The Limited Information Maximum Likelihood Approach to Dynamic Panel Structural Equation Models

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1 Supplementary Appendix

Proof of Lemma 2 : By using (9) and (20), we can show

$$\mathcal{E}[\mathbf{z}_{it-1}^{*(b)}\mathbf{w}_{it-1}^{'}] = b_t[\boldsymbol{\Gamma}_0 - \frac{1}{t}\boldsymbol{\Gamma}_0\boldsymbol{\Pi}^{*'}(\mathbf{I}_{K_*} - \boldsymbol{\Pi}^{*'})^{-1}(\mathbf{I}_{K_*} - \boldsymbol{\Pi}^{*' t+1})]$$
$$= b_t[\boldsymbol{\Gamma}_0 + O(\frac{1}{t})], \tag{69}$$

and

$$\mathcal{E}[\mathbf{z}_{it-1}^{*(b)}\mathbf{z}_{it-1}^{*(b)'}] = b_t^2[\boldsymbol{\Gamma}_0 + O(\frac{1}{t})] \ . \tag{70}$$

Then we find that $\lim_{t\to\infty}\mathcal{E}[\mathbf{z}_{i(t-1)}^{*(b)}\mathbf{z}_{i(t-1)}^{*(b)'}] = \boldsymbol{\Gamma}_0$ and

$$\mathcal{E}[\boldsymbol{\epsilon}_{it}^{(b)}\boldsymbol{\epsilon}_{it}^{(b)'}]$$
$$= \mathbf{J}'\mathcal{E}[\mathbf{w}_{it-1}\mathbf{w}_{it-1}']\mathbf{J} - 2\mathbf{J}'\mathcal{E}[\mathbf{w}_{it-1}\mathbf{z}_{it-1}^{*(b)'}]\mathbf{J}(b_t^2\mathbf{J}'\boldsymbol{\Gamma}_0\mathbf{J})^{-1}\mathbf{J}'\mathcal{E}[\mathbf{z}_{it-1}^{*(b)}\mathbf{w}_{it-1}']\mathbf{J}$$
$$+\mathbf{J}'\mathcal{E}[\mathbf{w}_{it-1}\mathbf{z}_{it-1}^{*(b)'}]\mathbf{J}(b_t^2\mathbf{J}'\boldsymbol{\Gamma}_0\mathbf{J})^{-1}\mathbf{J}'\mathcal{E}[\mathbf{z}_{it-1}^{*(b)}\mathbf{z}_{it-1}^{*(b)'}]\mathbf{J}(b_t^2\mathbf{J}'\boldsymbol{\Gamma}_0\mathbf{J})^{-1}\mathbf{J}'\mathcal{E}[\mathbf{z}_{it-1}^{*(b)}\mathbf{w}_{it-1}']\mathbf{J}$$
$$= \mathbf{J}'\boldsymbol{\Gamma}_0\mathbf{J} - \mathbf{J}'\boldsymbol{\Gamma}_0\mathbf{J} + O(\frac{1}{t}) \ ,$$

where $\boldsymbol{\epsilon}_{it}^{(b)} = (\epsilon_{it}^{(1,b)}, \cdots, \epsilon_{it}^{(K,b)})'$. By using the fact that $(\mathbf{I}_N - \mathbf{M}_t^{(b)})\mathbf{Z}_t^{*(b)}\mathbf{J} = \mathbf{O}$ and $\mathbf{W}_{t-1}'\mathbf{M}_t^{(b)}\ \mathbf{W}_{t-1} = \mathbf{W}_{t-1}'\mathbf{W}_{t-1} - \mathbf{E}_t^{(b)'}(\mathbf{I}_N - \mathbf{M}_t^{(b)})\mathbf{E}_t^{(b)}$, we have

$$\frac{1}{N_0 T}\sum_{t=1}^{T-1}\mathcal{E}[\boldsymbol{\epsilon}_t^{(k,b)'}(\mathbf{I}_{N_0} - \mathbf{M}_t^{(b)})\boldsymbol{\epsilon}_t^{(k,b)}] \leq \frac{1}{T}\sum_{t=1}^{T-1}\mathcal{E}[\boldsymbol{\epsilon}_{it}^{(k,b)2}] = \frac{O(\log T)}{T}$$

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because of the Markov inequality. For $j \neq k$, we apply the Cauchy-Schwarz inequality and we have that $(1/N_0T)\sum_t \mathbf{W}'_{t-1}\mathbf{W}_{t-1} \xrightarrow{p} \boldsymbol{\Gamma}_0$ as $T \to \infty$. **Q.E.D.**

Proof of Lemma 3 : For $k = 1, \cdots, K$, by using the fact $\mathbf{z}^*_{i(h-1)} = \mathbf{w}_{i(h-1)} + \boldsymbol{\mu}_i$,

$$\epsilon_{it}^{(k,a)} = \mu_i^{(k)} - \sum_{h=1}^{t} \mathbf{z}_{i(h-1)}^{(a)'}\gamma_{th}^{*(k,a)} = \left[\mu_i^{(k)} - \frac{1}{t}\sum_{h=1}^{t}\mu_i^{(k)}\right] - \frac{1}{t}\sum_{h=1}^{t} w_{i(h-1)}^{[k]} , \quad (71)$$

because by constructing the K_*-variables there is one variable in each $\mathbf{z}^{(a)}_{i(h-1)}$ $(h = 1, ..., t)$ such that $l = k$. (For convenience we use the notation that $w^{[k]}_{i(h-1)}$ is the $k-$th element of $\mathbf{w}_{i(h-1)}$.) Then $\mathcal{E}[\epsilon_{it}^{(k,a)2}] = Var[(1/t)\sum_{h=1}^{t} w^{[k]}_{i(h-1)}] = O(1/t)$. Moreover, $\mathbf{J}'\mathbf{W}'_{t-1}\mathbf{M}^{(a)}_t\mathbf{W}_{t-1}\mathbf{J} = \mathbf{J}'\mathbf{W}'_{t-1}\mathbf{W}_{t-1}\mathbf{J} - \mathbf{E}_t^{(a)'}(\mathbf{I}_N - \mathbf{M}_t^{(a)})\mathbf{E}_t^{(a)}$ because $\mathbf{W}_{t-1}\mathbf{J} = \mathbf{Z}^*_{t-1}\mathbf{J} - \mathbf{Z}_t^{(a)}[\gamma_t^{*(1,a)}, ..., \gamma_t^{*(K,a)}] - \mathbf{E}_t^{(a)}$ and $(\mathbf{I}_N - \mathbf{M}_t^{(a)})(\mathbf{Z}^*_{t-1}\mathbf{J} - \mathbf{Z}_t^{(a)}[\gamma_t^{*(1,a)}, ..., \gamma_t^{*(K,a)}]) = \mathbf{O}_{N\times K}$ due to the fact that $\mathbf{M}_t^{(a)}\mathbf{Z}^*_{t-1}\mathbf{J} = \mathbf{Z}^*_{t-1}\mathbf{J}$ (we have used the $N \times K^*$ matrix $\mathbf{Z}^*_{t-1} = (\mathbf{z}^{*'}_{t-1})$). The rest of the proof is established by the same arguments used for *Lemma 2*. **Q.E.D.**

Proof of Lemma 4 : Define $t_i^{(t)} = \mathbf{h}'\mathbf{U}_t^{\perp'}\mathbf{e}_i$, $m_{ij}^{(t)} = \mathbf{e}_i'\mathbf{M}_t\mathbf{e}_j$ and re-write $u_j^{(t)} = u_{jt}$, then

$$\begin{aligned}
&\mathcal{E}_t\left[[\mathbf{h}'\mathbf{U}_t^{\perp'}\mathbf{M}_t\mathbf{u}_t]^4\right] \\
&= \sum_{i,i',i'',i'''}^{N}\sum_{j,j',j'',j''}^{N} m_{ij}^{(t)}m_{i'j'}^{(t)}m_{i''j''}^{(t)}m_{i'''j'''}^{(t)}\mathcal{E}_t[t_i^{(t)}t_{i'}^{(t)}t_{i''}^{(t)}t_{i'''}^{(t)}u_j^{(t)}u_{j'}^{(t)}u_{j''}^{(t)}u_{j'''}^{(t)}] \\
&= \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}t_{i''}^{(t)}t_{i'''}^{(t)}u_ju_{j'}u_{j''}u_{j'''}]\sum_{I_h} m_{ij}^{(t)}m_{i'j'}^{(t)}m_{i''j''}^{(t)}m_{i'''j'''}^{(t)} + O(N^2)
\end{aligned}$$

where the second equality follows from that the homogeneity of $\mathbf{v}_{it}$ over i, t, and the fact that $\mathcal{E}[t_iu_j] = 0$, $|m_{ij}| \leq 1$ for any i, j, t. Hence we shall check that the summation over the following index set I_h becomes also $O(N^2)$. In order to define I_h, put the terms which has more than three products of the moments

$$\begin{aligned}
\alpha_1 &= \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}u_ju_{j'}]\mathcal{E}[t_{i''}^{(t)}t_{i'''}^{(t)}]\mathcal{E}[u_{j''}u_{j'''}] , \ \alpha_2 = \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}t_{i''}^{(t)}t_{i'''}^{(t)}]\mathcal{E}[u_ju_{j'}]\mathcal{E}[u_{j''}u_{j'''}] , \\
\alpha_3 &= \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}]\mathcal{E}[t_{i''}^{(t)}t_{i'''}^{(t)}]\mathcal{E}[u_ju_{j'}u_{j''}u_{j'''}] , \ \alpha_4 = \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}]\mathcal{E}[t_{i''}^{(t)}t_{i'''}^{(t)}]\mathcal{E}[u_ju_{j'}]\mathcal{E}[u_{j''}u_{j'''}] , \\
\alpha_5 &= \mathcal{E}[t_i^{(t)}u_ju_{j'}]\mathcal{E}[t_{i'}^{(t)}t_{i''}^{(t)}t_{i'''}^{(t)}]\mathcal{E}[u_{j''}u_{j'''}] , \ \alpha_6 = \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}u_{j'''}]\mathcal{E}[t_{i''}^{(t)}t_{i'''}^{(t)}]\mathcal{E}[u_ju_{j'}u_{j''}] , \\
\alpha_7 &= \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}u_j]\mathcal{E}[t_{i''}^{(t)}t_{i'''}^{(t)}u_{j'}]\mathcal{E}[u_{j''}u_{j'''}] , \ \alpha_8 = \mathcal{E}[t_i^{(t)}t_{i'}^{(t)}]\mathcal{E}[t_{i''}^{(t)}u_ju_{j'}]\mathcal{E}[t_{i'''}^{(t)}u_{j''}u_{j'''}] .
\end{aligned}$$

Then we define the set $I_h = \{\{i, i', i'', i''', j, j', j'', j''\} | \, \mathcal{E}[t_i^{(t)} t_{i'}^{(t)} t_{i''}^{(t)} t_{i'''}^{(t)} u_j u_{j'} u_{j''} u_{j'''}] = \alpha_h\}$. By using the fact that $\mathbf{M}_t^2 = \mathbf{M}_t$, we have $m_{ij}^{(t)} = m_{ji}^{(t)}$, and

$$\sum_{j}^{N} m_{ij}^{(t)} m_{ji'}^{(t)} = m_{ii'}^{(t)} \ , \ \ \sum_{i,j}^{N} m_{i,j}^{(t)} \le N \ , \tag{72}$$

where the inequality can be shown by using the similar arguments as used for Lemma 2 in Anderson et al. (2010).

In effect, using these properties we can obtain three types order $O([\mathrm{tr}(\mathbf{M}_t)]^2)$, $O(\mathrm{tr}(\mathbf{M}_t)N)$, and $O(N^2)$ for $\sum_{I_h}$. Then the total number of the patterns which belong to some order type is finite, hence we may conclude $\sum_{I_h} = O(N^2)$. It is because for $h = 1, ..., 4$, we have the conditions $i = i'$ and $i'' = i'''$ regardless $i = i''$ or $i \neq i''$, and also $j = j', \ j'' = j'''$. Then $\sum_{I_h}$ is reduced to the double summation

$$\sum_{i=i', i''=i''', j=j', j''=j'''}^{N} m_{ij}^{(t)} m_{i'j'}^{(t)} m_{i''j''}^{(t)} m_{i'''j'''}^{(t)} = \sum_{i=i', i''=i'''}^{N} m_{ii'}^{(t)} m_{i''i'''}^{(t)} = [\mathrm{tr}(\mathbf{M}_t)]^2 \ .$$

For $h = 5, ..., 8$, we can have the type of conditions that $i = i', \ i'' = i'''$ or $j = j', \ j'' = j'''$, and at least $j = j'$ or $i = i'$, thus the summation is reduced to the triple summation. By using (72),

$$\sum_{i=i', i''=i''', j=j', j'', j'''}^{N} m_{ij}^{(t)} m_{i'j'}^{(t)} m_{i''j''}^{(t)} m_{i'''j'''}^{(t)} = \sum_{j=j', j'', j'''}^{N} m_{jj'}^{(t)} m_{j''j'''}^{(t)} = N \mathrm{tr}(\mathbf{M}_t).$$

For any N, t, it holds that $(1/N)\mathrm{tr}(\mathbf{M}_t) < 1$ and then the existence of 8-th order moments is sufficient to have (68). **Q.E.D.**