Ann Inst Stat Math (2016) 68:469–490
DOI 10.1007/s10463-014-0502-9

Second-order asymptotic comparison of the MLE and MCLE of a natural parameter for a truncated exponential family of distributions

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Received: 7 November 2013 / Revised: 18 December 2014 / Published online: 21 February 2015
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Abstract For a truncated exponential family of distributions with a natural parameter θ and a truncation parameter γ as a nuisance parameter, it is known that the maximum likelihood estimators (MLEs) $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ and $\hat{\theta}_{\mathrm{ML}}$ of θ for known γ and unknown γ, respectively, and the maximum conditional likelihood estimator $\hat{\theta}_{\mathrm{MCL}}$ of θ are asymptotically equivalent. In this paper, the stochastic expansions of $\hat{\theta}_{\mathrm{ML}}^{\gamma}$, $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$ are derived, and their second-order asymptotic variances are obtained. The second-order asymptotic loss of a bias-adjusted MLE $\hat{\theta}_{\mathrm{ML}}^{*}$ relative to $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ is also given, and $\hat{\theta}_{\mathrm{ML}}^{*}$ and $\hat{\theta}_{\mathrm{MCL}}$ are shown to be second-order asymptotically equivalent. Further, some examples are given.

Keywords Truncated exponential family · Natural parameter · Truncation parameter · Maximum likelihood estimator · Maximum conditional likelihood estimator · Stochastic expansion · Asymptotic variance · Second-order asymptotic loss

1 Introduction

The first-order asymptotic theory in regular parametric models with nuisance parameters was discussed by Barndorff-Nielsen and Cox (1994). In higher order asymptotics, under suitable regularity conditions, the concept of asymptotic deficiency discussed by Hodges and Lehmann (1970) is useful in comparing asymptotically efficient estimators in the presence of nuisance parameters. Indeed, the asymptotic deficiencies of some asymptotically efficient estimators relative to the maximum likelihood estimator

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(MLE) based on the pooled sample were obtained in the presence of nuisance parameters [see, e.g. Akahira and Takeuchi (1982) and Akahira (1986)]. On the other hand, in statistical estimation in multiparameter cases, the conditional likelihood method is well known as a way of eliminating nuisance parameters [see, e.g. Basu (1977)]. The consistency, asymptotic normality and asymptotic efficiency of the maximum conditional likelihood estimator (MCLE) were discussed by Andersen (1970), Huque and Katti (1976), Bar-Lev and Reiser (1983), Bar-Lev (1984), Liang (1984) and others. Further, in higher order asymptotics, asymptotic properties of the MCLE of an interest parameter in the presence of nuisance parameters were also discussed by Cox and Reid (1987) and Ferguson (1992) in the regular case. However, in the non-regular case when the regularity conditions do not necessarily hold, the asymptotic comparison of asymptotically efficient estimators has not been discussed enough in the presence of nuisance parameters in higher order asymptotics yet.

For a truncated exponential family of distributions which is regarded as a typical non-regular case, we consider a problem of estimating a natural parameter θ in the presence of a truncation parameter γ as a nuisance parameter. Let $\hat{\theta}_{ML}^{\gamma}$ and $\hat{\theta}_{ML}$ be the MLEs of θ based on a sample of size n when γ is known and γ is unknown, respectively. Let $\hat{\theta}_{MCL}$ be the MCLE of θ. Then it was shown by Bar-Lev (1984) that the MLEs $\hat{\theta}_{ML}^{\gamma}$, $\hat{\theta}_{ML}$ and the MCLE $\hat{\theta}_{MCL}$ have the same asymptotic normal distribution, hence they are shown to be asymptotically equivalent in the sense of having the same asymptotic variance. A similar result can be derived from the stochastic expansions of the MLEs $\hat{\theta}_{ML}^{\gamma}$ and $\hat{\theta}_{ML}$ in Akahira and Ohyauchi (2012). But, $\hat{\theta}_{ML}^{\gamma}$ for known γ may be asymptotically better than $\hat{\theta}_{ML}$ for unknown γ in the higher order, because $\hat{\theta}_{ML}^{\gamma}$ has the full information on γ. Otherwise, the existence of a truncation parameter γ as a nuisance parameter is meaningless. So, it is a quite interesting problem to compare asymptotically them up to the higher order.

In this paper, we compare them up to the second order, i.e. the order n^{-1}, in the asymptotic variance. We show that a bias-adjusted MLE $\hat{\theta}_{ML}^{*}$ and $\hat{\theta}_{MCL}$ are second-order asymptotically equivalent, but they are asymptotically worse than $\hat{\theta}_{ML}^{\gamma}$ in the second order. We thus calculate the second-order asymptotic losses on the asymptotic variance among them.

2 Formulation and assumptions

In a similar way to Bar-Lev (1984), we have the formulation as follows. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to $P_{\theta,\gamma}$, having a density

$$f(x;\theta,\gamma) = \begin{cases} \frac{a(x)e^{\theta u(x)}}{b(\theta,\gamma)} & \text{for } c < \gamma \leq x < d, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with respect to the Lebesgue measure, where $-\infty \leq c < d \leq \infty$, $a(\cdot)$ is non-negative and continuous almost surely, and $u(\cdot)$ is absolutely continuous with $\mathrm{d}u(x)/\mathrm{d}x \not\equiv 0$ over the interval (γ, d). Let

$$\Theta(\gamma) := \left\{\theta \,\middle|\, 0 < b(\theta, \gamma) := \int_{\gamma}^{d} a(x)e^{\theta u(x)}\mathrm{d}x < \infty\right\}$$

for $\gamma \in (c, d)$. Then, it is shown that for any $\gamma_1, \gamma_2 \in (c, d)$ with $\gamma_1 < \gamma_2$, $\Theta(\gamma_1) \subset \Theta(\gamma_2)$. Assume that for any $\gamma \in (c, d)$, $\Theta \equiv \Theta(\gamma)$ is a non-empty open interval. A family $\mathcal{P} := \{P_{\theta,\gamma} \mid \theta \in \Theta, \gamma \in (c, d)\}$ of distributions $P_{\theta,\gamma}$ [see (1)] with a natural parameter θ and a truncation parameter γ is called a truncated exponential family of distributions.

In Bar-Lev (1984), the asymptotic behavior of the MLE $\hat{\theta}_{\mathrm{ML}}$ and MCLE $\hat{\theta}_{\mathrm{MCL}}$ of a parameter θ in the presence of γ as a nuisance parameter was compared and also done with that of the MLE $\hat{\theta}^{\gamma}_{\mathrm{ML}}$ of θ when γ is known. As the result, it was shown there that, for a sample of size $n(\geq 2)$, the $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$ of θ exist with probability 1 and are given as the unique roots of the appropriate maximum likelihood equations. These two estimators were also shown to be strongly consistent for θ with the limiting distribution coinciding with that of the MLE $\hat{\theta}^{\gamma}_{\mathrm{ML}}$ of θ when γ is known.

In the subsequent sections, we obtain the stochastic expansions of $\hat{\theta}^{\gamma}_{\mathrm{ML}}$, $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$ up to the second order, i.e. the order $o_p(n^{-1})$. We get their second-order asymptotic variances, and derive the second-order asymptotic losses on the asymptotic variance among them. The proofs of theorems are located in appendixes.

3 The MLE $\hat{\theta}^{\gamma}_{\mathrm{ML}}$ of θ when γ is known

Denote a random vector $(X_1, \ldots, X_n)$ by $\boldsymbol{X}$, and let $X_{(1)} \leq \cdots \leq X_{(n)}$ be the corresponding order statistics of a random vector $\boldsymbol{X}$. Here, we consider the case when γ is known. Then, the density (1) is considered to belong to a regular exponential family of distributions with a natural parameter θ, hence $\log b(\theta, \gamma)$ is strictly convex and infinitely differentiable in $\theta \in \Theta$ and

$$\lambda_j(\theta, \gamma) := \frac{\partial^j}{\partial \theta^j} \log b(\theta, \gamma) \tag{2}$$

is the jth cumulant corresponding to (1) for $j = 1, 2, \ldots$. For given $\boldsymbol{x} = (x_1, \ldots, x_n)$ satisfying $\gamma < x_{(1)} := \min_{1 \leq i \leq n} x_i$ and $x_{(n)} := \max_{1 \leq i \leq n} x_i < d$, the likelihood function of θ is given by

$$L^{\gamma}(\theta; \boldsymbol{x}) := \frac{1}{b^n(\theta, \gamma)} \left\{\prod_{i=1}^{n} a(x_i)\right\} \exp\left\{\theta \sum_{i=1}^{n} u(x_i)\right\}.$$

Then, the likelihood equation is

$$\frac{1}{n}\sum_{i=1}^{n} u(x_i) - \lambda_1(\theta, \gamma) = 0. \tag{3}$$

Since there exists a unique solution on θ of (3), we denote it by $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ which is the MLE of θ [see, e.g. Barndorff-Nielsen (1978) and Bar-Lev (1984)]. Let $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 2, 3, 4)$ and put

$$Z_1 := \frac{1}{\sqrt{\lambda_2 n}} \sum_{i=1}^{n} \{u(X_i) - \lambda_1\}, \quad U_\gamma := \sqrt{\lambda_2 n} \left(\hat{\theta}_{\mathrm{ML}}^{\gamma} - \theta \right).$$

Then, we have the following.

Theorem 1 *For the truncated exponential family $\mathcal{P}$ of distributions with a density (1) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ be the MLE of θ when γ is known. Then, the stochastic expansion of U_γ is given by*

$$U_\gamma = Z_1 - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} Z_1^2 + \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2} \right) Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}} \right),$$

and the second-order asymptotic mean and variance are given by

$$E_\theta\left(U_\gamma\right) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right),$$

$$V_\theta\left(U_\gamma\right) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2} \right) + O\left(\frac{1}{n\sqrt{n}}\right),$$

respectively.

Since $U_\gamma = Z_1 + o_p(1)$, it is seen that U_γ is asymptotically normal mean with mean 0 and variance 1, which coincides with the result of Bar-Lev (1984).

4 The MLE $\hat{\theta}_{\mathrm{ML}}$ of θ when γ is unknown

For given $\boldsymbol{x} = (x_1, \ldots, x_n)$ satisfying $\gamma < x_{(1)}$ and $x_{(n)} < d$, the likelihood function of θ and γ is given by

$$L(\theta, \gamma; \boldsymbol{x}) = \frac{1}{b^n(\theta, \gamma)} \left\{ \prod_{i=1}^{n} a(x_i) \right\} \exp\left\{ \theta \sum_{i=1}^{n} u(x_i) \right\}. \tag{4}$$

Let $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\gamma}_{\mathrm{ML}}$ be the MLEs of θ and γ, respectively. From (4) it is seen that $\hat{\gamma}_{\mathrm{ML}} = X_{(1)}$ and $L\left(\hat{\theta}_{\mathrm{ML}}, X_{(1)}; \boldsymbol{X}\right) = \sup_{\theta \in \Theta} L\left(\theta, X_{(1)}; \boldsymbol{X}\right)$, hence $\hat{\theta}_{\mathrm{ML}}$ satisfies the likelihood equation

$$0 = \frac{1}{n} \sum_{i=1}^{n} u(X_i) - \lambda_1\left(\hat{\theta}_{\mathrm{ML}}, X_{(1)}\right), \tag{5}$$

where $\boldsymbol{X} = (X_1, \ldots, X_n)$. Let $\lambda_2 = \lambda_2(\theta, \gamma)$ and put $\hat{U} := \sqrt{\lambda_2 n}(\hat{\theta}_{\mathrm{ML}} - \theta)$ and $T := n(X_{(1)} - \gamma)$. Then, we have the following.

Theorem 2 *For the truncated exponential family $\mathcal{P}$ of distributions with a density (1) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}_{\mathrm{ML}}$ be the MLE of θ when γ is unknown, and $\hat{\theta}^*_{\mathrm{ML}}$ be a bias-adjusted MLE such that $\hat{\theta}_{\mathrm{ML}}$ has the same asymptotic bias as that of $\hat{\theta}^{\gamma}_{\mathrm{ML}}$, i.e.*

$$\hat{\theta}^*_{\mathrm{ML}} = \hat{\theta}_{\mathrm{ML}} + \frac{1}{k\left(\hat{\theta}_{\mathrm{ML}}, X_{(1)}\right)\lambda_2\left(\hat{\theta}_{\mathrm{ML}}, X_{(1)}\right)n}\left\{\frac{\partial \lambda_1}{\partial \gamma}\left(\hat{\theta}_{\mathrm{ML}}, X_{(1)}\right)\right\}, \tag{6}$$

where $k(\theta, \gamma) := a(\gamma)e^{\theta u(\gamma)}/b(\theta, \gamma)$. Then, the stochastic expansion of $\hat{U}^* := \sqrt{\lambda_2 n}(\hat{\theta}^*_{\mathrm{ML}} - \theta)$ is given by

$$\hat{U}^* = \hat{U} + \frac{1}{k\sqrt{\lambda_2 n}}\left(\frac{\partial \lambda_1}{\partial \gamma}\right) - \frac{1}{k\lambda_2 n}\left\{\delta + \frac{1}{k}\left(\frac{\partial k}{\partial \theta}\frac{\partial \lambda_1}{\partial \gamma}\right)\right\}Z_1 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where $k = k(\theta, \gamma)$,

$$\delta = \frac{\lambda_3}{\lambda_2}\left(\frac{\partial \lambda_1}{\partial \gamma}\right) - \frac{\partial \lambda_2}{\partial \gamma},$$

$$\hat{U} = Z_1 - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}}Z_1^2 - \frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial \lambda_1}{\partial \gamma}\right)T + \frac{\delta}{\lambda_2 n}Z_1 T + \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right)Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

and the second-order asymptotic mean and variance are given by

$$E_{\theta,\gamma}(\hat{U}^*) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right),$$

$$V_{\theta,\gamma}(\hat{U}^*) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\{\lambda_1 - u(\gamma)\}^2 + O\left(\frac{1}{n\sqrt{n}}\right),$$

respectively.

Since $\hat{U} = \hat{U}^* = Z_1 + o_p(1)$, it is seen that $\hat{U}$ and $\hat{U}^*$ are asymptotically normal with mean 0 and variance 1, which coincides with the result of Bar-Lev (1984). But, it is noted from Theorems 1 and 2 that there is a difference between $V_\theta(U_\gamma)$ and $V_{\theta,\gamma}(\hat{U}^*)$ in the second order, i.e. the order n^{-1}, which is discussed in Sect. 6.

5 The MCLE $\hat{\boldsymbol{\theta}}_{\mathbf{MCL}}$ of $\boldsymbol{\theta}$ when $\boldsymbol{\gamma}$ is unknown

First, it is seen from (1) that there exists a random permutation, say $Y_2, \ldots, Y_n$ of the $(n-1)!$ permutations of $\left(X_{(2)}, \ldots, X_{(n)}\right)$ such that conditionally on $X_{(1)} = x_{(1)}$, the $Y_2, \ldots, Y_n$ are i.i.d. random variables with a density

$$g\left(y;\theta,x_{(1)}\right)=\frac{a(y)e^{\theta u(y)}}{b(\theta,x_{(1)})}\quad \text{for } x_{(1)}<y<d$$

[see Quesenberry (1975) and Bar-Lev (1984)]. For given $X_{(1)}=x_{(1)}$, the conditional likelihood function of θ for $\boldsymbol{y}=(y_2,\ldots,y_n)$ satisfying $x_{(1)}<y_i<d$ $(i=2,\ldots,n)$ is

$$L(\theta;\boldsymbol{y}|x_{(1)})=\frac{1}{b^{n-1}(\theta,x_{(1)})}\left\{\prod_{i=2}^{n}a(y_i)\right\}\exp\left\{\theta\sum_{i=2}^{n}u(y_i)\right\}.$$

Then, the likelihood equation is

$$\frac{1}{n-1}\sum_{i=2}^{n}u(y_i)-\lambda_1(\theta,x_{(1)})=0. \tag{7}$$

Since there exists a unique solution on θ of (7), we denote it by $\hat{\theta}_{\mathrm{MCL}}$, i.e. the value of θ for which $L\left(\theta;\boldsymbol{y}\ x_{(1)}\right)$ attains supremum. Let $\tilde{\lambda}_i:=\lambda_i(\theta,x_{(1)})$ $(i=1,2,3,4)$ and put

$$\tilde{Z}_1:=\frac{1}{\sqrt{\tilde{\lambda}_2(n-1)}}\sum_{i=2}^{n}\left\{u(Y_i)-\tilde{\lambda}_1\right\},\quad \tilde{U}_0:=\sqrt{\lambda_2 n}\left(\hat{\theta}_{\mathrm{MCL}}-\theta\right).$$

Then, we have the following.

Theorem 3 *For the truncated exponential family $\mathcal{P}$ of distributions with a density (1) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}_{\mathrm{MCL}}$ be the MCLE of θ when γ is unknown. Then, the stochastic expansion of $\tilde{U}_0$ is given by*

$$\tilde{U}_0=\tilde{Z}_1-\frac{\tilde{\lambda}_3}{2\tilde{\lambda}_2^{3/2}\sqrt{n}}\tilde{Z}_1^2+\frac{1}{2n}\left\{1-\frac{1}{\lambda_2}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T\right\}\tilde{Z}_1$$
$$+\frac{1}{2n}\left(\frac{\tilde{\lambda}_3^2}{\tilde{\lambda}_2^3}-\frac{\tilde{\lambda}_4}{3\tilde{\lambda}_2^2}\right)\tilde{Z}_1^3+O_p\left(\frac{1}{n\sqrt{n}}\right),$$

and the second-order asymptotic mean and variance are given by

$$E_{\theta,\gamma}\left(\tilde{U}_0\right)=-\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}}+O\left(\frac{1}{n\sqrt{n}}\right),$$
$$V_{\theta,\gamma}\left(\tilde{U}_0\right)=1+\frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3}-\frac{\lambda_4}{\lambda_2^2}\right)+\frac{1}{\lambda_2 n}\{\lambda_1-u(\gamma)\}^2+O\left(\frac{1}{n\sqrt{n}}\right).$$

Remark 1 From Theorems 2 and 3, it is seen that the second-order asymptotic mean and variance of $\tilde{U}_0$ are the same as those of $\hat{U}^*=\sqrt{\lambda_2 n}(\hat{\theta}^*_{\mathrm{ML}}-\theta)$. It is noted that $\hat{\theta}_{\mathrm{MCL}}$ has an advantage over $\hat{\theta}_{\mathrm{ML}}$ in the sense of no need of the bias adjustment.

Remark 2 As is seen from Theorems 1, 2 and 3, the first term of order $1/n$ in $V_\theta(U_\gamma)$, $V_{\theta,\gamma}(\hat{U}^*)$ and $V_{\theta,\gamma}(\tilde{U}_0)$ results from the regular part of the density (1), which coincides with the fact that the distribution with (1) is considered to belong to a regular exponential family of distributions when γ is known. The second term of order $1/n$ in $V_{\theta,\gamma}(\hat{U}^*)$ and $V_{\theta,\gamma}(\tilde{U}_0)$ follows from the non-regular (i.e. truncation) part of (1) when γ is unknown, which means a ratio of the variance $\lambda_2 = V_{\theta,\gamma}(u(X)) = E_{\theta,\gamma}\big[\{u(X) - \lambda_1\}^2\big]$ to the distance $\big\{\lambda_1 - u(\gamma)\big\}^2$ from the mean λ_1 of $u(X)$ to $u(x)$ at $x = \gamma$.

6 The second-order asymptotic comparison among $\hat{\theta}^{\gamma}_{\mathrm{ML}}$, $\hat{\theta}^*_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$

From the results in the previous sections, we can asymptotically compare the estimators $\hat{\theta}^{\gamma}_{\mathrm{ML}}$, $\hat{\theta}^*_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$ using their second-order asymptotic variances as follows.

Theorem 4 *For the truncated exponential family $\mathcal{P}$ of distributions with the density (1) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}^{\gamma}_{\mathrm{ML}}$, $\hat{\theta}^*_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$ be the MLE of θ when γ is known, the bias-adjusted MLE of θ when γ is unknown and the MCLE of θ when γ is unknown, respectively. Then, the bias-adjusted MLE $\hat{\theta}^*_{\mathrm{ML}}$ and the MCLE $\hat{\theta}_{\mathrm{MCL}}$ are second-order asymptotically equivalent in the sense that*

$$d_n(\hat{\theta}^*_{\mathrm{ML}}, \hat{\theta}_{\mathrm{MCL}}) := n\left\{V_{\theta,\gamma}(\hat{U}^*) - V_{\theta,\gamma}(\tilde{U}_0)\right\} = o(1) \tag{8}$$

as $n \to \infty$, and they are second-order asymptotically worse than $\hat{\theta}^{\gamma}_{\mathrm{ML}}$ with the second-order asymptotic losses of $\hat{\theta}^*_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$ relative to $\hat{\theta}^{\gamma}_{\mathrm{ML}}$

$$d_n(\hat{\theta}^*_{\mathrm{ML}}, \hat{\theta}^{\gamma}_{\mathrm{ML}}) := n\left\{V_{\theta,\gamma}(\hat{U}^*) - V_\theta(U_\gamma)\right\} = \frac{\{\lambda_1 - u(\gamma)\}^2}{\lambda_2} + o(1), \tag{9}$$

$$d_n(\hat{\theta}_{\mathrm{MCL}}, \hat{\theta}^{\gamma}_{\mathrm{ML}}) := n\left\{V_{\theta,\gamma}(\tilde{U}_0) - V_\theta(U_\gamma)\right\} = \frac{\{\lambda_1 - u(\gamma)\}^2}{\lambda_2} + o(1) \tag{10}$$

as $n \to \infty$, respectively.

The proof is straightforward from Theorems 1, 2 and 3.

7 Examples

Some examples on the second-order asymptotic loss of the estimators are given for a truncated exponential distribution, a truncated normal distribution and the Pareto distribution.

Example 1 **(Truncated exponential distribution)** Let $c = -\infty$, $d = \infty$, $a(x) \equiv 1$ and $u(x) \equiv -x$ for $-\infty < \gamma \leq x < \infty$ in the density (1). Since $b(\theta, \gamma) = e^{-\theta\gamma}/\theta$, it follows from (2) that $\Theta = (0, \infty)$,

$$\lambda_1 = \frac{\partial}{\partial\theta}\log b(\theta,\gamma) = -\gamma - \frac{1}{\theta},$$
$$\lambda_2 = \frac{\partial^2}{\partial\theta^2}\log b(\theta,\gamma) = \frac{1}{\theta^2}, \quad k(\theta,\gamma) = \theta.$$

From (3), (5), (6) and (7) we have

$$\hat{\theta}_{\mathrm{ML}}^{\gamma} = 1/(\bar{X}-\gamma), \quad \hat{\theta}_{\mathrm{ML}} = 1/\left(\bar{X} - X_{(1)}\right),$$
$$\hat{\theta}_{\mathrm{ML}}^{*} = \hat{\theta}_{\mathrm{ML}} - \frac{1}{n}\hat{\theta}_{\mathrm{ML}}, \quad \hat{\theta}_{\mathrm{MCL}} = 1\bigg/\left(\frac{1}{n-1}\sum_{i=2}^{n} X_{(i)} - X_{(1)}\right).$$

Note that $\hat{\theta}_{\mathrm{ML}}^{*} = \hat{\theta}_{\mathrm{MCL}}$. In this case, the first part in Theorem 4 is trivial, since $d_n(\hat{\theta}_{\mathrm{ML}}^{*}, \hat{\theta}_{\mathrm{MCL}}) = 0$. From Theorem 4, we obtain the second-order asymptotic loss

$$d_n(\hat{\theta}_{\mathrm{ML}}^{*}, \hat{\theta}_{\mathrm{ML}}^{\gamma}) = d_n(\hat{\theta}_{\mathrm{MCL}}, \hat{\theta}_{\mathrm{ML}}^{\gamma}) = 1 + o(1)$$

as $n \to \infty$.

Example 2 (**Truncated normal distribution**) Let $c = -\infty$, $d = \infty$, $a(x) = e^{-x^2/2}$ and $u(x) = x$ for $-\infty < \gamma \le x < \infty$ in the density (1). Since

$$b(\theta,\gamma) = \sqrt{2\pi}e^{\theta^2/2}\Phi(\theta-\gamma),$$

it follows from (2) and Theorem 2 that $\Theta = (-\infty, \infty)$,

$$\lambda_1(\theta,\gamma) = \theta + \rho(\theta-\gamma), \quad \frac{\partial\lambda_1}{\partial\gamma}(\theta,\gamma) = (\theta-\gamma)\rho(\theta-\gamma) + \rho^2(\theta-\gamma),$$
$$\lambda_2(\theta,\gamma) = 1 - (\theta-\gamma)\rho(\theta-\gamma) - \rho^2(\theta-\gamma),$$
$$k(\theta,\gamma) = \rho(\theta-\gamma),$$

where $\rho(t) := \phi(t)/\Phi(t)$ with

$$\Phi(x) = \int_{-\infty}^{x}\phi(t)dt, \quad \phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2} \quad \text{for } -\infty < t < \infty.$$

Then, it follows from (3), (5) and (7) that the solutions of θ of the following equations

$$\theta + \rho(\theta-\gamma) = \bar{X}, \quad \theta + \rho(\theta - X_{(1)}) = \bar{X},$$
$$\theta + \rho(\theta - X_{(1)}) = \frac{1}{n-1}\sum_{i=2}^{n} X_{(i)}$$

become $\hat{\theta}_{\mathrm{ML}}^{\gamma}$, $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$, respectively, where $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$. From (6), the bias-adjusted MLE is given by

$$\hat{\theta}_{\mathrm{ML}}^{*} = \hat{\theta}_{\mathrm{ML}} + \frac{\hat{\theta}_{\mathrm{ML}} - X_{(1)} + \rho\left(\hat{\theta}_{\mathrm{ML}} - X_{(1)}\right)}{1 - \left(\hat{\theta}_{\mathrm{ML}} - X_{(1)}\right)\rho\left(\hat{\theta}_{\mathrm{ML}} - X_{(1)}\right) - \rho^2\left(\hat{\theta}_{\mathrm{ML}} - X_{(1)}\right)}.$$

From Theorem 4, we obtain the second-order asymptotic losses

$$d_n(\hat{\theta}_{\mathrm{ML}}^{*}, \hat{\theta}_{\mathrm{MCL}}) = o(1),$$
$$d_n(\hat{\theta}_{\mathrm{ML}}^{*}, \hat{\theta}_{\mathrm{ML}}^{\gamma}) = d_n(\hat{\theta}_{\mathrm{MCL}}, \hat{\theta}_{\mathrm{ML}}^{\gamma}) = \frac{\{\theta - \gamma + \rho(\theta - \gamma)\}^2}{1 - (\theta - \gamma)\rho(\theta - \gamma) - \rho^2(\theta - \gamma)} + o(1)$$

as $n \to \infty$.

Example 3 **(Pareto distribution)** Let $c = 0, d = \infty, a(x) = 1/x$ and $u(x) = -\log x$ for $0 < \gamma \le x < \infty$ in the density (1). Then, $b(\theta, \gamma) = 1/(\theta\gamma^{\theta})$ for $\theta \in \Theta = (0, \infty)$. Letting $t = \log x$ and $\gamma_0 = \log \gamma$, we see that (1) becomes

$$f(t; \theta, \gamma_0) = \begin{cases} \theta e^{\theta\gamma_0} e^{-\theta t} & \text{for } t > \gamma_0, \\ 0 & \text{for } t \le \gamma_0. \end{cases}$$

Hence, the Pareto case is reduced to the truncated exponential one in Example 1.

8 Concluding remarks

In a truncated exponential family of distributions with a two-dimensional parameter (θ, γ), we considered the estimation problem of a natural parameter θ in the presence of a truncation parameter γ as a nuisance parameter. In the paper of Bar-Lev (1984), it was shown that the MLE $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ of θ for known γ, the MLE $\hat{\theta}_{\mathrm{ML}}$ and the MCLE $\hat{\theta}_{\mathrm{MCL}}$ of θ for unknown γ are asymptotically equivalent in the sense that they have the same asymptotic normal distribution. In this paper, we derived the stochastic expansions of $\hat{\theta}_{\mathrm{ML}}^{\gamma}$, $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{MCL}}$. We also obtained the second-order asymptotic loss of the bias-adjusted MLE $\hat{\theta}_{\mathrm{ML}}^{*}$ relative to $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ from their second-order asymptotic variances and showed that $\hat{\theta}_{\mathrm{ML}}^{*}$ and $\hat{\theta}_{\mathrm{MCL}}$ are second-order asymptotically equivalent in the sense that their asymptotic variances are same up to the order $o(1/n)$ as in (8). It seems to be natural that $\hat{\theta}_{\mathrm{ML}}^{\gamma}$ is second-order asymptotically better than $\hat{\theta}_{\mathrm{ML}}^{*}$ after adjusting the bias of $\hat{\theta}_{\mathrm{ML}}$ such that $\hat{\theta}_{\mathrm{ML}}$ has the same as that of $\hat{\theta}_{\mathrm{ML}}^{\gamma}$. The values of the second-order asymptotic losses of $\hat{\theta}_{\mathrm{ML}}^{*}$ and $\hat{\theta}_{\mathrm{MCL}}$ given by (9) and (10) are quite simple, which result from the truncated exponential family $\mathcal{P}$ of distributions.

The results of Theorems 1, 2, 3 and 4 can be extended to the case of a two-sided truncated exponential family of distributions with a natural parameter θ and two truncation parameters γ and ν as nuisance parameters, including an upper-truncated Pareto distribution which is important in applications [see Akahira et al. (2014)]. Further, they may be similarly extended to the case of a more general truncated family of distributions from the truncated exponential family $\mathcal{P}$. In relation to Theorem 2, if two different bias adjustments are introduced, i.e. $\hat{\theta}_{\mathrm{ML}} + (1/n)c_i(\hat{\theta}_{\mathrm{ML}})$ $(i = 1, 2)$, then the problem whether or not the admissibility result holds may be interesting.

Appendix A

The proof of Theorem 1 Let $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 1, 2, 3, 4)$. Since

$$Z_1 = \frac{1}{\sqrt{\lambda_2 n}} \sum_{i=1}^{n} \{u(X_i) - \lambda_1\}, \quad U_\gamma := \sqrt{\lambda_2 n}\left(\hat{\theta}_{\mathrm{ML}}^{\gamma} - \theta\right),$$

by the Taylor expansion we obtain from (3)

$$0 = \sqrt{\frac{\lambda_2}{n}} Z_1 - \sqrt{\frac{\lambda_2}{n}} U_\gamma - \frac{\lambda_3}{2\lambda_2 n} U_\gamma^2 - \frac{\lambda_4}{6\lambda_2^{3/2} n\sqrt{n}} U_\gamma^3 + O_p\left(\frac{1}{n^2}\right),$$

which implies that the stochastic expansion of U_γ is given by

$$U_\gamma = Z_1 - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} Z_1^2 + \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right) Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right). \tag{11}$$

Since

$$E_\theta(Z_1) = 0, \quad V_\theta(Z_1) = E_\theta(Z_1^2) = 1,$$
$$E_\theta(Z_1^3) = \frac{\lambda_3}{\lambda_2^{3/2}\sqrt{n}}, \quad E_\theta(Z_1^4) = 3 + \frac{\lambda_4}{\lambda_2^2 n}, \tag{12}$$

it follows that

$$E_\theta(U_\gamma) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{13}$$

$$E_\theta(U_\gamma^2) = 1 + \frac{1}{n}\left(\frac{11\lambda_3^2}{4\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + O\left(\frac{1}{n\sqrt{n}}\right), \tag{14}$$

hence, by (13) and (14)

$$V_\theta(U_\gamma) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + O\left(\frac{1}{n\sqrt{n}}\right). \tag{15}$$

From (11), (13) and (15) we have the conclusion of Theorem 1.

Before proving Theorem 2, we prepare three lemmas (the proofs are given in Appendix B).

Lemma 1 *The second-order asymptotic density of T is given by*

$$f_T(t) = k(\theta,\gamma)e^{-k(\theta,\gamma)t} + \frac{k(\theta,\gamma)}{a(\gamma)b(\theta,\gamma)n}\left\{c_\theta(\gamma)b(\theta,\gamma) + a^2(\gamma)e^{\theta u(\gamma)}\right\} \cdot \left\{t - \frac{k(\theta,\gamma)}{2}t^2\right\} e^{-k(\theta,\gamma)t} + O\left(\frac{1}{n^2}\right) \tag{16}$$

for $t > 0$, where $k(\theta,\gamma) := a(\gamma)e^{\theta u(\gamma)}/b(\theta,\gamma)$, and

$$E_{\theta,\gamma}(T) = \frac{1}{k(\theta,\gamma)} + \frac{A(\theta,\gamma)}{n} + O\left(\frac{1}{n^2}\right), \quad E_{\theta,\gamma}(T^2) = \frac{2}{k^2(\theta,\gamma)} + O\left(\frac{1}{n}\right), \tag{17}$$

where

$$A(\theta,\gamma) := -\frac{1}{k^2(\theta,\gamma)}\left\{\frac{c_\theta(\gamma)}{a(\gamma)} + k(\theta,\gamma)\right\}$$

with $c_\theta(\gamma) = a'(\gamma) + \theta a(\gamma)u'(\gamma)$.

Lemma 2 *It holds that*

$$E_{\theta,\gamma}(Z_1T) = \frac{1}{k\sqrt{\lambda_2 n}}\left\{u(\gamma) - \lambda_1 + \frac{2}{k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{18}$$

where $k = k(\theta,\gamma)$ and $\lambda_i = \lambda_i(\theta,\gamma)$ $(i = 1, 2)$.

Lemma 3 *It holds that*

$$E_{\theta,\gamma}(Z_1^2T) = \frac{1}{k} + O\left(\frac{1}{n}\right), \tag{19}$$

where $k = k(\theta,\gamma)$.

The proof of Theorem 2 Since, for $(\theta, \gamma) \in \Theta \times (c, X_{(1)})$

$$\begin{aligned}
&\lambda_1\left(\hat{\theta}_{\mathrm{ML}}, X_{(1)}\right) \\
&= \lambda_1(\theta,\gamma) + \left\{\frac{\partial}{\partial\theta}\lambda_1(\theta,\gamma)\right\}(\hat{\theta}_{\mathrm{ML}}-\theta) + \left\{\frac{\partial}{\partial\gamma}\lambda_1(\theta,\gamma)\right\}\left(X_{(1)}-\gamma\right) \\
&\quad + \frac{1}{2}\left\{\frac{\partial^2}{\partial\theta^2}\lambda_1(\theta,\gamma)\right\}(\hat{\theta}_{\mathrm{ML}}-\theta)^2 + \left\{\frac{\partial^2}{\partial\theta\partial\gamma}\lambda_1(\theta,\gamma)\right\}(\hat{\theta}_{\mathrm{ML}}-\theta)\left(X_{(1)}-\gamma\right) \\
&\quad + \frac{1}{2}\left\{\frac{\partial^2}{\partial\gamma^2}\lambda_1(\theta,\gamma)\right\}(X_{(1)}-\gamma)^2 + \frac{1}{6}\left\{\frac{\partial^3}{\partial\theta^3}\lambda_1(\theta,\gamma)\right\}(\hat{\theta}_{\mathrm{ML}}-\theta)^3 \\
&\quad + \frac{1}{2}\left\{\frac{\partial^2}{\partial\theta^2}\lambda_1(\theta,\gamma)\right\}\left\{\frac{\partial}{\partial\gamma}\lambda_1(\theta,\gamma)\right\}(\hat{\theta}_{\mathrm{ML}}-\theta)^2\left(X_{(1)}-\gamma\right) + \cdots, \qquad (20)
\end{aligned}$$

noting $\hat{U} = \sqrt{\lambda_2 n}\left(\hat{\theta}_{\mathrm{ML}} - \theta\right)$ and $T = n\left(X_{(1)} - \gamma\right)$, we have from (5) and (20)

$$\begin{aligned}
0 &= \sqrt{\frac{\lambda_2}{n}}Z_1 - \sqrt{\frac{\lambda_2}{n}}\hat{U} - \frac{1}{n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)T - \frac{\lambda_3}{2\lambda_2 n}\hat{U}^2 - \frac{1}{\sqrt{\lambda_2 n}n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)\hat{U}T \\
&\quad - \frac{\lambda_4}{6\lambda_2^{3/2}n\sqrt{n}}\hat{U}^3 + O_p\left(\frac{1}{n^2}\right),
\end{aligned}$$

where $\lambda_j = \lambda_j(\theta,\gamma)$ $(j = 1, 2, 3, 4)$ are defined by (2), hence the stochastic expansion of $\hat{U}$ is given by

$$\begin{aligned}
\hat{U} &= Z_1 - \frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)T - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}}\hat{U}^2 - \frac{1}{\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)\hat{U}T \\
&\quad - \frac{\lambda_4}{6\lambda_2^2 n}\hat{U}^3 + O_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= Z_1 - \frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)T - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}}Z_1^2 + \frac{\delta}{\lambda_2 n}Z_1 T \\
&\quad + \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right)Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right). \qquad (21)
\end{aligned}$$

It follows from (12) and (21) that

$$E_{\theta,\gamma}(\hat{U}) = -\frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)E_{\theta,\gamma}(T) - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + \frac{\delta}{\lambda_2 n}E_{\theta,\gamma}(Z_1 T) + O\left(\frac{1}{n\sqrt{n}}\right). \qquad (22)$$

Substituting (17) and (18) for (22), we obtain

$$E_{\theta,\gamma}(\hat{U}) = -\frac{1}{\sqrt{\lambda_2 n}}\left\{\frac{1}{k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) + \frac{\lambda_3}{2\lambda_2}\right\} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{23}$$

where $k = k(\theta, \gamma)$ is defined in Lemma 1. We have from (21)

$$\begin{aligned} E_{\theta,\gamma}(\hat{U}^2) = E_{\theta,\gamma}(Z_1^2) &- \frac{1}{\sqrt{\lambda_2 n}}\left\{2\left(\frac{\partial\lambda_1}{\partial\gamma}\right)E_{\theta,\gamma}(Z_1 T) + \frac{\lambda_3}{\lambda_2}E_{\theta,\gamma}(Z_1^3)\right\} \\ &+ \frac{1}{\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)^2 E_{\theta,\gamma}\left(T^2\right) + \frac{1}{\lambda_2 n}\left\{\frac{\lambda_3}{\lambda_2}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) + 2\delta\right\}E_{\theta,\gamma}(Z_1^2 T) \\ &+ \frac{1}{n}\left(\frac{5\lambda_3^2}{4\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right)E_{\theta,\gamma}(Z_1^4) + O\left(\frac{1}{n\sqrt{n}}\right). \end{aligned} \tag{24}$$

Substituting (12), (17), (18) and (19) for (24), we have

$$\begin{aligned} E_{\theta,\gamma}(\hat{U}^2) = 1 &- \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_1 + \frac{1}{k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + \frac{11\lambda_3^2}{4\lambda_2^3 n} \\ &+ \frac{3\lambda_3}{k\lambda_2^2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right) - \frac{\lambda_4}{\lambda_2^2 n} + O\left(\frac{1}{n\sqrt{n}}\right). \end{aligned} \tag{25}$$

Since

$$\begin{aligned} &\frac{\sqrt{\lambda_2}\frac{\partial\lambda_1}{\partial\gamma}(\hat{\theta}_{\mathrm{ML}}, X_{(1)})}{k(\hat{\theta}_{\mathrm{ML}}, X_{(1)})\lambda_2(\hat{\theta}_{\mathrm{ML}}, X_{(1)})\sqrt{n}} \\ &= \frac{\frac{\partial\lambda_1}{\partial\gamma}(\theta,\gamma)}{k\sqrt{\lambda_2 n}} + \frac{1}{k\lambda_2 n}\left\{\frac{\partial\lambda_2}{\partial\gamma}(\theta,\gamma) - \left(\frac{\lambda_3}{\lambda_2} + \frac{1}{k}\frac{\partial k}{\partial\theta}\right)\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\}\hat{U} + O_p\left(\frac{1}{n\sqrt{n}}\right), \end{aligned}$$

it follows from (6) that the stochastic expansion of $\hat{U}^*$ is given by

$$\begin{aligned} \hat{U}^* := \sqrt{\lambda_2 n}(\hat{\theta}^*_{\mathrm{ML}} - \theta) &= \sqrt{\lambda_2 n}(\hat{\theta}_{\mathrm{ML}} - \theta) + \frac{\sqrt{\lambda_2}\frac{\partial\lambda_1}{\partial\gamma}(\hat{\theta}_{\mathrm{ML}}, X_{(1)})}{k(\hat{\theta}_{\mathrm{ML}}, X_{(1)})\hat{\lambda_2}\sqrt{n}} \\ &= \hat{U} + \frac{1}{k\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) - \frac{1}{k\lambda_2 n}\left\{\delta + \frac{1}{k}\left(\frac{\partial k}{\partial\theta}\right)\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\}Z_1 + O_p\left(\frac{1}{n\sqrt{n}}\right), \end{aligned} \tag{26}$$

where $\hat{U}$ is given by (21), $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 1, 2, 3)$ and $k = k(\theta, \gamma)$. From (12) and (23), we have

$$E_{\theta,\gamma}(\hat{U}^*) = -\frac{1}{\sqrt{\lambda_2 n}}\left\{\frac{1}{k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) + \frac{\lambda_3}{2\lambda_2}\right\} + \frac{1}{k\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) + O\left(\frac{1}{n\sqrt{n}}\right)$$
$$= -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right). \tag{27}$$

It follows from (23), (25) and (26) that

$$E_{\theta,\gamma}\left(\hat{U}^{*2}\right) = 1 - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_1 + \frac{3}{2k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + \frac{11\lambda_3^2}{4\lambda_2^3 n}$$
$$- \frac{\lambda_4}{\lambda_2^2 n} - \frac{2}{k^2\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left(\frac{\partial k}{\partial\theta}\right) + O\left(\frac{1}{n\sqrt{n}}\right),$$

hence, by (27)

$$V_{\theta,\gamma}(\hat{U}^*) = E_{\theta,\gamma}(\hat{U}^{*2}) - \left\{E_{\theta,\gamma}(\hat{U}^*)\right\}^2$$
$$= 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_1 + \frac{1}{k}\left(\frac{\partial k}{\partial\theta}\right)\right\}$$
$$- \frac{3}{k^2\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)^2 + O\left(\frac{1}{n\sqrt{n}}\right). \tag{28}$$

Since, by (2)

$$\lambda_1(\theta, \gamma) = \frac{\partial}{\partial\theta}\log b(\theta, \gamma) = \frac{1}{b(\theta, \gamma)}\int_\gamma^d a(x)u(x)e^{\theta u(x)}\mathrm{d}x,$$

it follows that

$$\frac{\partial\lambda_1(\theta, \gamma)}{\partial\gamma} = \frac{a(\gamma)e^{\theta u(\gamma)}}{b(\theta, \gamma)}\left\{\lambda_1(\theta, \gamma) - u(\gamma)\right\} = k(\theta, \gamma)\left\{\lambda_1(\theta, \gamma) - u(\gamma)\right\}. \tag{29}$$

Since

$$\frac{\partial k}{\partial\theta}(\theta, \gamma) = k(\theta, \gamma)\{u(\gamma) - \lambda_1(\theta, \gamma)\}, \tag{30}$$

it is seen from (28), (29) and (30) that

$$V_{\theta,\gamma}(\hat{U}^*) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\left\{\lambda_1 - u(\gamma)\right\}^2 + O\left(\frac{1}{n\sqrt{n}}\right). \tag{31}$$

From (26), (27) and (31) we have the conclusion of Theorem 2.
The proof of Theorem 3 Since, from (7)

$$0 = \frac{1}{n-1}\sum_{i=2}^{n}\{u(Y_i) - \lambda_1(\theta, x_{(1)})\} - \frac{1}{\sqrt{n}}\lambda_2(\theta, x_{(1)})\sqrt{n}(\hat{\theta}_{\mathrm{MCL}} - \theta)$$
$$- \frac{1}{2n}\lambda_3(\theta, x_{(1)})n(\hat{\theta}_{\mathrm{MCL}} - \theta)^2$$
$$- \frac{1}{6n\sqrt{n}}\lambda_4(\theta, x_{(1)})n\sqrt{n}(\hat{\theta}_{\mathrm{MCL}} - \theta)^3 + O_p\left(\frac{1}{n^2}\right),$$

letting

$$\tilde{Z}_1 = \frac{1}{\sqrt{\tilde{\lambda_2}(n-1)}}\sum_{i=2}^{n}\left\{u(Y_i) - \lambda_1(\theta, x_{(1)})\right\},$$
$$\tilde{U} = \sqrt{\tilde{\lambda_2}n}(\hat{\theta}_{\mathrm{MCL}} - \theta),$$

where $\tilde{\lambda_i} := \lambda_i(\theta, x_{(1)})$ $(i = 1, 2, 3, 4)$, we have

$$0 = \sqrt{\frac{\tilde{\lambda_2}}{n-1}}\tilde{Z}_1 - \sqrt{\frac{\tilde{\lambda_2}}{n}}\tilde{U} - \frac{\tilde{\lambda_3}}{2\tilde{\lambda_2}n}\tilde{U}^2 - \frac{\tilde{\lambda_4}}{6\tilde{\lambda_2}^{3/2}n\sqrt{n}}\tilde{U}^3 + O_p\left(\frac{1}{n^2}\right),$$

hence the stochastic expansion of $\tilde{U}$ is given by

$$\tilde{U} = \sqrt{\frac{n}{n-1}}\tilde{Z}_1 - \frac{\tilde{\lambda_3}}{2\tilde{\lambda_2}^{3/2}\sqrt{n}}\tilde{U}^2 - \frac{\tilde{\lambda_4}}{6\tilde{\lambda_2}^{2}n}\tilde{U}^3 + O_p\left(\frac{1}{n\sqrt{n}}\right)$$
$$= \tilde{Z}_1 - \frac{\tilde{\lambda_3}}{2\tilde{\lambda_2}^{3/2}\sqrt{n}}\tilde{Z}_1^{\,2} + \frac{1}{2n}\tilde{Z}_1 + \frac{1}{2n}\left(\frac{\tilde{\lambda_3}^2}{\tilde{\lambda_2}^3} - \frac{\tilde{\lambda_4}}{3\tilde{\lambda_2}^2}\right)\tilde{Z}_1^{\,3} + O_p\left(\frac{1}{n\sqrt{n}}\right). \tag{32}$$

Since

$$\tilde{\lambda_2} = \lambda_2(\theta, X_{(1)}) = \lambda_2(\theta, \gamma) + \frac{1}{n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T + O_p\left(\frac{1}{n^2}\right),$$

we obtain

$$\tilde{U} = \sqrt{\tilde{\lambda_2}n}(\hat{\theta}_{\mathrm{MCL}} - \theta)$$
$$= \sqrt{\lambda_2 n}(\hat{\theta}_{\mathrm{MCL}} - \theta)\left\{1 + \frac{1}{2n\lambda_2}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T + O_p\left(\frac{1}{n^2}\right)\right\}, \tag{33}$$

where $T = n(X_{(1)} - \gamma)$ and $\lambda_2 = \lambda_2(\theta, \gamma)$. Then, it follows from (32) and (33) that

$$\begin{aligned}\tilde{U_0} &= \sqrt{\lambda_2 n}(\hat{\theta}_{\mathrm{MCL}} - \theta)\\ &= \tilde{Z_1} - \frac{\tilde{\lambda_3}}{2\tilde{\lambda_2}^{3/2}\sqrt{n}}\tilde{Z_1}^2 + \frac{1}{2n}\left\{1 - \frac{1}{\lambda_2}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T\right\}\tilde{Z_1}\\ &\quad + \frac{1}{2n}\left(\frac{\tilde{\lambda_3}^2}{\tilde{\lambda_2}^3} - \frac{\tilde{\lambda_4}}{3\tilde{\lambda_2}^2}\right)\tilde{Z_1}^3 + O_p\left(\frac{1}{n\sqrt{n}}\right). \qquad (34)\end{aligned}$$

For given $X_{(1)} = x_{(1)}$, i.e. $T = t := n(x_{(1)} - \gamma)$, the conditional expectation of $\tilde{Z_1}$ and $\tilde{Z_1}^2$ is

$$\begin{aligned}E_{\theta,\gamma}(\tilde{Z_1}|t) &= \frac{1}{\sqrt{\tilde{\lambda_2}(n-1)}}\sum_{i=2}^{n}\left\{E_{\theta,\gamma}[u(Y_i)|t] - \lambda_1(\theta, x_{(1)})\right\} = 0,\\ E_{\theta,\gamma}(\tilde{Z_1}^2|t) &= \frac{1}{\tilde{\lambda_2}(n-1)}\Bigg[\sum_{i=2}^{n}E_{\theta,\gamma}\left[\{u(Y_i) - \lambda_1(\theta, x_{(1)})\}^2|t\right]\\ &\quad + \sum_{\substack{i\neq j\\ 2\leq i,j\leq n}}\sum E_{\theta,\gamma}\left[\{u(Y_i) - \lambda_1(\theta, x_{(1)})\}\{u(Y_j) - \lambda_1(\theta, x_{(1)})\}\,|\,t\right]\Bigg]\\ &= 1, \qquad (35)\end{aligned}$$

hence the conditional variance of $\tilde{Z_1}$ is equal to 1, i.e. $V_{\theta,\gamma}(\tilde{Z_1}|t) = 1$. In a similar way to the above, we have

$$E_{\theta,\gamma}\left(\tilde{Z_1}^3|t\right) = \frac{\tilde{\lambda_3}}{\tilde{\lambda_2}^{3/2}\sqrt{n-1}}, \quad E_{\theta,\gamma}\left(\tilde{Z_1}^4|t\right) = 3 + \frac{\tilde{\lambda_4}}{\tilde{\lambda_2}^2(n-1)}. \qquad (36)$$

Since, by (34), (35) and (36)

$$\begin{aligned}E_{\theta,\gamma}(\tilde{U_0}|T) &= E_{\theta,\gamma}(\tilde{Z_1}|T) - \frac{\tilde{\lambda_3}}{2\tilde{\lambda_2}^{3/2}\sqrt{n}}E_{\theta,\gamma}(\tilde{Z_1}^2|T)\\ &\quad + \frac{1}{2n}E_{\theta,\gamma}(\tilde{Z_1}|T) + \frac{1}{2n}\left(\frac{\tilde{\lambda}_3^2}{\tilde{\lambda}_2^3} - \frac{\tilde{\lambda_4}}{3\tilde{\lambda_2}^2}\right)E_{\theta,\gamma}(\tilde{Z_1}^3|T)\\ &\quad - \frac{1}{2n\lambda_2}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T\,E_{\theta,\gamma}(\tilde{Z_1}|T) + O_p\left(\frac{1}{n\sqrt{n}}\right)\\ &= -\frac{\tilde{\lambda_3}}{2\tilde{\lambda_2}^{3/2}\sqrt{n}} + O_p\left(\frac{1}{n\sqrt{n}}\right), \qquad (37)\\ E_{\theta,\gamma}(\tilde{U}_0^2|T) &= E_{\theta,\gamma}(\tilde{Z_1}^2|T) - \frac{\tilde{\lambda_3}}{\tilde{\lambda_2}^{3/2}\sqrt{n}}E_{\theta,\gamma}(\tilde{Z_1}^3|T)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n}E_{\theta,\gamma}(\tilde{Z_1}^2|T)+\frac{1}{n}\left(\frac{5\tilde{\lambda_3}^2}{4\tilde{\lambda_2}^3}-\frac{\tilde{\lambda_4}}{3\tilde{\lambda_2}^2}\right)E_{\theta,\gamma}(\tilde{Z_1}^4|T)\\
&-\frac{1}{\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)TE_{\theta,\gamma}(\tilde{Z_1}^2|T)+O_p\left(\frac{1}{n\sqrt{n}}\right)\\
=&\ 1+\frac{1}{n}+\frac{1}{n}\left(\frac{11\tilde{\lambda_3}^2}{4\tilde{\lambda_2}^3}-\frac{\tilde{\lambda_4}}{\tilde{\lambda_2}^2}\right)-\frac{1}{\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T\\
&+O_p\left(\frac{1}{n\sqrt{n}}\right), \qquad (38)
\end{aligned}
$$

where $\tilde{\lambda_i}=\lambda_i(\theta,X_{(1)})$ $(i=2,3,4)$. Since, for $i=2,3,4$

$$
\begin{aligned}
\tilde{\lambda_i}&=\lambda_i(\theta,X_{(1)})=\lambda_i(\theta,\gamma)+\frac{1}{n}\left(\frac{\partial\lambda_i}{\partial\gamma}\right)n(X_{(1)}-\gamma)+O_p\left(\frac{1}{n^2}\right)\\
&=\lambda_i(\theta,\gamma)+O_p\left(\frac{1}{n}\right)=\lambda_i+O_p\left(\frac{1}{n}\right), \qquad (39)
\end{aligned}
$$

it follows from (37) that

$$
\begin{aligned}
E_{\theta,\gamma}(\tilde{U}_0)&=E_{\theta,\gamma}\left[E_{\theta,\gamma}(\tilde{U}_0|T)\right]=-\frac{1}{2\sqrt{n}}E_{\theta,\gamma}\left(\frac{\tilde{\lambda_3}}{\tilde{\lambda_2}^{3/2}}\right)+O\left(\frac{1}{n\sqrt{n}}\right)\\
&=-\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}}+O\left(\frac{1}{n\sqrt{n}}\right). \qquad (40)
\end{aligned}
$$

It is noted from (13), (27) and (40) that

$$
E_{\theta,\gamma}(U_\gamma)=E_{\theta,\gamma}(\hat{U}^*)=E_{\theta,\gamma}(\tilde{U}_0)=-\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}}+O\left(\frac{1}{n\sqrt{n}}\right).
$$

In a similar way to the above, we obtain from (17), (38) and (39)

$$
E_{\theta,\gamma}(\tilde{U}_0^2)=1+\frac{1}{n}+\frac{11\lambda_3^2}{4\lambda_2^3 n}-\frac{\lambda_4}{\lambda_2^2 n}-\frac{1}{k\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)+O\left(\frac{1}{n\sqrt{n}}\right). \qquad (41)
$$

Since, by (29) and (30)

$$
\begin{aligned}
\frac{1}{k}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)&=\frac{1}{k}\left(\frac{\partial^2\lambda_1}{\partial\theta\partial\gamma}\right)=\frac{1}{k}\frac{\partial}{\partial\theta}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)=\frac{1}{k}\frac{\partial}{\partial\theta}\left\{k\left(\lambda_1-u(\gamma)\right)\right\}\\
&=\frac{1}{k}\left\{\frac{\partial k}{\partial\theta}(\lambda_1-u(\gamma))+k\left(\frac{\partial\lambda_1}{\partial\theta}\right)\right\}=-\left(\lambda_1-u(\gamma)\right)^2+\lambda_2,
\end{aligned}
$$

it follows from (41) that

$$E_{\theta,\gamma}(\tilde{U}_0^2) = 1 + \frac{11\lambda_3^2}{4\lambda_2^3 n} - \frac{\lambda_4}{\lambda_2^2 n} + \frac{1}{\lambda_2 n}\{\lambda_1 - u(\gamma)\}^2 + O\left(\frac{1}{n\sqrt{n}}\right),$$

hence, by (40)

$$V_{\theta,\gamma}(\tilde{U}_0) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\{\lambda_1 - u(\gamma)\}^2 + O\left(\frac{1}{n\sqrt{n}}\right). \tag{42}$$

From (34), (40) and (42), we have the conclusion of Theorem 3.

Appendix B

The proof of Lemma 1 Since the second-order asymptotic cumulative distribution function of T is given by

$$\begin{aligned}
F_T(t) &= P_{\theta,\gamma}\{T \le t\} = P_{\theta,\gamma}\left\{n(X_{(1)} - \gamma) \le t\right\} \\
&= 1 - \left\{1 - \int_\gamma^{\gamma+\frac{t}{n}} \frac{1}{b(\theta,\gamma)} a(x) e^{\theta u(x)} dx\right\}^n \\
&= 1 - \left[\exp\left\{-\frac{a(\gamma)e^{\theta u(\gamma)}}{b(\theta,\gamma)} t\right\}\right] \\
&\quad \cdot \left[1 - \frac{e^{\theta u(\gamma)} t^2}{2b^2(\theta,\gamma) n}\left\{c_\theta(\gamma) b(\theta,\gamma) + a^2(\gamma) e^{\theta u(\gamma)}\right\} + O\left(\frac{1}{n^2}\right)\right]
\end{aligned}$$

for $t > 0$, where $c_\theta(\gamma) := a'(\gamma) + \theta a(\gamma) u'(\gamma)$, we obtain (16). From (16), we also get (17) by a straightforward calculation.

The proof of Lemma 2 As is seen from the beginning of Sect. 5, the $Y_2, \ldots, Y_n$ are i.i.d. random variables with a density

$$g(y; \theta, x_{(1)}) = \frac{a(y) e^{\theta u(y)}}{b(\theta, x_{(1)})} \quad \text{for} \quad x_{(1)} < y < d. \tag{43}$$

Then, the conditional expectation of Z_1 given T is obtained by

$$\begin{aligned}
E_{\theta,\gamma}(Z_1|T) &= \frac{1}{\sqrt{\lambda_2 n}} \sum_{i=1}^n \left\{E_{\theta,\gamma}[u(X_i)|T] - \lambda_1\right\} \\
&= \frac{1}{\sqrt{\lambda_2 n}} \left\{u(X_{(1)}) + \sum_{i=2}^n E_{\theta,\gamma}[u(Y_i)|T] - n\lambda_1\right\},
\end{aligned} \tag{44}$$

where $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 1, 2)$. Since, for each $i = 2, \ldots, n$, by (43)

$$E_{\theta,\gamma}[u(Y_i)|T] = \int_{X_{(1)}}^{d} u(y)\frac{a(y)e^{\theta u(y)}}{b(\theta, X_{(1)})}\mathrm{d}y$$
$$= \frac{\partial}{\partial\theta}\log b(\theta, X_{(1)}) = \lambda_1(\theta, X_{(1)}) =: \hat{\lambda_1} \quad \text{(say)},$$

it follows from (44) that

$$E_{\theta,\gamma}(Z_1|T) = \frac{1}{\sqrt{\lambda_2 n}}\left\{u(X_{(1)}) + (n-1)\hat{\lambda_1}\right\} - \frac{\lambda_1\sqrt{n}}{\sqrt{\lambda_2}},$$

hence, from (17) and (44)

$$\begin{aligned} E_{\theta,\gamma}(Z_1 T) &= E_{\theta,\gamma}[T\, E_{\theta,\gamma}(Z_1|T)] \\ &= \frac{1}{\sqrt{\lambda_2 n}}\left\{E_{\theta,\gamma}[u(X_{(1)})T] + (n-1)E_{\theta,\gamma}(\hat{\lambda_1}T)\right\} \\ &\quad - \sqrt{\frac{n}{\lambda_2}}\lambda_1\left\{\frac{1}{k} + \frac{A(\theta,\gamma)}{n} + O\left(\frac{1}{n^2}\right)\right\}, \end{aligned} \tag{45}$$

where $k = k(\theta, \gamma)$. Since, by the Taylor expansion

$$u(X_{(1)}) = u(\gamma) + \frac{u'(\gamma)}{n}T + \frac{u''(\gamma)}{2n^2}T^2 + O_p\left(\frac{1}{n^3}\right),$$
$$\hat{\lambda}_1 = \lambda(\theta, X_{(1)}) = \lambda_1(\theta,\gamma) + \frac{1}{n}\left\{\frac{\partial}{\partial\gamma}\lambda_1(\theta,\gamma)\right\}T + \frac{1}{2n^2}\left\{\frac{\partial^2}{\partial\gamma^2}\lambda_1(\theta,\gamma)\right\}T^2 + O_p\left(\frac{1}{n^3}\right),$$

it follows from (17) that

$$E_{\theta,\gamma}\left[u(X_{(1)})T\right] = \frac{u(\gamma)}{k} + \frac{1}{n}\left\{Au(\gamma) + \frac{2u'(\gamma)}{k^2}\right\} + O\left(\frac{1}{n^2}\right), \tag{46}$$
$$E_{\theta,\gamma}(\hat{\lambda_1}T) = \frac{\lambda_1}{k} + \frac{1}{n}\left\{\lambda_1 A + \frac{2}{k^2}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + O\left(\frac{1}{n^2}\right), \tag{47}$$

where $k = k(\theta, \gamma)$, $A = A(\theta, \gamma)$ and $\lambda_1 = \lambda_1(\theta, \gamma)$. From (45), (46) and (47), we obtain (18).

The proof of Lemma 3 First, we have

$$\begin{aligned}
E_{\theta,\gamma}(Z_1^2|T) &= E_{\theta,\gamma}\left[\frac{1}{\lambda_2 n}\left\{\sum_{i=1}^n (u(X_i)-\lambda_1)\right\}^2 \Big| T\right] \\
&= \frac{1}{\lambda_2 n}\left\{u(X_{(1)})-\lambda_1\right\}^2 \\
&\quad + \frac{2}{\lambda_2 n}\left\{u(X_{(1)})-\lambda_1\right\}\sum_{i=2}^n E_{\theta,\gamma}\left[u(Y_i)-\lambda_1|T\right] \\
&\quad + \frac{1}{\lambda_2 n}\sum_{i=2}^n E_{\theta,\gamma}\left[\{u(Y_i)-\lambda_1\}^2|T\right] \\
&\quad + \frac{1}{\lambda_2 n}\sum_{\substack{i\neq j \\ 2\le i,j\le n}}\sum E_{\theta,\gamma}\left[\{u(Y_i)-\lambda_1\}\left\{u(Y_j)-\lambda_1\right\} \mid T\right]. \qquad (48)
\end{aligned}$$

For $2 \le i \le n$, we have

$$\begin{aligned}
E_{\theta,\gamma}\left[u(Y_i)-\lambda_1|T\right] &= E_{\theta,\gamma}\left[u(Y_i)|T\right]-\lambda_1 = \lambda_1(\theta, X_{(1)})-\lambda_1(\theta,\gamma) \\
&= \left(\frac{\partial\lambda_1}{\partial\gamma}\right)\frac{T}{n} + O_p\left(\frac{1}{n^2}\right) = O_p\left(\frac{1}{n}\right), \qquad (49)
\end{aligned}$$

and for $i \neq j$ and $2 \le i, j \le n$

$$\begin{aligned}
&E_{\theta,\gamma}\left[\{u(Y_i)-\lambda_1\}\left\{u(Y_j)-\lambda_1\right\}|T\right] \\
&\quad = E_{\theta,\gamma}\left[u(Y_i)-\lambda_1 \mid T\right] E_{\theta,\gamma}\left[u(Y_j)-\lambda_1|T\right] \\
&\quad = \left(\frac{\partial\lambda_1}{\partial\gamma}\right)^2\frac{T^2}{n^2} + O_p\left(\frac{1}{n^3}\right) = O_p\left(\frac{1}{n^2}\right). \qquad (50)
\end{aligned}$$

Since, for $i = 2, \ldots, n$

$$\begin{aligned}
E_{\theta,\gamma}[u^2(Y_i)|T] &= \int_{X_{(1)}}^d u^2(y)\frac{a(y)e^{\theta u(y)}}{b(\theta, X_{(1)})}dy \\
&= \frac{1}{b(\theta, X_{(1)})}\frac{\partial^2}{\partial\theta^2}b(\theta, X_{(1)}) \\
&= \lambda_1^2(\theta, X_{(1)}) + \lambda_2(\theta, X_{(1)}) \\
&= \hat{\lambda_1}^2 + \hat{\lambda_2},
\end{aligned}$$

where $\hat{\lambda_i} = \lambda_i(\theta, X_{(1)})$ $(i = 1, 2)$, we have for $i = 2, \ldots, n$

$$\begin{aligned}
&E_{\theta,\gamma}\left[\{u(Y_i) - \lambda_1\}^2 \,|T\right] \\
&\quad = E_{\theta,\gamma}[u^2(Y_i)|T] - 2\lambda_1 E_{\theta,\gamma}[u(Y_i)|T] + \lambda_1^2 \\
&\quad = \hat{\lambda_1}^2 + \hat{\lambda_2} - 2\lambda_1\hat{\lambda_1} + \lambda_1^2 \\
&\quad = \lambda_2 + \frac{1}{n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T + O_p\left(\frac{1}{n^2}\right) = \lambda_2 + O_p\left(\frac{1}{n}\right).
\end{aligned} \tag{51}$$

From (48), (49), (50) and (51), we obtain

$$\begin{aligned}
E_{\theta,\gamma}(Z_1^2|T) &= \frac{1}{\lambda_2 n}\left\{u(X_{(1)}) - \lambda_1\right\}^2 \\
&\quad + \frac{2}{\lambda_2}\left\{u(X_{(1)}) - \lambda_1\right\}\left(1 - \frac{1}{n}\right)\left\{O_p\left(\frac{1}{n}\right)\right\} \\
&\quad + \frac{1}{\lambda_2}\left(1 - \frac{1}{n}\right)\left\{\lambda_2 + O_p\left(\frac{1}{n}\right)\right\} \\
&\quad + \frac{n}{\lambda_2}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left\{O_p\left(\frac{1}{n^2}\right)\right\} \\
&= 1 + O_p\left(\frac{1}{n}\right),
\end{aligned}$$

hence, by (17)

$$E_{\theta,\gamma}(Z_1^2 T) = E_{\theta,\gamma}[T E_{\theta,\gamma}(Z_1^2|T)] = E_{\theta,\gamma}(T) + O\left(\frac{1}{n}\right) = \frac{1}{k} + O\left(\frac{1}{n}\right).$$

Thus we get (19).

Acknowledgments The author thanks the referees for their careful reading and valuable comments, especially one of them for pointing out the mistake of omitting certain term in the stochastic expansions of the estimators. He also thanks the associate editor for the comment.

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