Supplementary Material

1 Proof of Proposition 1

Notice that $\hat{T}([a,b];X(n))$ is the sample mean of i.i.d. random variables $Y_i : \Omega \to \mathbb{R}$ defined as:

$$Y_i = \begin{cases} 1, & \text{if } X_i \cap [a,b] \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Therefore, an application of the strong law of large numbers in the classical case yields:

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \overset{a.s.}{\to} EY_1 = P\left(X_1 \cap [a,b] \neq \emptyset\right) = T_{\boldsymbol{\theta}_0}\left([a,b]\right), \text{ as } n \to \infty,$$

$\forall a, b : -\infty < a \leq b < \infty$, and assuming $\boldsymbol{\theta}_0$ is the true parameter value. That is,

$$\hat{T}\left([a,b];X(n)\right) \overset{a.s.}{\to} T_{\boldsymbol{\theta}_0}\left([a,b]\right),$$

as $n \to \infty$. It follows immediately that

$$\left[\hat{T}([a,b];X(n)) - T_{\boldsymbol{\theta}_0}\left([a,b]\right)\right]^2 W(a,b) \overset{a.s.}{\to} 0.$$

Notice that $\forall a, b : -\infty < a \leq b < \infty$, $\left[\hat{T}([a,b];X(n)) - T_{\boldsymbol{\theta}_0}\left([a,b]\right)\right]^2 W(a,b)$ is uniformly bounded by $4C$. By the bounded convergence theorem,

$$\iint_S \left[\hat{T}([a,b];X(n)) - T_{\boldsymbol{\theta}_0}\left([a,b]\right)\right]^2 W(a,b)\mathrm{d}a\mathrm{d}b \overset{a.s.}{\to} \iint_S 0 \cdot \mathrm{d}a\mathrm{d}b = 0,$$

given any $S \subset \mathbb{R}^2$ with finite Lebesgue measure. This verifies that

$$P_{\boldsymbol{\theta}}\left\{\omega : \lim_{n\to\infty} H\left(X(n);\boldsymbol{\theta}\right) = 0\right\} = 1. \tag{37}$$

Similarly, we also get

$$P_{\boldsymbol{\theta}}\left\{\omega : \lim_{n\to\infty} H\left(X(n);\boldsymbol{\zeta}\right) = \iint_S \left[T_{\boldsymbol{\theta}}([a,b]) - T_{\boldsymbol{\zeta}}([a,b])\right]^2 W(a,b)\mathrm{d}a\mathrm{d}b\right\} = 1, \tag{38}$$

$\forall \boldsymbol{\theta}, \boldsymbol{\zeta} \in \Theta$. Equations (37) and (38) together imply

$$N(\boldsymbol{\theta}, \boldsymbol{\zeta}) = \iint_S \left[T_{\boldsymbol{\theta}}([a,b]) - T_{\boldsymbol{\zeta}}([a,b])\right]^2 W(a,b)\mathrm{d}a\mathrm{d}b, \ \boldsymbol{\theta}, \boldsymbol{\zeta} \in \Theta. \tag{39}$$

By Assumption 2, $T_{\boldsymbol{\theta}}([a,b]) \neq T_{\boldsymbol{\zeta}}([a,b])$, for $\boldsymbol{\theta} \neq \boldsymbol{\zeta}$, except on a Lebesgue set of measure 0. This together with (39) gives

$$N(\boldsymbol{\theta},\boldsymbol{\theta}) < N(\boldsymbol{\theta},\boldsymbol{\zeta}),\ \forall\ \boldsymbol{\theta} \neq \boldsymbol{\zeta},\ \ \boldsymbol{\theta},\boldsymbol{\zeta} \in \Theta,$$

which proves that $H(X(n);\boldsymbol{\theta})$, $\theta \in \Theta$ is a family of contrast functions. To see the equicontinuity of $H(X(n);\boldsymbol{\theta})$, notice that $\forall \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \Theta$, we have

$$\begin{aligned}
& |H(X(n);\boldsymbol{\theta}_1) - H(X(n);\boldsymbol{\theta}_2)| \\
= & \ |\iint\limits_S \left(T_{\boldsymbol{\theta}_1}([a,b]) - \hat{T}([a,b];X(n))\right)^2 W(a,b)\mathrm{d}a\mathrm{d}b \\
& - \iint\limits_S \left(T_{\boldsymbol{\theta}_2}([a,b]) - \hat{T}([a,b];X(n))\right)^2 W(a,b)\mathrm{d}a\mathrm{d}b| \\
= & \ |\iint\limits_S (T_{\boldsymbol{\theta}_1}([a,b]) - T_{\boldsymbol{\theta}_2}([a,b])) \left(T_{\boldsymbol{\theta}_1}([a,b]) + T_{\boldsymbol{\theta}_2}([a,b]) - 2\hat{T}([a,b];X(n))\right) W(a,b)\mathrm{d}a\mathrm{d}b| \\
\leq & \ 4C \iint\limits_S |T_{\boldsymbol{\theta}_1}([a,b]) - T_{\boldsymbol{\theta}_2}([a,b])| \,\mathrm{d}a\mathrm{d}b,
\end{aligned}$$

since, by definition (18), $|W(a,b)|$ is uniformly bounded by C, $\forall a,b : -\infty < a \leq b$. Then the equicontinuity of $H(X(n);\boldsymbol{\theta})$ follows from the continuity of $T_{\boldsymbol{\theta}}([a,b])$.

2 Lemma 1

Let $H(X(n);\boldsymbol{\theta})$ be the contrast function defined in (18). Under the hypothesis of Assumption 4,

$$\sqrt{n}\left[\frac{\partial H}{\partial \theta_i}\left(X\left(n\right);\boldsymbol{\theta}_0\right)\right] \xrightarrow{\mathcal{D}} N\left(0,\Delta_i\right),\ \text{as } n \to \infty,$$

for $i = 1, \cdots, p$, where

$$\begin{aligned}
\Delta_i \ = \ & 4 \iiiint\limits_{S\times S} \left\{P\left(X_1 \cap [a,b] \neq \emptyset, X_1 \cap [c,d] \neq \emptyset\right) - T_{\boldsymbol{\theta}_0}\left([a,b]\right) T_{\boldsymbol{\theta}_0}\left([c,d]\right)\right\} \\
& \times \frac{\partial T_{\boldsymbol{\theta}_0}}{\partial \theta_i}\left([a,b]\right) \frac{\partial T_{\boldsymbol{\theta}_0}}{\partial \theta_i}\left([c,d]\right) W(a,b)W(c,d)\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}d.
\end{aligned}$$

Proof. We will write $\frac{\partial T_{\boldsymbol{\theta}_0}([a,b])}{\partial \theta_i} = T^i_{\boldsymbol{\theta}_0}\left(a,b\right)$ to simplify notations. Exchanging differentiation and integration by the bounded convergence theorem, we get

$$\begin{aligned}
& \frac{\partial H}{\partial \theta_i}\left(X\left(n\right);\boldsymbol{\theta}_0\right) \\
= & \frac{\partial}{\partial \theta_i}\iint\limits_S \left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-\hat{T}\left([a,b];X(n)\right)\right)^2 W(a,b)\mathrm{d}a\mathrm{d}b \\
= & \iint\limits_S \frac{\partial}{\partial \theta_i}\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-\hat{T}\left([a,b];X(n)\right)\right)^2 W(a,b)\mathrm{d}a\mathrm{d}b \\
= & \iint\limits_S 2\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-\hat{T}\left([a,b];X(n)\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a,b\right)W(a,b)\mathrm{d}a\mathrm{d}b.
\end{aligned} \tag{40}$$

Define $Y_i(a,b)$ as in (36). Then,

$$\begin{aligned}
(40) \quad = & \iint\limits_S 2\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-\frac{1}{n}\sum_{k=1}^{n}Y_k\left(a,b\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a,b\right)W(a,b)\mathrm{d}a\mathrm{d}b \\
= & \frac{2}{n}\iint\limits_S \sum_{k=1}^{n}\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-Y_k\left(a,b\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a,b\right)W(a,b)\mathrm{d}a\mathrm{d}b \\
= & \frac{1}{n}\sum_{k=1}^{n}2\iint\limits_S \left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-Y_k\left(a,b\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a,b\right)W(a,b)\mathrm{d}a\mathrm{d}b \qquad (41) \\
:= & \frac{1}{n}\sum_{k=1}^{n}R_k.
\end{aligned}$$

Notice that R_k's are i.i.d. random variables: $\Omega \to \mathbb{R}$.

Let $\{\Delta s_1, \Delta s_2, \cdots, \Delta s_m\}$ be a partition of S, and (a_j, b_j) be any point in Δs_j, $j = 1, \cdots, m$. Let $\lambda = \max_{1\le j\le m}\{\mathrm{diam}\Delta s_j\}$. Denote by $\Delta\sigma_j$ the area of Δs_j. By the definition of the double integral,

$$\begin{aligned}
R_k = & 2\iint\limits_S \left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-Y_k\left(a,b\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a,b\right)W(a,b)\mathrm{d}a\mathrm{d}b \\
= & \lim_{\lambda\to 0}\left\{\sum_{j=1}^{m}\left(T_{\boldsymbol{\theta}_0}\left([a_j,b_j]\right)-Y_k\left(a_j,b_j\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a_j,b_j\right)W(a_j,b_j)\Delta\sigma_j\right\}.
\end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}
& ER_k \\
= \; & 2E \lim_{\lambda\to 0}\left\{\sum_{j=1}^{m}\left(T_{\boldsymbol{\theta}_0}\left([a_j,b_j]\right)-Y_k\left(a_j,b_j\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a_j,b_j\right)W(a_j,b_j)\Delta\sigma_j\right\} \\
= \; & 2\lim_{\lambda\to 0}\left\{\sum_{j=1}^{m}\left[E\left(T_{\boldsymbol{\theta}_0}\left([a_j,b_j]\right)-Y_k\left(a_j,b_j\right)\right)\right]T^i_{\boldsymbol{\theta}_0}\left(a_j,b_j\right)W(a_j,b_j)\Delta\sigma_j\right\} \\
= \; & 2\lim_{\lambda\to 0}\left\{\sum_{j=1}^{m}0\right\}=0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& Var(R_k)=ER_k^2 \\
= \; & 4E\left\{\lim_{\lambda\to 0}\left\{\sum_{j=1}^{m}\left(T_{\boldsymbol{\theta}_0}\left([a_j,b_j]\right)-Y_k\left(a_j,b_j\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a_j,b_j\right)W(a_j,b_j)\Delta\sigma_j\right\}\right\}^2 \\
= \; & 4E\lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0}\left\{\sum_{j_1=1}^{m_1}\left(T_{\boldsymbol{\theta}_0}\left([a_{j_1},b_{j_1}]\right)-Y_k\left(a_{j_1},b_{j_1}\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a_{j_1},b_{j_1}\right)W(a_{j_1},b_{j_1})\Delta\sigma_{j_1}\right\} \\
& \left\{\sum_{j_2=1}^{m_2}\left(T_{\boldsymbol{\theta}_0}\left([a_{j_2},b_{j_2}]\right)-Y_k\left(a_{j_2},b_{j_2}\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a_{j_2},b_{j_2}\right)W(a_{j_2},b_{j_2})\Delta\sigma_{j_2}\right\} \\
= \; & 4E\lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\left(T_{\boldsymbol{\theta}_0}\left([a_{j_1},b_{j_1}]\right)-Y_k\left(a_{j_1},b_{j_1}\right)\right)\left(T_{\boldsymbol{\theta}_0}\left([a_{j_2},b_{j_2}]\right)-Y_k\left(a_{j_2},b_{j_2}\right)\right) \\
& T^i_{\boldsymbol{\theta}_0}\left(a_{j_1},b_{j_1}\right)T^i_{\boldsymbol{\theta}_0}\left(a_{j_2},b_{j_2}\right)W(a_{j_1},b_{j_1})W(a_{j_2},b_{j_2})\Delta\sigma_{j_1}\Delta\sigma_{j_2} \\
= \; & 4\lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}E\left(T_{\boldsymbol{\theta}_0}\left([a_{j_1},b_{j_1}]\right)-Y_k\left(a_{j_1},b_{j_1}\right)\right)\left(T_{\boldsymbol{\theta}_0}\left([a_{j_2},b_{j_2}]\right)-Y_k\left(a_{j_2},b_{j_2}\right)\right) \\
& T^i_{\boldsymbol{\theta}_0}\left(a_{j_1},b_{j_1}\right)T^i_{\boldsymbol{\theta}_0}\left(a_{j_2},b_{j_2}\right)W(a_{j_1},b_{j_1})W(a_{j_2},b_{j_2})\Delta\sigma_{j_1}\Delta\sigma_{j_2} \\
= \; & 4\lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}Cov\left(Y_k\left(a_{j_1},b_{j_1}\right),Y_k\left(a_{j_2},b_{j_2}\right)\right) \\
& T^i_{\boldsymbol{\theta}_0}\left(a_{j_1},b_{j_1}\right)T^i_{\boldsymbol{\theta}_0}\left(a_{j_2},b_{j_2}\right)W(a_{j_1},b_{j_1})W(a_{j_2},b_{j_2})\Delta\sigma_{j_1}\Delta\sigma_{j_2} \\
= \; & 4\iiiint\limits_{S\times S}Cov\left(Y_k\left(a,b\right),Y_k\left(c,d\right)\right)T^i_{\boldsymbol{\theta}_0}\left(a,b\right)T^i_{\boldsymbol{\theta}_0}\left(c,d\right)W(a,b)W(c,d)\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}d \\
= \; & 4\iiiint\limits_{S\times S}\left\{P\left(X_k\cap[a,b]\neq\emptyset,X_k\cap[c,d]\neq\emptyset\right)-T_{\boldsymbol{\theta}_0}\left([a,b]\right)T_{\boldsymbol{\theta}_0}\left([c,d]\right)\right\} \\
& T^i_{\boldsymbol{\theta}_0}\left(a,b\right)T^i_{\boldsymbol{\theta}_0}\left(c,d\right)W(a,b)W(c,d)\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}d.
\end{aligned}$$

From the central limit theorem for i.i.d. random variables, the desired result follows. □

3 Proof of Proposition 2

By the Cramér-Wold device, it suffices to prove

$$\sqrt{n}\sum_{i=1}^{p}\lambda_i\frac{\partial H}{\partial\theta_i}\left(X(n);\boldsymbol{\theta}_0\right)\xrightarrow{\mathcal{D}}N\left(0,\sum_{1\leq i,j\leq p}\lambda_i\lambda_j\Xi(i,j)\right),\tag{42}$$

for arbitrary real numbers $\lambda_i, i=1,\cdots,p$. It is easily seen from (41) in the proof of Lemma 1 that

$$\begin{aligned}&\sum_{i=1}^{p}\lambda_i\frac{\partial H}{\partial\theta_i}\left(X(n);\boldsymbol{\theta}_0\right)\\=\;&\frac{1}{n}\sum_{k=1}^{n}\left(2\sum_{i=1}^{p}\lambda_i\iint\limits_{S}\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-Y_k\left(a,b\right)\right)\frac{\partial T_{\boldsymbol{\theta}_0}}{\partial\theta_i}\left([a,b]\right)W(a,b)\mathrm{d}a\mathrm{d}b\right)\\:=\;&\frac{1}{n}\sum_{k=1}^{n}\left(2\sum_{i=1}^{p}\lambda_iQ_k^i\right).\end{aligned}$$

By Lemma 1,

$$E\left(2\sum_{i=1}^{p}\lambda_iQ_k^i\right)=2\sum_{i=1}^{p}\lambda_i\cdot 0=0.$$

In view of the central limit theorem for i.i.d. random variables, (42) is reduced to proving

$$Var\left(2\sum_{i=1}^{p}\lambda_iQ_k^i\right)=\sum_{1\leq i,j\leq p}\lambda_i\lambda_j\Xi(i,j).\tag{43}$$

By a similar argument as in Lemma 1, together with some algebraic calculations, we obtain

$$\begin{aligned}&Var\left(2\sum_{i=1}^{p}\lambda_iQ_k^i\right)\\=\;&4\sum_{1\leq i,j\leq p}\lambda_i\lambda_jCov\left(Q_k^i,Q_k^j\right)\\=\;&4\sum_{1\leq i,j\leq p}\lambda_i\lambda_jE\left(\iint\limits_{S}\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-Y_k\left(a,b\right)\right)\frac{\partial T_{\boldsymbol{\theta}_0}}{\partial\theta_i}\left([a,b]\right)W(a,b)\mathrm{d}a\mathrm{d}b\right)\\&\left(\iint\limits_{S}\left(T_{\boldsymbol{\theta}_0}\left([a,b]\right)-Y_k\left(a,b\right)\right)\frac{\partial T_{\boldsymbol{\theta}_0}}{\partial\theta_j}\left([a,b]\right)W(a,b)\mathrm{d}a\mathrm{d}b\right)\\=\;&4\sum_{1\leq i,j\leq p}\lambda_i\lambda_j\iiiint\limits_{S\times S}\left\{P\left(X_1\cap[a,b]\neq\emptyset,X_1\cap[c,d]\neq\emptyset\right)-T_{\boldsymbol{\theta}_0}\left([a,b]\right)T_{\boldsymbol{\theta}_0}\left([c,d]\right)\right\}\\&\frac{\partial T_{\boldsymbol{\theta}_0}}{\partial\theta_i}\left([a,b]\right)\frac{\partial T_{\boldsymbol{\theta}_0}}{\partial\theta_j}\left([c,d]\right)W(a,b)W(c,d)\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}d.\end{aligned}$$

This validates (43), and hence finishes the proof.