Supplementary Material

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Proof of Theorem 2.

The proof differs from that of Theorem 1 only in proving convergence (14) of the main paper. As shown in [1], using Theorem 2 of [2], and condition (26) of the main paper, we have

$$\max_{\alpha} \frac{\left|\mathbf{e}_n' P_n(\alpha)\mathbf{e}_n - \sigma^2 P_n(\alpha)\right|}{nR_n(\alpha)} \xrightarrow{p} 0 \tag{1}$$

and

$$\max_{\alpha} \frac{\left|\mu_n' \left(I_n - P_n(\alpha)\right)\mathbf{e}_n\right|}{nR_n(\alpha)} \xrightarrow{p} 0. \tag{2}$$

Now, $nL_n(\alpha) = 2\sigma^2 D_n(\alpha) + \mathbf{e}_n' P_n(\alpha)\mathbf{e}_n$ and

$$nR_n(\alpha) = E\left(nL_n(\alpha)\right) = 2\sigma^2 D_n(\alpha) + \sigma^2 p_n(\alpha). \tag{3}$$

Then from (20) or (23) of the main paper (depending on the case),

$$\max_{\alpha} \frac{\left|\sigma^2 p_n(\alpha)\right|}{nR_n(\alpha)} \to 0 \tag{4}$$

and therefore, from (1) we have

$$\max_{\alpha} \frac{\mathbf{e}_n' P_n(\alpha)\mathbf{e}_n}{nR_n(\alpha)} \xrightarrow{p} 0. \tag{5}$$

Also, from (3) and (20) or (23) of the main paper,

$$\max_{\alpha} \left| \frac{nR_n(\alpha)}{2\sigma^2 D_n(\alpha)} - 1 \right| \leq \frac{p_n}{\max_{\alpha} 2D_n(\alpha)} \to 0. \tag{6}$$

It now follows from (2), (5) and (6) that

$$\max_{\alpha} \frac{\mathbf{e}_n' P_n(\alpha)\mathbf{e}_n}{2\sigma^2 D_n(\alpha)} \xrightarrow{p} 0 \tag{7}$$

and

$$\max_{\alpha} \frac{\left|\mu_n' (I_n - P_n(\alpha))\mathbf{e}_n\right|}{2\sigma^2 D_n(\alpha)} \xrightarrow{p} 0 \tag{8}$$

which imply (14). □

Proof of Theorem 4.

The proof differs from that of the result stated in Remark 2 only in proving that $\xi_n \xrightarrow{p} 0$ where $\xi_n = \max_\alpha |\xi_n(\alpha)|$ and

$$\xi_n(\alpha) = \frac{2\mu_n'(I_n - P_n(\alpha))\mathbf{e}_n - 2\mathbf{e}_n' P_n(\alpha)\mathbf{e}_n}{nL_n(\alpha)}.$$

As $nL_n(\alpha) = 2\sigma^2 D_n(\alpha) + \mathbf{e}_n' P_n(\alpha)\mathbf{e}_n > 2\sigma^2 D_n(\alpha)$, the result follows from (7) and (8). □

Proof of the Result Stated in Remark 3.

Consider the 'model true' case when $M_{\alpha_c} = M_N$ and each candidate model is assigned equal probability. It was noted, while proving Theorem 5, that in this case

$$\frac{m_\alpha(\mathbf{y}_n)}{m_{\alpha_c}(\mathbf{y}_n)} = \frac{1}{(1+g_n)^{p_n(\alpha)/2}} \left(\frac{(1-a_n)\sum_{i=1}^n (y_i - \bar{y})^2 + a_n \mathbf{e}_n'\mathbf{e}_n - a_n \mathbf{e}_n' P_n(\alpha)\mathbf{e}_n}{\sum_{i=1}^n (y_i - \bar{y})^2} \right)^{-(n-1)/2}.$$

Noting that $a_n = g_n/(1+g_n)$, simple algebra shows that

$$\sum_{\alpha \in \mathcal{A}_1} \frac{m_\alpha(\mathbf{y}_n)}{m_{\alpha_c}(\mathbf{y}_n)} = \sum_{\alpha \in \mathcal{A}} \frac{1}{(1+g_n)^{-(n-p_n(\alpha)-1)/2}} \left(1 + g_n \frac{\mathbf{e}_n' (I_n - P_n(\alpha))\, \mathbf{e}_n}{\sum_{i=1}^n (y_i - \bar{y})^2}\right)^{-(n-1)/2}.$$

Noting that $0 < (\mathbf{e}_n' (I_n - P_n(\alpha))\, \mathbf{e}_n)/(\sum_{i=1}^n (y_i - \bar{y})^2) < 1$ with probability 1, it is immediate that last summation is larger than

$$\sum_{\alpha \in \mathcal{A}_1} \frac{1}{(1+g_n)^{p_n(\alpha)/2}} = \sum_{q=1}^{p_n} \binom{p_n}{q} (1+g_n)^{-q/2} = \left(1 + \frac{1}{\sqrt{1+g_n}}\right)^{p_n} - 1,$$

with probability 1. When $p_n = n^b$, $0 < b < 1$ and $g_n = kn^r$ for $k > 0$, $r > 0$ then the quantity is bounded away from zero asymptotically if $r \le 2b$. It can go to zero if $r > 2b$. □

Proof of the Result Stated in Remark 4.

By assumption (A.1), $\mu_n'(I - P_n(\alpha))\mu_n/n \le M$ for some $M > 0$. We show that when $M_{\alpha_c} = M_F$ then $P(M_{\alpha_c}|\mathbf{y}_n)$ does not converge to 1 in probability if $g_n = D^{n/p_n}$ where $D > 1 + M/\sigma^2$ and $p_n = n^b$. Recalling (29) and noting that $\mathcal{A}_1$ is a null set in this case, it suffices to show that for such a choice of g_n, $\sum_{\alpha \in \mathcal{A}_2} m_\alpha(\mathbf{y}_n)/m_{\alpha_c}(\mathbf{y}_n)$ does not converge to zero in probability. We recall from (49) that

$$\frac{m_\alpha(\mathbf{y}_n)}{m_{\alpha_c}(\mathbf{y}_n)} = (1+g_n)^{(p_n(\alpha_c) - p_n(\alpha))/2} \left[\frac{1 + a_n \mu_n'(I_n - P_n(\alpha))\mu_n/nC_n + U_{n\alpha}}{1 - a_n \mathbf{e}_n' P_n(\alpha_c)\mathbf{e}_n/nC_n} \right]^{-(n-1)/2},$$

where C_n and $U_{n\alpha}$ are as in (50) and (51) respectively. Using (54) and (55) it follows that $U_{n\alpha} = o_p(1)$ uniformly in α as $n \to \infty$ since $p_n/n \to 0$. Using these along with the facts that

$\mu_n'(I-P_n(\alpha))\mu_n/n \le M$, $a_n \to 1$, $C_n \xrightarrow{p} \sigma^2$ (vide (52)), $a_n\mathbf{e}_n'P_n(\alpha)\mathbf{e}_n/nC_n = o_p(1)$, one concludes that given any $0<\varepsilon<1$,

$$\sum_{\alpha\in\mathcal{A}_2}\frac{m_\alpha(\mathbf{y}_n)}{m_{\alpha_c}(\mathbf{y}_n)} > \sum_{\alpha\in\mathcal{A}_2}(1+g_n)^{(p_n(\alpha_c)-p_n(\alpha))/2}\left(\frac{1-\varepsilon}{1-\varepsilon+M/\sigma^2(1-\varepsilon)}\right)^{(n-1)/2}$$

with probability tending to 1 as $n\to\infty$. The expression on the right hand side above can be greater than

$$\left(\frac{1-\varepsilon}{1-\varepsilon+M/\sigma^2(1-\varepsilon)}\right)^{(n-1)/2}\left(\sum_{r=1}^{p_n-1}(1+g_n)^{r/2}\binom{p_n-1}{r}\right),$$

which, in turn, equals

$$\left(\frac{1-\varepsilon}{1-\varepsilon+M/\sigma^2(1-\varepsilon)}\right)^{(n-1)/2}\left(1+\sqrt{1+g_n}\right)^{p_n-1}.$$

Now suppose $p_n = n^b$, $0<b<1$. Then the above expression cannot converge to *zero* if $g_n = D^{n/p_n}$ where $D > \left(1+M/\sigma^2\right)$. This completes the proof. □

References

[1] Ker-Chau Li. Asymptotic optimality for C_p, C_L, cross-validation and generalized cross-validation: discrete index set. *The Annals of Statistics*, 15(3):958–975, 1987.

[2] P. Whittle. Bounds for the moments of linear and quadratic forms in independent variables. *Teor. Verojatnost. i Primenen.*, 5:331–335, 1960.