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Estimation of two ordered normal means under modified Pitman nearness criterion

Yuan-Tsung Chang · Nobuo Shinozaki

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Abstract The problem of estimating two ordered normal means is considered under the modified Pitman nearness criterion in the presence and absence of the order restriction on variances. When variances are not ordered, a class of estimators is considered that reduce to the estimators of a common mean when the unbiased estimators violate the order restriction. It is shown that the most critical case for uniform improvement with regard to the unbiased estimators is the one when two means are equal. When variances are ordered, a class of estimators is considered, taking the order restriction on variances into consideration. The proposed estimators of the mean with a larger variance improve upon the estimators that do not take the order restriction on variances into consideration. Although a similar improvement is not possible in estimating the mean with a smaller variance, a domination result is given in the simultaneous estimation.

Keywords Order restriction · Common mean · Restricted MLE · Unbiased estimator · Pitman nearness · Modified Pitman nearness · Uniform improvement

1 Introduction

Estimation of restricted normal means (simple order, tree order or orthant restrictions etc.) has been considered by many authors and some estimators improving upon the

Y.-T. Chang (✉)
Department of Social Information, Faculty of Studies on Contemporary Society,
Mejiro University, 4-31-1 Nakaochiai, Shinjuku-ku, Tokyo 161-8539, Japan
e-mail: chogenso@gmail.com; cho@mejiro.ac.jp

N. Shinozaki
Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi,
Kohoku-ku, Yokohama 223-8522, Japan

unbiased estimators have been proposed. Most works related to the inference about the restricted parameters are reviewed by Barlow et al. (1972), Robertson et al. (1988), Silvapulle and Sen (2004) and Van Eeden (2006). As a criterion to evaluate the goodness of estimators, mean squared error (MSE) is usually used and in some cases stochastic domination as well.

A possible alternative criterion, which was introduced by Pitman (1937), is Pitman nearness. Let T_1 and T_2 be two estimators of θ. Then, Pitman nearness of T_1 relative to T_2 is defined by

$$\mathrm{PN}_\theta(T_1, T_2) = P\{|T_1 - \theta| < |T_2 - \theta|\}$$

and T_1 is said to be closer to θ than T_2 if $\mathrm{PN}_\theta(T_1, T_2) \geq 1/2$ for any parameter value. We refer the reader to Rao (1981), Keating and Mason (1985, 1986), Peddada (1985), Rao et al. (1986), Peddada and Khattree (1986) for its discussions. Many works related to Pitman nearness were published in the special issue of Communications in Statistics—Theory and Methods A20 (11) in 1992 and were unified in the monograph by Keating et al. (1993). Although the Pitman nearness has been severely criticized as a measure of comparing estimators (see Robertson et al. 1993), we believe that it is a useful criterion in comparing two estimators and understanding the nature of the estimators.

Here, we consider the estimation of two ordered normal means when order restriction on the unknown variances is present and not present. We propose some estimators of the means which improve upon some baseline estimators which do not take into account of the order restrictions in terms of modified Pitman nearness criterion suggested by Gupta and Singh (1992). We first state some fundamental results on the estimation of a common mean and ordered means when MSE, stochastic domination or Pitman nearness is concerned. Let $X_{ij},\ i = 1, 2,\ j = 1, \ldots, n_i$ be independent observations from normal distribution with mean μ_i and variance σ_i^2, where both μ_i and σ_i^2 are unknown. Let

$$\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i \quad \text{and} \quad s_i^2 = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/(n_i - 1)$$

be the unbiased estimators of μ_i and σ_i^2, respectively.

For the common mean problem ($\mu_1 = \mu_2 = \mu$) when two variances are unknown and there is no order restriction on the two variances, Graybill and Deal (1959) proposed the estimator

$$\hat{\mu}^{\mathrm{GD}} = \frac{n_1 s_2^2}{n_1 s_2^2 + n_2 s_1^2}\bar{X}_1 + \frac{n_2 s_1^2}{n_1 s_2^2 + n_2 s_1^2}\bar{X}_2$$

and gave a necessary and sufficient condition on n_1 and n_2 for $\hat{\mu}^{\mathrm{GD}}$ to have a smaller variance than both $\bar{X}_1$ and $\bar{X}_2$. Later, many authors, including Brown and Cohen (1974), Khatri and Shah (1974), and Bhattacharya (1980), have given a class of improved estimators of the form

$$\hat{\mu}(\gamma) = \gamma \bar{X}_1 + (1-\gamma)\bar{X}_2,$$

where γ is a function of s_1^2, s_2^2 and possibly $(\bar{X}_1 - \bar{X}_2)^2$. As for Pitman nearness criterion, Kubokawa (1989) has given a broad class of estimators of a common mean with γ given by

$$\gamma_\psi = 1 - \frac{a}{1 + R\psi(s_1^2, s_2^2, (\bar{X}_1 - \bar{X}_2)^2)}$$

where $R = (bs_2^2 + c(\bar{X}_1 - \bar{X}_2)^2)/s_1^2$, ψ is a positive valued function and a, b, and c are nonnegative constants. For suitably chosen ψ, a, b, and c, Kubokawa (1989) has given a sufficient condition on n_1 and n_2 so that $\hat{\mu}(\gamma_\psi)$ is closer to μ than $\bar{X}_1$. Thus, Kubokawa (1989) has shown that $\hat{\mu}^{\mathrm{GD}}$ is closer to μ than both $\bar{X}_1$ and $\bar{X}_2$ if $n_1 \geq 5$ and $n_2 \geq 5$. Misra and Van der Meulen (1997) have discussed estimating a common mean of $k(\geq 2)$ normal distributions when order restriction is given on variances and proposed an estimator improving upon $\hat{\mu}^{\mathrm{GD}}$ in terms of stochastic dominance and Pitman nearness.

For the case when k normal means satisfy simple order restriction and variances are known, Lee (1981) has shown that restricted MLE (maximum likelihood estimator) uniformly improves upon sample means in terms of MSE. Kelly (1989) and Hwang and Peddada (1994) have proven that restricted MLE universally dominates sample means. For the case when σ_i^2's are unknown and two means satisfy the order restriction, $\mu_1 \leq \mu_2$, Oono and Shinozaki (2005) have proposed the estimators of μ_i, $i = 1, 2$

$$\hat{\mu}_1^{\mathrm{OS}} = \min\left\{\bar{X}_1, \hat{\mu}^{\mathrm{GD}}\right\} \quad \text{and} \quad \hat{\mu}_2^{\mathrm{OS}} = \max\left\{\bar{X}_2, \hat{\mu}^{\mathrm{GD}}\right\},$$

and have shown that $\hat{\mu}_i^{\mathrm{OS}}$ uniformly improves upon $\bar{X}_i$ in terms of MSE if and only if MSE of $\hat{\mu}_i^{\mathrm{OS}}$ is not larger than that of $\bar{X}_i$ for the case when $\mu_1 = \mu_2$. Further, it has been shown that $\hat{\mu}_i^{\mathrm{OS}}$ improves upon $\bar{X}_i$ if and only if $\hat{\mu}^{\mathrm{GD}}$ improves upon $\bar{X}_i$ in the common mean problem. It should be mentioned that Garren (2000) has proposed a similar estimator with s_i^2 replaced by the MLE of σ_i^2 in $\hat{\mu}_i^{\mathrm{GD}}$ and has given a condition on n_1 and n_2 for the estimator to have larger (or smaller) MSE when $\mu_1 = \mu_2$ and σ_2^2/σ_1^2 is sufficiently large.

Next, we consider the estimation of two ordered means when order restriction, $\sigma_1^2 \leq \sigma_2^2$, is given on two variances. Such a situation can occur, for example, if there are two kinds of motor engines, one of which is developed by a new method, and has higher power but has larger variation than the other one developed by a standard method. Shi (1994) and Ma and Shi (2002) have discussed the order restricted MLE of μ_i and σ_i^2. Chang et al. (2012) have considered a class of estimators of μ_i, $i = 1, 2$ of the form

$$\hat{\mu}_1(\gamma) = \min\{\bar{X}_1, \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2\}, \quad \hat{\mu}_2(\gamma) = \max\{\bar{X}_2, \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2\}, \quad (1)$$

where γ is a function of s_1^2, s_2^2, and $(\bar{X}_1 - \bar{X}_2)^2$. γ may be considered to be an estimator of $n_1\sigma_2^2/(n_1\sigma_2^2 + n_2\sigma_1^2)$ which is not larger than $n_1/(n_1 + n_2)$ when $\sigma_1^2 \leq \sigma_2^2$. Thus, as in the common mean problem (see, for example, Seshadri 1966; Nair 1982; Elfessi and Pal 1992), we may improve upon $\hat{\mu}_i(\gamma)$ by replacing γ with

$$\gamma^+ = \begin{cases} \gamma, & \text{if } \gamma \geq \frac{n_1}{n_1+n_2}, \\ \tilde{\gamma}, & \text{if } \gamma < \frac{n_1}{n_1+n_2}, \end{cases} \tag{2}$$

and choosing $\tilde{\gamma}$ appropriately. Actually, Chang et al. (2012) have shown that $\hat{\mu}_2(\gamma^+)$ stochastically dominates $\hat{\mu}_2(\gamma)$ if $\tilde{\gamma}$ is chosen to satisfy

$$\frac{n_1}{n_1+n_2} \leq \tilde{\gamma} \leq 2\frac{n_1}{n_1+n_2} - \gamma, \quad \text{when } \gamma < \frac{n_1}{n_1+n_2}. \tag{3}$$

Further, Chang et al. (2012) have shown that although $\hat{\mu}_1(\gamma)$ has smaller MSE than $\hat{\mu}_1(\gamma^+)$ for sufficiently large $\mu_2 - \mu_1$, $(\hat{\mu}_1(\gamma^+), \hat{\mu}_2(\gamma^+))$ dominates $(\hat{\mu}_1(\gamma), \hat{\mu}_2(\gamma))$ in the sense that

$$P\left\{\sum_{i=1}^{2}\left(\frac{\hat{\mu}_i(\gamma^+) - \mu_i}{\tau_i}\right)^2 \leq d\right\} \geq P\left\{\sum_{i=1}^{2}\left(\frac{\hat{\mu}_i(\gamma) - \mu_i}{\tau_i}\right)^2 \leq d\right\}$$

for any $d > 0$, where $\tau_i^2 = \sigma_i^2/n_i$.

Now, we state a modification of Pitman nearness to compare the estimators when they are equal with positive probability. Nayak (1990) defined modified Pitman nearness of an estimator T_1 of θ relative to the other estimator T_2 by

$$\text{MPN}_\theta(T_1, T_2) = P\{|T_1 - \theta| < |T_2 - \theta| \mid T_1 \neq T_2\}.$$

If $\text{MPN}_\theta(T_1, T_2) \geq 1/2$ for any parameter value, then T_1 is said to be closer to θ than T_2. Gupta and Singh (1992) have applied modified Pitman nearness to the estimation of ordered means of two normal population with common variance and have shown that MLE is closer than the unbiased estimator.

Here, we consider the estimation problems under modified Pitman nearness for the following two cases

Case-I: estimation of two ordered means when σ_1^2 and σ_2^2 are unrestricted,

Case-II: estimation of two ordered means when $\sigma_1^2 \leq \sigma_2^2$.

We are much interested in whether or not the similar results can be obtained under the modified Pitman nearness criterion to those obtained by Oono and Shinozaki (2005) and Chang et al. (2012) under MSE or stochastic domination criterion.

We first treat Case-I in Sect. 2 of this paper. With respect to modified Pitman nearness, we show that $\hat{\mu}_i(\gamma)$ improves upon $\bar{X}_i$ if and only if $\text{MPN}_{\mu_i}(\hat{\mu}_i(\gamma), \bar{X}_i) \geq 1/2$ when $\mu_1 = \mu_2$, which is the most critical case for uniform improvement. Further, it is shown that $\hat{\mu}_i(\gamma)$ improves upon $\bar{X}_i$ if and only if $\hat{\mu}(\gamma)$ improves upon $\bar{X}_i$ for the same γ in estimating a common mean. Thus, the problem of improving upon the unbiased estimators of two ordered means essentially reduces to the one of a common mean. This conclusion also applies even when variances are ordered. However, we show that when $\sigma_1^2 \leq \sigma_2^2$, $\hat{\mu}^{\text{GD}}$ improves upon $\bar{X}_2$ for a wide range of (n_1, n_2) in Sect. 3. In Sect. 4, for Case-II, we treat the problem of improving upon $\hat{\mu}_i(\gamma)$ by replacing γ with γ^+ satisfying (2) and (3) with respect to modified Pitman nearness when the restrictions $\mu_1 \leq \mu_2$ and $\sigma_1^2 \leq \sigma_2^2$ are given. We show that $\hat{\mu}_2(\gamma^+)$ improves upon $\hat{\mu}_2(\gamma)$ for suitably chosen $\tilde{\gamma}$, but $\hat{\mu}_1(\gamma^+)$ does not improve upon $\hat{\mu}_1(\gamma)$ even if $\tilde{\gamma}$ is chosen appropriately. Simultaneous estimation of μ_1 and μ_2 is also discussed with respect to generalized Pitman nearness based on sum of normalized squared errors (Rao et al. 1986; Peddada and Khattree 1986). In Sect. 5, we give some results of the numerical study to evaluate the performance of the proposed estimators for the choice $\gamma = n_1 s_2^2/(n_1 s_2^2 + n_2 s_1^2)$ in terms of modified Pitman nearness. In Sect. 6, we make a conclusion.

Throughout this paper, we assume that γ is a function of s_1^2 and s_2^2 satisfying $0 \leq \gamma \leq 1$ and that in some places γ may depend on $(\bar{X}_1 - \bar{X}_2)^2$ as well.

2 Case-I: improving unbiased estimators of ordered means when variances are unrestricted

In this section, we treat the estimation problem of ordered means $\mu_1 \leq \mu_2$ when variances are not ordered. We consider estimators of μ_i of the form (1) and compare them with the unbiased estimator $\bar{X}_i$. We first show that for the case when γ is a function of s_1^2 and s_2^2 only, the most critical case for $\hat{\mu}_i(\gamma)$ to be closer to μ_i than $\bar{X}_i$ is the one when $\mu_1 = \mu_2$. Further, it is shown that $\hat{\mu}_i(\gamma)$ improves upon $\bar{X}_i$ if and only if $\hat{\mu}(\gamma)$ dominates $\bar{X}_i$ in the estimation problem of a common mean.

Theorem 1 *Suppose that* $0 \leq \gamma \leq 1$ *is a function of* s_1^2 *and* s_2^2.

- (i) $\text{MPN}_{\mu_i}(\hat{\mu}_i(\gamma), \bar{X}_i) \geq 1/2$ *for all* $\mu_1 \leq \mu_2$ *and for all* σ_1^2 *and* σ_2^2 *if and only if for all* σ_1^2 *and* σ_2^2, $\text{MPN}_{\mu_i}(\hat{\mu}_i(\gamma), \bar{X}_i) \geq 1/2$ *when* $\mu_1 = \mu_2$.
- (ii) $\text{MPN}_{\mu_i}(\hat{\mu}_i(\gamma), \bar{X}_i) \geq 1/2$ *for all* $\mu_1 \leq \mu_2$ *and for all* σ_1^2 *and* σ_2^2 *if and only if for all* σ_1^2 *and* σ_2^2, $\text{PN}_{\mu}(\hat{\mu}(\gamma), \bar{X}_i) \geq 1/2$ *to estimate* μ *when* $\mu_1 = \mu_2 = \mu$.

Remark 1 In the estimation problem of a common mean, Kubokawa (1989) has given a sufficient condition on sample sizes n_1 and n_2 for $\hat{\mu}(\gamma)$ to be closer to μ than $\bar{X}_i$ for some specified class of γ. For example, for the choice of $\gamma = n_1 s_2^2/(n_1 s_2^2 + n_2 s_1^2)$, $\hat{\mu}^{\text{GD}}$ is closer to μ than both $\bar{X}_1$ and $\bar{X}_2$ if $n_1 \geq 5$ and $n_2 \geq 5$.

Proof We need only to give a proof for the case of μ_1.

(i) Since $\hat{\mu}_1(\gamma) \neq \bar{X}_1$ if and only if $\bar{X}_1 > \bar{X}_2$ and $\gamma < 1$, we have

$$\begin{aligned}
&\text{MPN}_{\mu_1}(\hat{\mu}_1(\gamma), \bar{X}_1) \\
&\quad = P\{|\hat{\mu}_1(\gamma) - \mu_1| < |\bar{X}_1 - \mu_1| \mid \hat{\mu}_1(\gamma) \neq \bar{X}_1\} \\
&\quad = P\{|\gamma \bar{X}_1 + (1-\gamma)\bar{X}_2 - \mu_1| < |\bar{X}_1 - \mu_1| \mid \bar{X}_1 > \bar{X}_2, \gamma < 1\} \\
&\quad = P\{2\mu_1 < (1+\gamma)\bar{X}_1 + (1-\gamma)\bar{X}_2 \mid \bar{X}_1 > \bar{X}_2, \gamma < 1\} \\
&\quad = P\{0 < (1+\gamma)Z_1 + (1-\gamma)Z_2 \mid Z_1 > Z_2, \gamma < 1\}, \qquad (4)
\end{aligned}$$

where $Z_i = \bar{X}_i - \mu_1, i = 1, 2$ are distributed as $N(0, \tau_1^2)$ and $N(\Delta, \tau_2^2)$, respectively, $\Delta = \mu_2 - \mu_1$ and $\tau_i^2 = \sigma_i^2/n_i$. Now, we fix the values of s_1^2 and s_2^2 for which $\gamma < 1$ and consider the conditional probability

$$P\{0 < (1+\gamma)Z_1 + (1-\gamma)Z_2 | Z_1 > Z_2, s_1^2, s_2^2\} \equiv f(\Delta)$$

as a function of Δ. We need only to show that $f(\Delta) \geq f(0)$. Putting $d = (1+\gamma)/(1-\gamma)$, we define the sets

$$A = \{(z_1, z_2) | z_1 \geq z_2, z_2 \geq -\mathrm{d}z_1\}, \quad B = \{(z_1, z_2) | z_1 \geq z_2, z_2 < -\mathrm{d}z_1\},$$
$$A_1 = \{(z_1, z_2) | z_1 \geq z_2, z_2 \geq 0\} \quad \text{and} \quad A_2 = \{(z_1, z_2) | z_2 \geq -\mathrm{d}z_1, z_2 < 0\}.$$

Since A_1 and A_2 are disjoint and $A = A_1 \cup A_2$, we have

$$\begin{aligned} f(\Delta) - f(0) &= \frac{P_\Delta(A)}{P_\Delta(A) + P_\Delta(B)} - \frac{P_0(A)}{P_0(A) + P_0(B)} \\ &= \frac{\{P_\Delta(A_1)P_0(B) - P_0(A_1)P_\Delta(B)\} + \{P_\Delta(A_2)P_0(B) - P_0(A_2)P_\Delta(B)\}}{\{P_\Delta(A) + P_\Delta(B)\}\{P_0(A) + P_0(B)\}}, \end{aligned}$$

where, for example, $P_0(A)$ denotes the probability of the set A when $\Delta = 0$. We first show that $P_\Delta(A_1)P_0(B) - P_0(A_1)P_\Delta(B) > 0$ for $\Delta > 0$. For that purpose, we note that

$$\begin{aligned} P_\Delta(B) &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\tau_2} \exp\left\{-\frac{(z_2 - \Delta)^2}{2\tau_2^2}\right\} \int_{z_2}^{-z_2/d} \frac{1}{\tau_1}\phi(z_1/\tau_1)\mathrm{d}z_1\mathrm{d}z_2 \\ &< \exp\left\{-\frac{\Delta^2}{2\tau_2^2}\right\} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\tau_2} \exp\left\{-\frac{z_2^2}{2\tau_2^2}\right\} \int_{z_2}^{-z_2/d} \frac{1}{\tau_1}\phi(z_1/\tau_1)\mathrm{d}z_1\mathrm{d}z_2 \\ &= \exp\left\{-\frac{\Delta^2}{2\tau_2^2}\right\} P_0(B). \end{aligned} \tag{5}$$

Similarly, we have

$$\begin{aligned} P_\Delta(A_1) &= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\tau_2} \exp\left\{-\frac{(z_2 - \Delta)^2}{2\tau_2^2}\right\} \int_{z_2}^{\infty} \frac{1}{\tau_1}\phi(z_1/\tau_1)\mathrm{d}z_1\mathrm{d}z_2 \\ &> \exp\left\{-\frac{\Delta^2}{2\tau_2^2}\right\} P_0(A_1). \end{aligned} \tag{6}$$

From (5) and (6), we see that $P_\Delta(A_1)P_0(B) - P_0(A_1)P_\Delta(B) > 0$.

Next, we show that $P_\Delta(A_2)P_0(B) - P_0(A_2)P_\Delta(B) > 0$ for $\Delta > 0$. We express $P_\Delta(A_2)$ as

$$
\begin{aligned}
P_\Delta(A_2) &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\tau_2} \exp\left\{-\frac{(z_2-\Delta)^2}{2\tau_2^2}\right\} \int_{-z_2/d}^{\infty} \frac{1}{\tau_1}\phi(z_1/\tau_1)\mathrm{d}z_1\mathrm{d}z_2 \\
&= P_\Delta\{Z_2 < 0\} E_\Delta[g(Z_2)|Z_2 < 0],
\end{aligned}
$$

where $g(z_2) = \int_{-z_2/d}^{\infty} \phi(z_1/\tau_1)/\tau_1 \mathrm{d}z_1$. Since $g(z_2)$ is an increasing function and the conditional distribution of $Z_2 < 0$ is stochastically smallest when $\Delta = 0$, we have for $\Delta > 0$

$$
P_\Delta(A_2) > P_\Delta\{Z_2 < 0\} E_0[g(Z_2)|Z_2 < 0] = P_0(A_2) P_\Delta\{Z_2 < 0\}/P_0\{Z_2 < 0\}. \quad (7)
$$

Similarly, since $h(z_2) = \int_{z_2}^{-z_2/d} \phi(z_1/\tau_1)/\tau_1 \mathrm{d}z_1$ is a decreasing function, we have

$$
\begin{aligned}
P_\Delta(B) &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\tau_2} \exp\left\{-\frac{(z_2-\Delta)^2}{2\tau_2^2}\right\} \int_{z_2}^{-z_2/d} \frac{1}{\tau_1}\phi(z_1/\tau_1)\mathrm{d}z_1\mathrm{d}z_2 \\
&= P_\Delta\{Z_2 < 0\} E_\Delta[h(Z_2)|Z_2 < 0] \\
&< P_\Delta\{Z_2 < 0\} E_0[h(Z_2)|Z_2 < 0] \\
&= P_0(B) P_\Delta\{Z_2 < 0\}/P_0\{Z_2 < 0\}. \quad (8)
\end{aligned}
$$

From (7) and (8), we have $P_\Delta(A_2)P_0(B) - P_0(A_2)P_\Delta(B) > 0$ and we have shown that $f(\Delta) > f(0)$ for $\Delta > 0$.

(ii) In the estimation problem of a common mean, as is stated in Kubokawa (1989, p. 481), Eq. (8), $\hat{\mu}(\gamma)$ is closer to μ than $\bar{X}_1$ if and only if

$$
P\{(1-\gamma)(U_2-U_1)^2 + 2U_1(U_2-U_1) \le 0\} \ge 1/2, \quad (9)
$$

where $U_i = \bar{X}_i - \mu, i = 1, 2$. Since

$$
(1-\gamma)(U_2-U_1)^2 + 2U_1(U_2-U_1) = (U_2-U_1)\{(1+\gamma)U_1 + (1-\gamma)U_2\},
$$

the LHS of (9) is expressed as

$$
\begin{aligned}
&P\{U_1 \le U_2\} P\{(1+\gamma)U_1 + (1-\gamma)U_2 < 0 \mid U_1 \le U_2\} \\
&\quad + P\{U_1 > U_2\} P\{(1+\gamma)U_1 + (1-\gamma)U_2 > 0 \mid U_1 > U_2\}.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&P\{(1+\gamma)U_1 + (1-\gamma)U_2 < 0 \mid U_1 \le U_2\} \\
&\quad = P\{(1+\gamma)U_1 + (1-\gamma)U_2 > 0 \mid U_1 > U_2\},
\end{aligned}
$$

since U_1 and U_2 are symmetrically distributed about the origin. Thus, noting that $P\{U_1 \le U_2\} = P\{U_1 > U_2\} = 1/2$, we see that the LHS of (9) is equal to

$$
P\{(1+\gamma)U_1 + (1-\gamma)U_2 > 0 \mid U_1 > U_2\},
$$

which is $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1(\gamma), \bar{X}_1)$ given by (4) for the case $\mu_1 = \mu_2$. Therefore, we see from (i) that $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1(\gamma), \bar{X}_1)) \geq 1/2$ for all $\mu_1 \leq \mu_2$ and for all σ_1^2 and σ_2^2 if and only if $\mathrm{PN}_{\mu}(\hat{\mu}(\gamma), \bar{X}_i) \geq 1/2$ for all μ and for all σ_1^2 and σ_2^2. We complete the proof. □

Remark 2 We should mention about the general case when γ is a function of s_1^2, s_2^2 and $(\bar{X}_1 - \bar{X}_2)^2$. We first consider the case when we estimate μ_1 and suppose that $\hat{\mu}_1(\gamma_0)$ is closer to μ_1 than $\bar{X}_1$, where γ_0 is a function of s_1^2, s_2^2 and possibly $(\bar{X}_1 - \bar{X}_2)^2$. For any γ satisfying $\gamma_0 \leq \gamma < 1$ if $\gamma_0 < 1$, $\hat{\mu}_1(\gamma)$ is closer to μ_1 than $\bar{X}_1$. This is seen from (4), since (4) is true even when γ depends on $(\bar{X}_1 - \bar{X}_2)^2$ and (4) is an increasing function of γ. Next, we consider the case when we estimate μ_2 and suppose that $\hat{\mu}_2(\gamma_0)$ is closer to μ_2 than $\bar{X}_2$, where γ_0 is a function of s_1^2, s_2^2 and possibly $(\bar{X}_1 - \bar{X}_2)^2$. For any γ satisfying $0 < \gamma \leq \gamma_0$ if $\gamma_0 > 0$, $\hat{\mu}_2(\gamma)$ is closer to μ_2 than $\bar{X}_2$.

3 Case-II: improving unbiased estimators of ordered means when variances are ordered

Now, we treat the estimation problem of ordered means $\mu_1 \leq \mu_2$ when the order restriction $\sigma_1^2 \leq \sigma_2^2$ on variances is present. It is necessary for us to modify the estimators $\hat{\mu}_i(\gamma)$, $i = 1, 2$ treated in Sect. 2, taking into account of the order restriction $\sigma_1^2 \leq \sigma_2^2$. We discuss such modifications of estimators in the next section. Before that we discuss a sufficient condition on the sample sizes n_1 and n_2 for $\hat{\mu}_i(\gamma)$ to improve upon $\bar{X}_i$ when the order restriction $\sigma_1^2 \leq \sigma_2^2$ is present.

It is easily seen that Theorem 1 is true even for the case when the restriction $\sigma_1^2 \leq \sigma_2^2$ is present, if we replace "for all σ_1^2 and σ_2^2" by "for all $\sigma_1^2 \leq \sigma_2^2$". That is, the most critical case for $\hat{\mu}_i(\gamma)$ to improve upon $\bar{X}_i$ is the case when $\mu_1 = \mu_2$, and $\hat{\mu}_i(\gamma)$ improves upon $\bar{X}_i$ if and only if $\hat{\mu}(\gamma)$ improves upon $\bar{X}_i$ for the same γ in the estimation of a common mean. Thus, the problem reduces to the estimation problem of a common mean when the restriction $\sigma_1^2 \leq \sigma_2^2$ is present, which we are concerned with in the rest of this section.

For the case when the restriction $\sigma_1^2 \leq \sigma_2^2$ is not present, Kubokawa (1989) has given a sufficient condition on n_1 and n_2 which guarantees that $\hat{\mu}(\gamma)$ improves upon $\bar{X}_i$ for some specified class of γ. We modify Kubokawa (1989)'s discussion for the case when the restriction $\sigma_1^2 \leq \sigma_2^2$ is present. We first note that there is no room for modifying the sufficient condition for $\hat{\mu}(\gamma)$ to improve upon $\bar{X}_1$ which is associated with smaller variance σ_1^2, since the case $\sigma_1^2/\sigma_2^2 \to 0$ is critical as is seen from the proof of Kubokawa (1989). However, we show that $\hat{\mu}(\gamma)$ improves upon $\bar{X}_2$ which is associated with larger variance for a wide range of (n_1, n_2). Unfortunately, we have not succeeded in showing generally the monotonicity of $\mathrm{PN}_{\mu}(\hat{\mu}(\gamma), \bar{X}_2)$ in σ_1^2/σ_2^2. Here, we restrict ourselves to the estimator $\hat{\mu}(\gamma) = \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2$ with γ of the form

$$\gamma = \frac{n_1 s_2^2}{n_1 s_2^2 + c n_2 s_1^2}, \tag{10}$$

where c is a positive constant to be chosen properly. From Remark 2, it is seen that if $\gamma = n_1 s_2^2/(n_1 s_2^2 + c n_2 s_1^2)$ gives an improved estimator, then any γ satisfying $0 < \gamma \le n_1 s_2^2/(n_1 s_2^2 + c n_2 s_1^2)$ also gives an improvement even when γ depends on $(\bar{X}_1 - \bar{X}_2)^2$.

We essentially follow the notations given in the proof of Kubokawa (1989) and put $\tau = (n_1\sigma_2^2)/(n_2\sigma_1^2)$ and $Z = \tau + cF$, where F is a random variable having F-distribution with degrees of freedom $(n_1 - 1, n_2 - 1)$. Putting $G = \sqrt{\tau}\{1 - (1 + \tau)/(2Z)\}$, $\mathrm{PN}_\mu(\hat{\mu}(\gamma), \bar{X}_2) \ge 1/2$ if

$$h(\tau) = E\left[\int_0^{(G+\sqrt{G^2+1})^2} f_{1,1}(v)\mathrm{d}v\right] \ge 1/2,$$

where $f_{1,1}(\cdot)$ is a density of an F-distribution with degrees of freedom $(1,\ 1)$ and the expectation is taken with respect to the random variable F. Further, the derivative of $h(\tau)$ is given in Kubokawa (1989) as

$$h'(\tau) = \frac{C_0}{2\sqrt{\tau}(1+\tau)} E\left[\frac{2Z^2 - (1+3\tau)Z + 2\tau(1+\tau)}{4Z^2 - 4\tau Z + \tau(1+\tau)}\right], \tag{11}$$

where C_0 is a positive constant.

First, we see that for any $c > 0$, $h(\tau) \ge 1/2$ for $\tau \ge 1$, since $\tau \ge 1$ implies $G \ge 0$. $\tau \ge 1$ for any $\sigma_2^2 \ge \sigma_1^2$ if and only if $n_1 \ge n_2$. Thus, we see that $\hat{\mu}(\gamma)$ with γ given by (10) improves upon $\bar{X}_2$ if $n_1 \ge n_2$.

Kubokawa (1989) has shown that if $c \ge (n_1 - 1)/\{2(n_1 - 3)\}$ for $n_1 \ge 4$, $h'(\tau) \ge 0$ for any $\tau \ge 0$ and thus $h(\tau) \ge h(0) = 1/2$, for any $\tau \ge 0$. Therefore, we see that under the same condition on c, $h(\tau) \ge 1/2$ for any $\tau \ge n_1/n_2$. Since $(n_1 - 1)/\{2(n_1 - 3)\} \le 1$ for $n_1 \ge 5$, $\hat{\mu}^{\mathrm{GD}}$ improves upon $\bar{X}_2$ for $n_1 \ge 5$.

Even when $n_1 \le 4$, we can show that $\hat{\mu}(\gamma)$ improves upon $\bar{X}_2$ for a wide range of n_2. From (11), we see that $h'(\tau) \ge 0$ if $2z^2 - (1+3\tau)z + 2\tau(1+\tau) \ge 0$ for any $z \ge \tau$, which is true if $\tau \ge (-5 + 4\sqrt{2})/7 = 0.0938$. Therefore, $h'(\tau) \ge 0$ for $\tau \ge n_1/n_2$ if $n_1/n_2 \ge 0.0938$. Thus, we have seen that if $n_1/n_2 \ge 0.0938$ and $h(n_1/n_2) \ge 1/2$, $\hat{\mu}(\gamma)$ improves upon $\bar{X}_2$. Setting $c = 1$, we have numerically evaluated $h(n_1/n_2)$ for some values of (n_1, n_2) as given in Table 1. From Table 1, we see that $\hat{\mu}^{\mathrm{GD}}$ is closer to μ than $\bar{X}_2$ for $(n_1 = 2, n_2 \le 7)$, $(n_1 = 3, n_2 \le 26)$ and $(n_1 = 4, n_2 \le 31)$, and is not closer to μ than $\bar{X}_2$ for $(n_1 = 2, 8 \le n_2 \le 31)$ and $(n_1 = 3, 27 \le n_2 \le 31)$.

4 Case-II: a class of improved estimators of ordered means when variances are also ordered

In Sect. 2, we have compared $\hat{\mu}_i(\gamma)$ with $\bar{X}_i$ for Case-I, where the order restriction is given on two means and σ_1^2 and σ_2^2 are unrestricted. Although we also compared them for Case-II, where order restrictions are given on means and variances in Sect. 3, $\hat{\mu}_i(\gamma)$ is not constructed so that the order restriction on variances is taken into consideration. Since the inequality $\sigma_1^2 \le \sigma_2^2$ implies $n_1\sigma_2^2/(n_1\sigma_2^2 + n_2\sigma_1^2) \ge n_1/(n_1 + n_2)$, we may modify $\hat{\mu}_i(\gamma)$ so that $\gamma \ge n_1/(n_1 + n_2)$ is satisfied. As is given in (3), we replace γ by

Table 1 $h(n_1/n_2)$

n_2/n_1	2	3	4	5	6	7	8	9	10
2	0.636	0.690	0.720	0.740	0.756	0.769	0.780	0.789	0.797
3	0.583	0.640	0.671	0.693	0.710	0.723	0.735	0.744	0.753
4	0.552	0.610	0.642	0.663	0.680	0.694	0.706	0.716	0.724
5	0.531	0.590	0.621	0.643	0.659	0.673	0.685	0.695	0.703
6	0.516	0.575	0.606	0.627	0.644	0.657	0.668	0.678	0.687
7	0.505	0.563	0.594	0.615	0.631	0.644	0.655	0.665	0.674
8	0.495	0.554	0.584	0.605	0.621	0.634	0.645	0.654	0.663
9	0.488	0.546	0.576	0.597	0.612	0.625	0.635	0.645	0.653
10	0.482	0.540	0.570	0.590	0.605	0.617	0.628	0.637	0.645
11	0.476	0.535	0.564	0.584	0.598	0.611	0.621	0.630	0.638
12	0.472	0.530	0.559	0.578	0.593	0.605	0.615	0.624	0.632
13	0.468	0.526	0.555	0.574	0.588	0.600	0.610	0.618	0.626
14	0.464	0.522	0.551	0.570	0.584	0.595	0.605	0.614	0.621
15	0.461	0.519	0.548	0.566	0.580	0.591	0.601	0.609	0.617
16	0.459	0.517	0.545	0.563	0.577	0.588	0.597	0.605	0.613
17	0.456	0.514	0.542	0.560	0.573	0.584	0.594	0.602	0.609
18	0.454	0.512	0.539	0.557	0.571	0.581	0.590	0.598	0.606
19	0.452	0.510	0.537	0.555	0.568	0.579	0.587	0.595	0.602
20	0.450	0.508	0.535	0.553	0.566	0.576	0.585	0.593	0.599
21	0.449	0.506	0.533	0.551	0.563	0.574	0.582	0.590	0.597
22	0.447	0.505	0.532	0.549	0.561	0.571	0.580	0.587	0.594
23	0.446	0.503	0.530	0.547	0.559	0.569	0.578	0.585	0.592
24	0.445	0.502	0.529	0.545	0.558	0.567	0.576	0.583	0.590
25	0.444	0.501	0.527	0.544	0.556	0.566	0.574	0.581	0.588
26	0.443	0.500	0.526	0.542	0.554	0.564	0.572	0.579	0.586
27	0.442	0.499	0.525	0.541	0.553	0.562	0.571	0.578	0.584
28	0.441	0.498	0.524	0.540	0.552	0.561	0.569	0.576	0.582
29	0.440	0.497	0.523	0.539	0.550	0.560	0.567	0.574	0.580
30	0.439	0.496	0.522	0.538	0.549	0.558	0.566	0.573	0.579
31	0.438	0.495	0.521	0.537	0.548	0.557	0.565	0.571	0.577

$\tilde{\gamma} \geq n_1/(n_1+n_2)$ when $\gamma < n_1/(n_1+n_2)$. Using γ^+ given in (2), we have $\hat{\mu}_i(\gamma^+)$ and compare it with $\hat{\mu}_i(\gamma)$ in this Section. In Theorem 2, for the estimation of the mean μ_2 of the population with larger variance, we show that $\hat{\mu}_2(\gamma^+)$ is closer to μ_2 than $\hat{\mu}_2(\gamma)$ for suitably chosen $\tilde{\gamma}$. In Theorem 3, for the estimation of the mean μ_1 of the population with smaller variance, we show that $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1(\gamma^+), \hat{\mu}_1(\gamma)) < 1/2$ for some parameter values even when $\tilde{\gamma}$ is chosen appropriately. In Theorem 4, we show that improvement is possible when we estimate means (μ_1, μ_2) simultaneously under generalized Pitman nearness based on sum of normalized squared errors. Here, we assume that γ may depend on $(\bar{X}_1 - \bar{X}_2)^2$ as well.

4.1 Estimation of individual means when means and variances are ordered

Theorem 2 *Suppose that* $P\{\bar{X}_1 - \bar{X}_2 > 0, \gamma < n_1/(n_1+n_2)\} > 0$ *for any* $\mu_1 \le \mu_2$ *and* $\sigma_1^2 \le \sigma_2^2$*. If* $\tilde{\gamma}$ *is chosen so that*

$$\frac{n_1}{n_1+n_2} \le \tilde{\gamma} \le 2\frac{n_1}{n_1+n_2} - \gamma \quad \text{when } \gamma < \frac{n_1}{n_1+n_2}, \tag{12}$$

then for any $\mu_1 \le \mu_2$ *and* $\sigma_1^2 \le \sigma_2^2$, $\mathrm{MPN}_{\mu_2}(\hat{\mu}_2(\gamma^+), \hat{\mu}_2(\gamma)) \ge 1/2$.

Proof We show that

$$\begin{aligned} &P\left\{|\hat{\mu}_2(\gamma^+) - \mu_2| < |\hat{\mu}_2(\gamma) - \mu_2|,\ \hat{\mu}_2(\gamma^+) \ne \hat{\mu}_2(\gamma)\right\} \\ &\quad \ge P\left\{\hat{\mu}_2(\gamma^+) \ne \hat{\mu}_2(\gamma)\right\}/2. \end{aligned} \tag{13}$$

From (12), we see that $\hat{\mu}_2(\gamma^+) \ne \hat{\mu}_2(\gamma)$ if and only if $\bar{X}_1 > \bar{X}_2$ and $\gamma < n_1/(n_1+n_2)$. In this case, $\hat{\mu}_2(\gamma^+) = \tilde{\gamma}\bar{X}_1 + (1-\tilde{\gamma})\bar{X}_2 > \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2 = \hat{\mu}_2(\gamma)$. Thus, the LHS of (13) is expressed as

$$\begin{aligned} &P\left\{|\hat{\mu}_2(\gamma^+) - \mu_2| < |\hat{\mu}_2(\gamma) - \mu_2|, \bar{X}_1 > \bar{X}_2, \gamma < n_1/(n_1+n_2)\right\} \\ &\quad = P\left\{\hat{\mu}_2(\tilde{\gamma}) + \hat{\mu}_2(\gamma) < 2\mu_2, \bar{X}_1 > \bar{X}_2, \gamma < n_1/(n_1+n_2)\right\} \\ &\quad = P\left\{(\gamma+\tilde{\gamma})(\bar{X}_1 - \mu_2) + \{2-(\gamma+\tilde{\gamma})\}(\bar{X}_2 - \mu_2) < 0,\right. \\ &\qquad \left.\bar{X}_1 > \bar{X}_2,\ \gamma < n_1/(n_1+n_2)\right\}. \end{aligned} \tag{14}$$

Now, we make the variable transformation

$$V_1 = (\bar{X}_1 - \mu_2) - (\bar{X}_2 - \mu_2), \quad V_2 = (\tau_2^2/\tau_1^2)(\bar{X}_1 - \mu_2) + (\bar{X}_2 - \mu_2).$$

Then, V_1 and V_2 are independently distributed as $N\left(-\Delta, \tau_1^2 + \tau_2^2\right)$ and $N(-(\tau_2^2/\tau_1^2)\Delta, \tau_2^2(\tau_1^2+\tau_2^2)/\tau_1^2)$, respectively, where $\Delta = \mu_2 - \mu_1 \ge 0$. Noting that

$$\bar{X}_1 - \mu_2 = \frac{\tau_1^2(V_1+V_2)}{\tau_1^2+\tau_2^2} \quad \text{and} \quad \bar{X}_2 - \mu_2 = \frac{\tau_1^2 V_2 - \tau_2^2 V_1}{\tau_1^2+\tau_2^2},$$

and denoting the indicator functions of the sets $V_1 > 0$ and $\gamma < n_1/(n_1+n_2)$ by $I_{V_1>0}$ and $I_{\gamma<n_1/(n_1+n_2)}$, respectively, (14) becomes

$$\begin{aligned} &P\left\{\left[(\gamma+\tilde{\gamma})\tau_1^2 - \{2-(\gamma+\tilde{\gamma})\}\tau_2^2\right]V_1 + 2\tau_1^2 V_2 < 0, V_1 > 0, \gamma < n_1/(n_1+n_2)\right\} \\ &\quad = E\left[\Phi\left(\frac{[\{2-(\gamma+\tilde{\gamma})\}\tau_2^2 - (\gamma+\tilde{\gamma})\tau_1^2]V_1 + 2\tau_2^2\Delta}{2\tau_1\tau_2\sqrt{\tau_1^2+\tau_2^2}}\right) I_{V_1>0} I_{\gamma<n_1/(n_1+n_2)}\right], \end{aligned} \tag{15}$$

where expectation is taken with respect to V_1, s_1^2 and s_2^2. Noting that $\{2-(\gamma+\tilde{\gamma})\}\tau_2^2 - (\gamma+\tilde{\gamma})\tau_1^2 \geq 0$ since $\gamma+\tilde{\gamma} \leq 2n_1/(n_1+n_2)$, we have

$$(15) \geq P\{V_1 > 0, \gamma < n_1/(n_1+n_2)\}/2,$$

which is RHS of (13). Since the inequality (13) implies that $\mathrm{MPN}_{\mu_2}(\hat{\mu}_2(\gamma^+), \hat{\mu}_2(\gamma)) \geq 1/2$, this completes the proof. □

Remark 3 We may mention that if we choose $\tilde{\gamma} = 2n_1/(n_1+n_2)-\gamma$, $|\gamma - n_1/(n_1+n_2)| = |\tilde{\gamma} - n_1/(n_1+n_2)|$. We note that $2n_1/(n_1+n_2)-\gamma$ may be larger than 1 for some values of s_1^2 and s_2^2 if $n_1 > n_2$. In such cases, we should replace $\tilde{\gamma}$ by $\tilde{\gamma}' = \min(\tilde{\gamma}, 1)$, which leads to a further improvement.

Remark 4 For the case when $\gamma \equiv 0$, we apply Theorem 2 and we obtain a class of estimators improving upon $\bar{X}_2$. We see that for any γ satisfying $n_1/(n_1+n_2) \leq \gamma \leq 2n_1/(n_1+n_2)$, $\hat{\mu}_2(\gamma)$ improves upon $\bar{X}_2$. From Remark 2, we can broaden the class of improved estimators. We see that when $n_1 \geq n_2$, $\hat{\mu}_2(\gamma)$ improves upon $\bar{X}_2$ for any γ satisfying $0 \leq \gamma \leq 1$.

Remark 5 Suppose that $\hat{\mu}_2(\gamma)$ improves upon $\bar{X}_2$ under modified Pitman nearness criterion. Although $\hat{\mu}_2(\gamma^+)$ improves upon $\hat{\mu}_2(\gamma)$ for suitably chosen $\tilde{\gamma}$ from Theorem 2, this does not imply that $\hat{\mu}_2(\gamma^+)$ improves upon $\bar{X}_2$ since transitivity does not hold for Pitman nearness criterion. As a matter of fact, $\mathrm{MPN}_{\mu_2}(\hat{\mu}_2(\gamma), \bar{X}_2) > \mathrm{MPN}_{\mu_2}(\hat{\mu}_2(\gamma^+), \bar{X}_2)$ since $|\hat{\mu}_2(\gamma^+) - \mu_2| < |\bar{X}_2 - \mu_2|$ implies $|\hat{\mu}_2(\gamma) - \mu_2| < |\bar{X}_2 - \mu_2|$ for the case when $\bar{X}_1 > \bar{X}_2$ and $\gamma < n_1/(n_1+n_2)$. Thus, $\hat{\mu}_2(\gamma^+)$ is less likely to improve upon $\bar{X}_2$ than $\hat{\mu}_2(\gamma)$. We need to examine whether $\hat{\mu}_2(\gamma^+)$ improves upon $\bar{X}_2$ or not in estimating a common mean, from the results given in Sects. 2 and 3 for the case when $n_2 > n_1$.

Remark 6 When we consider the estimation problem of a common mean of two normal distributions when variances satisfy the order restriction $\sigma_1^2 \leq \sigma_2^2$ and consider the estimator of the form $\hat{\mu}(\gamma) = \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2$, then $\mathrm{PN}_{\mu}(\hat{\mu}(\gamma^+), \hat{\mu}(\gamma)) \geq 1/2$ if $\tilde{\gamma}$ satisfies the condition (12). We can show this by treating the cases $\bar{X}_1 \leq \bar{X}_2$ and $\bar{X}_1 > \bar{X}_2$ separately along the same line as the proof of Theorem 2.

In the next Theorem, we treat the estimation problem of the mean μ_1 which is associated with smaller variance and show that $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1(\gamma^+), \hat{\mu}_1(\gamma)) < 1/2$ when the difference of the two means, $\Delta = \mu_2 - \mu_1$, is sufficiently large. To prove it, we need the following.

Lemma 1 *Let $g(\cdot)$ be the probability density function of the normal distribution $N(-\Delta, \sigma^2)$. For any $c > 0$ and $d > 0$*

$$\int_0^\infty \Phi(cy - d\Delta)g(y)\mathrm{d}y < \int_0^\infty g(y)\mathrm{d}y/2$$

for sufficiently large Δ.

A proof is given in Appendix A.

Theorem 3 *Suppose that γ is a non-decreasing function of $\bar{X}_1 - \bar{X}_2 > 0$ and that $P\{\bar{X}_1 - \bar{X}_2 > 0, \gamma < n_1/(n_1+n_2)\} > 0$ for any $\mu_1 \le \mu_2$ and $\sigma_1^2 \le \sigma_2^2$. If $\tilde{\gamma}$ is chosen so that (12) is satisfied, then for sufficiently large Δ,*

$$\mathrm{MPN}_{\mu_1}(\hat{\mu}_1(\gamma^+), \hat{\mu}_1(\gamma)) < 1/2.$$

Proof Noticing again that $\hat{\mu}_1(\gamma^+) \ne \hat{\mu}_1(\gamma)$ if and only if $\bar{X}_1 > \bar{X}_2$ and $\gamma < n_1/(n_1+n_2)$ and that in this case $\hat{\mu}_1(\gamma^+) > \hat{\mu}_1(\gamma)$, we have

$$\begin{aligned} &P\{|\hat{\mu}_1(\gamma^+) - \mu_1| < |\hat{\mu}_1(\gamma) - \mu_1|, \hat{\mu}_1(\gamma^+) \ne \hat{\mu}_1(\gamma)\} \\ &\quad = P\{\hat{\mu}_1(\gamma) + \hat{\mu}_1(\tilde{\gamma}) < 2\mu_1, \bar{X}_1 > \bar{X}_2, \gamma < n_1/(n_1+n_2)\} \\ &\quad = P\left\{(\gamma+\tilde{\gamma})(\bar{X}_1 - \mu_1) + \{2 - (\gamma+\tilde{\gamma})\}(\bar{X}_2 - \mu_1) < 0, \right. \\ &\qquad \left. \bar{X}_1 > \bar{X}_2, \gamma < n_1/(n_1+n_2)\right\}. \end{aligned} \tag{16}$$

We make the variable transformation

$$Y_1 = (\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_1) \quad \text{and} \quad Y_2 = (\bar{X}_1 - \mu_1) + (\tau_1^2/\tau_2^2)(\bar{X}_2 - \mu_1).$$

Then, Y_1 and Y_2 are independently distributed as $N(-\Delta, \tau_1^2 + \tau_2^2)$ and $N((\tau_1^2/\tau_2^2)\Delta, \tau_1^2(\tau_1^2+\tau_2^2)/\tau_2^2)$, respectively. Noting that

$$\bar{X}_1 - \mu_1 = \frac{\tau_1^2 Y_1 + \tau_2^2 Y_2}{\tau_1^2 + \tau_2^2} \quad \text{and} \quad \bar{X}_2 - \mu_1 = \frac{\tau_2^2(Y_2 - Y_1)}{\tau_1^2 + \tau_2^2},$$

(16) is rewritten as

$$\begin{aligned} &P\left\{[(\gamma+\tilde{\gamma})\tau_1^2 - \{2-(\gamma+\tilde{\gamma})\}\tau_2^2]Y_1 + 2\tau_2^2 Y_2 < 0, Y_1 > 0, \gamma < n_1/(n_1+n_2)\right\} \\ &\quad = E\left[\int_0^\infty \Phi\left(\frac{[\{2-(\gamma+\tilde{\gamma})\}\tau_2^2 - (\gamma+\tilde{\gamma})\tau_1^2]y_1/2 - \tau_1^2\Delta}{\tau_1\tau_2\sqrt{\tau_1^2+\tau_2^2}}\right) I_{\gamma<n_1/(n_1+n_2)}\, g(y_1)\mathrm{d}y_1\right], \end{aligned} \tag{17}$$

where expectation is taken with respect to s_1^2 and s_2^2 and $g(\cdot)$ is the density function of $N(-\Delta, \tau_1^2+\tau_2^2)$. Setting

$$c = \tau_2/(\tau_1\sqrt{\tau_1^2+\tau_2^2}) \quad \text{and} \quad d = \tau_1/(\tau_2\sqrt{\tau_1^2+\tau_2^2}),$$

we have

$$\Phi\left(\frac{[\{2-(\gamma+\tilde{\gamma})\}\tau_2^2 - (\gamma+\tilde{\gamma})\tau_1^2]y_1/2 - \tau_1^2\Delta}{\tau_1\tau_2\sqrt{\tau_1^2+\tau_2^2}}\right) \le \Phi(cy_1 - d\Delta).$$

Fixing s_1^2 and s_2^2, we define

$$t(s_1^2, s_2^2) = \sup\{Y_1 = \bar{X}_1 - \bar{X}_2 > 0 \mid \gamma < n_1/(n_1+n_2)\}.$$

Since γ is a non-decreasing function of $\bar{X}_1 - \bar{X}_2$,

$$\int_0^\infty \Phi(cy_1 - d\Delta) I_{\gamma < n_1/(n_1+n_2)} g(y_1)\mathrm{d}y_1 = \int_0^{t(s_1^2,s_2^2)} \Phi(cy_1 - d\Delta) g(y_1)\mathrm{d}y_1.$$

Thus, we need only to show that

$$\int_0^{t(s_1^2,s_2^2)} \Phi(cy_1 - d\Delta) g(y_1)\mathrm{d}y_1 < 1/2 \int_0^{t(s_1^2,s_2^2)} g(y_1)\mathrm{d}y_1.$$

From Lemma 1, we see that for sufficiently large Δ,

$$\int_0^\infty \Phi(cy_1 - d\Delta) g(y_1)\mathrm{d}y_1 < \int_0^\infty g(y_1)\mathrm{d}y_1/2,$$

which implies

$$\int_0^t \Phi(cy_1 - d\Delta) g(y_1)\mathrm{d}y_1 < \int_0^t g(y_1)\mathrm{d}y_1/2$$

for any $t > 0$ since $\Phi(cy_1 - d\Delta)$ is an increasing function of $y_1 > 0$. This completes the proof. □

Remark 7 Although $\hat{\mu}_1(\gamma^+)$ does not dominate $\hat{\mu}_1(\gamma)$ from Theorem 3, we see that if $\hat{\mu}_1(\gamma)$ dominates $\bar{X}_1$ and $P\{\bar{X}_1 > \bar{X}_2, \gamma < n_1/(n_1+n_2)\} > 0$, then $\hat{\mu}_1(\gamma^+)$ dominates $\bar{X}_1$ for any $\tilde{\gamma}$ satisfying the condition (12). This is seen from (4) since the probability of (4) is an increasing function of γ.

Remark 8 From the expression (17) of the probability $P\{|\hat{\mu}_1(\gamma^+) - \mu_1| < |\hat{\mu}_1(\gamma) - \mu_1|, \hat{\mu}_1(\gamma^+) \neq \hat{\mu}_1(\gamma)\}$, we see that the probability is larger than $P\{\hat{\mu}_1(\gamma^+) \neq \hat{\mu}_1(\gamma)\}/2$ when $\Delta = 0$ and σ_1^2/σ_2^2 is sufficiently small. Thus, neither $\hat{\mu}_1(\gamma^+)$ nor $\hat{\mu}_1(\gamma)$ dominates the other.

4.2 Simultaneous estimation of ordered means when variances are also ordered

Here, we simultaneously estimate ordered means $\mu_1 \leq \mu_2$ when variances are also ordered. We show that $\hat{\boldsymbol{\mu}}(\gamma^+) = (\hat{\mu}_1(\gamma^+), \hat{\mu}_2(\gamma^+))$ improves upon $\hat{\boldsymbol{\mu}}(\gamma) = (\hat{\mu}_1(\gamma), \hat{\mu}_2(\gamma))$ under generalized Pitman nearness based on sum of normalized squared errors in the following.

Theorem 4 *Suppose that* $P\{\bar{X}_1 - \bar{X}_2 > 0, \gamma < n_1/(n_1+n_2)\} > 0$ *for any* $\mu_1 \leq \mu_2$ *and* $\sigma_1^2 \leq \sigma_2^2$. *If* $\tilde{\gamma}$ *is chosen so that* (12) *is satisfied, then for any* $\mu_1 \leq \mu_2$ *and* $\sigma_1^2 \leq \sigma_2^2$, $\mathrm{MPN}_{\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}(\gamma^+), \hat{\boldsymbol{\mu}}(\gamma)) > 1/2$, *where*

$$\begin{aligned}&\mathrm{MPN}_{\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}(\gamma^+), \hat{\boldsymbol{\mu}}(\gamma))\\&\quad = P\left\{\sum_{i=1}^{2}\{\hat{\mu}_i(\gamma^+) - \mu_i\}^2/\tau_i^2 \le \sum_{i=1}^{2}\{\hat{\mu}_i(\gamma) - \mu_i\}^2/\tau_i^2 \mid \hat{\boldsymbol{\mu}}(\gamma^+) \ne \hat{\boldsymbol{\mu}}(\gamma)\right\}.\end{aligned}$$

Proof Putting $Z_i = \bar{X}_i - \mu_1$, $i = 1, 2$ and $\Delta = \mu_2 - \mu_1$, we express sum of normalized squared errors for $\hat{\boldsymbol{\mu}}(\gamma)$ as

$$\begin{aligned}&\sum_{i=1}^{2}\{\hat{\mu}_i(\gamma) - \mu_i\}^2/\tau_i^2\\&\quad = \frac{1}{\tau_1^2}\left\{\gamma\bar{X}_1 + (1-\gamma)\bar{X}_2 - \mu_1\right\}^2 + \frac{1}{\tau_2^2}\left\{\gamma\bar{X}_1 + (1-\gamma)\bar{X}_2 - \mu_2\right\}^2\\&\quad = \frac{1}{\tau_1^2}\{\gamma Z_1 + (1-\gamma)Z_2\}^2 + \frac{1}{\tau_2^2}\{\gamma Z_1 + (1-\gamma)Z_2 - \Delta\}^2\\&\quad = \left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}\right)\{\gamma Z_1 + (1-\gamma)Z_2\}^2 - \frac{2\Delta}{\tau_2^2}\{\gamma Z_1 + (1-\gamma)Z_2\} + \frac{\Delta^2}{\tau_2^2}.\end{aligned} \tag{18}$$

We note that $\hat{\boldsymbol{\mu}}(\gamma^+) \ne \hat{\boldsymbol{\mu}}(\gamma)$ if and only if $\bar{X}_1 > \bar{X}_2$ and $\gamma < n_1/(n_1+n_2)$. In this case, by replacing γ by $\tilde{\gamma}$ in (18), we have sum of normalized squared errors for $\hat{\boldsymbol{\mu}}(\gamma^+)$ as

$$\begin{aligned}\sum_{i=1}^{2}\{\hat{\mu}_i(\gamma^+) - \mu_i\}^2/\tau_i^2 &= \left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}\right)\{\tilde{\gamma} Z_1 + (1-\tilde{\gamma})Z_2\}^2\\&\quad - \frac{2\Delta}{\tau_2^2}\{\tilde{\gamma} Z_1 + (1-\tilde{\gamma})Z_2\} + \frac{\Delta^2}{\tau_2^2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sum_{i=1}^{2}\{\hat{\mu}_i(\gamma) - \mu_i\}^2/\tau_i^2 - \sum_{i=1}^{2}\{\hat{\mu}_i(\gamma^+) - \mu_i\}^2/\tau_i^2\\&\quad = (\gamma - \tilde{\gamma})(Z_1 - Z_2)\left[\left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}\right)[(\gamma+\tilde{\gamma})Z_1 + \{(2 - (\gamma+\tilde{\gamma})\}Z_2] - \frac{2\Delta}{\tau_2^2}\right],\end{aligned}$$

which is non-negative if and only if

$$(\tau_1^2 + \tau_2^2)\left[(\gamma+\tilde{\gamma})Z_1 + \{2 - (\gamma+\tilde{\gamma})\}Z_2\right] - 2\tau_1^2\Delta \le 0$$

since $Z_1 - Z_2 = \bar{X}_1 - \bar{X}_2 > 0$ and $\tilde{\gamma} > \gamma$. Therefore, we need only to show that

$$P\left\{(\tau_1^2 + \tau_2^2)[(\gamma+\tilde{\gamma})Z_1 + \{2 - (\gamma+\tilde{\gamma})\}Z_2] - 2\tau_1^2\Delta \le 0,\right.$$

$$Z_1 > Z_2, \gamma < n_1/(n_1+n_2)\Big\} \geq P\left\{Z_1 > Z_2, \gamma < n_1/(n_1+n_2)\right\}/2.$$

Now, we make the variable transformation

$$Y_1 = Z_1 - Z_2, \quad Y_2 = Z_1 + (\tau_1^2/\tau_2^2)Z_2.$$

Then, Y_1 and Y_2 are the same with those given in the proof of Theorem 3. Since Y_1 and Y_2 are independently distributed as $N(-\Delta, \tau_1^2+\tau_2^2)$ and $N(\tau_1^2\Delta/\tau_2^2, \tau_1^2(\tau_1^2+\tau_2^2)/\tau_2^2)$, respectively, and $Z_1 = (\tau_1^2 Y_1 + \tau_2^2 Y_2)/(\tau_1^2+\tau_2^2)$ and $Z_2 = \tau_2^2(Y_2 - Y_1)/(\tau_1^2+\tau_2^2)$, we have

$$\begin{aligned}
&P\Big\{(\tau_1^2+\tau_2^2)[(\gamma+\tilde{\gamma})Z_1 + \{2-(\gamma+\tilde{\gamma})\}Z_2] - 2\tau_1^2\Delta \leq 0,\\
&\quad Z_1 > Z_2, \gamma < n_1/(n_1+n_2)\Big\}\\
&= P\Big\{[(\gamma+\tilde{\gamma})\tau_1^2 - \{2-(\gamma+\tilde{\gamma})\}\tau_2^2]Y_1 + 2\tau_2^2(Y_2 - \tau_1^2\Delta/\tau_2^2) \leq 0,\\
&\quad Y_1 > 0, \gamma < n_1/(n_1+n_2)\Big\}\\
&= E\left[\Phi\left(\frac{[\{2-(\gamma+\tilde{\gamma})\}\tau_2^2 - (\gamma+\tilde{\gamma})\tau_1^2]Y_1}{2\tau_1\tau_2\sqrt{\tau_1^2+\tau_2^2}}\right) I_{Y_1>0} I_{\gamma<n_1/(n_1+n_2)}\right],
\end{aligned}$$

where expectation is taken with respect to Y_1, s_1^2 and s_2^2. Since $\{2-(\gamma+\tilde{\gamma})\}\tau_2^2 - (\gamma+\tilde{\gamma})\tau_1^2 > 0$, we have the desired result. □

5 Numerical results

Here, we illustrate the behavior of the proposed estimators by numerical evaluation for Case-II, where both means and variances are ordered. We set $\gamma = n_1 s_2^2/(n_1 s_2^2 + n_2 s_1^2)$ and $\tilde{\gamma} = n_1/(n_1+n_2)$ when $\gamma < n_1/(n_1+n_2)$. Then, $\hat{\mu}_i(\gamma^+)$ is given by

$$\hat{\mu}_1^{\mathrm{CS}} = \begin{cases} \hat{\mu}_1^{\mathrm{OS}}, & \text{if } s_1^2 \leq s_2^2 \\ \min\{\bar{X}_1, \frac{n_1}{n_1+n_2}\bar{X}_1 + \frac{n_2}{n_1+n_2}\bar{X}_2\}, & \text{if } s_1^2 > s_2^2 \end{cases}$$

and

$$\hat{\mu}_2^{\mathrm{CS}} = \begin{cases} \hat{\mu}_2^{\mathrm{OS}}, & \text{if } s_1^2 \leq s_2^2 \\ \max\{\bar{X}_2, \frac{n_1}{n_1+n_2}\bar{X}_1 + \frac{n_2}{n_1+n_2}\bar{X}_2\}, & \text{if } s_1^2 > s_2^2, \end{cases}$$

which are proposed by Chang and Shinozaki (2012). For various values of $n_1, n_2, \Delta, \sigma_2^2 \geq 1$ and $\sigma_1^2 = 1$, we compare $\hat{\mu}_i^{\mathrm{CS}}$ with $\hat{\mu}_i^{\mathrm{OS}}$, and have numerically evaluated $\mathrm{MPN}_{\mu_i}(\hat{\mu}_i^{\mathrm{CS}}, \hat{\mu}_i^{\mathrm{OS}})$, $i = 1, 2$. To compare $\hat{\boldsymbol{\mu}}^{\mathrm{CS}}$ with $\hat{\boldsymbol{\mu}}^{\mathrm{OS}}$, we have numerically evaluated $\mathrm{MPN}_{\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}^{\mathrm{CS}}, \hat{\boldsymbol{\mu}}^{\mathrm{OS}})$, and also $P\{\bar{X}_1 > \bar{X}_2, s_1^2 > s_2^2\}$, which are all given in Table 2.

Table 2 $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1^{\mathrm{CS}}, \hat{\mu}_1^{\mathrm{OS}})$, $\mathrm{MPN}_{\mu_2}(\hat{\mu}_2^{\mathrm{CS}}, \hat{\mu}_2^{\mathrm{OS}})$, $\mathrm{MPN}_{\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}^{\mathrm{CS}}, \hat{\boldsymbol{\mu}}^{\mathrm{OS}})$ and $P(\bar{X}_1 > \bar{X}_2, s_1^2 > s_2^2)$

n_1	n_2	Δ	$\sigma_2^2 = 1$				$\sigma_2^2 = 2$				$\sigma_2^2 = 4$			
			a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*
5	2	0	0.623	0.623	0.623	(0.313)	0.709	0.709	0.709	(0.241)	0.782	0.782	0.782	(0.178)
		0.2	0.556	0.741	0.612	(0.254)	0.670	0.813	0.696	(0.206)	0.761	0.867	0.772	(0.159)
		0.4	0.488	0.839	0.602	(0.198)	0.629	0.892	0.684	(0.172)	0.738	0.928	0.761	(0.140)
		0.6	0.421	0.910	0.594	(0.148)	0.588	0.945	0.673	(0.141)	0.715	0.966	0.752	(0.122)
		0.8	0.356	0.955	0.586	(0.106)	0.547	0.975	0.663	(0.112)	0.692	0.986	0.742	(0.105)
		1	0.296	0.980	0.579	(0.073)	0.506	0.990	0.654	(0.087)	0.669	0.995	0.733	(0.089)
5	5	0	0.558	0.558	0.558	(0.250)	0.651	0.651	0.651	(0.130)	0.735	0.735	0.735	(0.052)
		0.2	0.426	0.670	0.551	(0.188)	0.568	0.758	0.636	(0.103)	0.688	0.828	0.720	(0.044)
		0.4	0.303	0.770	0.545	(0.132)	0.484	0.846	0.623	(0.079)	0.639	0.900	0.705	(0.036)
		0.6	0.200	0.851	0.539	(0.086)	0.402	0.912	0.612	(0.057)	0.589	0.948	0.692	(0.029)
		0.8	0.121	0.911	0.535	(0.051)	0.326	0.954	0.602	(0.039)	0.539	0.976	0.680	(0.022)
		1	0.068	0.951	0.532	(0.028)	0.256	0.979	0.593	(0.025)	0.488	0.990	0.669	(0.017)
5	10	0	0.538	0.538	0.538	(0.228)	0.626	0.626	0.626	(0.089)	0.710	0.710	0.710	(0.020)
		0.2	0.333	0.633	0.533	(0.163)	0.489	0.722	0.610	(0.067)	0.629	0.800	0.691	(0.016)
		0.4	0.169	0.721	0.528	(0.106)	0.355	0.807	0.597	(0.047)	0.544	0.873	0.674	(0.012)
		0.6	0.069	0.798	0.524	(0.062)	0.238	0.875	0.586	(0.031)	0.458	0.927	0.660	(0.009)
		0.8	0.022	0.861	0.521	(0.033)	0.147	0.926	0.577	(0.018)	0.376	0.962	0.646	(0.006)
		1	0.006	0.909	0.519	(0.015)	0.083	0.959	0.570	(0.010)	0.299	0.983	0.635	(0.004)
10	2	0	0.639	0.639	0.639	(0.328)	0.715	0.715	0.715	(0.251)	0.782	0.782	0.782	(0.185)
		0.2	0.585	0.807	0.627	(0.261)	0.683	0.859	0.702	(0.213)	0.763	0.899	0.771	(0.165)
		0.4	0.530	0.919	0.616	(0.199)	0.651	0.946	0.690	(0.177)	0.745	0.964	0.761	(0.145)
		0.6	0.476	0.974	0.606	(0.144)	0.618	0.985	0.679	(0.143)	0.726	0.990	0.752	(0.126)

Table 2 continued

n_1	n_2	Δ	$\sigma_2^2=1$				$\sigma_2^2=2$				$\sigma_2^2=4$			
			a	b	c	d	a	b	c	d	a	b	c	d
		0.8	0.423	0.994	0.597	(0.099)	0.586	0.997	0.669	(0.112)	0.707	0.998	0.743	(0.108)
		1	0.372	0.999	0.589	(0.065)	0.553	0.999	0.659	(0.086)	0.688	1.000	0.734	(0.091)
10	5	0	0.562	0.562	0.562	(0.272)	0.652	0.652	0.652	(0.132)	0.734	0.734	0.734	(0.049)
		0.2	0.452	0.740	0.553	(0.195)	0.584	0.813	0.636	(0.102)	0.695	0.869	0.718	(0.041)
		0.4	0.346	0.873	0.546	(0.127)	0.515	0.920	0.622	(0.075)	0.655	0.950	0.703	(0.033)
		0.6	0.253	0.949	0.540	(0.074)	0.448	0.973	0.610	(0.052)	0.614	0.985	0.690	(0.026)
		0.8	0.175	0.984	0.535	(0.039)	0.383	0.993	0.599	(0.034)	0.573	0.997	0.677	(0.019)
		1	0.114	0.996	0.531	(0.018)	0.323	0.999	0.591	(0.021)	0.532	0.999	0.666	(0.014)
10	10	0	0.541	0.541	0.541	(0.250)	0.634	0.634	0.634	(0.079)	0.722	0.722	0.722	(0.013)
		0.2	0.359	0.702	0.533	(0.164)	0.517	0.787	0.616	(0.057)	0.654	0.854	0.701	(0.010)
		0.4	0.206	0.832	0.528	(0.093)	0.401	0.897	0.600	(0.037)	0.583	0.938	0.682	(0.007)
		0.6	0.101	0.919	0.524	(0.045)	0.296	0.960	0.588	(0.022)	0.512	0.980	0.666	(0.005)
		0.8	0.042	0.967	0.520	(0.018)	0.206	0.987	0.578	(0.011)	0.442	0.995	0.651	(0.003)
		1	0.014	0.989	0.518	(0.006)	0.136	0.997	0.569	(0.005)	0.375	0.999	0.638	(0.002)

$a:\ \mathrm{MPN}_{\mu_1}(\hat{\mu}_1^{\mathrm{CS}}, \hat{\mu}_1^{\mathrm{OS}}),\ b:\ \mathrm{MPN}_{\mu_2}(\hat{\mu}_2^{\mathrm{CS}}, \hat{\mu}_2^{\mathrm{OS}}),\ c:\ \mathrm{MPN}_{\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}^{\mathrm{CS}}, \hat{\boldsymbol{\mu}}^{\mathrm{OS}}),\ d:\ P(\bar{X}_1 > \bar{X}_2, s_1^2 > s_2^2)$

From Table 2, we see that $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1^{\mathrm{CS}}, \hat{\mu}_1^{\mathrm{OS}}) > 1/2$ when $\Delta = 0$ and $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1^{\mathrm{CS}}, \hat{\mu}_1^{\mathrm{OS}})$ gets larger as σ_2^2 increases. However, $\mathrm{MPN}_{\mu_1}(\hat{\mu}_1^{\mathrm{CS}}, \hat{\mu}_1^{\mathrm{OS}})$ gets smaller fast as Δ increases. We see that $\mathrm{MPN}_{\mu_2}(\hat{\mu}_2^{\mathrm{CS}}, \hat{\mu}_2^{\mathrm{OS}})$ gets larger as Δ increases or σ_2^2 increases. We also see that $\mathrm{MPN}_{\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}^{\mathrm{CS}}, \hat{\boldsymbol{\mu}}^{\mathrm{OS}})$ gets larger as σ_2^2 increases, but gets smaller as Δ increases.

6 Conclusion

In this paper, we have dealt with the problem of estimating two ordered normal means under modified Pitman nearness criterion when the order restriction on variances is present and not present. For the case where the order restriction on variances is not present, it is shown that the most critical case for $\hat{\mu}_i(\gamma)$ to improve upon $\bar{X}_i$ is the one when $\mu_1 = \mu_2$ and that the problem of improving upon $\bar{X}_i$ reduces to the one of a common mean. This result is similar to the one given in Theorems 2.1 and 2.2 of Oono and Shinozaki (2005) when MSE is the criterion. For the case where the order restriction on variances, $\sigma_1^2 \le \sigma_2^2$, is present, it is shown that we can improve upon $\hat{\mu}_2(\gamma)$ by $\hat{\mu}_2(\gamma^+)$ by suitably choosing γ^+. Although such an improvement is not possible for $\hat{\mu}_1(\gamma)$, it is shown that $(\hat{\mu}_1(\gamma^+),\ \hat{\mu}_2(\gamma^+))$ improves upon $(\hat{\mu}_1(\gamma),\ \hat{\mu}_2(\gamma))$ for suitably chosen γ^+ in the simultaneous estimation of μ_1 and μ_2. These results are similar to those given in Theorems 3.1, 3.3 and 4.1 of Chang et al. (2012) when MSE or stochastic dominance is concerned. Thus, we have confirmed that similar results are obtained under modified Pitman nearness criterion to those obtained under MSE or stochastic dominance criterion. Although we have not succeeded in giving an illustrative example, one possible inconsistency among the criterions is suggested in Remark 5 which may occur due to nontransitivity of the Pitman nearness criterion.

7 Appendix A: A proof of Lemma 1

Put $k = d/(2c)$, then $ck\Delta - d\Delta = -d\Delta/2$. For any $0 < \epsilon < 1/2$, we choose sufficiently large $\Delta > 0$ so that $\Phi(-d\Delta/2) < 1/2 - \epsilon$ and

$$\int_{k\Delta}^{\infty} g(y)\mathrm{d}y < \epsilon \int_0^{k\Delta} g(y)\mathrm{d}y.$$

To see that this is possible, we only need to show that

$$\lim_{\Delta\to\infty} \frac{\int_{k\Delta}^{\infty} g(y)\mathrm{d}y}{\int_0^{k\Delta} g(y)\mathrm{d}y} = 0,$$

which we can verify by applying L'Hospital's rule. Therefore, for sufficiently large $\Delta > 0$, we have

$$\int_0^{\infty} \Phi(cy - d\Delta)g(y)\mathrm{d}y = \int_0^{k\Delta} \Phi(cy - d\Delta)g(y)\mathrm{d}y + \int_{k\Delta}^{\infty} \Phi(cy - d\Delta)g(y)\mathrm{d}y$$

$$
\begin{aligned}
&< \int_0^{k\Delta} \Phi(-d\Delta/2) g(y)\mathrm{d}y + \int_{k\Delta}^{\infty} g(y)\mathrm{d}y \\
&< (1/2-\epsilon)\int_0^{k\Delta} g(y)\mathrm{d}y + \epsilon \int_0^{k\Delta} g(y)\mathrm{d}y \\
&= 1/2\int_0^{k\Delta} g(y)\mathrm{d}y < 1/2\int_0^{\infty} g(y)\mathrm{d}y.
\end{aligned}
$$

This completes the proof. □

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