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A general sequential fixed-accuracy confidence interval estimation methodology for a positive parameter: illustrations using health and safety data

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Abstract Estimation of positive parameters is important in areas including ecology, biology, medicine, nuclear power, and study of cell membranes. We develop a general structure for a *fixed-accuracy* sequential confidence interval estimation methodology in the spirit of Mukhopadhyay and Banerjee (Sequ Anal, 33:251–285, 2014a) for a positive parameter of an arbitrary distribution which may be discrete or continuous. The confidence interval is constructed using a maximum likelihood (ML) estimator of the unknown parameter. The methodology enjoys attractive properties such as asymptotic consistency and asymptotic first-order efficiency (Theorem 1). Three specific illustrations are included. Comprehensive data analyses from large-scale simulations have been incorporated which substantiate encouraging performances of the proposed estimation methodology. These are followed by real data analyses corresponding to the Bernoulli distribution (odds ratio of poisonous mushrooms), Poisson distribution (radioactive decay of isotopes), and a Normal distribution with the same mean and variance (real-time 911 calls dispatch).

Keywords Ecology · Environmental statistics · First-order properties · Insect count · ML estimator · 911 calls · Odds ratio · Positive parameter · Radioactive decay · Sequential methods

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1 Introduction

This paper focuses on developing a general procedure for estimating a positive parameter using sequential confidence interval estimation methods. Before elaborating on the methodology, we first draw attention to the importance of estimating positive parameters in applications.

1.1 Motivation

The first application that comes to mind requires estimation of infestation in ecology. The United States Department of Agriculture (USDA) has performed extensive research and development in the area of infestation. Recent articles (http://www.fs.usda.gov/detail/prescott/landmanagement/?cid=fswdev3_009832) from USDA Forest Service show, for example, that (i) the Ips beetle (sometimes known as "engraver beetles" within a group of bark beetles) infestation in the Penderosa Pine vegetation type in Prescott National Forest, and (ii) the mountain pine beetle in the Black Hills National Forest have reached epidemic levels.

The latter has severely damaged over 1.5 million acres of forest in Northern Colorado and Southern Wyoming. Also, there has been extensive research by USDA Food Safety and Inspection Service on cockroach infestation. In these situations, to minimize financial investments and to achieve effective control of potential damages, a precise estimation of average infestation, which is necessarily positive, is very crucial. In a recent paper, Mukhopadhyay and Banerjee (2014a) have investigated a practical methodology of estimating average infestation in overdispersed count data.

Another example comes from the study of decay of a radioactive isotope of Caesium, Cs-137. Decay of Cs-137 yields barium which is used in food irradiation as well as for radiotherapy of cancer. Cesium is also extensively used in industry, nuclear medicine, nuclear power production, and cell and membrane biology Chakrabarti and Kanjilal (2010). With such extensive and critical utilities, one may observe the distribution of radiation of Cs-137. The parameter of interest here is often the average count of the number of γ-rays emitted from the source of radiation. In this paper, we will analyze such a real count dataset consisting of the number of γ-rays emitted where determination of the half-life of a radioactive isotope is required. There are other examples from medicine and bioinformatics where the primary parameter of interest may be the average number of genes affected by some disease or the average number of genes under mutation. A parameter of interest is necessarily positive under each scenario.

1.2 Ecology and negative binomial

Having emphasized the importance of estimation of positive parameters in applications, we should mention that count data generated due to infestation in the area of entomology are usually over-dispersed and modeled often by a negative binomial (NB) distribution Mulekar and Young (2004). Mukhopadhyay and Banerjee (2014a) developed a fixed-accuracy confidence interval estimation method for the mean μ having k known in an NB(μ, k) model using sequential sampling.

Such an estimate informs one of the magnitudes of average infestation with pre-assigned accuracy. More useful references are found in Mukhopadhyay and Banerjee (2014b).

1.3 Significance of present research

The kind of fixed-accuracy sequential confidence intervals briefly mentioned in Sect. 1.2 is not limited to estimation of the mean of a negative binomial model. We find that the Mukhopadhyay and Banerjee (2014a) methodology can be unified and the basic concepts can be usefully generalized. The idea is to propose a broad structure with relevant generalities so that we can ultimately handle as many distributions and problems as possible in estimating positive parameters of practical importance. The general methodological research presented in this paper (Sect. 3) is applicable for many specific distributions of practical importance.

1.4 Layout of this paper

In Sect. 2, we briefly review selected existing sequential methodologies in estimating parameters that may be of general interest. For brevity, we mention precisely the notions of asymptotic consistency and asymptotic first-order efficiency in (4).

In Sect. 3, we first explain a new notion of "fixed-accuracy" confidence interval, especially relevant for estimating a positive parameter, and then describe an appropriately general sequential methodology to achieve our goal asymptotically (to be made precise soon).

Section 3 does not specifically focus on any particular distribution (such as Bernoulli, Poisson, NB). Instead, we define our sequential estimation methodology using a maximum likelihood (ML) estimator of the parameter of interest under any particular distribution satisfying customary regularity conditions. This procedure works for both discrete and continuous distributions. We show that our developed methodology enjoys the asymptotic consistency and asymptotic first-order efficiency properties in Theorem 1.

Section 4 includes three illustrations of our proposed general methodology in the case of three particular distributions: Bernoulli (Sect. 4.1), Poisson (Sect. 4.2), and Normal with same mean and variance (Sect. 4.3). We explain both implementation and validity of our theoretical findings in the case of these distributions.

Section 5 presents extensive data analysis using large-scale simulations in the contexts of all three illustrations discussed in Sect. 4. Section 6 highlights analysis of real data corresponding to the Bernoulli distribution (Sect. 6.1: odds ratio of poisonous mushrooms), Poisson distribution (Sect. 6.2: radioactive decay of isotopes), and Normal distribution with the same mean and variance (Sect. 6.3: real-time 911 calls dispatch).

Section 7 summarizes some concluding thoughts. It provides additional insights to show that the general approach introduced here can easily include estimation of a parameter whose parameter space is R. This adds some additional breadth of the proposed formulation and methodology.

2 A brief review

A rich body of work has been developed over the years on both point and interval estimation using sequential methods. For a broad review, one may refer to many sources including Ghosh and Sen (1991), Ghosh et al. (1997), Young and Young (1998), Mukhopadhyay et al. (2004), Mukhopadhyay and de Silva (2005, 2009), and Mukhopadhyay and Banerjee (2014a, b).

Before mentioning any specific sequential procedure, we define the following: consider independent random samples $X_1, \ldots, X_n$ of size n with a common probability density function (p.d.f.) or a probability mass function (p.m.f.) $f(x; \theta)$ where x belongs to some appropriate space $\mathcal{X}$ and the parameter of interest θ may or may not be positive. One may customarily begin with a confidence set:

$$C_n = \{\theta : \theta \in [T_n - d, T_n + d]\}, d > 0 \tag{1}$$

for θ where $T_n \equiv T_n(X_1, \ldots, X_n)$ is a point estimator of θ such that $T_n \xrightarrow{P} \theta$ as $n \to \infty$. The width of the confidence interval C_n is $2d$ where d is fixed in advance.

Now, one may additionally require that such a *fixed-width* confidence interval C_n must also have its coverage probability approximately at least $1-\alpha$ where $0 < \alpha < 1$ is fixed in advance. Then, the optimal fixed sample size required may be determined as follows:

$$\text{the smallest } n \geq n_d^0 = z_{\alpha/2}^2 \sigma^2 / d^2, \tag{2}$$

assuming that $n^{1/2}(T_n - \theta) \xrightarrow{\mathcal{L}} N(0, \sigma^2)$ as $n \to \infty$ with finite $\sigma^2 \equiv \sigma^2(\theta) > 0$, some parametric function, and $z_{\alpha/2}$ is the upper $50\,\alpha\%$ point of a standard normal distribution. We tacitly disregard the fact that n_d^0 may not be a (positive) integer.

The unknown optimal fixed sample size n_d^0 is estimated by a stopping rule of the form

$$N \equiv N_d = \inf\{n \geq n_0 : n \geq z_{\alpha/2}^2 \widehat{\sigma}_n^2 / d^2\}, \tag{3}$$

where $\widehat{\sigma}_n^2 \equiv \widehat{\sigma}_n^2(X_1, \ldots, X_n)$ denotes an appropriate estimator for σ^2 and n_0 is the pilot sample size. After termination (under mild regularity conditions guaranteeing $P_\theta(N_d < \infty) = 1$ for all θ) of the sequential procedure (3), one would have on hand the final dataset $\{X_1, \ldots, X_{N_d}, N_d\}$. Then, one may finally propose to estimate θ with C_{N_d} in the light of (1).

The performance of such a sequential stopping estimation rule (N_d, C_{N_d}) is traditionally evaluated by the following criteria: under appropriate regularity conditions, for the estimation rule (N_d, C_{N_d}) associated with the stopping rule (3), one may expect to obtain the following properties for all fixed θ *as* $d \downarrow 0$:

$$\begin{aligned}
&\text{(i)} \quad N_d / n_d^0 \to 1 \text{w.p.1;} \\
&\text{(ii)} \quad E_\theta \left[N_d / n_d^0 \right] \to 1 \text{ [asymptotic (first-order) efficiency]; and,} \\
&\text{(iii)} \quad P_\theta\{\theta \in C_{N_d}\} \to 1 - \alpha \text{ [asymptotic consistency];}
\end{aligned} \tag{4}$$

where n_d^0 comes from (2) and $0 < \alpha < 1$ is fixed in advance. One may review from Chow and Robbins (1965), Ghosh and Mukhopadhyay (1981), and other sources.

Now, let us take a closer look at the fixed-width confidence interval C_N. Obviously we do not face any significant problem when the parameter space is R. But, when the parameter space is R^+, we will surely feel uneasy if one or more confidence limit(s) from C_{N_d} turn out negative. At this point, one may opt to make usual adjustment by proposing

$$C_{N_d}^* = \left[\max(T_{N_d} - d, 0), \max(T_{N_d} + d, 0)\right],$$

but then the asymptotic consistency property from (4) may be severely compromised for this estimation rule $(N_d, C_{N_d}^*)$.

To circumvent this methodological difficulty, one may adopt the *proportional closeness* criteria from Zacks (1966) and Nadas (1969). Instead of a fixed-width confidence interval $C_n = [T_n - d, T_n + d]$ for θ, one may work with the following confidence set:

$$H_n = \{\theta : |T_n - \theta| \leq \delta\,|\theta|\}, \delta > 0 \tag{5}$$

for θ where δ is fixed in advance. Having this prefixed $\delta(> 0)$, again, (5) may lead to awkward upper or lower confidence limit for θ, when θ is assumed positive.

In a large number of point estimation problems, the measure of precision is alternatively taken to be the coefficient of variation (CV). An extensive review of the existing methodologies in estimating parameters of a negative binomial distribution, in particular, was given by Mukhopadhyay and Banerjee (2014a, b). This elaborated both sequential and non-sequential procedures.

On the other hand, Khan (1969) proposed a general method of estimating parameters (not necessarily positive) using sequential methods. Under mild regularity conditions, Khan (1969) developed a fixed-width confidence interval approach based on the ML estimator $\widehat{\theta}_{\text{MLE}}$ for the unknown parameter θ. For a fixed number $d(>0)$, the author began with a confidence interval of the usual form $\left[\widehat{\theta}_{\text{MLE}} - d, \widehat{\theta}_{\text{MLE}} + d\right]$, proposed an appropriate sequential stopping rule to estimate the optimal fixed sample size, and proved asymptotic first-order efficiency and asymptotic consistency properties defined in (4). We should emphasize that our proposed confidence interval estimation methodology described in Sect. 3 is based on the ML estimator of the unknown parameter $\theta(>0)$ in the light of Mukhopadhyay and Banerjee (2014a).

3 Fixed-accuracy confidence intervals

3.1 The formulation

In Sects. 1.1 and 1.2, we emphasized the importance of estimation of average infestation. Recall that a fixed-width or fixed proportional accuracy confidence interval procedure may have undesirable properties with $d > 0$ or $\delta > 0$. Here, we develop a broad and general structure extending Mukhopadhyay and Banerjee (2014a) notion of

a *fixed-accuracy* confidence interval estimation approach in the case of an unknown parameter $\theta \in \Theta$ where $\Theta \equiv R^+$.

Let us assume that we have a sequence of independent and identically distributed (i.i.d.) observations $X_1, X_2, \ldots$, either discrete or continuous, following a common p.m.f. or p.d.f. $f(x;\theta)$ involving a single parameter θ. Having recorded $X_1, \ldots, X_n$, we let $T_n \equiv T_n(X_1, \ldots, X_n)$ be an arbitrary point estimator for θ with the following properties:

(a) $0 < T_n < \infty$w.p.1;

(b) $n^{1/2}(T_n - \theta) \xrightarrow{\mathcal{L}} N(0, \sigma^2)$ with some $0 < \sigma^2 < \infty$ as $n \to \infty$; where $\sigma^2 \equiv \sigma^2(\theta)$ is a continuous function of θ;

(c) Anscombe's (1952) uniform continuity in probability (u.c.i.p.) condition: For every $\varepsilon > 0$ and $0 < \gamma < 1$ there exists some $\delta > 0$ such that $\limsup_n P_\theta\{\max_{|n'-n| \le \delta n} n^{1/2} |T_{n'} - T_n| > \varepsilon\} < \gamma$. (6)

We will often require that T_n be some function of the ML estimator of θ. Observe that under very mild regularity conditions on $f(x;\theta)$, the ML estimator of θ would satisfy all three conditions (a)–(c) listed in (6).

Next, having fixed some preassigned $d > 1$, we propose to consider the following fixed-accuracy confidence interval for θ:

$$J_n = \{\theta : \theta \in [d^{-1}T_n, dT_n]\}, n \ge 1. \tag{7}$$

One may instead argue in favor of the confidence set H_n from (5) by deliberately fixing δ to lie in $(0, 1)$. When θ is a positive proportion, such a choice of δ may be meaningful, but we do not find such a choice particularly compelling for θ belonging to R. For example, if θ represents the daily average number of 911 calls or the average monthly income per family, then allowing any preassigned $\delta > 0$ sounds more appealing to us when it comes to (5).

One may note that our proposed fixed-accuracy confidence interval J_n is not symmetric around T_n. However, symmetry is achieved around $\ln T_n$ for the unknown parameter $\ln \theta$. Mukhopadhyay and Banerjee (2014a) illustrated J_n geometrically in the NB case.

3.2 The basic structure

Appealing to central limit theorem (CLT) for T_n, that is under (6) condition (b), we have

$$n^{1/2}(\ln T_n - \ln \theta) \xrightarrow{\mathcal{L}} N(0, \theta^{-2}\sigma^2(\theta)) \quad \text{as } n \to \infty,$$

in view of the delta method or equivalently Mann–Wald theorem (Rao (1973), pp. 385–386; Mukhopadhyay (2000), pp. 261–262). Now, for J_n to include θ with a

preassigned probability $1-\alpha$, $0<\alpha<1$, the required optimal fixed sample size will reduce to

$$\text{the smallest } n \geq n_d^* = \left(\frac{z_{\alpha/2}}{\ln d}\right)^2 \theta^{-2}\sigma^2(\theta). \tag{8}$$

This n_d^* is a function of the unknown parameter θ and thus it remains unknown. To propose a stopping rule, the parameter θ within the expression of n_d^* needs to be replaced by its estimator. Here, we denote the ML estimator of θ:

$$U_n \equiv U_n(X_1, \ldots, X_n) = \widehat{\theta}_{n,\mathrm{MLE}}, n \geq n_0,$$

where n_0 is the pilot sample size. Using U_n to estimate θ within the expression of n_d^*, we write down the following sequential stopping rule:

$$N \equiv N_d = \inf\left\{n \geq n_0 : nU_n^2\left(\sigma^2(U_n)\right)^{-1} \geq \left(\frac{z_{\alpha/2}}{\ln d}\right)^2\right\}. \tag{9}$$

Remark 1 It is entirely possible that $U_n^2(\sigma^2(U_n))^{-1}$ may be zero with a positive probability in the case of observations from a discrete population distribution. But we may note that (9) will not terminate sampling as long as $U_n^2(\sigma^2(U_n))^{-1}$ continues to be zero.

Remark 2 On the other hand, one may implement the following stopping time:

$$N = \inf\left\{n \geq n_0 : n \geq \left(\frac{z_{\alpha/2}}{\ln d}\right)^2\left(U_n^{-2}\sigma^2(U_n) + n^{-1}\right)\right\}, \tag{10}$$

so that $U_n^{-2}\sigma^2(U_n) + n^{-1}$ is positive with probability one and remains consistent for $\theta^{-2}\sigma^2(\theta)$.

The illustrations given in Sect. 4 fall within the category covered by Remark 1 and hence we continue to work under the stopping rule (9). Frequently, it may so happen that we may be able to express U_n as a sample mean of i.i.d. random variables for all $n \geq n_0$. Thus, we denote $W_i = q(X_i)$ with some appropriate function $q(.)$, $i = 1, \ldots, n$, and let $\overline{W}_n = n^{-1}\Sigma_{i=1}^n W_i$, $n \geq n_0$.

Now, we rewrite the stopping rule (9) as follows:

$$N \equiv N_d = \inf\left\{n \geq n_0 : ng(\overline{W}_n) \geq a\right\}, \tag{11}$$

with

$$a \equiv a(\alpha, d) = \left(z_{\alpha/2}/\ln d\right)^2,$$

where $g(.)$ is some appropriate function. We assume that $g(.)$ is twice differentiable and $g''(.)$ is continuous.

In the illustrations from Sect. 4, one will see explicit expressions of θ, T_n, $\sigma^2(\theta)$, U_n, $q(.)$ and $g(.)$ in each case under consideration. We will exploit the representation from (11) to prove Theorem 1 part (ii) using some of the established techniques from non-linear renewal theory along the lines of Woodroofe (1977, 1982), Siegmund (1985, pp. 188–208), and Ghosh et al. (1997, pp. 58–65).

Next, we may equivalently rewrite the stopping rule (11) as

$$N \equiv N_d = \inf\left\{n \geq n_0 : Z_n \equiv ng(\overline{W}_n) \geq a\right\}. \tag{12}$$

Comparing (11)–(12) and using Taylor's expansion, we have

$$Z_n = \sum_{i=1}^{n} Y_i + \xi_n,$$

where $Y_1, \ldots, Y_n$ are i.i.d. random variables defined as

$$Y_i = g\left(E_\theta\left[W_i\right]\right) + g'\left(E_\theta\left[W_i\right]\right)\left(W_i - E_\theta\left[W_i\right]\right). \tag{13}$$

Here, $\{\xi_n; n \geq 1\}$ is a sequence of random variables given by

$$\xi_n = \frac{1}{2} n g''(\nu_n)\left(\overline{W}_n - E_\theta\left[W_1\right]\right)^2, \tag{14}$$

where $\{\nu_n; n \geq 1\}$ is a sequence of random variables such that ν_n lies between $\overline{W}_n$ and $E_\theta\left[W_1\right]$.

Next, we make the following assumptions on the sequence $\{\xi_n; n \geq 1\}$:

$$\begin{aligned} &\text{(i)} \quad n^{-1}\max_{1\leq i\leq n}|\xi_i| \xrightarrow{P_\theta} 0 \text{ as } n \to \infty; \\ &\text{(ii)} \quad \Sigma_{n=1}^{\infty} P_\theta\left\{\xi_n \leq -n\epsilon\right\} < \infty \text{ for some } 0 < \epsilon < E_\theta\left[W_1\right]. \end{aligned} \tag{15}$$

Now, we restate the following lemma from Woodroofe (1982, p. 41) and Ghosh et al. (1997, p. 59) for completeness. We will use this lemma for verifying the conditions stated in (15) in the case of some specific distributions considered in Sect. 4.

Lemma 1 *Consider ξ_n from (14). If $n^{-1}\xi_n \xrightarrow{P} 0$ as $n \to \infty$, then $n^{-1}\max_{1\leq i\leq n}|\xi_i| \xrightarrow{P} 0$ as $n \to \infty$.*

We observe that $P_\theta\left\{N_d < \infty\right\} = 1$ for every fixed d, θ, α so that after termination of the sequential procedure (9), we will have on hand the final set of data $\{N_d, X_1, \ldots, X_{N_d}\}$. Then, we propose the fixed-accuracy sequential confidence interval

$$J_N \equiv J_{N_d} = \left[d^{-1}T_{N_d}, dT_{N_d}\right] \tag{16}$$

for estimating the unknown parameter $\theta(> 0)$. In Theorem 1 stated in Sect. 3.3, we prove some important asymptotic first-order properties for the estimation rule $\left(N_d, J_{N_d}\right)$ associated with (9).

3.3 The main result

Theorem 1 *With the assumptions stated in (6), (15), and that $g(.)$ is twice differentiable and $g''(.)$ is continuous, for the estimation rule $\left(N_d, J_{N_d}\right)$ under the purely sequential stopping time (9), for each fixed $\theta(>0)$ and $0<\alpha<1$, we have as $d \downarrow 1$:*

(i) $N_d/n_d^* \to 1$ *w.p.1;*
(ii) $E_\theta\left[N_d/n_d^*\right] \to 1$; *and*
(iii) $P_\theta\left\{\theta \in J_{N_d} : [d^{-1}T_{N_d}, dT_{N_d}]\right\} \to 1-\alpha$;

where n_d^* comes from (8).

Part (i) is needed in the proofs of parts (ii) and (iii). Part (ii) shows that the proposed purely sequential procedure (9) is *asymptotically efficient* or *asymptotically first-order efficient* according to Chow and Robbins (1965) or Ghosh and Mukhopadhyay (1981), respectively. Part (iii) verifies *asymptotic consistency* of the procedure.

We offer following practical interpretations: (a) Part (ii) indicates that the average sequential sample size may be expected to hover around the optimal fixed sample size n_d^*; (b) Part (iii) indicates that the coverage probability for the confidence interval J_N obtained upon termination of the sequential procedure may be expected to be in a close proximity of the prefixed target, $1-\alpha$. Next, we briefly outline a proof of the main result.

Proof of Theorem 1

Part (i):

It follows from Chow and Robbins (1965) after noting the following facts: $N_d \to \infty$ w.p.1, $U_{N_d} \to \theta$ w.p.1, $U_{N_d-1} \to \theta$ w.p.1., $\sigma^2(\theta)$ is a continuous function of θ, and $n_0/n_d^* \to 0$ as $d \downarrow 1$.

Part (ii):

We rely upon Woodroofe (1977, 1982) and Ghosh et al. (1997) using techniques available from nonlinear renewal theory. Under the stated conditions (6), (15), and that $g(.)$ is twice differentiable and $g''(.)$ is continuous, we may appeal to Woodroofe (1982) or (Ghosh et al., 1997, Theorem 2.9.3, p. 62) to claim that the sequence of random variables $\{n_d^{*-1}N_d; d>1\}$ is uniformly integrable. Combining this fact with part (i), our part (ii) follows immediately by applying the dominated convergence theorem.

Part (iii):

We use Anscombe (1952) random CLT for the ML estimator. For more recent work on Anscombe's random CLT, one may also look at Mukhopadhyay and Chattopadhyay (2012) and Gut (2012).

Under conditions (b)–(c) from (6), using Anscombe (1952) random CLT, we can claim

$$\frac{N_d^{1/2}(T_{N_d}-\theta)}{\sigma(\theta)} \xrightarrow{\mathcal{L}} N(0,1),$$

so that we have

$$\frac{n_d^{*1/2}(T_{N_d} - \theta)}{\sigma(\theta)} \xrightarrow{\mathcal{L}} N(0,1) \quad \text{as } d \downarrow 1, \tag{17}$$

in view of part (i) and Slutsky's theorem. Hence, using the Mann–Wald Theorem Rao (1973, pp. 385–386; Mukhopadhyay (2000), pp. 261–262) on top of (17), we can conclude the following

$$Q_{N_d} \equiv \frac{z_{\alpha/2}(\ln T_{N_d} - \ln \theta)}{\ln d} \xrightarrow{\mathcal{L}} N(0,1) \quad \text{as } d \downarrow 1. \tag{18}$$

Thus, we can express

$$P_\theta\left\{\theta \in J_{N_d}\right\} = P_\theta\left\{|\ln T_N - \ln \theta| < \ln d\right\} = P_\theta\left\{\left|Q_{N_d}\right| < z_{\alpha/2}\right\},$$

which converges to $1-\alpha$ as $d \downarrow 1$ using (18). The proof of part (iii) is now complete.

This completes our proof of Theorem 1. □

Remark 3 We have not yet handled a proof of the asymptotic (as $d \downarrow 1$) second-order efficiency property postulated by Ghosh and Mukhopadhyay (1981), but Sect. 3.2 does highlight a basic non-linear renewal-theoretic structure of Woodroofe (1977, 1982), Siegmund (1985, pp. 188–208), and Ghosh et al. (1997, pp. 58–65). We expect to show the asymptotic second-order efficiency property in connection with the illustrations from Sect. 4.

4 Illustrations

Thus far we have elaborated a general methodology of fixed-accuracy confidence interval estimation for an unknown positive parameter. In the sequential stopping rule (9), we used the ML estimator, $U_n \equiv \widehat{\theta}_{n,\mathrm{MLE}}$. Under very mild regularity conditions (see, for example, Rao (1973, pp. 348–350) or Sen and Singer (1993, pp. 202–210)), we claim the customary CLT for U_n:

$$n^{1/2}(U_n - \theta) \xrightarrow{\mathcal{L}} N(0, I_X^{-1}(\theta)) \quad \text{with} \quad 0 < I_X(\theta) < \infty \quad \text{as } n \to \infty, \tag{19}$$

where

$$I_X(\theta) = E\left[\left(\frac{\partial}{\partial\theta} \ln f(X;\theta)\right)^2\right], \tag{20}$$

denotes Fisher Information about θ in a single observation X.

Next, we give some illustrations from specific distributions. Under each scenario, we have provided analysis of simulated as well as real data in Sects. 5 and 6.

4.1 Bernoulli distribution: odds ratio estimation

In statistical ecology, many a times, one is interested in observing "presence or absence" of infestation. We may assume that the proportion (or probability) of infestation is an unknown quantity p, $0 < p < 1$, which stays the same for the duration of an experiment. Then, we are interested in estimating the odds ratio of infestation, namely the parameter $\theta \equiv p/(1-p)$. Odds ratio estimation is also common in other fields. See, for example, Robbins and Siegmund (1974) and Zacks and Mukhopadhyay (2007).

Consider a random sample $X_1, X_2, \ldots$ from a Bernoulli(p) population. The parameter of interest is θ and $\theta \in \Theta = (0, \infty)$.

4.1.1 Estimation of odds-ratio

Based on $X_1, \ldots, X_n$, we have

$$\widehat{p}_{n,\mathrm{MLE}} = \overline{X}_n.$$

Using the invariance property of MLE Zehna (1966), we let

$$U_n \equiv \widehat{\theta}_{n,\mathrm{MLE}} = \frac{\overline{X}_n}{1-\overline{X}_n}. \tag{21}$$

Since the probability of $\overline{X}_n = 0$ is positive, the probability of $\overline{X}_n = 1$ is positive, we define

$$T_n = \frac{\overline{X}_n + n^{-\gamma}}{1-\overline{X}_n + n^{-\gamma}} \tag{22}$$

where $\gamma > \frac{1}{2}$ is a constant. Next, with a preassigned level of accuracy $d(>1)$, we consider as in (7):

$$J_n = \left\{\theta : \theta \in [d^{-1}T_n, dT_n]\right\}.$$

One can verify easily that the expression of the variance in the asymptotic distribution of $n^{1/2}(T_n - \theta)$ will be given by $\sigma^2(\theta) \equiv \theta(\theta+1)^2$. Thus, for J_n to include θ with a preassigned probability $1-\alpha$, $0 < \alpha < 1$, the required optimal fixed sample size (8) reduces to

$$\text{the smallest } n \geq n_d^* = \left(\frac{z_{\alpha/2}}{\ln d}\right)^2 \theta^{-1}(\theta+1)^2. \tag{23}$$

This n_d^* is unknown and hence in the light of (9), we arrive at the stopping rule:

$$N \equiv N_d = \inf\left\{n \geq n_0 : nU_n(U_n+1)^{-2} \geq \left(\frac{z_{\alpha/2}}{\ln d}\right)^2\right\}, \tag{24}$$

where U_n comes from (21). Recall Remark 1.

Now, after implementing the sequential procedure (24), one has the final dataset $\{N_d, X_1, \ldots, X_{N_d}\}$ and estimates θ with the confidence interval $J_{N_d} \equiv [d^{-1}T_{N_d}, dT_{N_d}]$. At this point, we set out to check the sufficient conditions under which Theorem 1 holds.

4.1.2 Verification of conditions from (6), (15), and for $g(.)$

Note that we have $0 < T_n < \infty$ w.p.1 which is (6) part (a). Also, as T_n is a suitable function of U_n so that (6) part (b) holds with $\sigma^2(\theta) = \theta(\theta+1)^2$ which is continuous in θ. Anscombe (1952) proved the u.c.i.p. condition for the ML estimator. Thus, T_n from (22) satisfies the u.c.i.p. condition so that (6) part (c) holds.

The stopping time N_d from (24) can be rewritten as follows:

$$N \equiv N_d = \inf\left\{n \geq n_0 : Z_n \equiv ng(\overline{X}_n) \geq a(\alpha, d)\right\},$$

where $a(\alpha, d) = \left(\frac{z_{\alpha/2}}{\ln d}\right)^2$, $g(x) = x(1-x)$, $0 \leq x \leq 1$, $Z_n = \Sigma_{i=1}^n Y_i + \xi_n$ with

$$Y_i = p(1-p) + (1-2p)(X_i - p), \quad i = 1, \ldots, n,$$

and

$$\xi_n = 2^{-1} n g''(v_n)\left(\overline{X}_n - p\right)^2 = -n\left(\overline{X}_n - p\right)^2.$$

Note that the function $g(.)$ is twice differentiable, and $g''(.)$ is continuous; the function $q(.)$ from (11) is an identity map, that is, we have $W_i = X_i$ for all $i = 1, \ldots, n$ and thus $E_\theta[W_1] = p$.

Here, v_n lies between $\overline{X}_n$ and p in the spirits of (13)–(14) so that we get

$$\sum_{i=1}^n Y_i = np(1-p) + (1-2p)\left(\sum_{i=1}^n X_i - np\right).$$

Note that, as $n \to \infty$, $\overline{X}_n \to p$ w.p.1 so that $\xi_n/n \xrightarrow{P} 0$. Using Lemma 1, part (i) from (15) is verified.

Now, let $\mu_{4,n}$ stand for $E_p[(\Sigma_{i=1}^n X_i - np)^4]$ and $\xi_p(>0)$ for a constant depending only on p. Then, for $0 < \epsilon < p$, we may write

$$\begin{aligned}\sum_{n=1}^\infty P_p\left(\xi_n \leq -n\epsilon\right) &\leq \sum_{n=1}^\infty P_p\left\{\left(\overline{X}_n - p\right)^2 \geq \epsilon\right\} \leq \epsilon^{-2}\sum_{n=1}^\infty n^{-4}\mu_{4,n} \\ &\leq \xi_p \epsilon^{-2}\sum_{n=1}^\infty n^{-2} < \infty. \end{aligned} \tag{25}$$

This verifies (15) part (ii).

Hence, we conclude right away that our sequential confidence interval methodology (N_d, J_{N_d}) where (i) N_d comes from (24) and (ii) $J_{N_d} \equiv [d^{-1}T_{N_d}, dT_{N_d}]$ with T_{N_d} coming from (22) is asymptotically first-order efficient and asymptotically consistent.

Remark 4 An alternate way to verify (15) part (ii) may go as follows. For a sequence of random variables $\left\{Q_n^*; n \geq 1\right\}$ and Q^*, the following result is well known:

$$Q_n^* \to Q^* \text{ w.p.1 as } n \to \infty \text{ if and only if}$$
$$\sum_{n=1}^{\infty} P\left\{\left|Q_n^* - Q^*\right| \geq \delta\right\} < \infty \text{ for each } \delta > 0.$$

One may refer to Jacod and Protter (2003, p. 74). In our context, $\overline{X}_n \to p$ w.p.1 so that by choosing $\epsilon = \delta^2$, we have (25).

4.2 Poisson distribution: mean estimation

A Poisson model is heavily used to study radioactive decay in cancer therapy, biology, and bioinformatics. In such cases, a parameter of interest is the mean radiation count. Consider a random sample $X_1, X_2, \ldots$ from a Poisson(θ) population. The parameter of interest is θ and $\theta \in \Theta = (0, \infty)$.

4.2.1 Estimation of mean

Based on $X_1, \ldots, X_n$, we have

$$U_n \equiv \widehat{\theta}_{n,\text{MLE}} = \overline{X}_n. \tag{26}$$

Since the probability of $\overline{X}_n = 0$ is positive, we define

$$T_n = \overline{X}_n + n^{-\gamma} \tag{27}$$

where $\gamma > \frac{1}{2}$. Next, with a preassigned level of accuracy $d(>1)$, we consider as in (7):

$$J_n = \left\{\theta : \theta \in [d^{-1}T_n, dT_n]\right\}.$$

From the asymptotic distribution of $n^{1/2}(T_n - \theta)$ we know that $\sigma^2(\theta) \equiv \theta$. Thus, for J_n to include θ with a preassigned probability $1-\alpha$, $0 < \alpha < 1$, the required optimal fixed sample size (8) reduces to

$$\text{the smallest } n \geq n_d^* = \left(\frac{z_{\alpha/2}}{\ln d}\right)^2 \theta^{-1}. \tag{28}$$

This n_d^* is unknown and hence in the light of (9), we arrive at the stopping rule:

$$N \equiv N_d = \inf\left\{n \geq n_0 : nU_n \geq \left(\frac{z_{\alpha/2}}{\ln d}\right)^2\right\}. \tag{29}$$

where U_n comes from (26). Recall Remark 1.

Now, after implementing the sequential procedure (29), one has the final dataset $\{N_d, X_1, \ldots, X_{N_d}\}$ and estimates θ with the confidence interval $J_{N_d} \equiv [d^{-1}T_{N_d}, dT_{N_d}]$. At this point, we set out to check the sufficient conditions under which Theorem 1 holds.

4.2.2 *Verification of conditions from* (6), (15), *and for* $g(.)$

Note that we have $0 < T_n < \infty$ w.p.1 which is (6) part (a). Also, recall that (6) part (b) holds with $\sigma^2(\theta) = \theta$ which is continuous in θ. T_n from (27) satisfies Anscombe (1952) u.c.i.p. condition so that (6) part (c) holds.

Along the line of (11), the stopping time N_d from (29) can be rewritten as follows:

$$N \equiv N_d = \inf\left\{n \geq n_0 : Z_n \equiv ng(\overline{X}_n) \geq a(\alpha, d)\right\},$$

where $a(\alpha, d) = \left(\frac{z_{\alpha/2}}{\ln d}\right)^2$, $g(x) = x$, $0 \leq x < \infty$, $Z_n = \Sigma_{i=1}^n Y_i + \xi_n$ with $Y_i = X_i$, $i = 1, \ldots, n$ and $\xi_n = 0$ for all n. We obviously have $\Sigma_{i=1}^n Y_i = \Sigma_{i=1}^n X_i$. Note that the function $g(.)$ is twice differentiable, and $g''(.)$ is continuous. Clearly, the function $q(.)$ from (11) is an identity map. That is, we have $W_i = X_i$ for all $i = 1, \ldots, n$ and thus $E_\theta[W_1] = \theta$.

Thus, (15) part (ii) holds. Hence, we have verified all sufficient conditions under which Theorem 1 holds. We conclude right away that our sequential confidence interval methodology (N_d, J_{N_d}) where (i) N_d comes from (29) and (ii) $J_{N_d} \equiv [d^{-1}T_{N_d}, dT_{N_d}]$ with T_{N_d} coming from (27) is asymptotically first-order efficient and asymptotically consistent.

4.3 Normal distribution with equal mean and variance

A first attempt to model count data usually proceeds via Poisson distribution. But, we know that if the mean is very large, then a Poisson distribution may be approximated by a Normal distribution with equal mean and variance. One may refer to Mukhopadhyay and Cicconetti (2004), Bhattacharjee (2011), Mukhopadhyay and Bhattacharjee (2011, 2012), and other sources.

Now, consider a random sample $X_1, X_2, \ldots$ from an $N(\theta, \theta)$ population. The parameter of interest is θ and $\theta \in \Theta = (0, \infty)$.

4.3.1 Estimation of mean

Based on $X_1, \ldots, X_n$, we have

$$U_n \equiv \widehat{\theta}_{n,\mathrm{MLE}} = -\frac{1}{2} + \left\{ n^{-1} \sum_{i=1}^{n} X_i^2 + \frac{1}{4} \right\}^{1/2}. \tag{30}$$

Then, we define

$$T_n = U_n. \tag{31}$$

Next, with a preassigned level of accuracy $d(>1)$, we consider as in (7):

$$J_n = \left\{ \theta : \theta \in [d^{-1} T_n, d T_n] \right\}.$$

From the asymptotic distribution of $n^{1/2}(T_n - \theta)$ we know that $\sigma^2(\theta) \equiv 2\theta^2(2\theta+1)^{-1}$. Thus, for J_n to include θ with a preassigned probability $1-\alpha$, $0 < \alpha < 1$, the required optimal fixed sample size (8) reduces to

$$\text{the smallest } n \geq n_d^* = \left(\frac{z_{\alpha/2}}{\ln d} \right)^2 2(2\theta + 1)^{-1}. \tag{32}$$

This n_d^* is unknown and hence in the light of (9), we arrive at the stopping rule:

$$N \equiv N_d = \inf \left\{ n \geq n_0 : \frac{1}{2} n (2U_n + 1) \geq \left(\frac{z_{\alpha/2}}{\ln d} \right)^2 \right\}, \tag{33}$$

where U_n comes from (30).

Now, after implementing the sequential procedure (33), one has the final dataset $\{N_d, X_1, \ldots, X_{N_d}\}$ and estimates θ with the confidence interval $J_{N_d} \equiv [d^{-1} T_{N_d}, d T_{N_d}]$. At this point, we set out to check the sufficient conditions under which Theorem 1 holds.

4.3.2 Verification of conditions from (6), (15), and for $g(.)$

Note that we have $0 < T_n < \infty$ w.p.1 which is (6) part (a). Also, recall that (6) part (b) holds with $\sigma^2(\theta) = 2\theta^2(2\theta+1)^{-1}$ which is continuous in θ. T_n from (31) clearly satisfies Anscombe (1952) u.c.i.p. condition so that (6) part (c) holds.

Along the line of (11), the stopping time N_d from (33) can be rewritten as follows:

$$N \equiv N_d = \inf \left\{ n \geq n_0 : n g \left(n^{-1} \sum_{i=1}^{n} X_i^2 \right) \geq a(\alpha, d) \right\},$$

where $a(\alpha, d) = \left(\frac{z_{\alpha/2}}{\ln d}\right)^2$ and $g(x) = (x + \frac{1}{4})^{1/2}$, $0 < x < \infty$. Note that the function $g(.)$ is twice differentiable, $g''(.)$ is continuous, and the function $q(.)$ from (11) amounts to $W_i = q(X_i) = X_i^2$ for all $i = 1, \ldots, n$ so that $E_\theta[W_1] = \theta + \theta^2$.

Re-expressing our stopping rule along the line of (12), we have

$$N \equiv N_d = \inf\{n \geq n_0 : Z_n \geq a(\alpha, d)\},$$

where $Z_n = \Sigma_{i=1}^n Y_i + \xi_n$ with

$$Y_i = \left(\theta + \theta^2 + \frac{1}{4}\right)^{1/2} + \frac{1}{2}\left(\theta + \theta^2 + \frac{1}{4}\right)^{-1/2}\left(X_i^2 - \left(\theta + \theta^2\right)\right), \quad i = 1, \ldots, n,$$

and for all n,

$$\begin{aligned}\xi_n &= \frac{1}{2} n g''(\nu_n)\left(n^{-1}\sum_{i=1}^n X_i^2 - \left(\theta + \theta^2\right)\right)^2 \\ &= -\frac{1}{8} n \left(\nu_n + \frac{1}{4}\right)^{-3/2}\left(n^{-1}\sum_{i=1}^n X_i^2 - \left(\theta + \theta^2\right)\right)^2.\end{aligned}$$

Here, ν_n lies between $n^{-1}\Sigma_{i=1}^n X_i^2$ and $(\theta + \theta^2)$ in the spirits of (13)–(14) so that we get

$$\sum_{i=1}^n Y_i = n\left(\theta + \theta^2 + \frac{1}{4}\right)^{1/2} + \frac{1}{2}\left(\theta + \theta^2 + \frac{1}{4}\right)^{-1/2}\left(\sum_{i=1}^n X_i^2 - (\theta + \theta^2)\right).$$

Note that, as $n \to \infty$, $n^{-1}\Sigma_{i=1}^n X_i^2 \to \theta + \theta^2$ w.p.1 so that $\xi_n/n \xrightarrow{P} 0$. Using Lemma 1, part (i) from (15) is verified. Also, as $n \to \infty$, $\hat{\theta}_{n,\mathrm{MLE}} \to \theta$ w.p.1. Now, to verify the condition in part (ii) of (15), let $0 < \epsilon < \theta + \theta^2$ and $\xi_\theta (>0)$ be a constant depending only on θ. Then, we can express

$$\begin{aligned}&\sum_{n=1}^\infty P_\theta\{\xi_n \leq -n\epsilon\} \\ &= \sum_{n=1}^\infty P_\theta\left\{\left\{n^{-1}\sum_{i=1}^n X_i^2 - (\theta + \theta^2)\right\}^2 \geq 8\epsilon\left(\nu_n + \frac{1}{4}\right)^{3/2}\right\} \\ &\leq \epsilon^2 \sum_{n=1}^\infty E_\theta\left\{n^{-1}\sum_{i=1}^n X_i^2 - (\theta + \theta^2)\right\}^4 \\ &\leq \epsilon^2 \xi_\theta \sum_{n=1}^\infty n^{-2}, \end{aligned} \tag{34}$$

which is obviously finite. The last step in (34) follows readily from the moment-inequality of Sen and Ghosh (1981, Lemma 2.2). This verifies (15) part (ii).

We conclude right away that our sequential confidence interval methodology (N_d, J_{N_d}) where (i) N_d comes from (33) and (ii) $J_{N_d} \equiv [d^{-1}T_{N_d}, dT_{N_d}]$ with T_{N_d} coming from (31) is asymptotically first-order efficient and asymptotically consistent.

5 Data analysis from simulations

In support of the general sequential fixed-accuracy confidence interval methodology described in Sect. 3, we gave three distinct illustrations in Sect. 4. To evaluate the moderate sample performances of those procedures, we proceed to data analysis from extensive sets of simulations.

5.1 Bernoulli odds ratio estimation from Sect. 4.1

Tables 1 and 2 summarize simulation results corresponding to the purely sequential procedure (24) in the case of a Bernoulli distribution. Each row is a summary obtained from 10000 simulations for $p = 0.2, 0.4, 0.5, 0.6, 0.7, 0.9$, n_d^* correspondingly computed using (23), $\gamma = 0.7, 1.0$, $d = 1.10, 1.07$, $\alpha = 0.05$, and $n_0 = 10$, the pilot sample size. The choice of n_0 did not appear to impact findings since the required sample size updated itself in a sequential manner.

We ran similar simulations with a number of choices of α other than 0.05. But, the overall features remained similar whatever choices of α we had made. Hence, for brevity, our tables correspond to $\alpha = 0.05$.

The $\overline{n}$ column shows the average sequential sample size from 10000 runs along with $s(\overline{n})$, the estimated standard error of $\overline{n}$. The next two columns, that is, the ratio and difference of $\overline{n}$ and n_d^*, give us an idea of efficiency measure in practice. The ratio $\overline{n}/n_d^*$ is expected to be close to 1 and this is a measure of the first-order efficiency. One may refer to Theorem 1, parts (i) and (ii). The difference, $\overline{n} - n_d^*$, gives us an idea regarding the status of second-order efficiency in practice. Although we have not yet proved a theoretical result claiming second-order efficiency for the purely sequential procedure (24), we try to gauge its possible status via simulations.

Next, suppose that κ is an indicator variable which takes the value 1(0) if the confidence interval J_n at termination when $N = n$ obtained for each run includes (does not include) the true value of θ. Then, $\overline{\kappa}$ is the average from 10000 such 0/1 observed values of κ. Clearly, $\overline{\kappa}$ gives an idea about the achieved coverage probability which we hope to be close to the preset target, 0.95. The estimated standard error values for $\overline{\kappa}$, namely $s(\overline{\kappa})$, are also provided.

We are satisfied to observe that $\overline{n}/n_d^*$ and $\overline{\kappa}$ values are respectively very close to 1 and 0.95, with both estimated standard errors small. From columns 7 and 11 in Tables 1 and 2, we have a distinct feeling that perhaps $E_p[N_d - n_d^*]$ remains bounded for the purely sequential procedure (24) as $d \downarrow 1$. Recall Remark 3.

Table 3 shows fewer columns, but it highlights performances for a range of values of

Table 1 Simulation results for the sequential procedure (24) in a Bernoulli distribution when $\alpha = 0.05, d = 1.10$

p	θ	n_d^*	n_0	$\gamma = 0.7$				$\gamma = 1.0$			
				$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
0.20	0.25	2643.100	10	2646.263 (0.768)	1.001	3.163	0.948 (0.002)	2644.600 (0.768)	1.000	1.500	0.954 (0.002)
0.40	0.67	1762.067	10	1763.447 (0.172)	1.000	1.380	0.953 (0.002)	1763.253 (0.169)	1.000	1.187	0.955 (0.002)
0.50	1.00	1691.584	10	1693.032 (0.015)	1.000	1.448	0.951 (0.002)	1693.014 (0.015)	1.000	1.430	0.950 (0.002)
0.60	1.50	1762.067	10	1763.306 (0.173)	1.000	1.237	0.951 (0.002)	1763.511 (0.172)	1.001	1.444	0.952(0.002)
0.70	2.33	2013.790	10	2015.784 (0.395)	1.000	1.993	0.950 (0.002)	2016.169 (0.392)	1.001	2.379	0.954 (0.002)
0.90	9.00	4698.844	10	4701.684 (1.830)	1.000	2.840	0.948 (0.002)	4704.085 (1.843)	1.001	5.240	0.947 (0.002)

Table 2 Simulation results for the sequential procedure (24) in a Bernoulli distribution when $\alpha = 0.05, d = 1.07$

p	θ	n_d^*	n_0	$\gamma = 0.7$				$\gamma = 1.0$			
				$(s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
0.20	0.25	5245.000	10	5248.003 (1.080)	1.001	3.003	0.945 (0.002)	5248.042 (1.086)	1.000	3.042	0.948 (0.002)
0.40	0.67	3496.667	10	3498.044 (0.243)	1.000	1.337	0.948 (0.002)	3498.599 (0.244)	1.000	1.932	0.948 (0.002)
0.50	1.00	3356.800	10	3358.252 (0.015)	1.000	1.452	0.951 (0.002)	3358.234 (0.015)	1.000	1.434	0.953 (0.002)
0.60	1.50	3496.667	10	3498.452 (0.242)	1.000	1.785	0.950 (0.002)	3498.394 (0.242)	1.001	1.727	0.949 (0.002)
0.70	2.33	3996.191	10	3997.935 (0.558)	1.000	1.744	0.947 (0.002)	3998.334 (0.546)	1.000	2.143	0.954 (0.002)
0.90	9.00	9324.445	10	9327.109 (2.348)	1.000	2.664	0.949 (0.002)	9330.826 (2.367)	1.000	6.381	0.949 (0.002)

Table 3 Behavior of $\overline{n}/n_d^*$ and $\overline{\kappa}$ as d goes near 1 in simulations with 10000 replications from a Bernoulli($p = 0.5$) distribution ($\theta = 1$) when $\alpha = 0.05$

d	n_0	$\gamma = 0.7$			$\gamma = 1.0$		
		$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
1.11	10	1412.443 (0.015)	1.001	0.952 (0.001)	1412.423 (0.015)	1.001	0.951 (0.002)
1.10	10	1693.020 (0.015)	1.001	0.952 (0.002)	1693.025 (0.014)	1.001	0.952 (0.002)
1.09	10	2070.722 (0.014)	1.001	0.950 (0.002)	2070.723 (0.014)	1.001	0.952 (0.002)
1.08	10	2595.863 (0.015)	1.000	0.949 (0.002)	2595.852 (0.014)	1.000	0.951 (0.002)
1.07	10	3358.215 (0.015)	1.000	0.950 (0.002)	3358.229 (0.015)	1.000	0.949 (0.002)
1.06	10	4527.299 (0.015)	1.000	0.950 (0.002)	4527.271 (0.015)	1.000	0.949 (0.002)
1.05	10	6456.748 (0.013)	1.000	0.950 (0.002)	6456.733 (0.014)	1.000	0.950 (0.002)

$$d = 1.11, 1.10, 1.09, 1.08, 1.07, 1.06, 1.05$$

for a more comprehensive understanding of how some important characteristics vary within a very tight range.

5.2 Poisson mean estimation from Sect. 4.2

Tables 4 and 5 summarize simulation results corresponding to the purely sequential procedure (29) in the case of a Poisson distribution. Each row is a summary obtained from 10000 simulations for $\theta = 1(1)5$, n_d^* correspondingly computed using (28), $\gamma = 0.7, 1.0$, $d = 1.10, 1.07$, $\alpha = 0.05$, and $n_0 = 10$, the pilot sample size. Again the choice of n_0 did not appear to impact findings.

The overall features we have found remained rather similar whatever choices of α we had made. Hence, for brevity, our Tables 4, 5, and 6 correspond to $\alpha = 0.05$.

The columns in Tables 4 and 5 are to be interpreted in the same way we had interpreted the entries in Tables 1 and 2. The ratio $\overline{n}/n_d^*$ is expected to be close to 1. Clearly, the $\overline{\kappa}$ column gives an idea about the achieved coverage probability which we hope to be close to the preset target, 0.95. From column 6 and 10 in Tables 4 and 5, we observe that $\overline{n} - n_d^*$ values appear to vary within a tight range. That gives us a distinct feeling that perhaps $E_\theta[N_d - n_d^*]$ remains bounded under the purely sequential procedure (29).

Table 6 shows fewer columns as in Table 3, but it highlights performances for a range of values of

$$d = 1.11, 1.10, 1.09, 1.08, 1.07, 1.06, 1.05$$

for a more comprehensive understanding of how some important characteristics vary within a very tight range.

Table 4 Simulation results for the sequential procedure (29) in a Poisson distribution when $\alpha = 0.05$, $d = 1.10$

θ	n_d^*	n_0	$\gamma = 0.7$				$\gamma = 1.0$			
			$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
1	422.896	10	421.831 (0.208)	1.002	0.935	0.943 (0.002)	421.361 (0.208)	1.001	0.465	0.949 (0.002)
2	211.448	10	212.016 (0.102)	1.003	0.568	0.945 (0.002)	212.086 (0.102)	1.003	0.638	0.950 (0.002)
3	140.965	10	141.438 (0.069)	1.003	0.472	0.945 (0.002)	141.577 (0.069)	1.004	0.612	0.946 (0.002)
4	105.724	10	106.215 (0.052)	1.005	0.491	0.943 (0.002)	106.284 (0.052)	1.005	0.560	0.949 (0.002)
5	84.579	10	85.053 (0.041)	1.005	0.474	0.950 (0.002)	85.091 (0.041)	1.006	0.512	0.948 (0.002)

Table 5 Simulation results for the sequential procedure (29) in a Poisson distribution when $\alpha = 0.05$, $d = 1.07$

θ	n_d^*	n_0	$\gamma = 0.7$				$\gamma = 1.0$			
			$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
1	839.200	10	840.154 (0.287)	1.001	0.953	0.943 (0.002)	840.134 (0.290)	1.001	0.934	0.950 (0.002)
2	419.600	10	420.393 (0.144)	1.002	0.793	0.945 (0.002)	420.570 (0.144)	1.002	0.970	0.952 (0.002)
3	279.733	10	280.562 (0.097)	1.003	0.829	0.946 (0.002)	280.471 (0.097)	1.003	0.737	0.951 (0.002)
4	209.800	10	210.571 (0.072)	1.004	0.771	0.949 (0.002)	210.527 (0.073)	1.003	0.727	0.948 (0.002)
5	167.840	10	168.436 (0.058)	1.004	0.596	0.943 (0.002)	168.448 (0.057)	1.004	0.608	0.952 (0.002)

Table 6 Behavior of $\overline{n}/n_d^*$ and $\overline{\kappa}$ as d goes near 1 in simulations with 10000 replications from a Poisson($\theta = 5$) distribution with $\alpha = 0.05$

d	n_0	$\gamma = 0.7$			$\gamma = 1.0$		
		$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
1.11	10	71.044 (0.038)	1.007	0.948 (0.001)	71.155 (0.037)	1.008	0.951 (0.002)
1.10	10	85.056 (0.042)	1.005	0.945 (0.002)	85.054 (0.042)	1.006	0.952 (0.002)
1.09	10	104.106 (0.046)	1.006	0.946 (0.002)	104.062 (0.045)	1.005	0.952 (0.002)
1.08	10	130.315 (0.052)	1.004	0.948 (0.002)	130.340 (0.051)	1.004	0.949 (0.002)
1.07	10	168.474 (0.058)	1.003	0.948 (0.002)	168.547 (0.058)	1.004	0.948 (0.002)
1.06	10	226.876 (0.068)	1.002	0.949 (0.002)	226.908 (0.067)	1.002	0.951 (0.002)
1.05	10	323.244 (0.081)	1.001	0.950 (0.002)	323.301 (0.081)	1.001	0.950 (0.002)

Table 7 Simulation results for the sequential procedure (33) in a normal distribution when $\alpha = 0.05$, $d = 1.10$

θ	n_d^*	n_0	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
3	120.827	10	121.476 (0.050)	1.005	0.648	0.951 (0.002)
5	76.890	10	77.523 (0.034)	1.008	0.633	0.951 (0.002)
7	56.386	10	57.013 (0.026)	1.011	0.627	0.951 (0.002)
10	40.276	10	40.864 (0.019)	1.014	0.588	0.953 (0.002)
15	27.284	10	27.812 (0.013)	1.019	0.528	0.952 (0.002)

Table 8 Simulation results for the sequential procedure (33) in a normal distribution when $\alpha = 0.05$, $d = 1.07$

θ	n_d^*	n_0	$\overline{n}\ (s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{n} - n_d^*$	$\overline{\kappa}\ (s(\overline{\kappa}))$
3	239.771	10	240.446 (0.071)	1.003	0.674	0.951 (0.002)
5	152.582	10	153.186 (0.048)	1.004	0.604	0.950 (0.002)
7	111.893	10	112.518 (0.036)	1.006	0.625	0.950 (0.002)
10	79.924	10	80.525 (0.026)	1.008	0.601	0.952 (0.002)
15	54.142	10	54.707 (0.018)	1.010	0.565	0.953 (0.002)

5.3 Normal mean estimation from Sect. 4.3

In Tables 7 and 8, we have summarized simulation results for an $N(\theta, \theta)$ distribution with $\theta = 3, 5, 7, 10, 15$. All results correspond to 10000 runs. Recall that in this case, our estimator for θ does not involve any γ. The notions of asymptotic first-order efficiency and asymptotic consistency are again validated as d moves close to 1.

Table 9 shows fewer columns as in Table 3, but it highlights performances when $\theta = 5$ for a range of values of

$$d = 1.11, 1.10, 1.09, 1.08, 1.07, 1.06, 1.05$$

Table 9 Behavior of $\overline{n}/n_d^*$ and $\overline{\kappa}$ as d goes near 1 in simulations with 10000 replications from a Normal(θ, θ) distribution when $\theta = 5$, $\alpha = 0.05$

d	n_0	$\overline{n}$ $(s(\overline{n}))$	$\overline{n}/n_d^*$	$\overline{\kappa}$ $(s(\overline{\kappa}))$
1.11	10	64.735 (0.031)	1.008	0.951 (0.001)
1.10	10	77.531 (0.034)	1.007	0.947 (0.002)
1.09	10	94.655 (0.038)	1.006	0.948 (0.002)
1.08	10	118.632 (0.042)	1.006	0.948 (0.002)
1.07	10	153.200 (0.048)	1.004	0.950 (0.002)
1.06	10	206.321 (0.055)	1.002	0.951 (0.002)
1.05	10	294.109 (0.067)	1.002	0.949 (0.002)

for a more comprehensive understanding of how some important characteristics vary within a very tight range.

6 Real data analysis

Now, we summarize analysis from implementing the proposed methodologies from Sect. 4 with the help of some available real data. We will briefly explain what exactly we have done and what it is that we have found in these illustrations.

6.1 Bernoulli distribution: odds ratio of poisonous and edible mushrooms

Mushroom records were gathered from the Audubon Society of Field Guide to North American Mushrooms (1981). In the context of Sect. 4.1, under a Bernoulli distribution, we consider a dataset on poisonous and edible mushrooms. This dataset is publicly available from the following website: http://archive.ics.uci.edu/ml/datasets/Mushroom.

This dataset includes observations corresponding to 23 species of gilled mushrooms from the Agaricus and Lepiota family. We consider two particular groups—edible mushrooms and non-edible (poisonous) mushrooms. The parameter of interest is the odds ratio of poisonous mushrooms. The total size of this dataset is 8124. A Chi-square goodness-of-fit test gave the p value 1. Also, qq-plots were very encouraging.

To have some idea of p, the proportion of poisonous mushrooms, we found its ML estimator, $\widehat{p}_{\text{MLE}} = 0.482$, from the whole dataset. Hence, the ML estimator for our parameter of interest θ, the odds of poisonous mushrooms in the data set is $\widehat{\theta}_{\text{MLE}} = 0.931$ based on the whole dataset. For purposes of illustration alone, we may reasonably treat 0.931 as the " true value" of θ, especially since the size of this data (viewed as a Bernoulli population) is very large.

Table 10 summarizes our findings from implementing the sequential procedure (24) on the mushroom dataset. Each row corresponds to a single run. In column 2, we show n_d^* obtained using the most plausible value of θ, namely 0.931. Once this sequential procedure terminated when $N = n$ with the observations $(N = n, x_1, \ldots, x_n)$, we formed the corresponding estimator T_n from (22). We considered a number of accuracy values of $d\,(>1)$ near 1 and observed that in each case, except in the case $d = 1.11$, $\gamma = 0.7$, the constructed confidence interval J_n included the most plausible value of θ,

Table 10 Illustration using mushroom data with ML estimate 0.931 treated as "true" θ under the procedure (24) for a Bernoulli distribution when $\alpha = 0.05$ with T_N from (22)

d	$\widehat{n}_d^*$	n_0	N	$N/\widehat{n}_d^*$	$N - \widehat{n}_d^*$	$\left[d^{-1}T_N, dT_N\right]$
$\gamma = 0.7$						
1.11	1412.750	10	1412	0.999	−0.750	(0.932, 1.148)
1.10	1693.772	10	1693	1.000	−0.772	(0.877, 1.061)
1.09	2071.781	10	2071	1.000	−0.781	(0.869, 1.032)
1.08	2597.716	10	2598	1.000	0.284	(0.868, 1.013)
1.07	3361.142	10	3359	1.000	−2.142	(0.891, 1.020)
1.06	4531.686	10	4530	1.000	−1.686	(0.890, 1.000)
1.05	6463.519	10	6464	1.000	0.481	(0.885, 0.976)
$\gamma = \mathbf{1.0}$						
1.11	1412.750	10	1419	1.004	6.250	(0.777, 0.957)
1.10	1693.772	10	1696	1.001	2.228	(0.827, 1.001)
1.09	2071.781	10	2073	1.000	1.219	(0.845, 1.004)
1.08	2597.716	10	2601	1.001	3.284	(0.842, 0.983)
1.07	3361.142	10	3366	1.001	4.858	(0.844, 0.966)
1.06	4531.686	10	4538	1.001	6.314	(0.852, 0.957)
1.05	6463.519	10	6467	1.000	3.481	(0.875, 0.965)

namely 0.931. In the case when $d = 1.11$, $\gamma = 0.7$, the most plausible value of θ, namely 0.931, missed the confidence interval J_n narrowly.

All other columns in Table 10 have similar meanings as explained in Section 5.1. Results are shown for both $\gamma = 0.7$ and $\gamma = 1.0$. The ratios of the sequential and optimal fixed sample size, N/n_d^*, appear close to 1. For each value of d, we observe that the difference $n - n_d^*$ has a higher magnitude when $\gamma = 1.0$ than when $\gamma = 0.7$. This could be a good theoretical problem to precisely address in the future.

6.2 Poisson distribution: average decay of a radioactive isotope

To illustrate Poisson mean estimation in the context of Sect. 4.2, we considered a radioactive decay dataset. The importance of radioactive isotopes is widespread in health studies. Determination of the half-life of such isotopes using γ-ray emission is crucial in many areas including industry, medicine, and biology.

For determining the half-life of radioactive isotopes, it is important to know what the background radiation may be in a given detector over a period of time. The dataset analyzed here was recorded in a γ-ray detection experiment over 300 one-second intervals. The source of these data is Hogg and Tanis (2006, Exercise 2.6–12, pp. 122–123). A Chi-square goodness-of-fit test gives a p value 1. Also, qq-plots were very encouraging.

The mean parameter θ would denote the average count of γ-rays emitted from the source. Once again we found the ML estimator of the parameter of interest, namely

Table 11 Illustration using radioactive decay data with ML estimate 3.030 treated as " true" θ under the procedure (29) for a Poisson distribution when $\alpha = 0.05$ with T_N from (27)

d	$\widehat{n}_d^*$	n_0	N	$N/\widehat{n}_d^*$	$N - \widehat{n}_d^*$	$[d^{-1}T_N, dT_N]$
$\gamma = 0.7$						
1.11	116.413	10	116	0.996	−0.413	(2.774, 3.418)
1.10	139.570	10	138	0.989	−1.570	(2.835, 3.431)
1.09	170.718	10	171	1.002	0.282	(2.810, 3.338)
1.08	214.056	10	222	1.037	7.944	(2.740, 3.196)
1.07	276.964	10	278	1.004	1.036	(2.845, 3.258)
$\gamma = 1.0$						
1.11	116.413	10	117	1.005	0.587	(2.734, 3.368)
1.10	139.570	10	141	1.010	1.430	(2.740, 3.316)
1.09	170.718	10	169	0.990	−1.718	(2.834, 3.367)
1.08	214.056	10	209	0.976	−5.056	(2.884, 3.364)
1.07	276.964	10	274	0.989	2.964	(2.868, 3.284)

$\widehat{\theta}_{\text{MLE}} = 3.030$ from the whole dataset. For purposes of illustration alone, we may reasonably treat 3.030 as the " true value" of θ, especially since the size of this data (viewed as a Poisson population) is very large.

Table 11 summarizes our findings from implementing the sequential procedure (29) on the decay dataset. Each row corresponds to a single run. In column 2, we show n_d^* obtained using the most plausible value of θ, namely 3.030. Once this sequential procedure terminated with the observations $(N = n, x_1, \ldots, x_n)$, we formed the corresponding estimator T_n from (27). We considered a number of accuracy values of $d(> 1)$ near 1 and observed that in each case, the constructed confidence interval J_n included the most plausible value of θ, namely 3.030.

All the other columns in Table 11 have similar meanings as explained in Section 6.1. Results are shown for both $\gamma = 0.7$ and $\gamma = 1.0$. The ratios of the sequential and optimal fixed sample size, N/n_d^*, appear close to 1. For each value of d, we observe that the difference $n - n_d^*$ has similar magnitudes whether $\gamma = 1.0$ or 0.7. We hope to return to address this feature with some theoretical probe in the future.

6.3 Normal distribution: 911 calls dispatching emergency help

To illustrate mean θ estimation in the context of Sect. 4.3, we considered real-time 911 calls dispatch dataset. In the event of an emergency, one dials 911 to seek help. The time taken to dispatch help (for example, ambulance, police, fire-trucks, emergency medical technicians) after a 911 call to such callers is of great interest.

The data set is based on " Real-time 911 Dispatch" in Seattle, Washington. Records were maintained in Seattle for every day from August 25–30, 2009. In the case of each call, time taken to dispatch help was recorded. The size of the dataset is 144. We fitted a Normal distribution to this data. The p value from Anderson–Darling test came out

Table 12 Illustration using hourly 911 dispatch data with ML estimate 3.922 treated as " true" θ under the procedure (33) for a normal distribution when $\alpha = 0.05$ with T_N from (31)

d	$\widehat{n}_d^*$	n_0	N	$N/\widehat{n}_d^*$	$N - \widehat{n}_d^*$	$\left[d^{-1}T_N, dT_N\right]$
1.15	44.469	10	46	1.034	1.530	(3.380,4.470)
1.14	50.595	10	52	1.028	1.405	(3.357,4.363)
1.13	58.153	10	61	1.049	2.847	(3.410,4.355)
1.12	67.633	10	71	1.049	3.367	(3.319,4.164)
1.11	79.758	10	80	1.003	0.242	(3.586,4.418)
1.10	95.623	10	94	0.983	−1.623	(3.664,4.433)
1.09	116.963	10	115	0.983	−1.963	(3.695,4.340)

0.121. The qq-plot of the data showed a good fit. This datum is publicly available from http://www2.seattle.gov/fire/realTime911/getDatePubTab.asp.

For more details, one is referred to Bhattacharjee (2011). The dataset agreed with a $N(\xi, c\xi)$ distribution with $c = 2.5$. Bhattacharjee (2011) used a scaled version of the dataset (scaled by $c = 2.5$) for analysis. In other words, from the original data Y, we formed a scaled new data $X \equiv \frac{2}{5}Y$ which may be assumed to follow a $N(\theta, \theta)$ where $\theta \equiv \frac{2}{5}\xi$.

We proceed in the same way as earlier to implement the sequential procedure (33) on the dataset corresponding to X. We found the ML estimator of the parameter of interest, namely $\widehat{\theta}_{\text{MLE}} = 3.922$ from the whole dataset. For purposes of illustration alone, we may reasonably treat 3.922 as the " true value" of θ, especially since the size of these data (viewed as a normal population) is very large.

Table 12 summarizes our findings from implementing the sequential procedure (33). Each row corresponds to a single run. In column 2, we show n_d^* obtained using the most plausible value of θ, namely 3.922. Once this sequential procedure terminated with the observations $(N = n, x_1, \ldots, x_n)$, we formed the corresponding estimator T_n from (31). We considered a number of accuracy values of $d(>1)$ near 1 and observed that in each case, the constructed confidence interval J_n included the most plausible value of θ, namely 3.922.

All other columns in Table 12 have similar meanings as explained in Sect. 6.1. The ratios of the sequential and optimal fixed sample size, N/n_d^*, appear close to 1. We note that for each value of d, the difference $n - n_d^*$ has similar magnitudes. We hope to return to address this feature with some theoretical probe in the future.

Remark 5 We may emphasize couple of important points. In each illustration from Section 6, the real dataset were treated as a population for practical purposes. In the implementation of a sequential methodology, the knowledge of true θ (which remains unknown) or $\widehat{\theta}_{\text{MLE}}$ obtained from full data has played no role at all. However, in the expression of n_d^* (column 2 in Tables 10, 11, 12), $\widehat{\theta}_{\text{MLE}}$ was plugged in place of θ just so that n_d^* and n (from a single run) may be readily compared. We also checked whether $\widehat{\theta}_{\text{MLE}}$ belonged to each constructed confidence interval (column 7 in Tables

10, 11, 12) from the viewpoint of a purely suggestive guideline knowing fully well that $\widehat{\theta}_{\text{MLE}}$ may not coincide with the " true" unknown θ.

7 Final thoughts

Our main motivation for this research was to broaden the basic methodology of sequential fixed-accuracy confidence intervals, initially developed Mukhopadhyay and Banerjee (2014a) for an NB distribution to estimate mean infestation in statistical ecology. In doing so, we encountered a number of other practical situations where one would like to estimate a parameter that is positive. We saw the need to propose a general structure under which a large breadth of sequential fixed-accuracy confidence interval estimation problems could be addressed.

The proposed sequential methodology so developed is easy to implement and enjoys both asymptotic consistency and asymptotic first-order efficiency properties. The extensive analyses of data obtained from large-scale simulations and our analyses of real data from a number of health studies and safety studies assure us that the proposed sequential methodology is interesting and attractive in numerous applications of today's statistical science.

We are looking into developing appropriate two-stage (along the lines of Mukhopadhyay and de Silva (2005)) and accelerated sequential (along the lines of Mukhopadhyay (1996)) fixed-accuracy confidence interval estimation methodologies under a general structure analogous to what has been introduced in the present paper. It is our hope to wrap this up in the near future in a sequel.

7.1 A unified treatment

This brief outline brings out the true flexibility of the present formulation. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d. with a common p.m.f. or p.d.f. given by $f(x; \psi)$ where ψ is a single unknown parameter and belongs to R. We may want to estimate ψ, but our general methodology from Sect. 3 is not readily applicable.

Under very mild regularity conditions, however, let $\widehat{\psi}_n \equiv \widehat{\psi}_{n,\text{MLE}}$, the ML estimator for ψ. According to (20), under very mild regularity conditions, according to (19) we have $n^{1/2}\left(\widehat{\psi}_n - \psi\right) \xrightarrow{\pounds} N(0, I_X^{-1}(\psi))$, as $n \to \infty$ assuming that $0 < I_X(\psi) < \infty$.

Next, we define a new parameter $\theta \equiv \exp(\psi)$ which is a one–one function of ψ, and clearly θ is positive. Now, we let $T_n \equiv \exp(\widehat{\psi}_n)$, the ML estimator of θ. Thus, we can propose the fixed-accuracy confidence interval $J_n = [d^{-1}T_n, dT_n]$ from (7) for θ with $d > 1$. But, we draw attention to the following (using (19)):

$$P_\psi\{\theta \in J_n\} = P_\psi\left\{-\ln d \le \widehat{\psi}_{n,\text{MLE}} - \psi \le \ln d\right\} \approx 2\Phi\left(n^{1/2} I_X(\psi) \ln d\right) - 1, \tag{35}$$

for large n.

Hence, we can find an expression for the associated $n_d^*(\equiv z_{\alpha/2}^2 I_X^{-2}(\psi)(\ln d)^{-2})$ so that the coverage probability from (35) will be approximately at least $1 - \alpha$ with

$0 < \alpha < 1$ preassigned. Then, the associated sequential stopping rule can be easily incorporated in the spirit of Khan (1969). Theorem 1 would hold under appropriate conditions.

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