left[\prod_{(z,m) \in Y \cap (B(l,r) \times E)} \left(1 - \frac{\bar{\lambda}_E}{\lambda(z,m)} \right) \right] \right).
 \end{aligned}$$

5.2 Hypothesis testing

In Sect. 4, we encountered two interaction hypotheses: random labelling and independence. Such hypotheses are complex, depending as they do both on the marginal distribution of the components of interest and the marking structure or interactions between them. Nevertheless, it is possible to construct non-parametric Monte Carlo tests by proper conditioning, see Besag and Diggle (1977), Lotwick and Silverman (1982), Myllymaki et al. (2013) or Ripley (1977), based on a realisation of Y with locations in some compact window $W \subseteq \mathbb{R}^d$.

First consider the random labelling hypothesis of Definition 3. Since conditional on the ground process $Z \cap W = \{z_1, \dots, z_n\}$, the marks are independent and identically distributed, a Monte Carlo test may be based on random permutations of the n observed marks—in effect conditioning on the empirical mark distribution. Another approach would be to sample the marks according to the mark distribution, but the latter is typically unknown in practice.

In general, testing for independence is hard. For hyper-rectangular windows, a Lotwick and Silverman (1982) type test can be constructed. Recall that when Y is stationary, the key idea is to wrap Y onto a torus by identifying opposite sides of W , keeping $Y_C \cap (W \times \mathcal{M})$ fixed and translating $Y_E \cap (W \times \mathcal{M})$ randomly over the torus (or vice versa). Since the random translations leave the distribution of the E -component unchanged, they can be used for testing. Note that this approach is conditional on the marginal structures of the C -marked and E -marked patterns.

For inhomogeneous marked point processes, randomly translating Y_E may change its distribution. To compensate, we also translate the intensity. More specifically, consider the random measure

$$\Xi^Y = \sum_{(z,m) \in Y} \frac{\delta_{(z,m)}}{\lambda(z,m)} \tag{19}$$

and let Ξ_a^Y be its translation over $a \in \mathbb{R}^d$, that is,

$$\begin{aligned}
 \Xi_a^Y(A) &= \Xi^Y(A_{-a}) = \sum_{(z,m) \in Y} \frac{\mathbf{1}\{(z+a,m) \in A\}}{\lambda(z,m)} \\
 &= \sum_{(z,m) \in (Y+a)} \frac{\mathbf{1}\{(z,m) \in A\}}{\lambda(z-a,m)}.
 \end{aligned}$$

Note that Y is translated over a and $\mu(z, m) = \lambda(z - a, m)$ is a translation of λ over the spatial vector a . Moreover, if Y is intensity-reweighted moment stationary, Ξ is moment stationary. In other words, for any a , Ξ_a has the same factorial moment measures as (19).

Proposition 4 *Let the assumptions of Theorem 1 be satisfied and C, E be disjoint Borel mark sets with strictly positive ν -measure. If Y_C and Y_E are independent and Ξ is stationary, then $\widehat{D}_{\text{inhom}}^{CE}$, $\widehat{F}_{\text{inhom}}^E$ and $\widehat{J}_{\text{inhom}}^{CE}$ can be expressed in terms of (Ξ^{Y_C}, Ξ^{Y_E}) and their laws are invariant under translations of a component.*

Proof For $a \in \mathbb{R}^d$,

$$\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda}_E \mathbf{1}\{(z, m) \in B(a, r) \times E\}}{\lambda(z, m)} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \left[\sum_{(z_1, m_1), \dots, (z_n, m_n) \in Y}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{(z_i, m_i) \in B(a, r) \times E\}}{\lambda(z_i, m_i)} \right]$$

by the local finiteness of Y and an inclusion–exclusion argument. The inner summand is the n th order factorial power measure $\xi_{Y_E}^{[n]}$ of Ξ^{Y_E} evaluated at $B(a, r)^n$. Hence (18) is a function of Ξ^{Y_E} . Furthermore, as C and E are disjoint, (17) can be written as:

$$\int_{W_{\Theta_r} \times C} \left(1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \xi_{Y_E}^{[n]}(B(a, r)^n) \right) d\Xi^{Y_C}(a, m),$$

which is well-defined by the local finiteness of Y . Finally

$$\sum_{(z,m) \in Y \cap (W_{\Theta_r} \times C)} \lambda(z, m)^{-1} = \Xi^{Y_C}(W_{\Theta_r} \times C)$$

is a function of Ξ^{Y_C} only. Since $(\Xi^{Y_C}, \Xi_a^{Y_E}) \stackrel{d}{=} (\Xi^{Y_C}, \Xi^{Y_E})$ by the independence of Y_C and Y_E and the invariance under translation of the law of Ξ^{Y_E} , the proof is complete. \square

6 Application

6.1 Data description

In this section, we will apply our statistics to data which are presented in Turner (2009) and available in the R package `spatstat` described in Baddeley and Turner (2005). These data were collected by the New Brunswick Department of Natural Resources and cleaned by Professor Turner. They contain records of wildfires which occurred in New Brunswick, Canada, during the years 1987 through 2003; records for 1988 are missing.

More formally, the data $\{(z_i, m_i, t_i)\}_{i=1}^{n_0}$ consist of $n_0 = 7108$ recordings of spatial locations z_i of wildfires. Attached to each location are two marks: $t_i \in T = \{1987, 1989, \dots, 2003\}$ gives the year of occurrence and m_i indicates the fuel type. There are four types, of which the dominant one is ‘forest’ (accounting for some 65% of the fires). The other three types account for only about ten percent of the fires each.

Below, we will quantify interaction in a particular year (here 2000) using the data in other years to estimate the intensity function. We restrict the study area to the rectangular region $W = [245.4663, 682.2945] \times [301.0545, 838.6173]$, a subset of New Brunswick. Doing so, we obtain the data set $\{(z_i, m_i, t_i)\}_{i=1}^n$ containing $n = 3267$ records of which 147 occur in the year 2000. Since the number of fires occurring in 2000 and fuelled by, e.g. ‘dump’ is small, we use the mark space $\mathcal{M} = \{\text{forest, other}\}$. For further details, see [Turner \(2009\)](#).

6.2 Independence

To quantify the dependence between the various categories (cf. Proposition 2), we use J_{inhom}^{CE} in combination with the Lotwick–Silverman approach discussed in Sect. 5.2.

Following ([Turner 2009](#), page 205), we assume that for any given year $t \in T$, the intensity function is of the form

$$\lambda^*(z, m, t) = c_t \lambda(z, m), \quad (z, m) \in W \times \mathcal{M}, \tag{20}$$

where $c_t > 0$ is a year-dependent scaling of some overall intensity $\lambda(z, m)$. Since the mark set is finite, we take the product of Lebesgue and counting measure as reference measure so that for fixed mark m , $\lambda(z, m) = \lambda_m(z)$ is the overall intensity function of wildfires with fuel type m .

From now on, focus on the year 2000. Since the Lotwick–Silverman approach is based on torus translations of one of the component patterns as well as its corresponding intensity function, we use a torus edge correction for the intensity function. More precisely, we estimate $\lambda_m(z)$ by means of a Gaussian kernel estimator based on all observations with mark m that do not fall in the year 2000. Regarding the bandwidth, since we consider each mark separately, we use the larger of the bandwidths considered by [Turner \(2009\)](#) for the ground process Z , that is, $\sigma = 66$ which (approximately) is the square root of the area of New Brunswick multiplied by 0.10. The results are displayed in Fig. 1.

Recall that the summary statistics discussed in this paper assume that the ground process is simple. As the data pattern $\{(z_i, m_i, 2000)\}$ for the year 2000 contains duplicated points, we choose to follow [Turner \(2009\)](#) and discard them. In other words, we delete all pairs (z_i, m_i, t_i) and (z_j, m_j, t_j) satisfying $z_i = z_j$ and $t_i = t_j = 2000$. This results in a marked point pattern with $n_{2000} = 124$ points (see Figure 2).

To estimate the year-dependent constant in (20), we use a mass preservation property and equate

$$\hat{c}_{2000} \int_W \sum_{m \in \mathcal{M}} \hat{\lambda}_m(z) \, dz$$

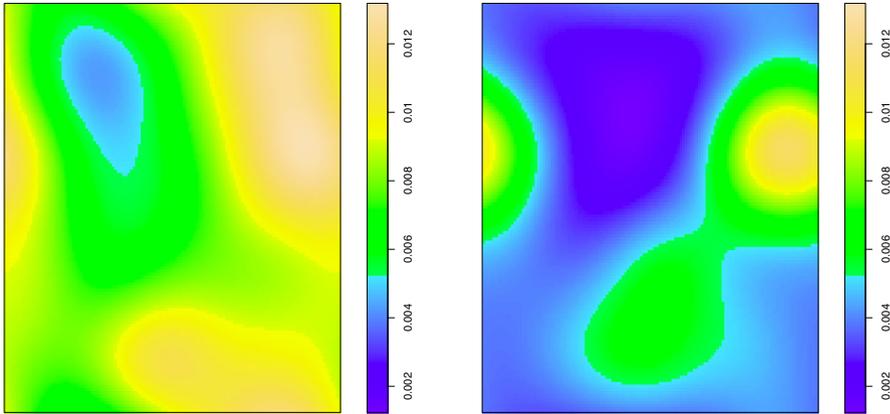
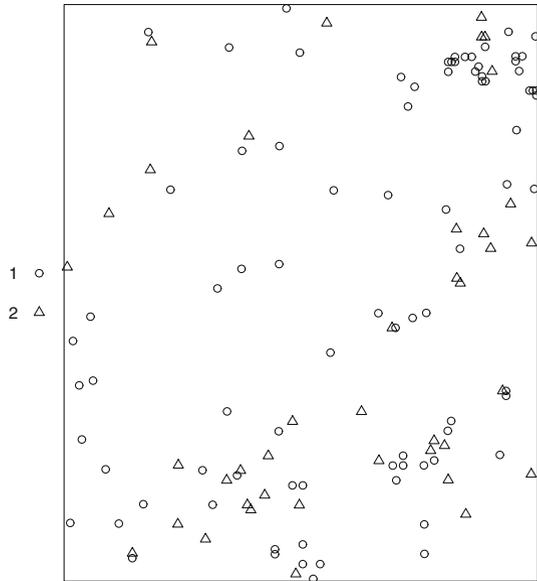


Fig. 1 Gaussian kernel estimator with bandwidth $\sigma = 66$ and torus edge correction calculated over all years except 2000. *Left* forest fuelled fires. *Right* other fires

Fig. 2 Cleaned wildfire data for the year 2000 (Turner 2009). Type 1 fires are those fuelled by ‘forest’; type 2 fires are fuelled by other materials



to n_{2000} to obtain $\hat{c}_{2000} = 124/3120 \approx 0.0397$. As an aside, an alternative model would be to replace the scaling in (20) by a mark dependent one. For $m = \text{‘forest’}$, this would lead to the value $\hat{c}_{2000}(m) \approx 0.0414$, which does not differ much from \hat{c}_{2000} .

Set $C = \{\text{forest}\}$ and write $E = \mathcal{M} \setminus C$. To assess whether the point process Y_C of forest fires occurring in 2000 is correlated with Y_E , we carry out the Lotwick–Silverman scheme developed in Sect. 5.2 and plot envelopes of $\hat{J}_{\text{inhom}}^{CE}$ as discussed in Sect. 5.1 based on 99 independent random torus translations. The results are shown in Fig. 3 and provide graphical evidence for positive correlation between the patterns Y_C

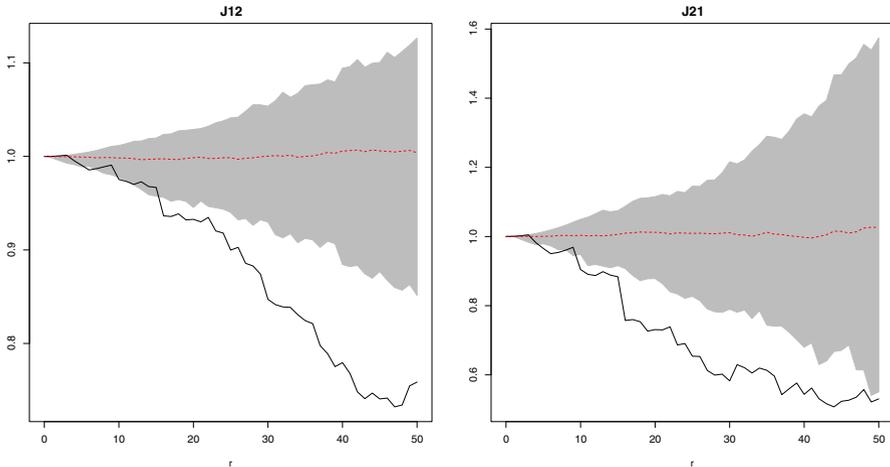


Fig. 3 Estimated inhomogeneous cross J -function for the pattern displayed in Fig. 2 (black line) with rank 5-envelopes (grey area) based on 99 independent translations of the pattern with fuel type ‘forest’ and their mean (dashed line). The plugged-in estimated intensity functions of the component patterns are proportional to those in Fig. 1. Left $\hat{J}_{\text{inhom}}^{CE}$ -function. Right $\hat{J}_{\text{inhom}}^{EC}$ -function

and Y_E , since the estimated inhomogeneous cross J -functions lie mostly below the grey envelope region. This could possibly be interpreted as sparks being transmitted from, say, a forest fire to some place further away, where the ignition takes place in some other matter, e.g. a field of grass. Furthermore, it may also be an indication that during certain periods particular regions are more dry and thus more likely to provide fuel for fires. However, since we do not have any specific temporal information connected to each point in the data set, besides in which year a fire occurs, such conclusions should be treated as speculative. A formal global test could be carried out (Myllymaki et al. 2013) but is computationally intensive.

7 Summary

In this paper, we defined cross D - and J -functions for inhomogeneous intensity-reweighted moment stationary marked point processes and indicated how they could be used to investigate various independence and marking assumptions. In practice, the intensity function tends to be unknown and must be estimated. This is not a problem when there are independent replicates or pseudo-replication in the form of covariates (Guan 2008). Otherwise, pragmatic model assumptions must be made. For example, in Sect. 6, we worked under the assumption that the spatial trend in a given year is proportional to the long-term trend. When prior information about the data is available, a parametric model may also be used.

Finally, it is important to realise that not all point processes on product spaces are marked point processes, as the ground process need not be locally finite. An important special class is formed by spatio-temporal point processes. This class is the focus of a companion paper, Cronie and van Lieshout (2015), in which we define inhomogeneous

D - and J -functions directly by equipping the product space $\mathbb{R}^d \times \mathbb{R}$ with a suitable metric.

Appendix: Proof of Theorem 1

Consider the function $u_r^{a,E}$ of Definition 1 and let $S_r^E = B(0, r) \times E$. Then, the expansion (6) implies

$$\begin{aligned} G(1 - u_r^{a,E}) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d \times \mathcal{M}} \cdots \int_{\mathbb{R}^d \times \mathcal{M}} \prod_{i=1}^n \frac{\bar{\lambda}_E \mathbf{1}\{(z_i, m_i) \in a + S_r^E\}}{\lambda(z_i, m_i)} \\ &\quad \times \rho^{(n)}((z_1, m_1), \dots, (z_n, m_n)) \prod_{i=1}^n dz_i dv(m_i) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \int_{(a+S_r^E)^n} \frac{\rho^{(n)}((z_1, m_1), \dots, (z_n, m_n))}{\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)} \prod_{i=1}^n dz_i dv(m_i), \end{aligned}$$

which is absolutely convergent by assumption and does not depend on the choice of a by the IRMS-assumption on Y . Furthermore, for any $a \in \mathbb{R}^d$,

$$\begin{aligned} G_C^{1a}(1 - u_r^{a,E}) &= \frac{1}{\nu(C)} \int_C \mathbb{E}^{!(a,b)} \left[\prod_{(z,m) \in Y} \left(1 - \frac{\bar{\lambda}_E \mathbf{1}\{(z, m) \in a + S_r^E\}}{\lambda(z, m)} \right) \right] d\nu(b) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \int_C \mathbb{E}^{!(a,b)} \\ &\quad \times \left[\sum_{(z_1, m_1), \dots, (z_n, m_n) \in Y}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{(z_i, m_i) \in a + S_r^E\}}{\lambda(z_i, m_i)} \right] \frac{d\nu(b)}{\nu(C)} \end{aligned}$$

by the local finiteness of Y and an inclusion–exclusion argument.

Next, we show that for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} \int_B \left\{ \int_C \mathbb{E}^{!(a,b)} \left[\sum_{(z_1, m_1), \dots, (z_n, m_n) \in Y}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{(z_i, m_i) \in a + S_r^E\}}{\lambda(z_i, m_i)} \right] d\nu(b) \right\} da &= \\ = \int_B \left\{ \int_C \left(\int_{(S_r^E)^n} \frac{\rho^{(n+1)}((0, b), (z_1, m_1), \dots, (z_n, m_n))}{\lambda(0, b)\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)} \prod_{i=1}^n dz_i dv(m_i) \right) d\nu(b) \right\} da \end{aligned}$$

so that the integrands in between the curly brackets are ℓ -almost everywhere equal and consequently the integrand on the left hand side is constant as a function of $a \in \mathbb{R}^d$. To do so, define

$$g_r^B((a, b), \varphi) = \frac{\mathbf{1}\{(a, b) \in B \times C\}}{\lambda(a, b)} \sum_{(z_1, m_1), \dots, (z_n, m_n) \in \varphi}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{(z_i, m_i) \in a + S_r^E\}}{\lambda(z_i, m_i)},$$

which is non-negative and measurable. By rewriting the expression for $g_r^B((a, b), Y \setminus \{(a, b)\})$, (2) and the translation invariance of the ξ_n , we obtain

$$\begin{aligned} & \mathbb{E} \left[\sum_{(a,b) \in Y} g_r^B((a, b), Y \setminus \{(a, b)\}) \right] \\ &= \mathbb{E} \left[\sum_{(a,b), (z_1, m_1), \dots, (z_n, m_n) \in Y}^{\neq} \frac{\mathbf{1}\{(a, b) \in B \times C\}}{\lambda(a, b)} \prod_{i=1}^n \frac{\mathbf{1}\{(z_i, m_i) \in a + S_r^E\}}{\lambda(z_i, m_i)} \right] \\ &= \int_B \int_C \left(\int_{(S_r^E)^n} \frac{\rho^{(n+1)}((0, b), (z_1, m_1), \dots, (z_n, m_n))}{\lambda(0, b)\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)} \prod_{i=1}^n dz_i dv(m_i) \right) dadv(b). \end{aligned}$$

At the same time, the reduced Campbell–Mecke formula (4) implies that

$$\begin{aligned} & \mathbb{E} \left[\sum_{(a,b) \in Y} g_r^B((a, b), Y \setminus \{(a, b)\}) \right] \\ &= \int_B \int_C \mathbb{E}^{!(a,b)} \left[\sum_{(z_1, m_1), \dots, (z_n, m_n) \in Y}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{(z_i, m_i) \in a + S_r^E\}}{\lambda(z_i, m_i)} \right] dadv(b) \end{aligned}$$

and the required equality of the two expressions follows. Hence, for ℓ -almost all $a \in \mathbb{R}^d$, provided the series is absolutely convergent,

$$\begin{aligned} G_C^{!a}(1 - u_r^{a,E}) &= 1 + \frac{1}{v(C)} \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \int_C \\ & \left(\int_{S_r^E} \cdots \int_{S_r^E} \frac{\rho^{(n+1)}((0, b), (z_1, m_1), \dots, (z_n, m_n))}{\lambda(0, b)\lambda(z_1, m_1) \cdots \lambda(z_n, m_n)} \prod_{i=1}^n dz_i dv(m_i) \right) dv(b) \\ &= 1 + \frac{1}{v(C)} \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \int_C \\ & \left(\int_{S_r^E} \cdots \int_{S_r^E} \sum_{k=1}^{n+1} \sum_{E_1, \dots, E_k} \prod_{j=1}^k \xi_{|E_j|}(\{(z_i, m_i) : i \in E_j\}) \prod_{i=2}^{n+1} dz_i dv(m_i) \right) dv(b), \end{aligned}$$

where $(z_1, m_1) \equiv (0, b)$. By splitting this expression into terms based on whether the index sets E_j contain the index 1 (i.e. on whether $\xi_{|E_j|}$ includes $(z_1, m_1) \equiv (0, b)$), under the convention that $\sum_{k=1}^0 = 1$, we obtain

$$\begin{aligned}
 G_C^{la}(1 - u_r^{a,E}) &= 1 + \frac{1}{\nu(C)} \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} \sum_{\Pi \in \mathcal{P}_n} J_{|\Pi|}^{CE}(r) \\
 &\quad \times \sum_{k=1}^{n-|\Pi|} \sum_{\substack{E_1, \dots, E_k \neq \emptyset \text{ disjoint} \\ \cup_{j=1}^k E_j = \{1, \dots, n\} \setminus \Pi}} \prod_{j=1}^k I_{|E_j|}, \\
 I_n &= \int_{S_r^E} \cdots \int_{S_r^E} \xi_n((z_1, m_1), \dots, (z_n, m_n)) \prod_{i=1}^n dz_i d\nu(m_i),
 \end{aligned}$$

where $J_0^{CE}(r) \equiv \nu(C)$, $|\cdot|$ denotes cardinality and \mathcal{P}_n the power set of $\{1, \dots, n\}$. Finally, by noting that the expansion contains terms of the form $J_k^{CE} I_1^{m_1} \cdots I_n^{m_n}$ multiplied by a scalar and basic combinatorial arguments, we conclude that

$$\begin{aligned}
 G_C^{la}(1 - u_r^{a,E}) &= \frac{1}{\nu(C)} \left(\nu(C) + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda}_E)^n}{n!} J_n^{CE}(r) \right) \\
 &\quad \times \left(1 + \sum_{l=1}^{\infty} \frac{(-\bar{\lambda}_E)^l}{l!} \sum_{k=1}^l \sum_{\substack{E_1, \dots, E_k \neq \emptyset \text{ disjoint} \\ \cup_{j=1}^k E_j = \{1, \dots, l\}}} \prod_{j=1}^k I_{|E_j|} \right) \\
 &= J_{\text{inhom}}^{CE}(r) G(1 - u_r^{0,E}).
 \end{aligned}$$

The right hand side is absolutely convergent as a product of absolutely convergent terms; therefore, so is the series expansion for $G_C^{la}(1 - u_r^{a,E})$. □

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