The sinh-arcsinhed logistic family of distributions: properties and inference
Supplementary Material

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Abstract In Section SM1 we provide a proof of the modality of the sinh-arcsinhed logistic density (9). In Section SM2, we present results for the moments of its skew-logistic ($\delta = 1$) and symmetric ($\varepsilon = 0$) subfamilies.

Keywords Modality · Moments

SM1: Proof of the modality of the SAS-logistic density

The logarithm of density (9) is

$$\ell = \log f(x) = \log S'_{\varepsilon,\delta}(x) - 2\log\{\cosh(\tfrac{1}{2}S_{\varepsilon,\delta}(x))\} - \log 4,$$

with first partial derivative

$$\begin{aligned}
\frac{\partial \ell}{\partial x} &= \frac{S''_{\varepsilon,\delta}(x)}{S'_{\varepsilon,\delta}(x)} - \tanh(\tfrac{1}{2}S_{\varepsilon,\delta}(x))S'_{\varepsilon,\delta}(x) \\
&= \frac{1}{\sqrt{x^2+1}}\left(\delta\frac{S_{\varepsilon,\delta}(x)}{C_{\varepsilon,\delta}(x)} - \frac{x}{\sqrt{x^2+1}}\right) - \tanh(\tfrac{1}{2}S_{\varepsilon,\delta}(x))\frac{\delta C_{\varepsilon,\delta}(x)}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}\left(\frac{\delta}{\sqrt{1+S^2_{\varepsilon,\delta}(x)}}\left[S_{\varepsilon,\delta}(x) - \tanh(\tfrac{1}{2}S_{\varepsilon,\delta}(x))\{1+S^2_{\varepsilon,\delta}(x)\}\right] - \frac{x}{\sqrt{x^2+1}}\right).
\end{aligned}$$

In order to identify the number of solutions to $\partial\ell/\partial x = 0$, set $u = -\varepsilon + \delta\sinh^{-1}(x)$. Then

$$\begin{aligned}
\sqrt{x^2+1}\frac{\partial \ell}{\partial x} &= \frac{\delta}{\sqrt{1+S^2_{\varepsilon,\delta}(x)}}\left[S_{\varepsilon,\delta}(x) - \tanh(\tfrac{1}{2}S_{\varepsilon,\delta}(x))\{1+S^2_{\varepsilon,\delta}(x)\}\right] - \frac{x}{\sqrt{x^2+1}} \\
&= \delta\{\tanh u - \tanh(\tfrac{1}{2}\sinh u)\cosh u\} - \tanh((u+\varepsilon)/\delta) = 0.
\end{aligned}$$

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Defining the odd functions $h(u)$ and $g_\delta(u)$ as

$$h(u) = \tanh u - \tanh(\tfrac{1}{2}\sinh u)\cosh u \quad \text{and} \quad g_\delta(u) = -u + \delta \tanh^{-1}(\delta h(u)),$$

the modality of the SAS-logistic density is determined by the number of solutions to the equation $\varepsilon = g_\delta(u)$.

If $\delta \leq \sqrt{2}$ then, since the second derivative

$$h''(u) = \tfrac{1}{2}\big[-4\mathrm{sech}^2 u \tanh u - \{3\sinh u + \sinh(\sinh u)\}\,\mathrm{sech}^2(\tfrac{1}{2}\sinh u)\cosh u \\ + \cosh^3 u \tanh(\tfrac{1}{2}\sinh u)\mathrm{sech}^2(\tfrac{1}{2}\sinh u)\big]$$

is negative for all $u > 0$, the odd function $h(u)$ is a convex function for all $u > 0$. Noting that $\tanh(x) \geq x$ for $|x| \leq \pi/2$, it holds that

$$\delta h(u) \leq \delta h'(0)u = \delta u/2 \leq \tanh(\delta u/2).$$

Thus,

$$g_\delta(u) = -u + \delta \tanh^{-1}\{\delta h(u)\} \leq (-1 + \delta^2/2)u \leq 0$$

for all $u > 0$, and therefore the density is always unimodal regardless of the value of ε.

If $\delta \geq \delta^\dagger$ there exists a unique solution of the equation $\varepsilon = g_\delta(u)$ for $u > 0$, and therefore the density is always bimodal.

Finally, consider the case when $\sqrt{2} < \delta < \delta^\dagger$. As the second derivative of $h(u)$ is negative, $h'(0) = 1/2$ and $h'(u) \to -\infty$ as u tends to infinity, there exists a unique u_0 such that $h'(u_0) = 0$. Let $u^\dagger$ be the maximum argument of h for which $h(u) = 0$. To four decimal places, $u^\dagger = \max\{h^{-1}(0)\} = \max\{-0.9459, 0, 0.9459\} = 0.9459$. Then there exists a unique maximum of $h(u)$ on the interval $[0, u^\dagger]$ and its value is, to four decimal places,

$$\max_{0<u<u^\dagger} h(u) = 0.1758.$$

Therefore, if $\sqrt{2} < \delta < \delta^\dagger$, there exists a unique $v_0(> 0)$ for which $g_\delta(v_0) = 0$. Hence, the density is unimodal if there is no intersection of $y = \varepsilon$ and $y = g_\delta(u)$. Otherwise, the density is bimodal.

SM2 Moments for two subfamilies

Consider first the moments of the skew-logistic subfamily with $\delta = 1$.

$$\begin{aligned} E[X_{\varepsilon,1}^k] &= \int_{-\infty}^{\infty} x^k S'_{\varepsilon,1}(x) f_L(S_{\varepsilon,1}(x))dx \\ &= \int_{-\infty}^{\infty} \{y\cosh\varepsilon + (1+y^2)^{1/2}\sinh\varepsilon\}^k f_L(y)dy \\ &= \sum_{m=0}^{k} \binom{k}{m} \cosh^{k-m}(\varepsilon)\sinh^{m/2}(\varepsilon) \int_{-\infty}^{\infty} y^{k-m}(1+y^2)^{m/2} f_L(y)dy, \end{aligned}$$

where $y = S_{\varepsilon,1}(x)$. Using Equation (10) in Erdélyi (1954), for integer values of ν,

$$\begin{aligned}\int_0^\infty (1+t^2)^{\nu-1/2} f_L(t)dt &= \int_0^\infty (1+t^2)^{\nu-1/2} \frac{e^{-t}}{(1+e^{-t})^2} dt \\ &= \sum_{k=1}^\infty (-1)^{k-1} k \int_0^\infty (1+t^2)^{\nu-1/2} e^{-kt} dt \\ &= \sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right) 2^{\nu-1} \sum_{k=1}^\infty (-1)^{k-1} k^{1-\nu} (H_\nu(k) - Y_\nu(k)),\end{aligned}$$

where $Y_\nu(z) = \csc(\pi\nu)\{\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)\}$ and $J_\nu(z)$ denote the Bessel functions of the second and first kind, respectively, and

$$H_\nu(z) = \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{2m+\nu+1}}{\Gamma(m+3/2)\,\Gamma(\nu+m+3/2)}$$

is the Struve function. Thus, the first four moments of $X_{\varepsilon,1}$ are given by

$$\begin{aligned}E[X_{\varepsilon,1}] &= \int_{-\infty}^\infty \{y\cosh\varepsilon + (1+y^2)^{1/2}\sinh\varepsilon\} f_L(y)dy \\ &= \sinh\varepsilon \int_{-\infty}^\infty (1+y^2)^{1/2} f_L(y)dy \\ &= \pi\sinh\varepsilon \sum_{k=1}^\infty (-1)^{k-1}(H_1(k) - Y_1(k)), \\ &= 0.5816\pi\sinh\varepsilon,\end{aligned}$$

$$\begin{aligned}E[X_{\varepsilon,1}^2] &= \int_{-\infty}^\infty \{y\cosh\varepsilon + (1+y^2)^{1/2}\sinh\varepsilon\}^2 f_L(y)dy \\ &= \int_{-\infty}^\infty \{y^2\cosh^2\varepsilon + (1+y^2)\sinh^2\varepsilon\} f_L(y)dy \\ &= 2\pi^2|B_2|(\cosh^2\varepsilon + \sinh^2\varepsilon) + \sinh^2\varepsilon \\ &= \left(1+\frac{2\pi^2}{3}\right)\cosh^2\varepsilon - \left(1+\frac{\pi^2}{3}\right),\end{aligned}$$

$$\begin{aligned}E[X_{\varepsilon,1}^3] &= \int_{-\infty}^\infty \{y\cosh\varepsilon + (1+y^2)^{1/2}\sinh\varepsilon\}^3 f_L(y)dy \\ &= \sinh\varepsilon \int_{-\infty}^\infty \{3y^2(1+y^2)^{1/2}\cosh^2\varepsilon + (1+y^2)^{3/2}\sinh^2\varepsilon\} f_L(y)dy \\ &= \sinh\varepsilon \int_{-\infty}^\infty \{(1+y^2)^{3/2}(3\cosh^2\varepsilon + \sinh^2\varepsilon) - 3(1+y^2)^{1/2}\cosh^2\varepsilon\} f_L(y)dy \\ &= 3\pi\sinh\varepsilon \sum_{k=1}^\infty (-1)^{k-1}\left[\frac{1}{k}(3\cosh^2\varepsilon + \sinh^2\varepsilon)(H_2(k) - Y_2(k)) - \cosh^2\varepsilon(H_1(k) - Y_1(k))\right] \\ &= 3\pi\sinh\varepsilon\{1.4081(3\cosh^2\varepsilon + \sinh^2\varepsilon) - 0.5816\cosh^2\varepsilon\} \\ &= 3\pi\sinh\varepsilon\{5.0509\cosh^2\varepsilon - 1.4081\},\end{aligned}$$

$$
\begin{aligned}
E[X_{\varepsilon,1}^4] &= \int_{-\infty}^{\infty} \{y\cosh\varepsilon + (1+y^2)^{1/2}\sinh\varepsilon\}^4 f_L(y)dy \\
&= \int_{-\infty}^{\infty} \{y^4\cosh^4\varepsilon + 6y^2(1+y^2)\cosh^2\varepsilon\sinh^2\varepsilon + (1+y^2)^2\sinh^4\varepsilon\} f_L(y)dy \\
&= \int_{-\infty}^{\infty} \{y^4\cosh^4\varepsilon + 6y^2(1+y^2)\cosh^2\varepsilon\sinh^2\varepsilon + (1+y^2)^2\sinh^4\varepsilon\} f_L(y)dy \\
&= \int_{-\infty}^{\infty} \{(1+8\cosh^2\varepsilon\sinh^2\varepsilon)y^4 + 2\sinh^2\varepsilon(3\cosh^2\varepsilon+\sinh^2\varepsilon)y^2\} f_L(y)dy + \sinh^4\varepsilon \\
&= 14\pi^4|B_4|(1+8\cosh^2\varepsilon\sinh^2\varepsilon) + 4\pi^2|B_2|\sinh^2\varepsilon(3\cosh^2\varepsilon+\sinh^2\varepsilon) + \sinh^4\varepsilon \\
&= \left(1+\frac{2\pi^2}{3}+\frac{7\pi^4}{15}\right) - \left(2+\frac{10\pi^2}{3}+\frac{56\pi^4}{15}\right)\cosh^2\varepsilon + \left(1+\frac{8\pi^2}{3}+\frac{56\pi^4}{15}\right)\cosh^4\varepsilon,
\end{aligned}
$$

where $B_m, m = 0, 1, ...,$ denote the Bernoulli numbers and the numerical values in the first and third moments are quoted to four decimal places. Note how those two moments are multiples of the median, $\sinh(\varepsilon)$, of $X_{\varepsilon,1}$.

Now consider the symmetric subfamily obtained when $\varepsilon = 0$. The odd moments are all 0. The even moments can be expressed as

$$
\begin{aligned}
E[X^{2n}] &= \int_{-\infty}^{\infty} x^{2n} S'_{0,\delta}(x) f_L(S_{0,\delta}(x))dx \\
&= \frac{1+(-1)^{2n}}{2^{2n}} \int_0^{\infty} [\{(y^2+1)^{1/2}+y\}^{1/\delta} - (\{(y^2+1)^{1/2}-y\}^{1/\delta}]^{2n} f_L(y)dy \\
&= \frac{1}{2^{2n-1}} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_0^{\infty} \{(y^2+1)^{1/2}+y\}^{2(n-k)/\delta} f_L(y)dy,
\end{aligned}
$$

where the integral in the last line must be computed numerically.

References

Erdélyi, A. (Ed.) (1954). *Tables of integral transforms.* Vol. 1. New York: McGraw-Hill.