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Testing for positive expectation dependence

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Abstract In this paper, hypothesis testing for positive first-degree and higher-degree expectation dependence is investigated. Some tests of Kolmogorov–Smirnov type are constructed, which are shown to control type I error well and to be consistent against global alternative hypothesis. Further, the tests can also detect local alternative hypotheses distinct from the null hypothesis at a rate as close to the square root of the sample size as possible, which is the fastest possible rate in hypothesis testing.

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A nonparametric Monte Carlo test procedure is applied to implement the new tests because both sampling and limiting null distributions are not tractable. Simulation studies and a real data analysis are carried out to illustrate the performances of the new tests.

Keywords Expectation dependence · Nonparametric Monte Carlo · Test of Kolmogorov–Smirnov type

1 Introduction

Expectation dependence is an important issue in some scientific fields such as finance, insurance and asset pricing. There are a number of proposals in the literature to demonstrate the profound impact of expectation dependence in these fields. Wright (1987) first introduced the notion of first-degree expectation dependence and Li (2011) extended the concept to higher-degree expectation dependence. Wright (1987) and Hadar and Seo (1988) used expectation dependence to study portfolio problem and asset allocation. Tsetlin and Winkler (2005) and Li (2011) studied the demand for a risky asset in the presence of financial risk and background risk. Denuit et al. (2012) used first- and second-degree expectation dependence to obtain the sign of the equity premium in the consumption-based CAPM. Denuit et al. (2013) further extended the concept to an almost expectation dependence concept that is applied to portfolio diversification, the determination of the sign of the equity premium in the consumption-based CAPM and the optimal investment under a background risk. Denuit et al. (2005) provided the detailed account of dependence structure and explained its relationship with stochastic dominance.

Moreover, the notion of expectation dependence has been further developed recently. Hong et al. (2011) gave a brief description for several notions of dependence in economics, including quadrant dependence, regression dependence and expectation dependence, and proved that positive or negative expectation dependence is a necessary and sufficient condition for generalized Mossin's Theorem. Recently, several efforts were devoted to testing whether dependence between random variables holds or not. Scaillet (2005) proposed a test of Kolmogorov–Smirnov type for quadrant dependence. Denuit et al. (2007), in spirit similar to Scaillet's (2005) proposal, suggested a test of Kolmogorov–Smirnov type for shortfall dominance against parametric alternatives. Denuit and Scaillet (2004) introduced a nonparametric test for quadrant dependence and extended it to handle positive orthant dependence that is a higher dimensional case. Gijbels et al. (2010) used an approximate sampling null distribution of the test statistic to replace the limiting null distribution. Gijbels and Sznajder (2013) proposed a test for positive quadrant dependence under the null hypothesis of positive quadrant dependence via resampling from a constrained copula. Scaillet and Topaloglou (2010) developed consistent tests for stochastic dominance efficiency for portfolio choice. Cebrián et al. (2004) introduced some testing procedures to analyze concordance ordering.

It is well acknowledged that in the context of dependence, first-degree expectation dependence is a stronger dependence than the correlation between random variables.

The notion has received more attention in recent years. But the problem whether positive or negative expectation dependence holds or not is not all so clear-cut in practice. Directly assuming this type of dependence, without statistical evidence, can lead to devastating effects, resulting in adverse performance in equity premium and asset allocation. To the best of our knowledge, testing expectation dependence has not yet received much attention. In this paper, we first propose a consistent test for first-degree positive expectation dependence and extend it to the higher-degree cases. The related asymptotic properties are studied, which show that the proposed tests can control the type I error well and are consistent against global alternative hypotheses. Further, the tests can detect local alternative hypotheses distinct from the null hypothesis at a rate as close to $1/\sqrt{n}$ as possible. This rate is the fastest possible rate in hypothesis testing. To implement the proposed tests, a nonparametric Monte Carlo test procedure is suggested to simulate p values because of the intractability of the sampling and limiting null distributions.

The rest of the paper is structured as follows: Section 2 contains related definitions and notations, and the construction of consistent tests for expectation dependence. Section 2.1 presents the results of first-degree expectation dependence and the higher-degree extension is investigated in Sect. 2.2. The asymptotic properties are also presented in this section. Section 3 is devoted to the implementation. In Sect. 4, the performances of the proposed tests are examined through numerical studies. The real data analysis is discussed in Sect. 5. The technical proofs are relegated to Appendix.

2 Test statistics and asymptotic properties

We first give some notations. Without loss of generality, suppose that the random variables X_1 and X_2 take values in two sets Ω_1 and Ω_2, respectively.

Define

$$\begin{aligned} D_1(z) &= -\text{cov}[X_1, I(X_2 > z)]; \\ D_k(z) &= \text{cov}(X_1, (z - X_2)_+^{k-1}) \quad \text{for } k = 2, 3, \ldots\ ; \end{aligned} \tag{1}$$

where $(z - X_2)_+^{k-1} = (z - X_2)^{k-1} I(z - X_2 > 0)$ and $I(\cdot)$ denotes indicator function.

2.1 First-degree expectation dependence

Recall the concept of first-degree expectation dependence introduced by Wright (1987) as follows.

Definition 1 (Wright 1987) Let $ED_1(z) = E(X_1) - E(X_1|X_2 \leq z)$. If

$$ED_1(z) \geq 0 \quad \text{for all } z, \tag{2}$$

then the random variable X_1 is positive first-degree expectation dependent on X_2. Negative first-degree expectation dependence is defined analogously if we reverse the sign of the inequality in (2).

The hypothesis of interest for positive first-degree expectation dependence can now be stated as:

$$H_0 : E(X_1) \geq E(X_1|X_2 \leq z) \quad \text{for all } z; \tag{3}$$
$$H_1 : E(X_1) < E(X_1|X_2 \leq z) \quad \text{for some } z. \tag{4}$$

Note that for any given z, we have

$$E(X_1) = P(X_2 > z)E(X_1|X_2 > z) + P(X_2 \leq z)E(X_1|X_2 \leq z) \tag{5}$$

and

$$E(X_1) = P(X_2 > z)E(X_1) + P(X_2 \leq z)E(X_1). \tag{6}$$

Therefore, from both Eqs. (5) and (6), we have

$$P(X_2 > z)\Big(E(X_1|X_2 > z) - E(X_1)\Big) = P(X_2 \leq z)\Big(E(X_1) - E(X_1|X_2 \leq z)\Big).$$

It shows that the positive first-degree expectation dependence in (2) can be equivalently restated as

$$E(X_1|X_2 > z) \geq E(X_1).$$

We can clearly see that:

$$\begin{aligned} & P(X_2 > z)\Big(E(X_1|X_2 > z) - E(X_1)\Big) \geq 0 \\ \Longleftrightarrow\ & E(X_1 I(X_2 > z)) - E(X_1)E(I(X_2 > z)) \geq 0 \\ \Longleftrightarrow\ & \text{cov}(X_1, I(X_2 > z)) \geq 0. \end{aligned}$$

The above equivalent forms imply that the positive first-degree expectation dependence is equivalent to the positive correlation between X_1 and $I(X_2 > z)$ for all z. Correspondingly, we can similarly obtain that the negative first-degree expectation dependence is equivalent to the negative correlation between X_1 and $I(X_2 > z)$ for all z.

Thus, we can further rewrite the null and alternative hypotheses as follows:

$$\begin{aligned} H_0 &: -\text{cov}(X_1, I(X_2 > z)) \leq 0 \ \text{ for all } z; \\ H_1 &: -\text{cov}(X_1, I(X_2 > z)) > 0 \ \text{ for some } z. \end{aligned}$$

The minus sign used here is for the convenience of technical details. Let $\bar{D}_1 = \sup_{z \in \Omega_2} \sqrt{n}(D_1(z))$, where $D_1(\cdot)$ is defined in (1). Under the null hypothesis H_0,

$\bar{D}_1 \leq 0$. However, under the alternative hypothesis H_1, since there exists a constant z satisfying $-\text{cov}(X_1, I(X_2 > z)) > 0$, we have $\bar{D}_1 > 0$. It is obvious that $\bar{D}_1$ can be used as the base for constructing a test statistic, and the null hypothesis will be rejected for large positive values of $\bar{D}_1$. Suppose that $(X_{1i}, X_{2i})_{i=1}^n$ are independent identically distributed observations. A natural estimator $D_{1n}(z)$ of $D_1(z)$ is as follows:

$$D_{1n}(z) = -\frac{1}{n}\sum_{i=1}^{n}(X_{1i} - \bar{X_1})(I(X_{2i} > z) - \overline{I(X_2 > z)}),$$

where $\bar{X_1} = \sum_{i=1}^n X_{1i}/n$ and $\overline{I(X_2 > z)} = \sum_{i=1}^n I(X_{2i} > z)/n$. A test of Kolmogorov–Smirnov type can be defined as

$$T_{1n} = \sup_{z \in \Omega_2} \sqrt{n}(D_{1n}(z)).$$

The null hypothesis H_0 is rejected if the value of T_{1n} exceeds some critical value c. To obtain the asymptotic properties of this test statistic, we only need the following condition:

(A) The second moment of X_1 exists: $E(X_1^2) < \infty$.

Remark 1 This condition is standard, by which we can ensure that the defined empirical process converges to a Gaussian process, and the test statistic converges to a maximum functional of this Gaussian process.

Lemma 1 *Under Condition* (A), *the empirical process series* $\sqrt{n}\{D_{1n}(z) - D_1(z)\}$ *converges weakly to a Gaussian process G with mean zero and the covariance function given by*

$$\begin{aligned}\Omega(z_1, z_2) = {} & E((X_1 - E(X_1))^2 I(z_1 < X_2)) \\ & + E((X_1 - E(X_1))^2 E(I(z_1 < X_2))E(I(z_2 < X_2)) \\ & - E((X_1 - E(X_1))^2 I(z_1 < X_2))E(I(z_2 < X_2)) \\ & - E(I(z_1 < X_2))E((X_1 - E(X_1))^2 I(z_2 < X_2)) - D_1(z_1)D_1(z_2),\end{aligned}$$

for all $z_1 > z_2$. *Particularly, if* X_1 *and* X_2 *are independent random variables, the covariance function can be reduced to*

$$\Omega(z_1, z_2) = cov(X_1)cov(I(z_1 < X_2), I(z_2 < X_2))$$

for all z_1, z_2.

Therefore, we use the test statistic T_{1n} to construct the decision rule as:

$$\text{reject } H_0 \text{ if } T_{1n} > c,$$

where c is a critical value that will be determined later.

We get the following Propositions 1–2 that state the asymptotic properties of the test statistic T_{1n}.

Proposition 1 *Under Condition* (A), *we have*

$$\max_{z\in\Omega_2} \sqrt{n}\{D_{1n}(z) - D_1(z)\} \longrightarrow \bar{T}_1,$$

where $\bar{T}_1 = \max_{z\in\Omega_2} G(z)$ *and* G *is defined in Lemma 1. Let* c *be a positive finite constant. We have*

(I) *under* H_0,

$$\lim_{n\to\infty} P[\textit{reject } H_0] \le P[\bar{T}_1 > c] = \alpha(c);$$

(II) *under* H_1,

$$\lim_{n\to\infty} P[\textit{reject } H_0] = 1.$$

Part (I) of Proposition 1 shows that the type I error is not more than $\alpha(c)$ and thus, can be under control for our test. Part (II) demonstrates that the test is consistent against the global alternative hypothesis H_1. It is noteworthy to point out that the limiting null distribution of the test statistic is unknown. Therefore, it is difficult to obtain the critical value c satisfying $P(\bar{T}_1 > c) = \alpha$. Thus, in spirit similar to that in Neuhaus and Zhu (1998), Zhu and Neuhaus (2000) and Barrett and Donald (2003), we suggest a Monte Carlo approximation to determine the critical value c or p value to make the test operational. More details will be discussed in Sect. 3.

To further examine how sensitive the test is to the alternative hypothesis, we consider a sequence of local alternative hypotheses $D_1(z)$, which is written as $f_n(z)$:

$$H_{1n}: \ f_n(z_0) > 0 \ \text{ for some } z_0. \tag{7}$$

Then we have the following results.

Proposition 2 *Assuming the same condition in Lemma 1, we have that under* H_{1n},

(I) *if* $n^{1/2} f_n(z) \to f(z)$ *with* $f(z_0) > 0$, *then* $\sqrt{n}D_{1n}$ *converges to a Gaussian process* $G + f$ *where* G *is the Gaussian process in Lemma 1 and* f *is the shift function that has the positive value* $f(z_0)$ *at* z_0;

(II) *if* $n^{\gamma} f_n(z_0) \to f(z_0)$ *with* $f(z_0) > 0$ *and* $0 \le \gamma < 1/2$, *then*

$$\lim_{n\to\infty} P[\textit{reject } H_0] = 1.$$

Part (I) of Proposition 2 indicates that T_{1n} is able to detect the local alternative hypotheses distinct from the null hypothesis at a rate as close to $n^{-1/2}$ as possible. This rate is the fastest possible rate in hypothesis testing. Part (II) suggests that T_{1n} has also asymptotically power 1 against the local alternative hypotheses converging to the null hypothesis at any slower rate n^{γ} with $0 \le \gamma < 1/2$.

2.2 Higher-degree extension

Li (2011) extended the concept of first-degree expectation dependence to higher-degree expectation dependence. Let

$$ED_2(X_1|z) = \int_{-\infty}^{z} ED_1(X_1|s)P(X_2 \leq s)\mathrm{d}s.$$

The general case is defined recursively as

$$ED_k(X_1|z) = \int_{-\infty}^{z} ED_{k-1}(X_1|s)\mathrm{d}s \quad \text{for } k = 3, 4, \ldots.$$

Li (2011) defined the concept of expectation dependence by controlling the sign of $ED_k(X_1|z)$.

Definition 2 (Li 2011). If

$$ED_k(X_1|z) \geq 0 \quad \text{for all } z, \tag{8}$$

then the random variable X_1 is positive kth-degree expectation dependent on X_2. Similarly, the random variable X_1 is negative kth-degree expectation dependent on X_2 if the sign of the inequality in (8) is reversed.

Now we are in the position to test for positive kth-degree expectation dependence. The corresponding hypotheses are defined as

$$H_0^k : ED_k \geq 0 \quad \text{for all } z;$$
$$H_1^k : ED_k < 0 \quad \text{for some } z.$$

It is noteworthy that the equations (7) and (19) in Denuit et al. (2013) show that

$$ED_k(X_1|z) = -\frac{1}{(k-1)!}\mathrm{cov}[X_1, (z - X_2)_+^{k-1}], \quad \text{for } k = 2, \ldots. \tag{9}$$

The equality of (9) implies that if $\mathrm{cov}(X_1, (z - X_2)_+^{k-1}) \leq 0$ for all z, then X_1 is positive kth-degree expectation dependent on X_2. Therefore, the testing problem can be rewritten as:

$$H_0^k : \mathrm{cov}(X_1, (z - X_2)_+^{k-1}) \leq 0 \quad \text{for all } z;$$
$$H_1^k : \mathrm{cov}(X_1, (z - X_2)_+^{k-1}) > 0 \quad \text{for some } z.$$

Define $\bar{D}_k = \sup_{z \in \Omega_2} \sqrt{n}(D_k(z))$, where $D_k(\cdot)$ is defined in (1). As a result, under H_0^k, $\bar{D}_k \leq 0$; whereas under H_1^k, $\bar{D}_k > 0$. Consequently, a test statistic can be developed by the empirical version of $\bar{D}_k$.

To this end, we first estimate $D_{(k+1)}(z)$ by $D_{(k+1)n}(z)$:

$$D_{(k+1)n}(z) = \frac{1}{n}\sum_{i=1}^{n}\left\{(X_{1i} - \bar{X}_1)[(z - X_{2i})_+^k - \overline{(z - X_2)_+^k}\,]\right\},$$

where $\bar{X}_1 = \sum_{i=1}^{n} X_{1i}/n$, and $\overline{(z - X_2)_+^k} = \sum_{i=1}^{n} (z - X_{2i})_+^k/n$. We then construct a test statistic of Kolmogorov–Smirnov type as:

$$T_{kn} = \sup_{z\in\Omega_2} \sqrt{n} D_{kn}(z).$$

Analogously, the null hypothesis H_0^k is rejected if T_{kn} exceeds some critical value c_k.

To obtain the asymptotic properties of the test statistic T_{kn}, assume an additional condition:

(B) The random variable X_2 satisfies that $E(X_2^{2k}) < \infty$ for $k = 1, 2, \ldots d$.

Lemma 2 *Under Conditions* (A) *and* (B)*, for some fixed positive integer k, the empirical process* $\sqrt{n}\{D_{(k+1)n}(z) - D_{(k+1)}(z)\} := G_{kn}(z)$ *converges weakly to a Gaussian process* G_k *with mean zero and the covariance function given by*

$$\begin{aligned}\Omega(z_1, z_2) = {} & E((X_1 - E(X_1))^2 I(z_1 > X_2)(z_1 - X_2)^k(z_2 - X_2)^k) \\ & + E((X_1 - E(X_1))^2 E((z_1 - X_2)_+^k)E((z_2 > X_2)_+^k) \\ & - E((X_1 - E(X_1))^2(z_1 > X_2)_+^k)E((z_2 > X_2)_+^k) \\ & - E((z_1 > X_2)_+^k)E((X_1 - E(X_1))^2(z_2 > X_2)_+^k) \\ & - D_{k+1}(z_1)D_{k+1}(z_2),\end{aligned}$$

for all $z_1 \le z_2$*. Particularly, if* X_1 *and* X_2 *are independent, the covariance function can be reduced to*

$$\Omega(z_1, z_2) = cov(X_1)cov((z_1 - X_2)_+^k, (z_2 - X_2)^k),$$

for all z_1, z_2*.*

Similar to the first-degree case, use T_{kn} to build the decision rule as:

$$\text{reject } H_0^k \text{ if } T_{kn} > c_k,$$

where c_k is some critical value that will be determined later.

Then we have the following Propositions 3–4 which describe the behavior of the test statistic T_{kn}.

Proposition 3 *Under the same conditions in Lemma 2, we have*

$$\max_{z\in\Omega_2} G_{kn}(z) \longrightarrow \bar{T}_k,$$

where $\bar{T}_k = \max_{z\in\Omega_2} G_k(z)$ *and* G_k *is the Gaussian process in Lemma 2. Let* c_k *be a positive finite constant. Then*

(I) *if* H_0^k *is true,*

$$\lim_{n\to\infty} P[\textit{reject } H_0^k] \leq P[\bar{T}_k > c_k] = \alpha(c_k);$$

(II) *if* H_1^k *is true,*

$$\lim_{n\to\infty} P[\textit{reject } H_0^k] = 1.$$

Proposition 3 extends the results of Proposition 1 to higher-degree expectation dependence. The results are parallel, and thus, we do not make more explanations.

Parallel to the sequence of the local alternative hypotheses (7), we consider the following sequence as:

$$H_{1n}^k : \mathrm{cov}(X_1, (z_0 - X_2)_+^{k-1}) := f_{kn}(z_0) > 0 \ \text{ for some } z_0. \tag{10}$$

Then we obtain the following results which are similar to those those in Proposition 2.

Proposition 4 *Assuming the same conditions as Lemma 2, under* H_{1n}^k,

(I) *if* $n^{1/2} f_{kn} \to f_k$ *with* $f_k(z_0) > 0$, *then* $\sqrt{n} D_{kn}$ *converges to a Gaussian process* $G_k + f_k$ *where* G_k *is the Gaussian process in Lemma 2 and* f_k *is the shift function that has the positive value* $f_k(z_0)$ *at* z_0;

(II) *if* $n^{\gamma} f_{kn} \to f_k$ *with* $f_k(z_0) > 0$ *and* $0 \leq \gamma < 1/2$, *then*

$$\lim_{n\to\infty} P[\textit{reject } H_0^k] = 1.$$

In practice, analogous to the first-degree case, we also need to determine the critical value c_k satisfying $P(\bar{T}_k > c_k) = \alpha$ to make the test operational. Again because both the sampling and limiting null distributions of the test statistic are unknown, a Monte Carlo approximation is applied to simulate critical values or p values.

3 Implementation

We now suggest a Monte Carlo test procedure for implementation, which is in spirit similar to that in Neuhaus and Zhu (1998), Zhu and Neuhaus (2000) and Barrett and Donald (2003). Zhu (2005) is a relatively comprehensive reference for the similar techniques.

First, generate i.i.d. random variables $\mathbf{U} = \{U_i\}_{i=1}^n$ from $N(0, 1)$. Denote the following processes:

$$\tilde{\Delta}_1(z, D_{1n}(z), \mathbf{U}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ -(X_{1i} - \bar{X}_1)(I(X_{2i} > z) - \overline{I(X_2 > z)}) - D_{1n}(z) \right\} U_i$$

and

$$\tilde{\Delta}_k(z, D_{kn}(z), \mathbf{U}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ (X_{1i} - \bar{X}_1)((z - X_{2i})_+^{k-1} - \overline{(z - X_2)_+^{k-1}})) - D_{kn}(z) \right\} U_i$$

for $k = 2, 3, \ldots, d$. By Propositions 1–4, the p values can be computed to be: for given $\{(X_{1i}, X_{2i}), i = 1, \ldots, n\}$

$$\hat{p}_k = P_U(\sup_{z \in \Omega_2} \tilde{\Delta}_k(z, D_{kn}(z), \mathbf{U}) > T_{kn} | \{(X_{1i}, X_{2i}), i = 1, \ldots, n\}),$$

for $k = 1, 2, 3, \ldots, d$, where P_U denotes the conditional probability function of $\mathbf{U}$ given the sample (X_{1i}, X_{2i})'s. In the following proposition, we will find that for almost all sequences $\{(X_{1i}, X_{2i}), i = 1, \ldots, n, \ldots\}$, this conditional distribution is asymptotically the same as the distribution of G_{kn} under the null hypothesis H_0^k.

In practice, when the support of the random variable X_2 is bounded, we can compute the maximum over the interval from $\hat{a} = \min_{i=1}^n X_{2i}$ to $\hat{b} = \max_{i=1}^n X_{2i}$. When the support of the random variable X_2 is unbounded, we slightly modify $\hat{a}$ and $\hat{b}$ to make the tests more operational. Given a small value $\gamma > 0$, the modified values are:

$$\hat{a} = \max\{X_{2i}, F_n(X_{2i}) \leq \gamma\},$$
$$\hat{b} = \max\{X_{2i}, F_n(X_{2i}) \leq 1 - \gamma\}.$$

Then the p values can be approximated by

$$\hat{p}_k \approx \frac{1}{R} \sum_{j=1}^{R} \left\{ \max_{\hat{a} \leq z \leq \hat{b}} \tilde{\Delta}_k(z, D_{kn}(z), \mathbf{U}^j) > T_{kn} | \{(X_{1i}, X_{2i}), i = 1, \ldots, n\} \right\},$$

where the averaging is made on R replications by independently generating the random variable sets $\mathbf{U}^j = \{U_i\}^j$. When $\hat{p}_k \leq \alpha$ for a given significance level α, we then reject H_0^k.

Lemma 3 *Under Conditions* (A) *and* (B), *for almost all sequences* $\{(X_{1i}, X_{2i}), i = 1, \ldots, n, \ldots\}$, *we have for any* $c > 0$

$$\left| P\left(\max_{\hat{a} \leq z \leq \hat{b}} \tilde{\Delta}_k(z, D_{kn}(z), \mathbf{U}^j) \right) > c | \{(X_{1i}, X_{2i}), i = 1, \ldots, n\}) - P\left(\sup_{z \in \Omega_2} G_{kn}(z) > c \right) \right| \xrightarrow{p} 0.$$

The lemma means that the conditional distribution approximates in probability to the distribution of $\sup_{z \in \Omega_2} G_{kn}(z)$ whose limit is the distribution of $\bar{T}_k$. Thus, its p values converge to p values about $\bar{T}_k$. We can then have parallel results to those in Propositions 1– 4.

Proposition 5 *Under the same conditions in Lemma 3, assume that* $\alpha < 1/2$. *If the test for* ED_k *has the following rule:*

$$reject\ H_0^k\ if\ \hat{p}_k \leq \alpha.$$

Then, under H_0^k,

$$\lim_{n\to\infty} P(reject\ H_0^k) \leq \alpha.$$

Further, under the global alternative hypothesis H_1^k,

$$\lim_{n\to\infty} P(reject\ H_0^k) = 1;$$

and under the local alternative hypotheses H_{1n}^k, *if* $n^\gamma f_{kn} \to f_k$ *with the positive value* $f_k(z_0)$ *at* z_0 *and* $0 \leq \gamma < 1/2$,

$$\lim_{n\to\infty} P(reject\ H_0^k) = 1.$$

This proposition shows that the Monte Carlo test procedure can well simulate critical values or p values such that the tests can maintain the properties at the population level that we described in Propositions 1–4.

4 Numerical studies

In this section, we investigate the performance of the proposed tests by two numerical examples. A total of 2000 Monte Carlo test replications is considered to compute p values, and each experiment is repeated 1000 times to compute empirical size and power at the significance level $\alpha = 0.05$.

Model 1 Generate data from the following model with sample sizes 50, 100, and 200 respectively:

$$X_{1i} = 2Z_{1i} - 1,$$
$$X_{2i} = \theta X_{1i} + 2Z_{2i} - 1,$$

where both Z_{1i} and Z_{2i} are from the uniform distribution $U(0, 1)$ and mutually independent, θ is a parameter that varies across different experiments. Note that, expectation dependence has an important hierarchy property: positive lower-degree expectation dependence implies positive higher-degree expectation dependence. When $\theta = 0$, both X_{1i} and X_{2i} are mutually independent, and then $ED_1(z) \equiv 0$ for all z. Using the hierarchy property, X_1 is then positive for any degree expectation dependence on X_2. Hence, $\theta = 0$ implies that the model is under the null hypothesis. When $\theta \neq 0$, X_1 is not positive first-degree expectation dependent on X_2, which corresponds to the alternative hypotheses. We compute the empirical powers of the proposed tests T_{1n}, T_{2n} and T_{3n} under different combinations of sample sizes and alternative hypotheses.

Table 1 Empirical size and power under *Model 1*

θ	$n = 50$			$n = 100$			$n = 200$		
	FED	SED	TED	FED	SED	TED	FED	SED	TED
0	0.0440	0.0560	0.0560	0.0470	0.0440	0.0530	0.0480	0.0500	0.0480
−0.1	0.1630	0.1910	0.1860	0.1950	0.2610	0.2590	0.2930	0.4090	0.4080
−0.2	0.3180	0.3800	0.3810	0.4780	0.6600	0.6540	0.7380	0.8720	0.8740
−0.3	0.4860	0.6700	0.6660	0.7660	0.8940	0.8940	0.9620	0.9940	0.9920
−0.4	0.7640	0.8780	0.8790	0.9520	0.9910	0.9900	1.0000	1.0000	1.0000

The null hypothesis corresponds to $\theta = 0$

Table 2 Empirical size and power of the tests with *Model 2*

θ	$n = 50$			$n = 100$			$n = 200$		
	FED	SED	TED	FED	SED	TED	FED	SED	TED
0	0.0460	0.0570	0.0610	0.0470	0.0460	0.0460	0.0490	0.0480	0.0490
−0.1	0.1690	0.1760	0.1690	0.2350	0.2560	0.2610	0.3600	0.4240	0.4230
−0.2	0.3250	0.3800	0.3800	0.5310	0.6200	0.5880	0.7490	0.8680	0.8500
−0.3	0.5770	0.6670	0.6610	0.8280	0.9150	0.8870	0.9780	0.9960	0.9940
−0.4	0.7540	0.8420	0.8310	0.9670	0.9870	0.9850	0.9990	1.0000	1.0000

The null hypothesis corresponds to $\theta = 0$

The results are reported in Table 1. The results show that under the null hypothesis, the empirical size is larger than the significance level, but not much even when $n = 50$. This is what we expect when the Monte Carlo test procedure is applied in Sect. 3. Also, it is expectable to have higher power with larger distinction (larger $|\theta|$) from the null hypothesis and larger size of sample, which approaches 1 quickly. Also, the test for first-degree expectation dependence has lower empirical power than the other two tests have.

Model 2 Consider the following model with sample sizes 50, 100 and 200:

$$X_{1i} = Z_{1i},$$
$$X_{2i} = \theta Z_{1i} + Z_{2i},$$

where both Z_{1i} and Z_{2i} are from standard normal distribution and mutually independent. Again $\theta = 0$ corresponds to the null hypothesis. In this case, the support is infinite, we then consider a truncation with $\gamma = 0.01$. Then for ease of computation, consider the truncated interval with two ends as $\hat{a} = \max\{X_{2i}, F_n(X_{2i}) \le 0.005\}$, $\hat{b} = \max\{X_{2i}, F_n(X_{2i}) \le 0.995\}$.

The related results are listed in Table 2. From Table 2, the tests can still have resonable empirical size and high empirical powers. Reasonably, large size of sample helps on hypothesis testing with higher empirical power. We can also observe that the empirical power performance is as much as that with *Model 1*.

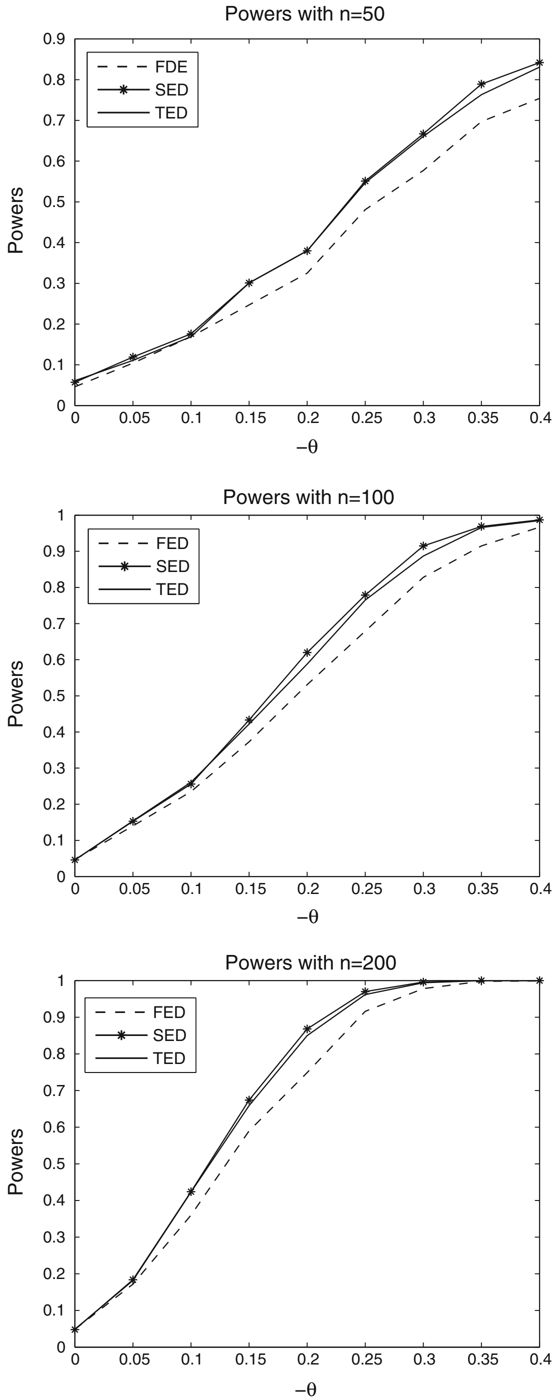

Fig. 1 Empirical power functions of the tests for the first- to third positive expectation dependence at the 5 % significance level for *Model 2*

Table 3 Approximated p values for first-degree expectation dependence for Danish fire insurance data, based on 10000 Monte Carlo runs

p value	(B, C)	(B, P)	(C, P)	(C, B)	(P, B)	(P, C)
Case 1	0.0601	0.6979	0.7683	0.2070	0.3360	0.8033
Case 2	0.8543	1.0000	1.0000	1.0000	1.0000	0.9731
Case 3	0.8620	1.0000	1.0000	1.0000	1.0000	0.9676

To more clearly show the empirical power performance of the tests, Fig. 1 furthermore draws the empirical power curves of the different tests.

From Fig. 1, we can see that again the test for first-degree expectation dependence has lower empirical power than the other two tests have. Together with the same phenomenon with *Model 1*, it seems that the first-degree positive expectation dependence would be slightly more difficult to detect than higher-degree ones.

5 Real data analysis

We now analyze the well-known Danish fire insurance data set for illustration that is available at http://www.ma.hw.ac.uk/~mcneil/data.html. This data set has been studied for testing positive quadrant dependence by Denuit and Scaillet (2004), Gijbels and Sznajder (2013) and Ledwina and Wyłupek (2014). The data set contains three categories of claims referring to losses in buildings (B), their contents (C) and the profit (P) which generated in the years 1980–1990. There exist 2167 observations which contain 1502, 529, and 604 strictly positive observations for the respective pairs of (B,C), (B,P) and (C,P). Further, the sample size is reduced to 517 with these three variables being strictly positive. Here we also consider three cases, Case 1: all the samples; Case 2: strictly positive observations for the respective pairs of (B,C), (B,P) and (C,P); and Case 3: reduced samples which are strictly positive observations for three variables. The p values for different pairs are presented in Table 3. We can see clearly that in all the cases, (B,P) and (C,P) are positive first-degree expectation dependent on each other with very large p values. It is known that positive quadrant dependence implies positive expectation dependence. Thus our results are consistent with those found in Ledwina and Wyłupek (2014). On the other hand, reducing to 517 observations does not significantly change the inference in Case 2. However, Case 1 is very different from others, especial for (B,C) and (C,B).

Appendix

In this paper, because the proofs about the first-degree one are analogous to those about the high-degree ones, we only give the proofs for Lemma 2, Propositions 3, 4 and 5.

Proof of Lemma 2 Decomposing $D_{(k+1)n}(z)$, we have

$$\begin{aligned}
D_{(k+1)n}(z) &= \frac{1}{n}\sum_{i=1}^{n}\left\{(X_{1i}-\bar{X}_1)((z-X_{2i})_+^k-\overline{(z-X_{2i})_+^k})\right\}\\
&= \frac{1}{n}\sum_{i=1}^{n}\left\{(X_{1i}-E(X_1))((z-X_{2i})_+^k-E(z-X_{2i})_+^k)\right\}\\
&\quad +(E(X_1)-\bar{X}_1)\frac{1}{n}\sum_{i=1}^{n}\left\{((z-X_{2i})_+^k-\overline{(z-X_{2i})_+^k})\right\}\\
&\quad +(E((z-X_{2i})_+^k)-\overline{(z-X_{2i})_+^k}))\}\frac{1}{n}\sum_{i=1}^{n}\{(X_{1i}-\bar{X}_1)\\
&\quad +(E(X_1)-\bar{X}_1)(E((z-X_{2i})_+^k)-\overline{(z-X_{2i})_+^k}))\}\\
&= \frac{1}{n}\sum_{i=1}^{n}\left\{(X_{1i}-E(X_1))((z-X_{2i})_+^k-E(z-X_{2i})_+^k)\right\}\\
&\quad +(E(X_1)-\bar{X}_1)(E((z-X_{2i})_+^k)-\overline{(z-X_{2i})_+^k}))\}\\
&\equiv: S_{k1n}(z)+S_{k2n}(z),
\end{aligned}$$

where

$$\begin{aligned}
S_{k1n}(z) &= \frac{1}{n}\sum_{i=1}^{n}\left\{(X_{1i}-E(X_1))((z-X_{2i})_+^k-E(z-X_{2i})_+^k)\right\},\\
S_{k2n}(z) &= \frac{1}{n}\sum_{i=1}^{n}\left\{(E(X_1)-\bar{X}_1)(E((z-X_{2i})_+^k)-\overline{(z-X_{2i})_+^k}))\right\}.
\end{aligned}$$

Define $g_{kn}(z)=\sqrt{n}(S_{k1n}(z)-D_{k+1}(z))$. Then we get

$$\begin{aligned}
g_{kn}(z) &= \sqrt{n}(S_{k1n}(z)-D_{k+1}(z))\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{(X_{1i}-E(X_1))((z-X_{2i})_+^k-E(z-X_{2i})_+^k))-D_{k+1}(z)\right\}.
\end{aligned}$$

It is easy to see that for $z_1<z_2$, we have

$$\begin{aligned}
H(z_1,z_2) &= \lim_{n\to\infty}\mathrm{P}g_{kn}(z_1)g_{kn}(z_2)\\
&= E\{[(X_1-E(X_1))((z_1-X_2)_+^k-E(z_1-X_2)_+^k)-D_{k+1}(z_1)]\\
&\quad [(X_1-E(X_1))((z_2-X_2)_+^k-E(z_2-X_2)_+^k)-D_{k+1}(z_2)]\}\\
&= E((X_1-E(X_1))^2I(z_1>X_2)(z_1-X_2)^k(z_2-X_2)^k)\\
&\quad +E((X_1-E(X_1))^2E((z_1-X_2)_+^k)E((z_2>X_2)_+^k)
\end{aligned}$$

$$-E((X_1 - E(X_1))^2(z_1 > X_2)_+^k)E((z_2 > X_2)_+^k)$$
$$-E((z_1 > X_2)_+^k)E((X_1 - E(X_1))^2(z_2 > X_2)_+^k)$$
$$+D_{k+1}(z_1)D_{k+1}(z_2).$$

Let $M_z(X_1, X_2, z) = (X_1 - E(X_1))((z - X_2)_+^k - E(z - X_2)_+^k)$. It is easy to see that the class of functions $\{M_z(X_1, X_2, z) : z$ is any real number$\}$ is a VC-class. Functional Central Limit Theorem 10.6 of Pollard (1990) yields that the empirical process $\sqrt{n}(S_{k1n}(z) - cov(X_1, (z - X_2)^k I(z - X_2 > 0)))$ converges weakly to a mean zero Gaussian process whose covariance kernel is given by $H(z_1, z_2)$ for all $z_1 < z_2$.

Further, note that $\overline{(z - X_{2i})_+^k}$ is consistent to $E((z-X_{2i})_+^k)$ and $\bar{X}_1$ is $\sqrt{n}$-consistent to $E(X_1)$. By Central Limit Theorem, we then have

$$\sqrt{n}(S_{k2n}(z)) = \sqrt{n}(E(X_1) - \bar{X}_1)(E((z - X_{2i})_+^k) - \overline{(z - X_{2i})_+^k}))\}$$
$$= \sqrt{n} \times O_p\left(\frac{1}{\sqrt{n}}\right) \times o_p(1) = o_p(1).$$

When X_1 is independent of X_2, $D_{k+1}(z_1) = D_{k+1}(z_2) = 0$ and $E((X_1 - E(X_1))^2(z_1 > X_2)_+^k)E((z_2 > X_2)_+^k) = E((X_1 - E(X_1))^2 E((z_1 - X_2)_+^k)E((z_2 > X_2)_+^k)$. Thus, simplifying $H(z_1, z_2)$, we get that

$$\begin{aligned} H(z_1, z_2) &= E((X_1 - E(X_1))^2 E\left\{I(z_1 > X_2)(z_1 - X_2)^k(z_2 - X_2)^k)\right\} \\ &\quad -E((z_1 > X_2)_+^k)E((X_1 - E(X_1))^2(z_2 > X_2)_+^k) \\ &= E((X_1 - E(X_1))^2(E\left\{I(z_1 > X_2)(z_1 - X_2)^k(z_2 - X_2)^k)\right\} \\ &\quad -E((z_1 > X_2)_+^k)E((z_2 > X_2)_+^k)) \\ &= cov(X_1)cov((z_1 > X_2)_+^k, (z_2 > X_2)_+^k). \end{aligned}$$

The proof of Lemma 2 is finished. □

Proof of Proposition 3 Lemma 2 implies that $\sqrt{n}\{D_{kn}(z) - D_k(z)\}$ converges to the Gaussian process G_k. As $\sup f(\cdot)$ is a continuous function about f, the continuous mapping theorem yields

$$\max_{z \in \Omega_2} \sqrt{n}\{D_{kn}(z) - D_k(z)\} \longrightarrow \bar{T}_k.$$

Proof of (I): From the definition of T_{kn} and the fact that under H_0^k, $D_k(z) \geq 0$ for all $z \in \Omega_2$, we get that

$$\begin{aligned} T_{kn} &\leq \sup_{z \in \Omega_2} \sqrt{n}\{D_{kn}(z) - D_k(z)\} + \sup_{z \in \Omega_2} \sqrt{n} D_k(z) \\ &\leq \sup_{z \in \Omega_2} \sqrt{n}\{D_{kn}(z) - D_k(z)\}. \end{aligned}$$

Hence the results follows from the week convergence of $\sqrt{n}\{D_{kn} - D_k\}$ and the definition of $\bar{T}_k$.

Proof of (II): Under the alternative hypothesis, there exists some $\bar{z} \in \Omega_2$ satisfied $D_k(\bar{z}) = \delta > 0$. Due to the inequality $T_{kn} \geq \sqrt{n}D_{kn}(\bar{z})$ and the fact $T_{kn} \geq \sqrt{n}D_{kn}(\bar{z}) \to \infty$ as $n \to \infty$, we obtain the result, i.e. $\lim_{n\to\infty} P[\text{reject } H_0^k] = 1$. □

Proof of Proposition 4 Proof of (I): By Lemma 2, $\sqrt{n}\{D_{(k+1)n}(z) - D_{k+1}(z)\}$ converges to the Gaussian process G_k with mean zero and the covariance given by

$$\begin{aligned}\Omega(z_1, z_2) = E((X_1 - E(X_1))^2 I(z_1 > X_2)(z_1 - X_2)^k(z_2 - X_2)^k) \\ +E((X_1 - E(X_1))^2 E((z_1 - X_2)_+^k) E((z_2 > X_2)_+^k) \\ -E((X_1 - E(X_1))^2 (z_1 > X_2)_+^k) E((z_2 > X_2)_+^k) \\ -E((z_1 > X_2)_+^k) E((X_1 - E(X_1))^2 (z_2 > X_2)_+^k) \\ -D_{k+1}(z_1) D_{k+1}(z_2).\end{aligned}$$

Note that $D_k(z) = f_{kn}$. Further, under the local alternative hypotheses H_{1n}^{k+1}, $n^{1/2} f_{kn} \to f_k$ in which $f_k(z_0) > 0$ at z_0. Thus, the shift function f_k is not a zero function whereas it is the case under the null hypothesis. Thus, the process converges to $G_k + f_k$.

Proof of (II): If the local alternative hypotheses hold, there exists some $\bar{z} \in \Omega_2$ satisfying $D_k(\bar{z}) \geq f_{kn}/2 > 0$. Since $n^\gamma f_{kn} \to f_k$ with $f_k > 0$ and $0 \leq \gamma < 1/2$, we can get $T_{kn} \geq \sqrt{n}D_n(\bar{z}) \geq \sqrt{n} f_{kn}/2 \longrightarrow \infty$. Then we have $\lim_{n\to\infty} P(\text{reject } H_0^k) = 1$. Therefor, we have completed the justification. □

Proof of Lemma 3 The proof is very similar to that in Zhu (2005), we then give an outline here. To obtain the results in this proposition, what we need is just to prove that the process $\tilde{\Delta}_k(\cdot, D_{kn}(\cdot))$ is asymptotically equivalent to $\sqrt{n}\{D_{kn}(\cdot) - D_k(\cdot)\}$. Rewrite $\tilde{\Delta}_k(z, D_{kn}(z))$ as

$$\begin{aligned}\tilde{\Delta}_k(z, D_{kn}(z)) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{(X_{1i} - \bar{X}_1)((z - X_{2i})_+^k - \overline{(z - X_{2i})_+^k)}) - D_{kn}(z)\right\} U_i \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{(X_{1i} - \bar{X}_1)((z - X_{2i})_+^k - \overline{(z - X_{2i})_+^k)}) - D_k(z)\right\} U_i \\ &\quad -(D_{kn}(z) - D_k(z)) \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i.\end{aligned}$$

Consider the second term $-(D_{kn}(z) - D_k(z))\frac{1}{\sqrt{n}}\sum_{i=1}^n U_i$. Lemma 2 implies that $\sqrt{n}(D_{kn}(z) - D_k(z))$ converges weakly to a mean zero Gaussian distribution, and then $\sup_z(D_{kn}(z) - D_k(z)) \xrightarrow{p} 0$. Note that U_i are i.i.d. $N(0, 1)$ random variables. $\frac{1}{\sqrt{n}}\sum_{i=1}^n U_i$ is still normal with mean zero and variance one. Thus, $\sup_{z\in\Omega_2} |(D_{kn}(z) - D_k(z))||\frac{1}{\sqrt{n}}\sum_{i=1}^n U_i|$ converges weakly to zero. This holds true in probability for almost all sequences of $\{(X_{1i}, X_{2i}), i = 1, \ldots, n, \ldots\}$. Similar idea can be applied to the first term to have the asymptotic equivalence in probability between it and the one when $\bar{X}_1$ and $\overline{(z - X_{2i})_+^k}$ are replaced by their corresponding expectations $E(X_1)$ and $E((z - X_{2i})_+^k)$. Thus, we only need to deal with the first term for p value computation. Therefore, we derive that $\tilde{\Delta}_k(z, D_{kn}(z))$ is asymptotically equivalent to

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ (X_{1i} - E(X_1))((z - X_{2i})_+^k - E((z - X_{2i})_+^k) - D_k(z) \right\} U_i.$$

It is clear that the kernel function of this process is, for almost all sequences of $\{(X_{1i}, X_{2i}), i = 1, \ldots, n, \ldots\}$, asymptotically the same as $\Omega(z_1, z_2)$ in Lemma 2. Then, the conditional distribution of $\sup_{z\in\Omega_2} \tilde{\Delta}_k(z, D_{kn}(z))$ given the sequence of $\{(X_{1i}, X_{2i}), i = 1, \ldots, n, \ldots\}$ can well approximate the limiting null distribution of $\sup_{z\in\Omega_2} \sqrt{n}\{D_{kn}(z) - D_k(z)\}$, namely,

$$P\left(\max_{\hat{a}\le z\le\hat{b}} \tilde{\Delta}_k(z, D_{kn}(z), \mathbf{U}^j \right) > c|\{(X_{1i}, X_{2i}), i = 1, \cdots, n\})$$
$$- P\left(\sup_{z\in\Omega_2} G_{kn}(z) > c \right) \xrightarrow{p} 0.$$

□

Proof of Proposition 5 Under null hypothesis, combining to Propositions 1 and 3, we use the results of Lemma 3 to get directly that $\lim_{n\to\infty} P(\text{reject } H_0^k) \le \alpha$. On the other hand, it is worthwhile to point out that under the local alternatives in (10) approaching the null hypothesis, $\sup_{z\in\Omega_2} \tilde{\Delta}_k(z, D_{kn}(z))$ still has the same limit as that under the null hypothesis. This means that under the local alternative hypothesis, we can still well approximate the limiting null distribution by using Monte Carlo test procedure with $\tilde{\Delta}_k(z, D_{kn}(z))$. The results in the proposition follows. □

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