On the convergence and consistency of the blurring mean-shift process

Ting-Li Chen

Abstract The mean-shift algorithm is a popular algorithm in computer vision and image processing. It can also be cast as a minimum gamma-divergence estimation. In this paper we focus on the "blurring" mean-shift algorithm, which is one version of the mean-shift process that successively blurs the dataset. The analysis of the blurring mean-shift is relatively more complicated compared to the nonblurring version, yet the algorithm convergence and the estimation consistency have not been well studied in the literature. In this paper we prove both the convergence and the consistency of the blurring mean-shift. We also perform simulation studies to compare the efficiency of the blurring and the nonblurring versions of the mean-shift algorithms. Our results show that the blurring mean-shift has more efficiency.

Keywords Mean-shift · Convergence · Consistency · Clustering · Super robustness · γ-Divergence

1 Introduction

The mean-shift algorithm is a popular algorithm in computer vision and image processing. It was initially designed for kernel density estimation (Fukunaga and Hostetler 1975), which iteratively uses the sample mean within a local region to estimate the gradient of a density function. The mean-shift algorithm was further extended and analyzed by Cheng (1995). Comaniciu and Meer (2002) later applied the mean-shift algorithm to the problem of image segmentation. Since then the algorithm has become

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more well known in the computer science community than in the statistics community. A vast amount of work has been devoted to the development and analysis of the mean-shift algorithm. For example, Fashing and Tomasi (2005) interpreted mean-shift as an optimization procedure. Carreira-Perpinan (2006, 2007) proposed a method to accelerate the Gaussian blurring mean-shift and showed that mean-shift is an EM algorithm when the kernel is Gaussian. Wu and Yang (2007) investigated the robust properties of the mean-shift on three famous kernels and proposed a graphical method of correlation comparisons to estimate parameters. Zhang et al. (2012) employed locality-sensitive hashing technique to reduce the computational complexity of the adaptive mean-shift algorithm. Liu et al. (2013) generalized the mean-shift algorithm by modeling the covariance matrices by the Gaussian mixtures of covariance matrices and presented properties of the proposed generalized mean-shift. Recent applications by mean-shift type of algorithm includes applications on circular data (Chang-Chien et al. 2012), remotely sensed data (Friedman et al. 2013) and visual tracker (Leichter 2012). Recent methods that use iterative processes on minimizing γ-divergence were proposed for robust parameter estimation (Fujisawa and Eguchi 2008) and for robust clustering (Chen et al. 2013). These methods can also be viewed as the mean-shift based approaches.

Suppose $S = \{x_1, \ldots, x_N\}$ are sample points and $T = \{y_1, \ldots, y_M\}$ are cluster centers. The nonblurring mean-shift updating rule can be defined as follows:

$$y_i^{(t+1)} = \sum_{j=1}^{N} \frac{f\left(x_j - y_i^{(t)}\right) w(x_j) x_j}{\sum_{k=1}^{N} f\left(x_k - y_i^{(t)}\right) w(x_k)}, \tag{1}$$

where f is a kernel function, w is a weight function, and $y_i^{(0)} = y_i$. The convergence of the nonblurring version of mean-shift was studied in Cheng (1995), Meer (2000); Comaniciu (2001), Li et al. (2007), and Liu et al. (2013).

When $T = S$, the updating rule becomes

$$x_i^{(t+1)} = \sum_{j=1}^{N} \frac{f\left(x_j^{(t)} - x_i^{(t)}\right) w(x_j) x_j^{(t)}}{\sum_{k=1}^{N} f\left(x_k^{(t)} - x_i^{(t)}\right) w(x_k)}, \tag{2}$$

where $x_i^{(0)} = x_i$. This is called the blurring mean-shift. Note that the weighted average is over the updated data points, instead of the original data. The convergence analysis on the blurring mean-shift is therefore more complicated than the nonblurring one. Cheng (1995) proved the convergence of the blurring mean-shift algorithm for the following two limited cases. When the mutual influence between each pair of data points is nonzero, Theorem 3 in Cheng (1995) shows that all data points eventually converge to a single cluster. When in practice the iterative process is simulated by a digital computer such that data points can never go arbitrarily close to each other, Theorem 4 in Cheng (1995) guarantees that the algorithm converges in a finite number of steps. In Sect. 2, we show that there is a gap in the proof of Theorem 4 by Cheng (1995). We also discuss related work and the condition on f and w.

In Sect. 3, we present a more general result on the convergence of the blurring mean-shift algorithm than Theorem 4 in Cheng (1995). The convergence of the blurring mean-shift is guaranteed under the general definition: data points eventually become arbitrarily close to some locations. Since the number of data points is always finite, there exists a common t^*, such that each data point is close enough to where it converges after the t^*th iteration. That is to say, the convergence under the general definition can imply the convergence in a finite number of steps subject to floating point precision. In addition, Theorem 3 in Cheng (1995) is an immediate implication of our result, which is listed in our Corollary 1.

While the mean-shift algorithm is originally designed for mode seeking using kernel density estimation, it is questioned whether this estimation produces results that converge to the true parameter values when the number of data points goes to infinity. Windham (1995) proposed a robust model fitting, which can be viewed as a nonblurring approach. Fujisawa and Eguchi (2008) proposed a robust estimation by minimizing γ-divergence and proved the consistency of their proposed estimation. This is also a nonblurring approach. In the literature, the consistency of blurring processes has not been well studied. We present the consistency of the blurring processes in Sect. 4.

In addition to convergence and consistency, the exponential rate of convergence of the nonblurring process has been shown by Cheng (1995) and Liu et al. (2013). In Sect. 5 we present simulation studies to compare the performance of the blurring and the nonblurring processes. Discussions and conclusions are given in Sect. 6.

2 Related work and conditions

Before we start the proof of convergence, it is necessary to bring out some of our comments on related works (Cheng 1995; Chen and Shiu 2007).

2.1 A GAP in the proof of theorem 4 in Cheng (1995)

As mentioned in the previous section, there is a gap in the proof of Theorem 4 in Cheng (1995). Quote from the proof of Theorem 4 in Cheng (1995):

> Lemma 2 says that the radius of data reaches its final value in finite number of steps. Lemma 2 also implies that those points at this final radius will not affect other data points or each other. Hence, they can be taken out from consideration for further process of the algorithm.

This implication of Cheng's Lemma 2 is questionable in two respects. First, when the radius of data points reaches its final value, it is not trivial to conclude that there do not exist two data points alternatively switching their locations to be at the final radius, meaning that data points in such a situation fail to converge. Although this situation will not happen during the mean-shift iterative process, it requires to be proven. See our Lemma 2 and its proof.

Second, the convergence of some points at the final radius does not imply that these points do not affect other points. Although these points no longer move, it is possible that they still receive influences from other points, which are just too small to induce a

move larger than the floating point precision. The accumulated influences from these converged data points at the same location may be large enough to affect other data points and to induce a different move in them. Therefore, these converged data points should not be immediately taken out for future process of the algorithm.

2.2 The weight function w

It was stated (Cheng 1995) that the weight function w can be either fixed through the process or re-evaluated after each iteration, and the convergence was only studied for the case when w was fixed. In fact, we found that the process does not converge for arbitrary ws that change over the iterations. The following example illustrates this.

Example 1 Assume the number of data points is 3. Let $x_1 = \delta_1$, $x_2 = 1/2 + \delta_2$, $x_3 = -1/2 - \delta_3$, where $0 < \delta_i < 1/4$. Let

$$f(d) = \begin{cases} 1 & d = 0, \\ 1/2 & 0 < d < 1, \\ 0 & 1 < d. \end{cases}$$

$x_2 - x_3 > 1$, $f(x_2 - x_3) = 0$, meaning that $x2$ and $x3$ do not influence each other in the next update. Let $w(x) = 1$ for $-1/2 < x < 1/2$. Therefore, $w(x_1) = 1$. Now, we can assign large value to $w(x_2)$ and $w(x_3)$ so that

$$x_2^{(1)} = \frac{w(x_2)x_2 + x_1/2}{w(x_2) + 1/2} > 1/2,$$
$$x_3^{(1)} = \frac{w(x_3)x_3 + x_1/2}{w(x_3) + 1/2} < -1/2.$$

We can also assign a large enough value to $w(x_3)$, so that

$$-1/2 < x_1^{(1)} = \frac{x1 + w(x_2)x_2/2 + w(x_3)x_3/2}{1 + w(x_2) + w(x_3)} < 0.$$

These inequalities show that after the first update, $x_1^{(1)}$ becomes negative, and $x_2^{(1)}$ and $x_3^{(1)}$ remain outside $[-1/2, 1/2]$.

At each iteration, we can assign large enough values to $w(x_2)$ and $w(x_3)$, so that $x_1^{(t)}$ is positive when t is even and is negative when t is odd. We can further control the absolute value of $x_1^{(t)}$ to be away from zero, so that $x_1^{(t)}$ and consequently the whole system do not converge. Note that $x_2^{(t)}$ and $x_3^{(t)}$ do converge in this case.

Having seen the above example, in the next section we only prove the convergence under the condition when $w(x_i^{(t)})$s are fixed throughout the process meaning that $w(x_i^{(t)})$s depend on i. It is worth noting that the convergence of the iterative process in fact also holds for varying $w(x_i^{(t)})$s with $\lim_t w(x_i^{(t)})$ existing for each i.

2.3 The influence function f

While the mean-shift algorithm was originally developed for kernel density estimation, it is natural to have f in (2) to be integrable. A weaker condition of f, however, suffices to guarantee the convergence of the iterative process.

Chen and Shiu (2007) proposed a self-updating process (SUP) for clustering as follows:

(i) $x_1^{(0)}, \ldots, x_N^{(0)} \in R^p$ are data points to be clustered.
(ii) At time $t + 1$, every point is updated to

$$x_i^{(t+1)} = \sum_{j=1}^{N} \frac{f\left(x_i^{(t)}, x_j^{(t)}\right)}{\sum_{k=1}^{N} f\left(x_i^{(t)}, x_k^{(t)}\right)} x_j^{(t)}, \tag{3}$$

where f is some function that measures the influence between two data points at time t.
(iii) Repeat (ii) until every point converges.

Although not specified in the notation, the f function in (3) is allowed to be inhomogeneous with respect to t. That is to say, it is more general compared to the f function in the mean-shift updating rule in (2). Chen and Shiu (2007) has demonstrated the use of inhomogeneous fs in several of their experiments. The f function in (3) does not require to be integrable. It is proposed to satisfy the following PDD condition.

Definition 1 The function f in (3) is positive and decreasing with respect to distance (PDD), if

(i) $0 \leq f(u, v) \leq 1$, and $f(u, v) = 1$ if and only if $u = v$.
(ii) $f(u, v)$ depends only on $\|u - v\|$, the distance from u to v.
(iii) $f(u, v)$ is decreasing with respect to $\|u - v\|$.

Note that f in (2) is already defined to be only depending on $u - v$. In the following, we will prove the convergence under (i) f is PDD and (ii) $w(x_i^{(t)})$ only depends on i.

3 Convergence

Theorem 1 *If the function f in (2) is PDD, and if the weight function $w(x_j^{(t)}) = w_j$ in (2) depends only on j, there exists $\{x_1^*, \ldots, x_N^*\}$, such that*

$$\lim_{t \to \infty} x_i^{(t)} = x_i^* \quad \forall i.$$

Below, we outline the proof for Theorem 1.

– First, consider the convex hull of all data points in each iteration. The convex hulls with respect to iterations are nested (Lemma 1) and converge.
– Next, for each vertex of the converged convex hull, there exists at least one sequence of the updated data points converging to this vertex (Lemma 2).

- The influence from the converged data points at the vertices of the converged convex hull goes down to zero to other data points (Lemma 3).
- Consider the convex hull of the rest data points (exclude those already converged). Using the same arguments again, we have a few more converged data points. We can repeat this process over and over again until all data points converge.

Definition 2 The convex hull $C(X)$ for a set of points X in a vector space $\mathcal{V}$ is the minimal convex set containing X.

Lemma 1 *Let $C_1^{(t)}$ be the convex hull of $\{x_1^{(t)}, \ldots, x_N^{(t)}\}$. Then*

$$C_1^{(0)} \supseteq \ldots \supseteq C_1^{(t)} \supseteq \ldots.$$

Proof The convex hull $C(X)$ for a set of points X is the minimal convex set containing X. Since

$$x_i^{(t+1)} = \frac{\sum_{j=1}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) w_j x_j^{(t)}}{\sum_{j=1}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) w_j},$$

$x_i^{(t+1)}$ is a weighted average of $x_j^{(t)}$ for $j = 1, \ldots, N$. Therefore, $x_i^{(t+1)} \in C_1^{(t)}$. Since the above is true for each i, we have

$$C_1^{(t)} \supseteq C\left(\left\{x_1^{(t+1)}, \ldots, x_N^{(t+1)}\right\}\right) = C_1^{(t+1)}.$$

□

Note that the nested structure presented in Lemma 1 ensures the convergence of convex hulls $\{C_1^{(t)}\}$. Let C_1 be the limit of $C_1^{(t)}$,

$$C_1 \equiv \lim_{t \to \infty} C_1^{(t)} = \bigcap_{t=0}^{\infty} C_1^{(t)}.$$

On the other hand, since the convex hull of any finite set of points in R^p is a polytope, each $C_1^{(t)}$ is a polytope. Each vertex of $C_1^{(t)}$ therefore must contain at least one $x_i^{(t)}$ for some i, otherwise the polytope would have been smaller. With the convergence of convex hulls $\{C_1^{(t)}\}$, Lemma 2 claims that for each vertex of C_1, there exists at least one sequence of $\{x_i^{(t)}\}$ which converges to this vertices.

Lemma 2 *If the function f in (2) is PDD, for each vertex $v_{1,i}$ of C_1, there exists at least one j, such that*

$$\lim_{t \to \infty} x_j^{(t)} = v_{1,i}. \tag{4}$$

Proof Since $C_1 = \lim_{t\to\infty} C_1^{(t)}$ for each i, there exists a sequence of $v_{1,i}^{(t)}$s (exchange vertex indices if necessary), such that $\lim_{t\to\infty} v_{1,i}^{(t)} = v_{1,i}$, where $v_{1,i}^{(t)}$ is a vertex of $C_1^{(t)}$. Since for any t and i, $v_{1,i}^{(t)} = x_k^{(t)}$ for at least one k, there exists j, such that $x_j^{(t)} = v_{1,i}^{(t)}$ for infinite many ts. Therefore, there exists an infinite time sequence t_ns, such that

$$x_j^{(t_n)} = v_{1,i}^{(t_n)} \qquad \forall n,$$

which leads to

$$\lim_{n\to\infty} x_j^{(t_n)} = v_{1,i}.$$

If $x_j^{(t)} = v_{1,i}^{(t)}$ except for any finite t, then Eq. (4) is established. Otherwise, there exists $j' \neq j$ and another infinite time sequence s_ns, such that

$$x_{j'}^{(s_n)} = v_{1,i}^{(s_n)} \qquad \forall n.$$

Without loss of generality, assume that $v_{1,i}^{(t)} = x_j^{(t)}$ or $x_{j'}^{(t)}$ for all $t > \tilde{t}$ and assume that $w_j \geq w_{j'}$. If $w_j < w_{j'}$, we can exchange the subscripts j and j'. From Eq. (3), if $x_j^{(s)} = x_{j'}^{(s)}$ for some s, $x_j^{(t)} = x_{j'}^{(t)}$ for all $t > s$. Therefore, for any $s > 0$, there exists $t > s$, such that $v_{1,i}^{(t)} = x_j^{(t)}$ and $v_{1,i}^{(t+1)} = x_{j'}^{(t+1)}$. We claim that this case, however, can never happen: when t is large enough, it is impossible that a data point inside the convex hull later becomes a new vertex, since it is closer to other points than the current vertex is. In the following, we prove this claim only for the one dimensional case. For higher dimensional cases, consider the supporting hyperplane contained $v_{1,i}$. Since $v_{1,i}$ is a vertex of a convex set, a supporting hyperplane can be chosen such that no other point is in the hyperplane. Now we can project all data points onto the straight line, which is perpendicular to the supporting hyperplane and pass through $v_{1,i}$. Then we can make the same argument on the projected data points. Furthermore, we can assume that $x_{j'}^{(t)}$ is the closest point to $x_j^{(t)}$. If there is another point $x_k^{(t)}$ which is the closest to $x_j^{(t)}$, all the points on the right side are closer to $x_{j'}^{(t)}$ and all the points on the left side are closer to $x_k^{(t)}$. Therefore, we can argue that $x_{j'}^{(t)}$ would not pass $x_k^{(t)}$ to become a new vertex. This argument is similar to the closest point not becoming a new vertex. The proof is essentially the same.

Without loss of generality, assume $v_{1,i} = 0$, $x_j^{(t)} \leq 0$, and $x_k^{(t)} > 0$ for $k \neq j$ or j'. If $x_{j'}^{(t+1)}$ becomes the new vertex, then

$$\frac{\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_k^{(t)}\right) w_k x_k^{(t)}}{\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_k^{(t)}\right) w_k} < \frac{\sum_{k=1}^{N} f\left(x_j^{(t)} - x_k^{(t)}\right) w_k x_k^{(t)}}{\sum_{k=1}^{N} f\left(x_j^{(t)} - x_k^{(t)}\right) w_k}. \tag{5}$$

Moreover, since $x_{j'}^{(t+1)}$ is the new vertex,

$$\frac{\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}}{\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k} \leq 0 \implies \sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)} \leq 0.$$

Since $x_j^{(t)}$ is the current vertex, $\|x_j^{(t)} - x_k^{(t)}\| > \|x_{j'}^{(t)} - x_k^{(t)}\|$ for all k, and hence $f(x_j^{(t)} - x_k^{(t)}) < f(x_{j'}^{(t)} - x_k^{(t)})$. Then,

$$\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}$$

$$= w_{j'} x_{j'}^{(t)} + f\left(x_{j'}^{(t)} - x_{j}^{(t)}\right) w_j x_j^{(t)} + \sum_{k \neq j, j'} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}$$

$$\geq w_j x_j^{(t)} + f\left(x_{j'}^{(t)} - x_{j}^{(t)}\right) w_{j'} x_{j'}^{(t)} + \sum_{k \neq j, j'} f\left(x_{j}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}$$

$$= \sum_{k=1}^{N} f\left(x_{j}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}.$$

Since

$$\sum_{k=1}^{N} f\left(x_{j}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)} \leq \sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)} < 0,$$

and

$$0 < \sum_{k=1}^{N} f\left(x_{j}^{(t)} - x_{k}^{(t)}\right) w_k < \sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k,$$

we have

$$\frac{\sum_{k=1}^{N} f\left(x_{j}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}}{\sum_{k=1}^{N} f\left(x_{j}^{(t)} - x_{k}^{(t)}\right) w_k} < \frac{\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k x_k^{(t)}}{\sum_{k=1}^{N} f\left(x_{j'}^{(t)} - x_{k}^{(t)}\right) w_k},$$

which is a contradiction to (5). □

Having shown that at least some points converge under the iterative updates, hereafter we consider the rest of the data points. Let Ω_1 be the set of points shown converging to the vertices of C_1. Let $C_2^{(t)}$ be the convex hull of $\{x_i^{(t)}\}_{i \notin \Omega_1}$. Note that $\{C_2^{(t)}\}$ may not be nested at the early stages of iterations: points not in Ω_1 may move outside the current convex hull $C_2^{(t)}$ due to the influence from Ω_1, the volume of the convex

hull therefore may increase by iteration. This nested property, however, would hold after some iteration when all data points in Ω_1 converge. Explicitly,

$$C_2^{(t)} \supseteq C_2^{(t+1)} \quad \forall t \geq \tilde{t} \text{ for some } \tilde{t},$$

which also implies the convergence of $\{C_2^{(t)}\}$,

$$C_2 \equiv \lim_{t \to \infty} C_2^{(t)}.$$

We introduce the following Lemma 3, which can lead to the nested property of $\{C_2^{(t)}\}$. It states that when all data points in Ω_1 converge, points in Ω_1 receive no influence from points not in Ω_1, otherwise they would have been attracted inward. That is to say, data points not in Ω_1 also no longer receive influence from points in Ω_1, meaning that the influence from points in Ω_1 goes down to zero.

Lemma 3 *For an arbitrary $x_i \in \Omega_1$, we have*

$$\lim_{t \to \infty} f\left(x_i^{(t)} - x_j^{(t)}\right) = 0,$$

for all j such that $\lim_{t \to \infty} x_j^{(t)} \neq \lim_{t \to \infty} x_i^{(t)}$.

Proof Without loss of generality, assume that $x_i^{(t)}$ is the only data point that converges to $v_{i,1}$.

$$\frac{\sum_{j=1}^N f\left(x_i^{(t)} - x_j^{(t)}\right) w_j x_j^{(t)}}{\sum_{j=1}^N f\left(x_i^{(t)} - x_j^{(t)}\right) w_j} = x_i^{(t+1)}$$

$$\Rightarrow \frac{\sum_{j=1}^N f\left(x_i^{(t)} - x_j^{(t)}\right) w_j \cdot \left(x_j^{(t)} - x_i^{(t+1)}\right)}{\sum_{j=1}^N f(x_i^{(t)} - x_j^{(t)}) w_j} = 0$$

$$\Rightarrow \sum_{\substack{j=1 \\ j \neq i}}^N f\left(x_i^{(t)} - x_j^{(t)}\right) w_j \cdot \left(x_j^{(t)} - x_i^{(t+1)}\right) = w_i \cdot \left(x_i^{(t+1)} - x_i^{(t)}\right). \quad (6)$$

Since $x_i^{(t)}$ converges to $v_{i,1}$, $x_i^{(t+1)}$ and $x_i^{(t)}$ become arbitrarily close to each other when t is large enough. That is, the right-hand side of (6) goes down to zero. On the other hand, since $x_j^{(t)}$ does not converge to $v_{i,1}$ for $j \neq i$, there is a gap between $x_j^{(t)}$ and $x_i^{(t+1)}$. To force the left-hand side of (6) to be zero, $f(x_i^{(t)} - x_j^{(t)})$ must go down to zero as well. This sketches the proof for Lemma 3. The precise details are given in the following.

Because $x_j^{(t)}$ does not converge to $v_{i,1}$ for $j \neq i$, there exists $\epsilon > 0$, and for any $t_0 > 0$, there exists $t > t_0$ such that $\|x_j^{(t)} - v_{i,1}\| > \epsilon$. In fact, $x_j^{(t)}$ cannot go arbitrarily

close to $v_{i,1}$ when t is large enough, otherwise the updating process will move $x_j^{(t)}$ and $x_i^{(t)}$ closer and closer to each other. That is, there exists $\epsilon_0 > 0$ and t_1 such that $\|x_j^{(t)} - v_{i,1}\| > \epsilon_1$ for all $t > t_1$. On the other hand, because $x_i^{(t)} \to v_{i,1}$, for any $\epsilon_2 > 0$, there exists t_2, such that $\|x_i^{(t)} - x_i^{(t+1)}\| < \epsilon_2$ for $t > t_2$.

Since $v_{1,i}$ is a vertex of the convex set C_1, there exists $x \in C_1$, such that the inner product of $x - v_{1,i}$ and $y - v_{1,i}$ is positive for any $y \in C_1$. Let

$$v_x = \frac{x - v_{1,i}}{\|x - v_{1,i}\|}.$$

There exists $\alpha > 0$ and $t_3 > t_1$ such that

$$\left\langle x_j^{(t)} - v_{1,i}, v_x \right\rangle \geq \alpha \left\| x_j^{(t)} - v_{1,i} \right\| \qquad \forall t > t_3 \text{ and } \forall j \neq i,$$

where $\langle , \rangle$ denotes the inner product. Take the inner product of both sides of (6) with v_x, we have

$$\left\langle \sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) w_j \cdot \left(x_j^{(t)} - x_i^{(t+1)}\right), v_x \right\rangle$$

$$= \sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) w_j \cdot \left\langle x_j^{(t)} - x_i^{(t+1)}, v_x \right\rangle$$

$$= \sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) w_j \cdot \left(\left\langle x_j^{(t)} - v_{1,i}, v_x \right\rangle + \left\langle v_{1,i} - x_i^{(t+1)}, v_x \right\rangle\right)$$

$$\geq \sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) w_j \alpha \left\| x_j^{(t)} - v_{1,i} \right\|$$

$$> \max_j w_j \cdot \alpha \epsilon_1 \sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right)$$

for $t > t_3$, and

$$\left\langle x_i^{(t+1)} - x_i^{(t)}, v_x \right\rangle \leq \left\| x_i^{(t+1)} - x_i^{(t)} \right\| < \epsilon_2$$

for $t > t_2$. Therefore, for $t > \max(t_3, t_2)$,

$$\max_j w_j \cdot \alpha \epsilon_1 \sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) < w_i \epsilon_2.$$

Since ϵ_2 can be arbitrarily small, the inequality above implies

$$\sum_{j \neq i}^{N} f\left(x_i^{(t)} - x_j^{(t)}\right) \to 0.$$

Since $f \geq 0$, $f(x_i^{(t)} - x_j^{(t)}) \to 0$ for all $j \neq i$. $\qquad\square$

From the above, we can claim a similar result for C_2 as Lemma 2 for C_1: each of the vertex of C_2 has at least one data point converging to it. The same argument can apply again and again to C_3, C_4, ..., until all data points converge. This completes the proof of Theorem 1.

Although Theorem 1 guarantees the convergence when f has PDD condition, there are some fs that produce trivial clustering results, in which all data points are clustered into one single group. We identify such fs in the following corollary.

Corollary 1 *Let $r_M \equiv \max_{i,j}\{||x_i - x_j||\}$. If f is PDD with $f(r_M) > 0$, then there exists c, such that*

$$\lim_{t \to \infty} x_i^{(t)} = c \qquad \forall i.$$

Proof Lemma 1 implies that $||x_i^{(t)} - x_j^{(t)}|| \leq r_M$ for every t, i and j. Since f is decreasing with respect to distance, $f(x_i^{(t)} - x_j^{(t)}) \geq f(r_M) > 0$. Lemma 3 shows that, however, the influence between any two points which do not converge to the same position tends to zero. Thus, $f(x_i^{(t)} - x_j^{(t)}) \geq f(r_M) > 0$ for every i and j, which implies that all data points converge to the same position. $\qquad\square$

For the purpose of clustering, it is not desirable to have all data points converging to the same position. To prevent trivial clustering results, f has to be zero on (r, ∞) for some $r < r_M$.

4 Consistency

In the previous section, we proved the convergence of the algorithm. In this section, we study the estimation consistency of the algorithm. We show the consistency for the normal case and remark on the more general cases. The difficulty of our consistency proof arises from the blurring process, i.e., the the iterative data shrinkage update.

Assume x_is $\in R^p$ are i.i.d. sampled from $N(0, \Sigma)$, and the mutual influence function f adopted is $\exp(-(x - y)^\top (x - y)/2\tau^2)$, where $(x - y)^\top$ is the transpose of vector $x - y$. Assume $w = 1$. The updating rule is:

$$x_{i,n}^{(t+1)} = \sum_{j=1}^{N} \frac{f\left(x_{i,n}^{(t)} - x_{j,n}^{(t)}\right)}{\sum_{j=1}^{N} f\left(x_{i,n}^{(t)} - x_{j,n}^{(t)}\right)} x_{j,n}^{(t)}, \qquad (7)$$

where $x_{i,n}^{(t)}$ denotes the updated x_i at tth iteration when considering only the first n samples. By Corollary 1 presented in the previous section, we know that for all i

$$\lim_{t \to \infty} x_{i,n}^{(t)} = c$$

for the same c. Here, we want to show that c will converge to zero almost surely, which we state as the following theorem:

Theorem 2

$$\lim_{n \to \infty} \lim_{t \to \infty} x_{i,n}^{(t)} = 0 \quad \text{a.s.}$$

Proof Let $G(x; \Sigma)$ be the CDF of $N(0, \Sigma)$, $G_n^{(t)}(x)$ be the empirical CDF of the n sample at tth iteration, and $G^{(t)}(x) = \lim_{n \to \infty} G_n^{(t)}(x)$. By Glivenko–Cantelli theorem,

$$\lim_{n \to \infty} \sup_x \left| G_n^{(0)}(x) - G(x, \Sigma) \right| = 0 \quad \text{a.s.}$$

We claim that the the empirical distribution of the updated data points of each iteration converges to a normal distribution. In the following, we show that

$$\lim_{n \to \infty} \sup_x \left| G_n^{(t)}(x) - G^{(t)}(x) \right| = 0 \quad \text{a.s.} \tag{8}$$

where $G^{(t)}(x) = G(x; \Sigma_t)$. This is true for $t = 0$. Assume that it is true for $t = s$, and we want to show that it is true for $t = s + 1$. Assume that

$$\sup_x \left| G_n^{(s)}(x) - G^{(s)}(x) \right| < \epsilon_s,$$

for $n > N_{\epsilon_s}$. Define

$$K_H(x) = \frac{\int_y f(x - y) \cdot y \cdot dH(y)}{\int_y f(x - y) \cdot dH(y)}.$$

With the assumption that $G^{(s)}(x) = G(x; \Sigma_s)$, we have

$$\begin{aligned}
f(x - y)dG^{(s)} &= c_s \exp\left(-\frac{(x - y)^\top (x - y)}{2\tau^2}\right) \cdot \exp\left(-\frac{y^\top \Sigma_s^{-1} y}{2}\right) dy \\
&= c_s \exp\left[-\frac{1}{2}\left\{\frac{1}{\tau^2}(x^\top x - 2x^\top y) + y^\top(I/\tau^2 + \Sigma_s^{-1})y\right\}\right] dy \\
&= c_s'(x) \exp\left[-\frac{1}{2}\left\{y - (I + \tau^2\Sigma_s^{-1})^{-1}x\right\}^\top (I/\tau^2 + \Sigma_s^{-1})\right. \\
&\quad \times \left. \left\{y - (I + \tau^2\Sigma_s^{-1})^{-1}x)\right\}\right] dy.
\end{aligned}$$

Therefore,

$$K_{G^{(s)}}(x) = (I + \tau^2 \Sigma_s^{-1})^{-1} x. \tag{9}$$

Since

$$\left| G_n^{(s)}(x) - G^{(s)}(x) \right| < \epsilon_s$$

and $f(x-y)y$ and $f(x-y)$ are bounded, we have

$$\left\| K_{G_n^{(s)}}(x) - K_{G^{(s)}}(x) \right\|_2 < \alpha_s \epsilon_s \tag{10}$$

for some positive number α_s where $|| \cdot ||_2$ is the L^2 norm. Since

$$
\begin{aligned}
K_{G_n^{(s)}}\left(x_{i,n}^{(s)}\right) &= \frac{\int_y f\left(x_{i,n}^{(s)} - y\right) \cdot y \cdot dG_n^{(s)}(y)}{\int_y f\left(x_{i,n}^{(s)} - y\right) \cdot dG_n^{(s)}(y)} \\
&= \frac{\sum_{j=1}^N f_s\left(x_{i,n}^{(s)} - x_{j,n}^{(s)}\right) x_{j,n}^{(s)}}{\sum_{j=1}^N f_s\left(x_{i,n}^{(s)} - x_{j,n}^{(s)}\right)} \\
&= x_{i,n}^{(s+1)},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left\| x_{i,n}^{(s+1)} - (I + \tau^2 \Sigma_s^{-1})^{-1} x_{i,n}^{(s)} \right\|_2 \\
&= \left\| K_{G_n^{(s)}}(x_{i,n}^{(s)}) - K_{G^{(s)}}(x_{i,n}^{(s)}) \right\|_2 \\
&< \alpha_s \epsilon_s.
\end{aligned}
$$

The empirical distribution of $x_{i,n}^{(s+1)}$ is $G_n^{(s+1)}(x)$, and that of $(I + \tau^2 \Sigma_s^{-1})^{-1} x_{i,n}^{(s)}$ is $G_n^{(s)}((I + \tau^2 \Sigma_s^{-1})x)$. Then,

$$
\begin{aligned}
&\left| G_n^{(s+1)}(x) - G^{(s)}((I + \tau^2 \Sigma_s^{-1})x) \right| \\
&\leq \max_{||\Delta x|| < \alpha_s \epsilon_s} \left| G_n^{(s)}((I + \tau^2 \Sigma_s^{-1})(x + \Delta x)) - G^{(s)}((I + \tau^2 \Sigma_s^{-1})x) \right| \\
&\leq \max_{||\Delta x|| < \alpha_s \epsilon_s} \left\{ \left| G_n^{(s)}((I + \tau^2 \Sigma_s^{-1})(x + \Delta x)) - G^{(s)}((I + \tau^2 \Sigma_s^{-1})(x + \Delta x)) \right| \right. \\
&\quad \left. + \left| G^{(s)}((I + \tau^2 \Sigma_s^{-1})(x + \Delta x)) - G^{(s)}((I + \tau^2 \Sigma_s^{-1})x) \right| \right\} \\
&< \epsilon_s + \max_{||\Delta x|| < \alpha_s \epsilon_s} \left| G^{(s)}((I + \tau^2 \Sigma_s^{-1})(x + \Delta x)) - G^{(s)}((I + \tau^2 \Sigma_s^{-1})x) \right|
\end{aligned}
$$

$$\leq \epsilon_s + \max_{||\Delta x|| < \alpha_s \epsilon_s} \left|\left|(I + \tau^2 \Sigma_s^{-1})\Delta x\right|\right|_2 \max_{||\Delta x|| < \alpha_s \epsilon_s} \left|\left|\frac{\partial}{\partial x} G^{(s)}(x + \Delta x)\right|\right|_2$$

$$\leq \epsilon_s + \lambda \alpha_s \epsilon_s \frac{1}{\sqrt{2\pi} |\det(I + \tau^2 \Sigma_s^{-1})|^{1/2}},$$

where λ is the largest eigenvalue of $I + \tau^2 \Sigma_s^{-1}$. Therefore, $|G_n^{(s+1)}(x) - G^{(s)}((I + \tau^2 \Sigma_s^{-1})x)|$ can be arbitrarily small by choosing a small enough ϵ_s. This completes the induction.

From (9), we have

$$\Sigma_{s+1} = (I + \tau^2 \Sigma_s^{-1})^{-1} \Sigma_s (I + \tau^2 \Sigma_s^{-1})^{-1}.$$

Since Σ_s is a covariance matrix, it is symmetric and positive definite. Then Σ_s can be factorized as

$$\Sigma_s = P \Lambda_s P^\top$$

where $P P^\top = I$ and Λ_s is a diagonal matrix. Then

$$\Sigma_s^{-1} = P \Lambda_s^{-1} P^\top,$$
$$I + \tau^2 \Sigma_s^{-1} = P(I + \tau^2 \Lambda_s^{-1}) P^\top,$$
$$\Sigma_{s+1} = (I + \tau^2 \Sigma_s^{-1})^{-1} \Sigma_s (I + \tau^2 \Sigma_s^{-1})^{-1}$$
$$= P(I + \tau^2 \Lambda_s^{-1})^{-1} \Lambda_s (I + \tau^2 \Lambda_s^{-1})^{-1} P^\top.$$

Therefore, Σ_s and Σ_{s+1} share the same eigenvectors. Assume that $\lambda_i^{(s)}$s are the eigenvalues of Σ_s and $\lambda_i^{(s+1)}$s are those of Σ_{s+1}. Then,

$$\lambda_i^{(s+1)} = \left(1 + \tau^2/\lambda_i^{(s)}\right)^{-1} \lambda_i^{(s)} \left(1 + \tau^2/\lambda_i^{(s)}\right)^{-1}$$

$$= \frac{\left(\lambda_i^{(s)}\right)^2}{\left(\lambda_i^{(s)} + \tau^2\right)^2} \lambda_i^{(s)}$$

$$\leq \frac{\left(\lambda_i^{(0)}\right)^2}{\left(\lambda_i^{(0)} + \tau^2\right)^2} \lambda_i^{(s)}$$

$$\leq \left\{\frac{\left(\lambda_i^{(0)}\right)^2}{\left(\lambda_i^{(0)} + \tau^2\right)^2}\right\}^{s+1} \lambda_i^{(0)}.$$

Therefore, $\lambda_i^{(s)} \to 0$ as $s \to \infty$. For any ϵ, there exists t_0 such that $\min_i \lambda_i^{(t_0)} < \epsilon^2/k$, where k is a large integer. From (8), almost surely

$$\sup_x \left| G_n^{(t_0)}(x) - G^{(t_0)}(x) \right| \to 0.$$

Equivalently,

$$\sup_A \left| G_n^{(t_0)}(A) - G^{(t_0)}(A) \right| \to 0,$$

where $G_n^{(t_0)}(A)$ and $G^{(t_0)}(A)$ denote the probabilities of $x \in A$. Therefore, for any $\delta > 0$, there exists n_{t_0} such that

$$\sup_A \left| G_n^{(t_0)}(A) - G^{(t_0)}(A) \right| < \delta$$

for all $n > n_{t_0}$. Then

$$
\begin{aligned}
\Pr\left(\left\| x_{i,n}^{(t_0)} \right\|_2 > \epsilon \right) &= G_n^{(t_0)}(x^\top x > \epsilon^2) \\
&< G^{(t_0)}(x^\top x > \epsilon^2) + \delta \\
&= G^{(t_0)}\left(\frac{1}{\min_i \lambda_i^{(t_0)}} x^\top x > \frac{\epsilon^2}{\min_i \lambda_i^{(t_0)}} \right) + \delta \\
&\leq G^{(t_0)}\left(x^\top \Sigma_{t_0}^{-1} x > \frac{\epsilon^2}{\min_i \lambda_i^{(t_0)}} \right) + \delta \\
&\leq G^{(t_0)}(x^\top \Sigma_{t_0}^{-1} x > k) + \delta \\
&= G(x^\top x > k; I_p) + \delta,
\end{aligned}
$$

where I_p is the identity matrix. This can be arbitrarily small by choosing k large enough and δ small enough. Therefore, almost all updated data points are in $B(0, \epsilon)$ at t_0th iteration, where $B(0, \epsilon) = \{x : \|x\|_2 > \epsilon\}$. For iteration $t > t_0$, all updated data points within $B(0, \epsilon)$ will not move outside $B(0, \epsilon)$, since there are more updated data points and hence more influence in the direction toward zero. Therefore, $|x_{i,n}^{(t)}| \leq \epsilon$ for almost all i and for all $t > t_0$ and $n > n_{t_0}$. By Corollary 1, all data points will converge to a single location. We have

$$\left\| \lim_{t \to \infty} x_{i,n}^{(t)} \right\|_2 \leq \epsilon.$$

for all i when $n > n_{t_0}$, which completes the proof. $\qquad \square$

Remark 1 In this section, we present the results under the assumption that both f and G are normal. The results can be generalized to general second-order symmetric kernel functions with translation invariance. For this type of kernel functions, the empirical distribution at each iteration still converges to some distribution, and the variance decreases through iterations. The shrunk distribution, however, may not have a nice form as that in the normal case.

Remark 2 If the weight function depends only on the value of $x - \mu$ where μ is the unobserved center, a necessary and sufficient condition for consistency is

$$\int \left(\frac{\int_y f(x-y) \cdot w(y-\mu) \cdot y \cdot dG(y)}{\int_y f(x-y) \cdot w(y-\mu) \cdot dG(y)} \right) dG(x) = 0. \tag{11}$$

The above equation would hold if w is a symmetric kernel function. However, since we do not know μ, μ is iteratively replaced by $\overline{x_{\cdot,n}^{(t)}} = \frac{1}{n} \sum_{i=1}^{n} x_{i,n}^{(t)}$. We can establish consistency by similar arguments since $\overline{x_{\cdot,n}^{(t)}}$ converges to μ as n goes to infinity.

Remark 3 If the data points are sampled from a finite mixture distribution, the locations which the data points converge to through the iterative process may not be consistent to the parameters. Take the mixture distribution $\alpha_1 N(\mu_1, 1) + (1 - \alpha_1) N(\mu_2, 1)$ as an example. By choosing a proper f, data points will be clustered into two groups. Since the domains of these two normal distribution are overlapped, the converged locations through the iterative process will not converge to μ_1 and μ_2.

5 Simulation

In this section we consider a one-dimensional case where the data are sampled from $N(0, \sigma_0^2)$. The f function in (2) is taken to be $f = \exp(-(x-y)^2/2\tau^2)$. We used three experiments to compare the blurring and the nonblurring processes in the following three aspects: the convergence rate, the efficiency, and the robustness to the outliers.

5.1 Convergence rate

Based on (9), we have shown that

$$K_{G^{(s)}}(x) = \frac{\int_y f(x-y) \cdot y \cdot dG^{(s)}(y)}{\int_y f(x-y) \cdot dG^{(s)}(y)} = \frac{\sigma_s^2}{\sigma_s^2 + \tau^2} x.$$

For the nonblurring process, the integration is over the original data, instead of updated data. The shrinkage ratio is therefore $\frac{\sigma_0^2}{\sigma_0^2 + \tau^2}$, meaning that the convergence rate of the blurring process is higher than that of the nonblurring process. Take $\sigma_0 = 1$ and $\tau = 2$ as an example. For the blurring process,

$$\sigma_1 = \sigma_0 \frac{\sigma_0^2}{\sigma_0^2 + \tau^2} = \frac{1^2}{1^2 + 2^2} = 0.2$$

$$\sigma_2 = \sigma_1 \frac{\sigma_1^2}{\sigma_1^2 + \tau^2} = 0.2 \frac{0.2^2}{0.2^2 + 2^2} \approx 0.002$$

$$\sigma_3 = \sigma_2 \frac{\sigma_2^2}{\sigma_2^2 + \tau^2} = 0.002 \frac{0.002^2}{0.002^2 + 2^2} \approx 0.000000002.$$
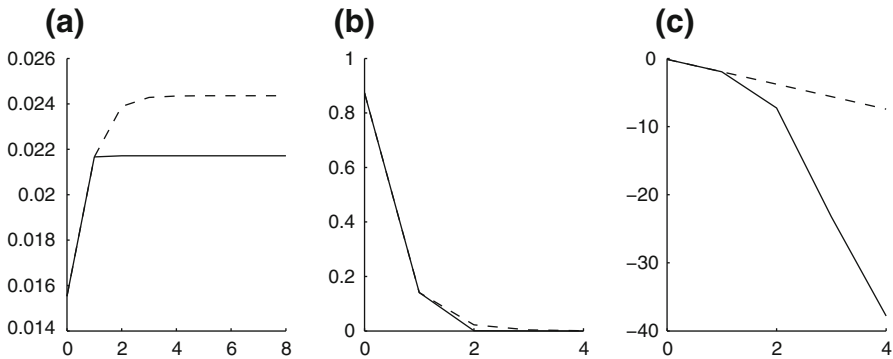
Fig. 1 The simulation results in 100 samplings from $N(0, 1)$. The *solid line* is from the blurring process, and the *dashed line* is from the nonblurring process. **a** Mean, **b** standard deviation and **c** log scale of std

For the nonblurring process,

$$\sigma_1' = \sigma_0' \frac{\sigma_0^2}{\sigma_0^2 + \tau^2} = \frac{1^2}{1^2 + 2^2} = 0.2$$

$$\sigma_2' = \sigma_1' \frac{\sigma_0^2}{\sigma_0^2 + \tau^2} = 0.2 \frac{1^2}{1^2 + 2^2} = 0.04$$

$$\sigma_3' = \sigma_2' \frac{\sigma_0^2}{\sigma_0^2 + \tau^2} = 0.04 \frac{1^2}{1^2 + 2^2} = 0.008.$$

In this experiment, we sampled 100 data points from $N(0, 1)$. Figure 1 presents the simulation results by the blurring and the nonblurring process. In detail, Fig. 1a shows that both processes converged to very close to the true mean of zero. Figure 1b shows that the standard deviations of the updated data points dropped way down at the first iteration and became nearly zero after the second iteration. This illustrates that both processes converged very fast, while the updated data points by the blurring process shrank even faster. Figure 1c further presents the shrinkage of the updated data points in terms of the log scale of the standard deviations in Fig. 1b.

5.2 Efficiency

In this experiment we consider τ to be 0.5, 1 or 2. For each τ value, we simulated 100,000 sets of 100 data points, which were again sampled from $N(0, 1)$. According to the simulated 100,000 sets, we summarized the means and the standard deviations of the following three statistics: the sample mean, the number each set of data points converged to by the blurring process and that by the nonblurring processes. The results are presented in Table 1.

Table 1 The mean and the standard deviation of the converged points

τ	Sample mean	Blurring	Nonblurring
0.5	-1.897×10^{-4} (0.1000)	-5.697×10^{-4} (0.1210)	-5.349×10^{-4} (0.2126)
1	1.260×10^{-4} (0.0997)	2.400×10^{-4} (0.1043)	4.185×10^{-4} (0.1239)
2	6.352×10^{-4} (0.0998)	5.842×10^{-4} (0.1008)	5.565×10^{-4} (0.1025)

In this experiment, we consider 100 data points sampled from $N(0, 1)$. Now, we experiment with $\tau = 0.5$, 1 and 2. For each parameter, we simulate 100,000 times. The means and the standard deviations of the sample mean and the converged numbers of blurring and nonblurring processes in these 100,1000 simulations are presented in Table 1. There is no noticeable difference between the means of three statistics. We did run multiple 100,000 sample sets, and the orders (with respect to the absolute value) are different for different sets. However, the standard deviations of the three statistics are clearly different. The standard deviations of the sample means are close to 0.1, which is the theoretic value. The standard deviations of the converged number from the blurring process are smaller than that from the nonblurring process. Therefore, the converged number from the blurring one seems to be a better estimator over that from the nonblurring one.

There is no noticeable difference between the means of the three statistics. We did run multiple 100,000 sample sets, and the orders (with respect to the absolute value) are different for different sets. However, the standard deviations of the three statistics were clearly different. The standard deviations of the sample means were close to 0.1, which is the theoretical value. The standard deviations of the numbers where the data points converged by the blurring process were closer to those of the sample mean and smaller than those by the nonblurring process. This suggests that the blurring process produced more efficient estimates than the nonblurring process.

5.3 Robustness to outliers

In this experiment, each data set has 95 data points sampled from $N(0, 1)$ and another 5 data points from $N(5, 1)$. We consider τ to be 0.5, 1, or 2. For each τ value, we simulated 100,000 data sets.

By Corollary 1, all data points should converge to a single number. However, due to the floating precision, the outliers which are far from most of the data points may converge to different numbers. For both the blurring and the nonblurring process, we take the number that most of the data points converged to as the statistic. The results are presented in Table 2. While the sample mean was no longer an unbiased estimator of the true mean when outliers are present, Table 2 shows that the numbers where most of data points converged by the blurring and the nonblurring processes were still very close to the true mean of zero. This suggests that both processes produced good estimates for the mean. The standard deviations produced by the blurring process were again smaller than those by the nonblurring process.

Table 2 The mean and the standard deviation of the converged points with 5 % outliers

τ	Sample mean	Blurring	Nonblurring
0.5	0.2495 (0.1003)	−0.0006 (0.1241)	−0.0038 (0.2167)
1	0.2495 (0.1000)	−0.0106 (0.1102)	0.0002 (0.1276)
2	0.2503 (0.0998)	0.0928 (0.1046)	0.0220 (0.1080)

6 Discussion and conclusion

In this paper, we first give a rigorous mathematical proof of the convergence of the blurring mean-shift process. Our result is under the condition that f is PDD and w depends only on data points.

We also prove the consistency of the blurring process, which ensures the estimation to converge to the true values of the parameters as the number of data points goes to infinity. Our consistency proof is for the normal case, in which we could show the explicit form of the shrinkage rate of the data points. The consistency for more general kernel functions can be proven in similar arguments.

From our simulation studies, both the blurring and the nonblurring processes have good robustness against outliers. The estimations by the blurring process usually yield smaller variances than those by the nonblurring process.

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