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Probabilistic properties of second order branching process

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Abstract The classical BGW process assumes first order dependence, whereas many real life datasets exhibit a second or higher order dependence. Further, in some situations, there is a need for a model which allows for simultaneous reproduction by a parent and its offspring. This paper proposes a second order branching process model to accommodate such situations and discusses its probabilistic properties such as extinction probability and limiting behaviour of the generation sizes. Estimation of offspring means and growth rate are also discussed. This model is further used to model the swine flu data for Pune, India, and La-Gloria, Mexico.

Keywords Almost sure convergence · Extinction probability · Generating functions · Higher order branching processes · L^2 convergence

1 Introduction

Branching processes have conventionally been used to model the spread of an infectious disease. The individuals who get infected from a person are supposed to be the offspring of that person. The classical Bienayme–Galton–Watson (BGW) model assumes that an individual can reproduce only once during its lifetime and then dies. This assumption is violated if the period of infectiousness is more than one time unit (see, for example, Sparks et al. 2010). This phenomenon is observed in the Swine Flu data, consisting of number of cases tested positive, on each day in Pune city, in India

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in 2009. When we model this dataset using time series approach, AR(2) model gives a better fit than AR(1) model, as discussed in Kanade and Rajarshi (2010).

This observation motivated us to propose a second order branching process (SOBP) model to fit such type of data. In more general terms, we propose a model, where an individual can produce offspring more than once, and simultaneous reproduction by an individual and its offspring is allowed. It is to be noted that this feature is not included in an age-dependent branching process and hence a higher order branching process is different from the classical age-dependent branching process. Thus, in a pth order branching process, an individual at nth generation can be an offspring of an individual from any of the p previous generations. It will be a suitable model in case of the spread of rumour and also in case of network marketing.

In this paper, to avoid complexity of notations, we restrict to SOBP. In Sect. 2, we define SOBP and study its probabilistic properties. Section 3 discusses the long term behaviour of the process in the supercritical case, Sect. 4 briefly reports the estimation of parameters of SOBP and Sect. 5 presents the simulation and data analysis results.

2 Second order branching process model

In SOBP, we assume that an individual reproduces at age 1 and also at age 2 and then dies. Hence, if Z_n is the number of offspring born at nth generation and $Z_0 = 1$, i.e. the process starts with one ancestor, we define,

$$Z_1 = \xi_1(0)\ , \quad Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i(n-1) + \sum_{j=1}^{Z_{n-2}} \psi_j(n-2)\ ; \qquad \forall\, n \geq 2\ , \tag{1}$$

where, $\xi_i(n-1)$ is a random variable indicating the number of offspring produced at time n by ith individual in $(n-1)$st generation and $\psi_j(n-2)$ is a random variable indicating the number of offspring produced at time n by jth individual in $(n-2)$nd generation. We assume that $\{\xi_i(n-1)\}$ and $\{\psi_j(n-2)\}$ are two independent sequences, each being a sequence of nonnegative integer valued independent and identically distributed random variables with $P[\xi_i(n-1) = r] = p_r$, for $r = 0, 1, 2, \ldots, \ \forall\, i, n$ and $P[\psi_j(n-2) = s] = q_s$, for $s = 0, 1, 2, \ldots, \ \forall\, j, n$. The process $\{Z_n, n \geq 0\}$ is then defined as SOBP. By definition of Z_n in Eq. (1), it follows that the process is a second order Markov chain with state space $\{0, 1, 2, \ldots\}$. If $V_n = (Z_n, Z_{n+1})$, $\{V_n, n \geq 1\}$ forms a first order Markov chain with (0,0) as an absorbing state. Furthermore, for $p_0 > 0, q_0 > 0$, $P[V_{n+2} = (0,0)|V_n = (i, j)] = q_0^i p_0^j q_0^j > 0$, implying that the states other than $(0, 0)$ are transient. Thus, SOBP is not ergodic. Suppose that μ_1, μ_2 are the means and σ_1^2, σ_2^2 are the variances of $\xi_i(n-1)$ and $\psi_j(n-2)$ respectively. We may view this process as a two-type process $\{(Z_n, Z_{n-1}), n \geq 1\}$. Type I individuals are the individuals of age 0 and type II individuals are those of age 1. Each type I individual produces exactly one offspring of type II and on an average μ_1 offspring of type I and each type II individual produces on an average μ_2 offspring of type I and no offspring of type II. The mean matrix (Athreya and Ney 1972, p.184) of the two-type process is given by

$$\begin{bmatrix} \mu_1 & 1 \\ \mu_2 & 0 \end{bmatrix}.$$

Probabilistic properties of SOBP can be obtained from the properties of a two-type branching process. But, we have adopted a more direct approach as is evident from the following sections.

2.1 Generating function and probability of extinction

Suppose that $P(s)$ and $Q(s)$ are the probability generating functions (pgfs) and $h_\xi(t)$ and $h_\psi(t)$ denote the cumulant generating functions (cgfs) of $\xi_i(n-1)$ and $\psi_j(n-2)$ respectively, with $p_0 \neq 0$, $q_0 \neq 0$. To ensure strict convexity of the function $P(s)Q(s)$, we need any one of the following three conditions to hold:

$$\text{(i) } p_0 + p_1 < 1 \text{ (ii) } q_0 + q_1 < 1 \text{ (iii) } p_1 q_1 > 0\,. \tag{2}$$

The following theorem gives the expression for the pgf (cgf) of Z_n, in terms of the pgfs (cgfs) of the two offspring distributions.

Theorem 1 *The pgf $R_n(s)$ of Z_n is given by $R_n(s) = P[R_{n-1}(s)]Q[R_{n-2}(s)]$. Further, if $g_n(t)$ is the cgf of Z_n, then, $g_n(t) = h_\xi(g_{n-1}(t)) + h_\psi(g_{n-2}(t))$.*

Proof It is easy to see that, $R_0(s) = s,\ R_1(s) = P(s)$,

$R_2(s) = P(R_1(s))Q(R_0(s))$. Similarly, $Z_3 = \sum_{i=1}^{Z_2} \xi_i(2) + \sum_{j=1}^{Z_1} \psi_j(1)$ yields

$R_3(s) = E\left[(Q(s))^{Z_1} E\left\{(P(s))^{Z_2} | Z_1, Z_0\right\}\right] = Q[R_1(s)]P\left[R_2(s)\right]$. Thus, in general,

$$R_n(s) = P\left[R_{n-1}(s)\right] Q\left[R_{n-2}(s)\right]\ . \tag{3}$$

Putting $s = e^{-t}$, in (3), we get,

$$E(e^{-tZ_n}) = E\left[\left\{E\left(e^{-tZ_{n-1}}\right)\right\}^{\xi_1(0)}\right] E\left[\left\{E\left(e^{-tZ_{n-2}}\right)\right\}^{\psi_1(0)}\right].$$

From the definition of the cgf, $E(e^{-tZ_{n-1}}) = e^{-g_{n-1}(t)}$. Therefore, we have, $E(e^{-tZ_n}) = E[e^{-g_{n-1}(t)\xi_1(0)}]E[e^{-g_{n-2}(t)\psi_1(0)}]$. Taking log on both the sides of the above equation and multiplying by -1, we get,

$$g_n(t) = h_\xi\left(g_{n-1}(t)\right) + h_\psi\left(g_{n-2}(t)\right). \qquad \square$$

As discussed earlier, (0, 0) is the absorbing state. Therefore, a second order branching process will become extinct if and only if $Z_{n-1} = 0,\ Z_n = 0$ for some n.

Theorem 2 (i) *The probability of ultimate extinction is the smallest nonnegative solution of $P(s)Q(s) = s$. (ii) The extinction probability is 1 if $\mu_1 + \mu_2 \leq 1$, while it is less than 1, if $\mu_1 + \mu_2 > 1$.*

Proof As noted earlier, if the state of the SOBP is denoted by $V_n = (Z_n, Z_{n+1})$, then $\{V_n\}$ forms a first order Markov chain. Let $T_n(s_1, s_2)$ be the pgf of V_n. Then,

$$\begin{aligned}
T_0(s_1, s_2) &= E\left(s_1^{Z_0} s_2^{Z_1}\right) = s_1 E\left(s_2^{Z_1}\right) = s_1 P(s_2).\\
T_1(s_1, s_2) &= E\left(s_1^{Z_1} s_2^{Z_2}\right) = E\left[s_1^{Z_1} E\left(s_2^{Z_2} | \mathbb{F}_1\right)\right] = E\left[s_1^{Z_1} \left(P(s_2)\right)^{Z_1} Q(s_2)\right]\\
&= P\left[T_0(s_1, s_2)\right] Q(s_2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
T_2(s_1, s_2) &= E\left[s_1^{Z_2} s_2^{Z_3}\right] = E\left[s_1^{Z_2} E\left(s_2^{Z_3} | \mathbb{F}_2\right)\right]\\
&= E\left[s_1^{Z_2} \left(P(s_2)\right)^{Z_2} \left(Q(s_2)\right)^{Z_1}\right]\\
&= E\left[\left(T_0(s_1, s_2)\right)^{Z_2} \left(Q(s_2)\right)^{Z_1}\right]\\
&= E\left[\left(Q(s_2)\right)^{Z_1} \left(P[T_0(s_1, s_2)]\right)^{Z_1} Q[T_0(s_1, s_2)]\right]\\
&= P[T_1\,(s_1, s_2)] Q[T_0(s_1, s_2)]\,.
\end{aligned}$$

Thus, in general we have,

$$T_n(s_1, s_2) = P\left[T_{n-1}(s_1, s_2)\right] Q\left[T_{n-2}(s_1, s_2)\right]\,. \tag{4}$$

In the bivariate setup, extinction corresponds to the visit to state (0, 0). Let π_n be the probability that the process has become extinct at or before the n^{th} generation. Note that $P[Z_n = 0, Z_{n+1} = 0 | Z_{n-1} = j, Z_n = 0] = q_0^j$. Thus,

$$\begin{aligned}
\pi_n &= P\left[Z_n = 0, Z_{n-1} = 0\right]\\
&= \sum_{j=0}^{\infty} P\left[Z_n = 0, Z_{n+1} = 0 | Z_{n-1} = j, Z_n = 0\right] P(Z_{n-1} = j, Z_n = 0)\\
&= \sum_{j=0}^{\infty} q_0^j P\left(Z_{n-1} = j, Z_n = 0\right).
\end{aligned}\tag{5}$$

But, the second factor in the summation can be written as,

$$
\begin{aligned}
&P(Z_{n-1} = j, Z_n = 0) \\
&\quad = \sum_{i=0}^{\infty} P\left[Z_{n-1} = j, Z_n = 0 | Z_{n-2} = i, Z_{n-1} = j\right] P\left(Z_{n-2} = i, Z_{n-1} = j\right) \\
&\quad = \sum_{i=0}^{\infty} q_0^i p_0^j P\left(Z_{n-2} = i, Z_{n-1} = j\right). \qquad (6)
\end{aligned}
$$

Substituting from (6) in (5), we get,

$$
\pi_n = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} q_0^i (p_0 q_0)^j P\left(Z_{n-2} = i, Z_{n-1} = j\right) = T_{n-2}(q_0, p_0 q_0). \qquad (7)
$$

Substituting from (4) in (7), we get,

$$
\begin{aligned}
\pi_n &= T_{n-2}(q_0, p_0 q_0) = P\left[T_{n-3}(q_0, p_0 q_0)\right] Q\left[T_{n-4}(q_0, p_0 q_0)\right] \\
&= P\left[\pi_{n-1}\right] Q\left[\pi_{n-2}\right].
\end{aligned}
$$

The sequence π_n is nondecreasing and bounded above by 1. Hence, its limit π exists. Since $P(s)$ and $Q(s)$ are continuous, taking limit on the both the sides of the above equation, we get that the extinction probability π is the solution to the equation $P(\pi)Q(\pi) = \pi$. In view of conditions in (2), we get $P(s)Q(s)$ to be strictly increasing and strictly convex. Thus, proceeding on the similar lines as in the case of first order branching process (FOBP) (cf. Guttorp 1991, p.8), we get extinction probability to be the smallest nonnegative solution of the equation $P(s)Q(s) = s$. Note that, the derivative of $P(s)Q(s)$, evaluated at $s = 1$ is $\mu_1 + \mu_2$. Thus, the extinction probability is 1 if $\mu_1 + \mu_2 \leq 1$ and it is less than one, when $\mu_1 + \mu_2 > 1$. This completes the proof. □

Consequently, as in FOBP, we classify SOBP in three types as follows:

(i) subcritical case: $\mu_1 + \mu_2 < 1$,
(ii) critical case: $\mu_1 + \mu_2 = 1$ and
(iii) supercritical case: $\mu_1 + \mu_2 > 1$.

3 Long run behaviour

3.1 Stability of the process in the supercritical case

To study the long run behaviour of a SOBP in the supercritical case, we first need to compute the moments of the generation sizes. The expected number of new births at any given time point is computed using Eq. (1) as,

$$
E[Z_n] = \mu_1 E[Z_{n-1}] + \mu_2 E[Z_{n-2}], \qquad \forall\, n \geq 2. \qquad (8)
$$

Using the theory of difference equations, we get,

$$E[Z_n] = \frac{a^{n+1} - b^{n+1}}{2^n(a-b)} = \sum_{j=0}^{n} \frac{a^{n-j}b^j}{2^n} = \frac{\left(\frac{a}{2}\right)^n}{1-\left(\frac{b}{a}\right)}\left[1-\left(\frac{b}{a}\right)^{n+1}\right],$$

where $a = \mu_1 + \sqrt{\mu_1^2 + 4\mu_2}$ and $b = \mu_1 - \sqrt{\mu_1^2 + 4\mu_2}$.

From Eq. (1), and the fact that the sequences $\{\xi_i(n-1)\}$ and $\{\psi_j(n-2)\}$ are independent, we get,

$$V(Z_1) = \sigma_1^2, \qquad V(Z_n|\mathbb{F}_{n-1}) = \sigma_1^2 Z_{n-1} + \sigma_2^2 Z_{n-2}, \qquad \forall\, n \geq 2\,. \tag{9}$$

From Eqs. (8) and (9),

$$V[Z_n] = \sigma_1^2 E[Z_{n-1}] + \sigma_2^2 E[Z_{n-2}] + \mu_1^2 V[Z_{n-1}] + \mu_2^2 V[Z_{n-2}] + 2\mu_1\mu_2 Cov(Z_{n-1}, Z_{n-2})\,.$$

To compute variances without computing covariances, we use, $R_n''(1)$ obtained from Theorem 1, in $V(Z_n) = R_n''(1) + E(Z_n) - [E(Z_n)]^2$, to get,

$$V[Z_{n+1}] = \sigma_1^2[E(Z_n)]^2 + \sigma_2^2[E(Z_{n-1})]^2 + \mu_1 V(Z_n) + \mu_2 V(Z_{n-1})\,.$$

Solving this difference equation by induction, we get the variance of Z_n as,

$$V(Z_n) = \sigma_1^2\left[E(Z_{n-1}) + \sum_{i=0}^{n-2} E(Z_{n-2-i})(E(Z_{i+1}))^2\right] + \sigma_2^2\left[\sum_{i=0}^{n-2} E(Z_{n-2-i})(E(Z_i))^2\right].$$

Thus the moments of the generation sizes are exactly same as those in case of second order branching process with continuous state space. Hence, if we define a new sequence $W_n = Z_n/E(Z_n)$, using the results in Kashikar and Deshmukh (2012), we obtain the following result.

Theorem 3 *In the supercritical SOBP, there exists a random variable W, such that $W_n \to W$, almost surely as well as in quadratic mean and if $\phi(s)$ and $\Lambda(s)$ are the cgf and pgf of W respectively, then, with $\rho = a/2$, we have*

$$\phi\left(\rho^2 s\right) = h_\xi(\phi(\rho s)) + h_\psi(\phi(s)) \quad \textit{and}$$
$$\Lambda\left(s^{\rho^2}\right) = P\left(s^\rho\right) Q(s).$$

Further, the sequence $\{Z_n/\rho^n\}$ converges almost surely to a random variable $W' = \frac{a}{a-b} W$.

Also, using the formula for ρ, we have,

$$\frac{\mu_1}{\rho} + \frac{\mu_2}{\rho^2} = 1. \tag{10}$$

3.2 Extinction: explosion theorem

For proving the extinction–explosion theorem, we need the following properties of pgf. Let X be any random variable and $P(s)$ be its pgf.

Property 1 $\frac{\log P(t)}{\log t}$ *is increasing in* t.

Proof Let $h(s)$ be the cgf of X. We know that $h(s)/s$ is decreasing in s. Thus, $\frac{-\log E(e^{-sX})}{s}$ is decreasing in s. Let $e^{-s} = t$. Thus, $0 < t < 1$ and $s = -\log t$. Further, we get,

$$\frac{h(s)}{s} = \frac{-\log E(t^X)}{-\log t} = \frac{\log P(t)}{\log t}.$$

Let s_1 and s_2 be two real numbers and $t_1 = e^{-s_1}$ and $t_2 = e^{-s_2}$. $s_1 < s_2$ implies $t_1 > t_2$. Using this and the above mentioned property of cgf, we get that $\log P(t)/\log t$ is an increasing function of t. □

Property 2 $\lim_{t\to 0} \frac{\log P(t)}{\log t} = a$ *and* $\lim_{t\to 1} \frac{\log P(t)}{\log t} = \mu$, *where* a *is the first point of increase of distribution of* X *and* μ *is the mean of* X.

Proof Using the same substitution as in Property 1, we get

$$\lim_{t\to 0} \frac{\log P(t)}{\log t} = \lim_{s\to\infty} \frac{h(s)}{s} = a.$$

By L'Hospital's rule,

$$\lim_{t\to 1} \frac{\log P(t)}{\log t} = \lim_{t\to 1} \frac{tP'(t)}{P(t)} = \mu.$$ □

Property 3 $s^{\mu} \le P(s) \le s^{a}$.

Proof Combining Properties 1 and 2, we get,

$$a \le \frac{\log P(t)}{\log t} \le \mu.$$

Since t is between 0 and 1, this gives,

$$a \log t \ge \log P(t) \ge \mu \log t,$$

which gives the desired result. □

To study the limiting behaviour of the generation sizes, we define a new sequence, $Y_n = Z_n + f(s_0)Z_{n-1}$, $n \geq 1$ where s_0 is the smallest nonnegative solution to the equation $P(s)Q(s) = s$, with $Y_0 = 1$. We define, $f(s_0)$ as,

$$\begin{aligned} f(s_0) &= \frac{\log Q(s_0)}{\log s_0} & 0 < s_0 < 1 \\ &= \lim_{s \to s_0} \frac{\log Q(s)}{\log s} & s_0 = 1 \text{ or } s_0 = 0 \end{aligned} \tag{11}$$

Let $J_n(s)$ be the pgf of Y_n. Then, $J_0(s) = E(s^{Y_0}) = s$.

$$\begin{aligned} J_1(s) &= E\left(s^{Y_1}\right) = E\left(s^{(Z_1+f(s_0))}\right) = s^{f(s_0)} E\left(s^{Z_1}\right) = s^{f(s_0)} P(s), \\ J_2(s) &= E\left(s^{Y_2}\right) = E\left(s^{(Z_2+f(s_0)Z_1)}\right) = E\left[s^{f(s_0)Z_1} E\left\{s^{Z_2}|\mathbb{F}_1\right\}\right] \\ &= E\left[s^{f(s_0)Z_1}(P(s))^{Z_1} Q(s)\right] = E\left[\left(s^{f(s_0)} P(s)\right)^{Z_1} Q(s)\right] = Q(s)P(J_1(s)). \end{aligned}$$

Continuing similarly, we get,

$$J_n(s) = P\left(J_{n-1}(s)\right) Q\left(J_{n-2}(s)\right) . \tag{12}$$

To establish the extinction–explosion theorem, we study the limiting behaviour of the sequence $\{J_n(s)\}$ which is presented in the next theorem. Let a_1 and a_2 be the first points of increase of ξ_1 and ψ_1 respectively.

Theorem 4 $\lim_{n\to\infty} J_n(s) = \pi$ *for all* $s \in (0, 1)$. *Furthermore, either of the following three holds:*

(a) $P(s)Q(s) < s$ *for all* $s \in (0, 1)$, *and* $\lim_{n\to\infty} J_n(s) = 0$.
(b) *There exists an* $s_0 \in (0, 1)$ *such that* $P(s_0)Q(s_0) = s_0$ *and* $\lim_{n\to\infty} J_n(s) = s_0$.
(c) $P(s)Q(s) > s$ *for all* $s \in (0, 1)$, *and* $\lim_{n\to\infty} J_n(s) = 1$.

Proof Due to convexity of $P(s)Q(s)$ and the fact that $\lim_{s\to 1} P(s)Q(s) = 1$, we note that the three cases mentioned above are exclusive and exhaustive.
Case I: $P(s)Q(s) < s$ for all s in $(0, 1)$
By the convexity of $P(s)Q(s)$, the smallest solution to the equation $P(s)Q(s) = s$ is zero. Hence, $s_0 = 0$. Thus, using the definition of $f(s_0)$ and the Property 2, $f(s_0) = a_2$. In this case, the extinction probability is zero. Therefore, at least one of p_0 and q_0 is zero and hence at least one of the a_1 and a_2 is strictly greater than zero, i.e. greater than or equal to 1 as $\xi_1(1)$ and $\psi_1(1)$ are integer valued. Thus, $a_1 + a_2 \geq 1$.

$$\begin{aligned} J_1(s) &= s^{a_2} P(s) \leq s^{a_1+a_2} \leq s = J_0(s), \\ J_2(s) &= P[J_1(s)]Q(s) \leq P(s)s^{a_2} = J_1(s). \end{aligned}$$

Thus, by induction, we get the sequence $\{J_n(s)\}$ to be nonincreasing in n for every s and hence its limit $J(s)$ exists. From Eq. (12), using the continuity of pgfs, it follows that $J(s)$ satisfies the equation $P(s)Q(s) = s$. Hence, $J(s) = 0 = \pi,\ \forall\, s \in (0, 1)$.
Case II: There exists $s_0 \in (0, 1)$, such that $P(s_0)Q(s_0) = s_0$
Due to convexity of $P(s)Q(s)$, $P(s)Q(s) > s$ for $s < s_0$ and $P(s)Q(s) < s$ for $s > s_0$.

Case (a): $s < s_0$

$$\begin{aligned}
\log J_1(s) &= \log[s^{f(s_0)}P(s)] = f(s_0)\log s + \log P(s)\\
&= \frac{\log Q(s_0)}{\log s_0}\log s + \log P(s) = \left[1 - \frac{\log P(s_0)}{\log s_0}\right]\log s + \log P(s)\\
&\geq \left[1 - \frac{\log P(s)}{\log s}\right]\log s + \log P(s) = \log s = \log J_0(s),\\
\log J_2(s) &= \log P[J_1(s)] + \log Q(s) \geq \log P(s) + \frac{\log Q(s_0)}{\log s_0}\log s = \log J_1(s).
\end{aligned}$$

Case (b): $s > s_0$

$$\begin{aligned}
\log J_1(s) &= \log[s^{f(s_0)}P(s)] = f(s_0)\log s + \log P(s) = \frac{\log Q(s_0)}{\log s_0}\log s + \log P(s)\\
&= \left[1 - \frac{\log P(s_0)}{\log s_0}\right]\log s + \log P(s) \leq \left[1 - \frac{\log P(s)}{\log s}\right]\log s + \log P(s)\\
&= \log s = \log J_0(s),\\
\log J_2(s) &= \log P[J_1(s)] + \log Q(s) \leq \log P(s) + \frac{\log Q(s_0)}{\log s_0}\log s = \log J_1(s).
\end{aligned}$$

Using induction in both the cases, and the fact that the limit $J(s)$ satisfies Eq. (12), we get that $J_n(s)$ increases to s_0 for $s < s_0$ and it decreases to s_0 for $s > s_0$. In this case, using the basic branching property and the fact that $P(s_0)Q(s_0) = s_0$, we have,

$$\begin{aligned}
E\left[s_0^{Y_n}\middle|\mathbb{F}_{n-1}\right] &= E\left[s_0^{Z_n + f(s_0)Z_{n-1}}|\mathbb{F}_{n-1}\right]\\
&= s_0^{f(s_0)Z_{n-1}}(P(s_0))^{Z_{n-1}}(Q(s_0))^{Z_{n-2}}.
\end{aligned}$$

Taking log on the right hand side of the above equation, we get,

$$\begin{aligned}
&f(s_0)Z_{n-1}\log s_0 + Z_{n-1}\log P(s_0) + Z_{n-2}\log Q(s_0)\\
&\quad = Z_{n-1}\log Q(s_0) + Z_{n-1}\log P(s_0) + Z_{n-2}\log Q(s_0)\\
&\quad = Z_{n-1}\log s_0 + Z_{n-2}\log Q(s_0)\\
&\quad = Z_{n-1}\log s_0 + Z_{n-2}f(s_0)\log s_0\\
&\quad = \log s_0^{Y_{n-1}}.
\end{aligned}$$

Thus, $\{s_0^{Y_n}\}$ is a bounded martingale and hence it converges almost surely to some random variable $X_\infty(s_0) \in [0, 1]$ as $n \to \infty$, i.e. $Y_n \stackrel{a.s.}{\to} \frac{\log(X_\infty(s_0))}{\log s_0}$. Hence, using the convergence of the corresponding pgfs, we have,

$$E\left[s^{\left(\frac{\log X_\infty}{\log s_0}\right)}\right] = \lim_{n\to\infty} J_n(s) = s_0 \text{ if } s \in (0, \infty).$$

This gives,

$$P\left(\frac{\log X_\infty}{\log s_0} = 0\right) = 1 - P\left(\frac{\log X_\infty}{\log s_0} = \infty\right) = s_0$$

which yields, $\pi = P(X_\infty(s_0) = 1) = s_0$.
Case III: $P(s)Q(s) > s$ for all s in $(0, 1)$
In this case, the smallest solution to the equation $P(s)Q(s) = s$ is 1. Hence, $s_0 = 1$. Thus, we have $f(s_0) = \mu_2$ and hence, $Y_n = Z_n + \mu_2 Z_{n-1}$. In this case, the extinction probability is 1. Therefore, $\mu_1 + \mu_2 \leq 1$.

$$\begin{aligned}
J_1(s) &= s^{\mu_2} P(s) \geq s^{\mu_1+\mu_2} \geq s = J_0(s),\\
J_2(s) &= P[J_1(s)]Q(s) \geq P(s)s^{\mu_2} = J_1(s).
\end{aligned}$$

Thus, proceeding in the similar manner as in Case I, we get the limit of $J_n(s)$ to be 1. Furthermore, we have, $\pi = 1$ and hence, $\lim_{n\to\infty} J_n(s) = \pi$.

$$\begin{aligned}
E(Y_n|\mathbb{F}_{n-1}) &= E(Z_n + \mu_2 Z_{n-1}|\mathbb{F}_{n-1})\\
&= (\mu_1 + \mu_2)Z_{n-1} + \mu_2 Z_{n-2}\\
&\leq Z_{n-1} + \mu_2 Z_{n-2}\\
&= Y_{n-1}.
\end{aligned}$$

Thus, the sequence $\{Y_n\}$ forms a nonnegative supermartingale and hence it converges almost surely to Y as $n \to \infty$. But, we know that $\lim_{n\to\infty} J_n(s) = 1$. Thus, the continuity of pgf gives, $P(Y = 0) = 1$. This completes the proof. □

From Theorem 4, we know that $\lim_{n\to\infty} J_n(s) = \pi$ for all $s \in (0, \infty)$. By the continuity of pgfs, this means that $Y_n \stackrel{d}{\to} Y$, where Y is a random variable with distribution $P(Y = 0) = 1 - P(Y = \infty) = \pi$. The following extinction–explosion theorem extends this to the almost sure convergence.

Theorem 5 *Y_n converges almost surely to a random variable Y such that,*

$$P(Y = 0) = 1 - P(Y = \infty).$$

Proof When $\pi > 0$, i.e. in Case II and III of Theorem 4, we have already established that the convergence of Y_n to Y is almost sure. In Case I, we have, since $a_1 + a_2 \geq 1$,

$$Y_n = Z_n + a_2 Z_{n-1} \geq a_1 Z_{n-1} + a_2 Z_{n-2} + a_2 Z_{n-1} \geq Z_{n-1} + a_2 Z_{n-2} = Y_{n-1}.$$

Thus, the sequence $\{Y_n\}$ is monotone increasing and converges almost surely. But, using Theorem 4, we get that Y_n converges in distribution to a random variable Y such that $P[Y = \infty] = 1 - \pi = 1$. Thus, Y_n converges to ∞ almost surely. □

4 Estimation

In this section, we discuss estimation of the offspring means and the growth rate from the generation sizes $\{Z_0, Z_1, \dots, Z_n\}$ by various methods and study their limiting behaviour in the supercritical case, on the non-extinction path ($W > 0$).

4.1 Conditional least squares estimation

From Eq. (1), it follows that $E(Z_t|\mathbb{F}_{t-1}) = \mu_1 Z_{t-1} + \mu_2 Z_{t-2}$, where $\mathbb{F}_{t-1} = \sigma\{Z_0, Z_1, \dots, Z_{t-1}\}$. This representation suggests the conditional least squares (CLS) method given by Klimko and Nelson (1978) for the estimation of μ_1 and μ_2. It involves minimizing the error sum of squares, $S_n = \sum_{t=2}^{n}(Z_t - E(Z_t|\mathbb{F}_{t-1}))^2$. Thus, the CLS equations are,

$$\sum_{t=2}^{n}(Z_t - \mu_1 Z_{t-1} - \mu_2 Z_{t-2})Z_{t-i} = 0 \quad \text{for } i = 1, 2. \tag{13}$$

Solving these equations, we get,

$$\begin{aligned} \hat{\mu}_1 &= \frac{\left(\sum Z_{t-2}^2\right)\left(\sum Z_t Z_{t-1}\right) - \left(\sum Z_t Z_{t-2}\right)\left(\sum Z_{t-1} Z_{t-2}\right)}{\left(\sum Z_{t-2}^2\right)\left(\sum Z_{t-1}^2\right) - \left(\sum Z_{t-1} Z_{t-2}\right)^2}, \\ \hat{\mu}_2 &= \frac{\left(\sum Z_{t-1}^2\right)\left(\sum Z_t Z_{t-2}\right) - \left(\sum Z_t Z_{t-1}\right)\left(\sum Z_{t-1} Z_{t-2}\right)}{\left(\sum Z_{t-2}^2\right)\left(\sum Z_{t-1}^2\right) - \left(\sum Z_{t-1} Z_{t-2}\right)^2}. \end{aligned} \tag{14}$$

The matrix of second derivatives of S_n, when multiplied by a norming constant or random variables, converges to a singular matrix. Hence the conditions required for the strong consistency of these estimators (Klimko and Nelson 1978) are not satisfied. But these are sufficient conditions only. Hence, we proceed to the direct evaluation of the limits. If we define A_{ij} as, $A_{ij} = \sum(Z_{t-i} Z_{t-j}),\ i, j = 0, 1, 2$, using the result $Z_n/\rho^n \xrightarrow{a.s.} W' = \frac{a}{a-b} W$ and Toeplitz lemma, we get,

$$\frac{A_{ij}}{\sum \rho^{2t-i-j}} \xrightarrow{a.s.} \lim \left[\frac{Z_{t-i} Z_{t-j}}{\rho^{t-i} \rho^{t-j}}\right] = W'^2.$$

But, $\sum_{t=2}^{n} \rho^{2t-i-j} = \rho^{-(i+j)} \sum_{t=2}^{n} \rho^{2t} = \rho^{-(i+j)} \frac{\rho^4(\rho^{2(n-1)}-1)}{\rho^2-1}$. Thus, by defining $K_n = \frac{\rho^4(\rho^{2(n-1)}-1)}{\rho^2-1}$, we get,

$$A_{ij}/K_n \overset{a.s.}{\rightarrow} \rho^{-(i+j)} W'^2, \quad i = 0, 1, 2, \quad j = 0, 1, 2. \tag{15}$$

Consequently, both the numerator and the denominator of each of $\hat{\mu}_1$ and $\hat{\mu}_2$, converge almost surely to zero. To investigate further, suppose that L_{1i}, L_{2i} are the liminfs and L_{1s}, L_{2s} are the limsups of $\hat{\mu}_1$ and $\hat{\mu}_2$ respectively. Dividing the first estimating equation in (13) by K_n and using the results regarding liminf and limsup, we get,

$$\liminf \frac{A_{01}}{K_n} \leq \limsup \hat{\mu}_1 \frac{A_{11}}{K_n} + \liminf \hat{\mu}_2 \frac{A_{12}}{K_n} \leq \limsup \left[\hat{\mu}_1 \frac{A_{11}}{K_n} + \hat{\mu}_2 \frac{A_{12}}{K_n} \right].$$

This gives, $W'^2 \leq W'^2 \frac{L_{1s}}{\rho} + W'^2 \frac{L_{2i}}{\rho^2} \leq W'^2$. Thus, we have, $\frac{L_{1s}}{\rho} + \frac{L_{2i}}{\rho^2} = 1$. Similarly, by reversing the roles of limsup and liminf, we get another identity as, $\frac{L_{1i}}{\rho} + \frac{L_{2s}}{\rho^2} = 1$. This implies that the limsup and liminf of both the estimators are finite and hence, using these two identities, we get,

$$L_{1s} - L_{1i} = \frac{L_{2s} - L_{2i}}{\rho}.$$

The second equation in (13) yields the same equation. Thus, we can not determine the values of limsups and liminfs uniquely. Therefore, the CLS estimators have indeterminate limits.

4.2 Maximum likelihood estimation

Suppose that $\xi_i(n-1)$ and $\psi_j(n-2)$ are Poisson with parameters μ_1 and μ_2 respectively. By the additive property of the Poisson distribution, the conditional distribution of $(Z_t|\mathbb{F}_{t-1})$ is Poisson with parameter $\mu_1 Z_{t-1} + \mu_2 Z_{t-2}$. Therefore, conditioned on Z_0 and Z_1 the likelihood can be written as,

$$L(\mu_1, \mu_2) = \prod_{t=2}^{n} \frac{e^{-(\mu_1 Z_{t-1} + \mu_2 Z_{t-2})} (\mu_1 Z_{t-1} + \mu_2 Z_{t-2})^{Z_t}}{Z_t!}.$$

The likelihood equations for the estimation of μ_1 and μ_2 are,

$$\sum_{t=2}^{n} \frac{Z_t Z_{t-1}}{\mu_1 Z_{t-1} + \mu_2 Z_{t-2}} = \sum_{t=2}^{n} Z_{t-1} \quad \text{and} \quad \sum_{t=2}^{n} \frac{Z_t Z_{t-2}}{\mu_1 Z_{t-1} + \mu_2 Z_{t-2}} = \sum_{t=2}^{n} Z_{t-2}.$$

These can also be written as,

$$\sum_{t=2}^{n} \frac{(Z_t - \mu_1 Z_{t-1} - \mu_2 Z_{t-2}) Z_{t-1}}{\mu_1 Z_{t-1} + \mu_2 Z_{t-2}} = 0 \qquad \sum_{t=2}^{n} \frac{(Z_t - \mu_1 Z_{t-1} - \mu_2 Z_{t-2}) Z_{t-2}}{\mu_1 Z_{t-1} + \mu_2 Z_{t-2}} = 0. \tag{16}$$

Suppose the almost sure limits of $\hat{\mu}_1$ and $\hat{\mu}_2$ exist and are denoted by θ_1 and θ_2 respectively. Consider, for $i = 0, 1, 2$ and $j = 0, 1, 2$,

$$B_{ij} = \sum \frac{Z_{t-i} Z_{t-j}}{\hat{\mu}_1 Z_{t-1} + \hat{\mu}_2 Z_{t-2}} = \sum \left[\frac{Z_{t-i} Z_{t-j}}{\theta_1 Z_{t-1} + \theta_2 Z_{t-2}} \left(\frac{\theta_1 Z_{t-1} + \theta_2 Z_{t-2}}{\hat{\mu}_1 Z_{t-1} + \hat{\mu}_2 Z_{t-2}} \right) \right].$$

Using the fact that $Z_n/\rho^n \stackrel{a.s.}{\to} W'$, we get,

$$\frac{\theta_1 Z_{t-1} + \theta_2 Z_{t-2}}{\hat{\mu}_1 Z_{t-1} + \hat{\mu}_2 Z_{t-2}} = \frac{\theta_1 Z_{t-1} + \theta_2 Z_{t-2}}{\rho^{t-1}} \frac{\rho^{t-1}}{\hat{\mu}_1 Z_{t-1} + \hat{\mu}_2 Z_{t-2}} \stackrel{a.s.}{\to} \frac{\theta_1 W' + \frac{\theta_2 W'}{\rho}}{\theta_1 W' + \frac{\theta_2 W'}{\rho}} = 1.$$

Furthermore,

$$\frac{\sum \frac{Z_{t-i} Z_{t-j}}{\theta_1 Z_{t-1} + \theta_2 Z_{t-2}}}{\sum \rho^{t-i-j+1}} \stackrel{a.s.}{\to} \lim \left[\frac{Z_{t-i} Z_{t-j}}{\rho^{t-i} \rho^{t-j}} \left(\frac{\rho^{t-1}}{\theta_1 Z_{t-1} + \theta_2 Z_{t-2}} \right) \right] = \frac{\rho W'}{\theta_1 \rho + \theta_2}.$$

If we define, $K'_n = \frac{\rho^2(\theta_1\rho + \theta_2)(\rho^{n-1} - 1)}{\rho - 1}$, we have using the above two results that,

$$\frac{B_{ij}}{K'_n} \stackrel{a.s.}{\to} W' \rho^{-(i+j)}. \tag{17}$$

Using this result in both the equations in (16), we get the identity

$$\frac{\theta_1}{\rho} + \frac{\theta_2}{\rho^2} = 1.$$

As both the likelihood equations lead to the same equation, the unique values of θ_1 and θ_2 cannot be computed. It is interesting to note that, from (10), (μ_1, μ_2) is one of the possible value of (θ_1, θ_2). Thus, maximum likelihood estimation also does not produce consistent estimators.

Further investigation leads to the fact that $\text{Corr}(Z_n, Z_{n+1}) \to 1$, as $n \to \infty$. This may be the reason for indeterminate limits of $\hat{\mu}_1$ and $\hat{\mu}_2$.

4.3 Ridge type adjustment

To address this problem of indeterminate limits, we slightly modify the estimating equations on the lines of ridge regression. The new estimating equations are,

$$A_{01} = \mu_1 A_{11}(1 + \lambda) + \mu_2 A_{12} \text{ and } A_{02} = \mu_1 A_{12} + \mu_2 A_{22}(1 + \lambda)$$

where, the ridge parameter λ is a nonzero constant. The new estimates of μ_1 and μ_2 are,

$$\hat{\mu}_{r1} = \frac{A_{22} A_{01}(1 + \lambda) - A_{02} A_{12}}{A_{22} A_{11}(1 + \lambda)^2 - (A_{12})^2} \quad \text{and} \quad \hat{\mu}_{r2} = \frac{A_{11} A_{02}(1 + \lambda) - A_{01} A_{12}}{A_{22} A_{11}(1 + \lambda)^2 - (A_{12})^2}.$$

Using Eq. (15), we get $\hat{\mu}_{r1} \stackrel{a.s.}{\to} \frac{\rho}{2+\lambda}$ and $\hat{\mu}_{r2} \stackrel{a.s.}{\to} \frac{\rho^2}{2+\lambda}$. Similar adjustment in the likelihood equations for Poisson offspring gives the following equations:

$$B_{01} = \hat{\mu}_1 B_{11}(1+\lambda) + \hat{\mu}_2 B_{12}, \quad B_{02} = \hat{\mu}_1 B_{12} + \hat{\mu}_2 B_{22}(1+\lambda). \tag{18}$$

Dividing these equations by K_n' and using Eq. (17), we get the two equations as,

$$1 = \frac{\theta_1}{\rho}(1+\lambda) + \frac{\theta_2}{\rho^2}, \quad 1 = \frac{\theta_1}{\rho} + \frac{\theta_2}{\rho^2}(1+\lambda).$$

Solving these equations simultaneously, we get the values of θ_1 and θ_2 as $\frac{\rho}{2+\lambda}$ and $\frac{\rho^2}{2+\lambda}$ respectively. Thus, ridge type adjustment in the estimating equations as well as in the likelihood equations gives similar results, and does not lead to consistent estimators of μ_1 and μ_2.

4.4 Growth rate

We know that $\frac{Z_n}{\rho^n} \stackrel{a.s.}{\to} W'$. Thus, ρ plays the role of growth rate for the second order branching process. Its estimation is therefore of interest. Following two functions of the generation sizes are proposed as the estimators for this growth rate:

$$\text{(i) } \bar{\rho} = \frac{Z_n}{Z_{n-1}}, \text{ if } Z_{n-1} > 0 \qquad \text{(ii) } \hat{\rho} = \frac{\sum_{t=1}^n Z_t}{\sum_{t=1}^n Z_{t-1}}.$$

Consistency of these estimators, follows immediately, using the fact that $\frac{Z_n}{\rho^n} \stackrel{a.s.}{\to} W'$ and Toeplitz lemma. In fact, using Toeplitz lemma, it can be shown that any estimator of the type $\sum_t Z_{t-i} / \sum_t Z_{t-i-1}$ or $\sum_t Z_{t-i} Z_{t-i-1} / \sum_t Z_{t-i-1}^2$, for $i = 1, 2, \ldots, n-1$ converges to ρ almost surely.

5 Simulations and data analysis

Simulations are carried out by taking offspring distributions as Geometric, Binomial and Poisson with $\mu_1 = 0.7$ and $\mu_2 = 0.4$. This gives $\rho = 1.0728$. In case of Binomial(m, p) distribution, m is taken to be 3. In Tables 1 and 2 titles of the type "Binomial–Geometric" indicate that the offspring distribution at lag 1 is Binomial and offspring distribution at lag 2 is Geometric. Number of generations n is taken to be 10, with $Z_0 = 1$ and the number of samples drawn is 5,000. Table 1 presents the means of estimates of μ_1 and μ_2, obtained using optimal estimating equations and conditional least squares along with their estimated Mean Squared Errors (MSE). In most of the cases, estimating functions give the best estimates. Table 2 gives various estimates of growth rates and their MSE's. The estimates are denoted by, $\hat{\rho}_1 = \frac{\sum Z_t}{\sum Z_{t-1}}$, $\hat{\rho}_2 = \frac{\sum Z_t Z_{t-1}}{\sum Z_{t-1}^2}$ and $\hat{\rho}_3 = \frac{\sum Z_{t-1} Z_{t-2}}{\sum Z_{t-2}^2}$. From the values of the estimated means and MSE's of the estimates, it can be seen that $\hat{\rho}_1$ performs better compared to the other estimates.

Table 1 Estimates by various methods for $n = 10$

Method		Poisson–Poisson		Geometric–Poisson	
		Mean	MSE	Mean	MSE
Optimal estimating equations	μ_1	0.7530	0.0818	1.1082	0.3887
	μ_2	0.6878	0.1922	0.5697	0.1722
Conditional least squares	μ_1	0.6034	0.1302	0.5879	0.1508
	μ_2	0.4725	0.1875	0.4757	0.2046
Method		Binomial–Binomial		Binomial–Geometric	
		Mean	MSE	Mean	MSE
Optimal estimating equations	μ_1	0.6775	0.0352	0.3179	0.2000
	μ_2	0.6671	0.1173	1.0936	0.5628
Conditional least squares	μ_1	0.5949	0.1252	0.5938	0.1265
	μ_2	0.4846	0.1692	0.4945	0.1909
Method		Geometric–Geometric		Poisson–Binomial	
		Mean	MSE	Mean	MSE
Optimal estimating equations	μ_1	0.8111	0.2764	0.9075	0.0800
	μ_2	0.9503	0.6867	0.3998	0.0309
Conditional least squares	μ_1	0.5894	0.1408	0.5924	0.1294
	μ_2	0.4835	0.2137	0.4840	0.1806

Table 2 Estimates of growth rate for $n=10$

	Poisson–Poisson		Binomial–Binomial		Geometric–Geometric	
	Mean	MSE	Mean	MSE	Mean	MSE
$\hat{\rho}_1$	1.1191	0.0200	1.1116	0.0171	1.1371	0.0279
$\hat{\rho}_2$	0.9114	0.0806	0.9241	0.0710	0.8786	0.1090
$\hat{\rho}_3$	0.9094	0.0860	0.9182	0.0783	0.8802	0.1169
	Geometric–Poisson		Binomial–Geometric		Poisson–Binomial	
	Mean	MSE	Mean	MSE	Mean	MSE
$\hat{\rho}_1$	1.1277	0.0265	1.1236	0.0192	1.1154	0.0202
$\hat{\rho}_2$	0.8677	0.1207	0.9249	0.0680	0.9098	0.0828
$\hat{\rho}_3$	0.8658	0.1311	0.9284	0.0736	0.9057	0.0897

Next we apply the SOBP model to the swine flu data. The data consist of number of cases tested positive on each day in Pune, India and La-Gloria, Mexico. For Pune, the data are recorded from July, 15 to August, 4, 2009. In case of La-Gloria, the data are recorded from March, 9 to April, 13, 2009. Pune dataset is from Kanade and Rajarshi

Table 3 Estimates for swine flu data

Method	Pune data			La-Gloria data		
	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\rho}$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\rho}$
Conditional least squares	0.3212	0.5858	0.9426	0.4480	0.4092	0.9018
MLE (Poisson)	0.4704	0.6187	1.0562	0.4656	0.5566	1.0143
Growth rate	1.0563			1.0069		
Vaccination required	5%			1%		

(2010), whereas La-Gloria dataset is taken from Fraser et al. (2009). We model both the datasets using SOBP and obtain the estimates of μ_1, μ_2 and ρ.

Maximum Likelihood estimates are computed assuming both the offspring distributions to be Poisson. For Pune data, by both the methods, we get $\hat{\mu}_2$ greater than $\hat{\mu}_1$, suggesting that the rate of infection is more on day 2 of the infection than on day 1. From Table 3, it can be seen that for Pune data, maximum likelihood method gives the sum of the estimates of μ_1 and μ_2 to be greater than 1. Also, the estimator of growth rates ($\hat{\rho}_1$) are greater than 1 for both the datasets. This indicates that both the processes are supercritical in nature and there is a positive probability that the epidemic will not die out unless some efforts are taken to curb its spread. In Pune as well as in Mexico, respective governments took some measures such as closing down the educational institutes, theatres, malls etc. which are justified, by our findings. The proportion of vaccination (ν) required to guarantee the elimination of the disease is computed using the result $\nu \geq 1 - 1/\rho$, which is proved using arguments similar to those in Becker (1976).

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