Supplement to "Varying coefficients partially linear models with randomly censored data"

1 Proofs

Let C denote a generic constant, $c_n = (\log n/(nh))^{1/2} + h^2$ respectively. “CLT”, “CMT”, ”LLN” stand, respectively, for “central limit theorem”, “continuous mapping theorem” and (possibly uniform) “law of large numbers”. Recall also that $\Sigma(U) = E(XX'|U)$ and $\Omega(U) = E(WX'|U)$.

1.1 Proofs of the auxiliary Lemmas

Proof of Lemma 14. The result follows combining Lemma A.2 of Fan and Huang (2005) with the results of Masry (1996). ■

Proof of Lemma 15. By Lemma 14 and LLN it follows that

$$\sup_{U_i \in \mathcal{U}} \left\| \frac{1}{n} \sum_{i=1}^{n} \widetilde{A}(W_i, X_i, U_i) \widetilde{A}(W_i, X_i, U_i)' - E\left[\overline{A}(W, X, U) \overline{A}(W, X, U)'\right] \right\| = O_p(c_n),$$

hence the conclusion follows by CMT. ■

Proof of Lemma 16. For (i) note that by Lemma 14

$$\widetilde{\xi}_i = \xi_i - \sum_{i=1}^{n} E(X|U)' \Sigma(U)^{-1} X_i O_p(c_n)$$

uniformly in $U \in \mathcal{U}$ hence the conclusion follows by CLT. For (ii) note that by Lemma 14, triangle inequality and LLN

$$\left\| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{A}(W_i, X_i, U_i) S_j(U_i) \xi_{1i} \xi_{2i} \right\| \leq \left\| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{A}(W_i, X_i, U_i) \xi_i \right\| +$$
$$n^{1/2} \sup_{\xi_{1i} \in \mathcal{S}_{\xi_1}} |\xi_{1i}| \frac{1}{n} \sum_{i=1}^{n} \left(W_i X_i' \Sigma(U)^{-1} \Omega(U)' - \Omega(U) \Sigma(U)^{-1} \Omega(U)' \right) \xi_{2i} (1 + O_p(c_n)) = o_p(1).$$

■

Proof of Lemma 17. The first result follows by Lemma 15 and assumption (ξ_2); the second follows by Lemma 16 and assumption (ξ_2). Finally the last result follows noting that $\widetilde{X_i \xi_{1i}} = X_i \xi_{1i} (1 + O_p(c_n))$ by Lemma 14 and hence by Lemma 16 and assumption (ξ_2) we obtain the result. ■

1.2 Proof of the main results

Proof of Theorem 1. Note that

$$n^{1/2}\left(\widehat{\beta} - \beta_0\right) = \left(\frac{1}{n} \sum_{i=1}^{n} \widetilde{W}_i \widetilde{W}_i'\right)^{-1} \sum_{i=1}^{n} \frac{\widetilde{W}_i \left(\widetilde{Z}_{i\widehat{G}} - \widetilde{W}_i' \beta_0\right)}{n^{1/2}}$$

and by Lemma 15

$$\left\| \left(\frac{1}{n} \sum_{i=1}^{n} \widetilde{W}_i \widetilde{W}_i' \right)^{-1} - \Gamma^{-1} \right\| = o_p(1)$$

uniformly in $\mathcal{U}$. Next

$$\begin{aligned} \frac{1}{n^{1/2}} \sum_{i=1}^{n} \widetilde{W}_i \left(\widetilde{Z}_{i_{\widehat{G}}} - \widetilde{W}_i' \beta_0 \right) &= \frac{1}{n^{1/2}} \sum_{i=1}^{n} \widetilde{W}_i \left(\widetilde{\varepsilon}_{iG_0} + \frac{\widehat{G}(Z_i) - G_0(Z_i)}{(1 - G_0((Z_i)))} \widetilde{Z}_{iG_0} \delta_i \right) - \\ \frac{1}{n^{1/2}} \sum_{i=1}^{n} \widetilde{W}_i \widetilde{X}_i' \alpha_0(U_i) &: = T_{1n} + T_{2n}. \end{aligned}$$

We first consider T_{1n} and note that by Lemma 16 (case i)

$$\sum_{i=1}^{n} \widetilde{W}_i \widetilde{\varepsilon}_{iG_0} = \sum_{i=1}^{n} \widetilde{W}_i \varepsilon_{iG_0} + O_p(c_n)$$

uniformly in $\mathcal{U}$. Furthermore since

$$\sup_{0 \le z \le \max_i Z_i} \left| \frac{\widehat{G}(z) - G_0(z)}{1 - G_0(z)} \right| = O_p\left(n^{-1/2}\right) \tag{31}$$

(Zhou 1991), Lemma 16 (case ii) applies yielding

$$\frac{1}{n^{1/2}} \sum_{i=1}^{n} \widetilde{W}_i \frac{\widehat{G}(Z_i) - G_0(Z_i)}{(1 - G_0(Z_i))} \widetilde{Z}_{iG_0} \delta_i = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \widetilde{W}_i \frac{\widehat{G}(Z_i) - G_0(Z_i)}{(1 - G_0(Z_i))} Z_{iG_0} \delta_i + o_p(1).$$

Next by Lemma 14 and LLN

$$\|T_{2n}\| = n^{1/2} \left\| \frac{1}{n} \sum_{i=1}^{n} \left(W_i - \Omega(U_i) \Sigma(U_i)^{-1} X_i \right) X_i' \alpha_0(U_i) (1 + O_p(c_n)) O_p(c_n) \right\| = O_p\left(n^{1/2} c_n^2\right)$$

uniformly in $\mathcal{U}$, and thus

$$n^{1/2}\left(\widehat{\beta} - \beta_0\right) = \frac{1}{n^{1/2}} \left(\sum_{i=1}^{n} \widetilde{W}_i \widetilde{W}_i' / n \right)^{-1} \sum_{i=1}^{n} \widetilde{W}_i \left(\varepsilon_{iG_0} + \frac{\widehat{G}(Z_i) - G_0(Z_i)}{(1 - G_0(Z_i))} Z_{iG_0} \delta_i \right). \tag{32}$$

Next using Lo and Singh (1986)'s representation of the Kaplan-Meier estimator, we have that

$$T_{1n} = T_{1n}^U + R_{11n} + R_{12n}$$

where

$$T_{1n}^U = \frac{1}{n^{3/2}} \sum_{i<j} h(U_i, X_i, W_i, Z_i, \delta_i, U_j, X_j, W_j, Z_j, \delta_j)$$

and

$$h\left(U_i, X_i, W_i, Z_i, \delta_i, U_j, X_j, W_j, Z_j, \delta_j\right) = \left(W_i - \Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1} X_i\right)\left(\varepsilon_{iG_0} + \frac{\eta\left(Z_j, \delta_j; Z_i\right)}{\left(1 - G_0\left(Z_i\right)\right)^2} Z_i\delta_i\right) +$$
$$\left(W_j - \Omega\left(U_j\right)\Sigma\left(U_j\right)^{-1} X_j\right)\left(\varepsilon_{jG_0} + \frac{\eta\left(Z_i, \delta_i; Z_j\right)}{\left(1 - G_0\left(Z_i\right)\right)^2} Z_j\delta_j\right),$$
$$R_{11n} = \frac{1}{n^{3/2}}\sum_{i=1}^{n}\left(W_i - \Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1} X_i\right)\varepsilon_{iG_0},$$
$$R_{12n} = -\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left(W_i - \Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1} X_i\right)\left(\frac{Z_i\delta_i}{\left(1 - G_0\left(Z_i\right)\right)^2}\sum_{j=1}^{n}\eta\left(Z_j, \delta_j; Z_i\right)\right).$$

Clearly $E\left[h\left(\cdot\right)\right] = 0$, $E\left[\eta\left(Z_1, \delta_1; Z_2\right)|Z_2\right] = 0$, $E\left[\eta\left(Z_2, \delta_2; Z_1\right)|Z_1\right] = 0$; furthermore

$$\begin{aligned} E\left[\eta^2\left(Z_1, \delta_1; Z_2\right)|Z_2\right] &= -\left(1 - G_0\left(Z_2\right)\right)^2\int_0^{Z_2}\frac{d\overline{H}_0\left(s\right)}{\left[H_0\left(s\right)\right]^2}, \\ E\left[\eta^2\left(Z_2, \delta_2; Z_1\right)|Z_1\right] &= -\left(1 - G_0\left(Z_1\right)\right)^2\int_0^{Z_1}\frac{d\overline{H}_0\left(s\right)}{\left[H_0\left(s\right)\right]^2}, \end{aligned}$$

hence, for any $k\times 1$ vector ξ, A3 implies that $E\left[\xi' h\left(U_1, X_1, W_1, Z_1, \delta_1, U_2, X_2, W_2, Z_2, \delta_2\right)\right]^2 < \infty$.

Let

$$h_1\left(U_1, X_1, W_1, Z_1, \delta_1\right) := E\left[\left(U_1, X_1, W_1, Z_1, \delta_1, U_2, X_2, W_2, Z_2, \delta_2\right)|U_1, X_1, W_1, Z_1, \delta_1\right] =$$
$$\left[W_1 - E\left(W_1X_1'|U_1\right)\left[E\left(W_1W_1'|U_1\right)\right]^{-1}\right] X_1\varepsilon_{1G_0} +$$
$$E\left[W_2 - E\left(W_2X_2'|U_2\right)\left[E\left(W_2W_2'|U_2\right)\right]^{-1} X_2\eta\left(Z_1, \delta_1; Z_2\right) Z_{2G_0}|Z_1, \delta_1\right],$$

and note that

$$Var\left(\xi' h\left(U_1, X_1, W_1, Z_1, \delta_1, U_2, X_2, W_2, Z_2, \delta_2\right)\right) = E\left(\xi' h_1\left(U_1, X_1, W_1, Z_1, \delta_1\right)\right) = \xi'\Xi\xi;$$

hence by a standard CLT for second-order U-statistics (see e.g. Serfling (1980))

$$U_n := \frac{n^2\xi' T_{1n}^U}{\begin{pmatrix} n \\ 2 \end{pmatrix}} \xrightarrow{d} N\left(0, 4\xi'\Xi\xi\right),$$

and therefore by the Cramer-Wold device

$$T_{1n}^U \xrightarrow{d} N\left(0, \Xi_\beta\right).$$

Note that by triangle and Cauchy-Schwarz inequalities

$$E\left\|R_{11n}\right\| \le \left[E\left\|W_i - \Omega\left(U_i\right)'\Sigma\left(U_i\right)^{-1} X_i\right\|^2\right]^{1/2}\left[E\left|\varepsilon_{iG_0}\right|^2\right]^{1/2}/n^{1/2} \to 0,$$

hence by Markov inequality $\|R_{11n}\| = o_p(1)$, and

$$\|R_{12n}\| \le n^{1/2} \sup_{0\le z\le \max_j Z_j} |\eta(Z_j, \delta_j; z)| \left\| \sum_{i=1}^n \frac{\left(W_i - \Omega(U_i)\Sigma(U_i)^{-1} X_i\right)}{n} \left(\frac{Z_i\delta_i}{\left(1-G_0(Z_i)\right)^2}\right)\right\| = o_p(1)$$

by LLN and $\sup_{0\le z\le \max_j Z_j} |\eta(Z_j,\delta_j;z)| = o_p\left(n^{-1/2}\right)$ (Lo and Singh 1986). ■

Proof of Theorem 2. Note that by the $n^{1/2}$ consistency of both $\widehat{\beta}$ and $\widehat{G}(\cdot)$

$$\begin{aligned}\widehat{\alpha}(u) &= \left(\sum_{i=1}^n X_i X_i' K_h(U_i-u)\right)^{-1} \sum_{i=1}^n X_i \left(Z_{i\widehat{G}} - W_i'\widehat{\beta}\right) K_h(U_i-u) = \\ &\left(\sum_{i=1}^n X_i X_i' K_h(U_i-u)\right)^{-1} \sum_{i=1}^n X_i \left(Z_{iG_0} - W_i'\beta_0\right) K_h(U_i-u) + o_p\left(n^{-1/2}\right)\end{aligned}$$

so that as in Li et al. (2002)

$$\widehat{\alpha}(u) - \alpha_0(u) = \left(\sum_{i=1}^n X_i X_i' K_h(U_i-u)/n\right)^{-1} \sum_{i=1}^n X_i \left[X_i'(\alpha(U_i) - \alpha_0(u)) + \varepsilon_{iG_0}\right] K_h(U_i-u)/n.$$

By Lemma 15

$$\left\| \left(\sum_{i=1}^n X_i X_i' K_h(U_i-u)/n\right)^{-1} - \Delta(u)^{-1} \right\| = o_p(1)$$

uniformly in $u \in \mathcal{U}$, whereas by a standard kernel calculation

$$\begin{aligned}&E\left[XX'(\alpha(U) - \alpha_0(u)) K_h(U-u)\right] = \int XX'\left[\alpha^{(1)}(u)hv + \alpha^{(2)}(u)(hv)^2\right] \times \\ &\left[f(X,u) + f^{(1)}(X,u)hv\right] K(v)\, dX dv + O\left(h^3\right), \\ &Var\left[XX'(\alpha(U) - \alpha_0(u)) K_h(U-u)\right] = O\left(1/nh + h^4\right), \\ &Var\left(\left(\frac{h}{n}\right)^{1/2} \sum_{i=1}^n X_i \varepsilon_{iG_0} K_h(U_i-u)\right) = \int K^2(v)\, dv \sigma_{G_0}^2 \int XX' f(X,u)\, dX + O(h) = \\ &\Xi_\alpha(u) + O(h).\end{aligned}$$

Thus by CLT

$$\left(\frac{h}{n}\right)^{1/2} \sum_{i=1}^n X_i \varepsilon_{iG_0} K_h(U_i-u) \xrightarrow{d} N(0, \Xi_\alpha(u)),$$

and the result follows by CMT. ■

Proof of Proposition 3. We first show that $\left|RSS_1/n - \sigma_{G_0}^2\right| = o_p(1)$. By the same arguments as those used in the proof of Theorem 5.1 of Fan and Huang (2005)

$$
\begin{aligned}
\frac{RSS_1}{n} &= \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{\varepsilon}_{iG_0}\right)^2+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)}{\left(1-G_0\left(\left(Z_i\right)\right)\right)}\widetilde{Z}_{iG_0}\delta_i\right]^2+\frac{2}{n}\sum_{i=1}^{n}\frac{\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)}{\left(1-G_0\left(\left(Z_i\right)\right)\right)}\widetilde{Z}_{iG_0}\delta_i\widetilde{\varepsilon}_{iG_0}+\\
&\quad \frac{2}{n}\sum_{i=1}^{n}\frac{\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)}{\left(1-G_0\left(\left(Z_i\right)\right)\right)}\widetilde{X}_i'\alpha_0\left(U_i\right)+\frac{2}{n}\sum_{i=1}^{n}\frac{\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)}{\left(1-G_0\left(\left(Z_i\right)\right)\right)}\widetilde{Z}_{iG_0}\delta_i\widetilde{W}_i'\left(\widehat{\beta}-\beta_0\right)+o_p\left(1\right)\\
: &= \sum_{j=1}^{5}T_{2jn}.
\end{aligned}
$$

By LLN $\left|T_{21n}-\sigma_{G_0}^2\right| = o_p\left(1\right)$, while by Lemma17 $\left|T_{22n}\right| = O_p\left(n^{-1}\right)$, $\left|T_{23n}\right| = O_p\left(n^{-1/2}\right)$, $\left|T_{24n}\right| = O_p\left(c_n\right)$ and $\left|T_{25n}\right| = O_p\left(n^{-1/2}c_n\right)$. Note that $\widehat{\varepsilon}_{0i\widehat{G}} = \widehat{\varepsilon}_{i\widehat{G}}+\widetilde{W}_i'\left(\widehat{\beta}_0-\widehat{\beta}\right)$ hence by LLN and simple algebra it follows that

$$
\left|\frac{RSS_0}{n}-\sigma_{G_0}^2-\frac{\left(\widehat{\beta}_0-\widehat{\beta}\right)'}{n}\sum_{i=1}^{n}\widetilde{W}_i\widetilde{W}_i'\left(\widehat{\beta}_0-\widehat{\beta}\right)\right| = o_p\left(1\right),
$$

so that the profile least squares ratio

$$
\begin{aligned}
PL_n &= n\frac{\left(\widehat{\beta}_0-\widehat{\beta}\right)'}{\sigma_{G_0}^2}\Gamma\left(\widehat{\beta}_0-\widehat{\beta}\right)+o_p\left(1\right) = \\
&\quad \left(R\widehat{\beta}-r\right)'\left(\sigma_{G_0}^2R\Gamma^{-1}R'\right)^{-1}\left(R\widehat{\beta}-r\right)+o_p\left(1\right),
\end{aligned} \tag{33}
$$

and the conclusion follows by the result of Kent (1982). ■

Proof of Theorem 4. Note that

$$
\begin{aligned}
&\frac{\widehat{\sigma}_{\widehat{G}}^2}{n}\sum_{i=1}^{n}W_iW_i' = \frac{1}{n}\sum_{i=1}^{n}\left[\left(Z_{iG_0}\delta_i\frac{\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)}{1-G_0\left(Z_i\right)}\right)^2+\right.\\
&\left[\left(\widehat{\alpha}\left(U_i\right)-\alpha_0\left(U_i\right)\right)'X_i\right]^2+\left[\left(\widehat{\beta}-\beta_0\right)'W_i\right]^2+\varepsilon_{iG_0}^2+2\left(Z_{iG_0}\delta_i\frac{\widehat{G}\left(Z_i\right)-G_0\left(Z_i\right)}{1-G_0\left(Z_i\right)}\right)\times\\
&\left.\left[\left(\widehat{\alpha}\left(U_i\right)-\alpha_0\left(U_i\right)\right)'X_i+\left(\widehat{\beta}-\beta_0\right)'W_i+\varepsilon_{iG_0}\right]\right]\frac{1}{n}\sum_{i=1}^{n}W_iW_i' := \sum_{j=1}^{5}T_{3jn},
\end{aligned}
$$

and by (31), LLN, and consistency of $\widehat{\alpha}\left(U_i\right)$ and $\widehat{\beta}$ $\left\|T_{31n}\right\| = O\left(n^{-1}\right)$, $\left\|T_{3jn}\right\| = o_p\left(1\right)$ $(j=2,3,5)$, whereas $\left\|\sum_{i=1}^{n}\varepsilon_{iG_0}^2W_iW_i'/n-\sigma_{G_0}^2E\left[W_iW_i'\right]\right\| = o_p\left(1\right)$ by LLN and iterated expectations. Also by Lemma 14, LLN and triangle inequality

$$\left\| \frac{\widehat{\sigma}_{\widehat{G}}^2}{n} \sum_{i=1}^{n} \widehat{E}\left(W_i X_i' | U_i\right) \left[\widehat{E}\left(X_i X_i' | U_i\right)\right]^{-1} \widehat{E}\left(X_i W_i' | U_i\right) - \right.$$

$$\left. \sigma_{G_0}^2 E\left[\Omega\left(U_i\right) \Sigma^{-1}\left(U_i\right) \Omega\left(U_i\right)'\right] \right\| \leq \left|\widehat{\sigma}_{\widehat{G}}^2 - \sigma_{G_0}^2\right| \left\| \frac{1}{n} \sum_{i=1}^{n} \Omega\left(U_i\right) \Sigma^{-1}\left(U_i\right) \Omega\left(U_i\right)' \right\| +$$

$$\left|\widehat{\sigma}_{\widehat{G}}^2 - \sigma_{G_0}^2\right| \left(\frac{1}{n} \sum_{i=1}^{n} \left\| \widehat{E}\left(W_i X_i' | U_i\right) - \Omega\left(U_i\right) \right\|^2 \left\| \Sigma^{-1}\left(U\right) \right\| \right) \left(1 + O_p\left(c_n\right)\right) +$$

$$2\left|\widehat{\sigma}_{\widehat{G}}^2 - \sigma_{G_0}^2\right| \left(\frac{1}{n} \sum_{i=1}^{n} \left\| \Omega\left(U_i\right) \Sigma^{-1}\left(U_i\right) \Omega\left(U_i\right)' \right\|^2 \right)^{1/2} \times$$

$$\left(\frac{1}{n} \sum_{i=1}^{n} \left\| \widehat{E}\left(W_i X_i' | U_i\right) - \Omega\left(U_i\right) \right\|^2 \left\| \Sigma^{-1}\left(U\right) \right\|^2 \right)^{1/2} \left(1 + O_p\left(c_n\right)\right) = o_p(1)$$

uniformly in $\mathcal{U}$. Also by repeated applications of Lemmas 14 and 15

$$\left\| \widehat{E}\left[W_j - \widehat{E}\left(W_j X_j' | U_j\right) \left[\widehat{E}\left(X_i X_i' | U_i\right)\right]^{-1} X_j \eta\left(Z_i, \delta_i; Z_j\right) Z_{j\widehat{G}} | Z_i \right] - \right.$$

$$\left. E\left[W_2 - \Omega\left(U_2\right) \Sigma\left(U_2\right)^{-1} X_2 \eta\left(Z_1, \delta_1; Z_2\right) Z_{2G_0} | Z_1, \delta_1\right] \right\| = o_p(1),$$

and similarly for all the other terms appearing in $\widehat{\Xi}_\beta$; hence by CMT $\left\|\widehat{\Xi}_\beta - \Xi_\beta\right\| = o_p(1)$ and the conclusion follows by CLT and CMT, since $\left\|\widehat{\Gamma}^{-1} - \Gamma^{-1}\right\| = o_p(1)$. ■

Proof of Theorem 5. We first show that $\max_i \left\|\widetilde{W}_i \widehat{\widetilde{\varepsilon}}_{i\widehat{G}}\right\| = o_p\left(n^{1/2}\right)$. Note that by triangle inequality

$$\begin{aligned} \max_i \left\|\widetilde{W}_i \widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right\| &\leq \max_i \left\|\widetilde{W}_i\right\| \left|\widetilde{Z}_{iG_0}\right| + \max_i \left\|\widetilde{W}_i\right\| \left|\widetilde{Z}_{i\widehat{G}} - \widetilde{Z}_{iG_0}\right| + \\ & \max_i \left\|\widetilde{W}_i\right\|^2 \left\|\widehat{\beta}_0 - \beta_0\right\| := \sum_{j=1}^{3} T_{4jn}. \end{aligned}$$

Again by triangle inequality

$$\begin{aligned} T_{41n} &\leq \max_i \left\|\left(W_i - \Omega\left(U_i\right) \Sigma\left(U_i\right)^{-1} X_i\right) Z_{iG_0}\right\| + \max_i \left\|\widehat{E}\left(W_i X_i' | U_i\right) - \Omega\left(U_i\right)\right\| \left\|\Sigma\left(U_i\right)^{-1}\right\| \times \\ & \left\|X_i Z_{iG_0}\right\| + \max_i \left\|\Omega\left(U_i\right)\right\| \left\|\widehat{\Sigma}\left(U_i\right)^{-1} - \Sigma\left(U_i\right)^{-1}\right\| \left\|X_i Z_{iG_0}\right\| = \sum_{j=1}^{3} T_{41jn}, \end{aligned}$$

and $T_{411n} = o_p\left(n^{1/2}\right)$ by the Borel-Cantelli lemma whereas by Masry (1996) and Borel-Cantelli lemma $T_{41jn} = o_p\left(c_n n^{1/2}\right)$ $(j = 2, 3)$. By the Borel Cantelli lemma and CMT

$$T_{42n} \leq \max_i \left| \frac{\widehat{G}\left(Z_i\right) - G_0\left(Z_i\right)}{\left(1 - G_0\left(Z_i\right)\right)} \right| \left|\widetilde{Z}_{iG_0}\right| = O_p(1) \left(o_p\left(n^{1/2}\right) + O_p\left(c_n\right)\right),$$

and similarly $T_{43n} = o_p\left(n^{1/4}\right) O_p\left(n^{-1/2}\right) = o_p\left(n^{1/2}\right)$. Next, noting that the constrained estimator $\widehat{\beta}_0$ can be expressed as

$$\widehat{\beta}_0 - \beta_0 = \Gamma^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\widetilde{W}_i\varepsilon_{0i\widehat{G}}\right) - \Gamma^{-1}R'\left(R\Gamma^{-1}R'\right)^{-1}R\left(\widehat{\beta} - \beta_0\right) + o_p\left(n^{-1/2}\right),$$

it follows by Lemma 16 that

$$\frac{1}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}} = \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{W}_i\varepsilon_{0i\widehat{G}} - \widetilde{W}_i\widetilde{W}_i'\left(\widehat{\beta}_0 - \beta_0\right)\right) = R'\left(R\Gamma^{-1}R'\right)^{-1}R\left(\widehat{\beta} - \beta_0\right) + o_p\left(n^{-1/2}\right). \tag{34}$$

Note also that

$$\frac{1}{n}\sum_{i=1}^{n}\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}^2\widetilde{W}_i\widetilde{W}_i' = \frac{1}{n}\sum_{i=1}^{n}\left[\left(Z_{iG_0}\delta_i\frac{\widehat{G}(Z_i) - G_0(Z_i)}{1 - G_0(Z_i)}\right)^2 + \left[\left(\widehat{\beta}_0 - \beta_0\right)'\widetilde{W}_i\right]^2 + \widehat{\widetilde{\varepsilon}}_{0iG_0}^2 + 2\left(Z_{iG_0}\delta_i\frac{\widehat{G}(Z_i) - G_0(Z_i)}{1 - G_0(Z_i)}\right)\left(\widehat{\beta}_0 - \beta_0\right)'\widetilde{W}_i + \widehat{\widetilde{\varepsilon}}_{0iG_0}\right]\widetilde{W}_i\widetilde{W}_i' := \sum_{j=1}^{5}T_{5jn},$$

and by (31), LLN, Lemmas 14, 16, 17 and the consistency of $\widehat{\beta}_0$, $\|T_{51n}\| = O\left(n^{-1}\right)$, $\|T_{5jn}\| = o_p(1)$ $(j = 2, 3, 5)$, whereas by Lemmas 15 and 16

$$\left\|\sum_{i=1}^{n}\widetilde{\varepsilon}_{0iG_0}^2\widetilde{W}_i\widetilde{W}_i'/n - \sigma_{G_0}^2\Gamma\right\| = o_p(1).$$

Next by Owen (1990)'s arguments it follows that

$$\left\|\widehat{\lambda}\right\|\left(\iota'\sigma_{G_0}^2\Gamma\iota - \max_i\left\|\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right\|\frac{\iota'}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right) \leq \frac{\iota'}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}$$

where ι is a k dimensional vector of ones. Hence by (34) and CLT $\left\|\widehat{\lambda}\right\|\left(\rho_\Gamma^m + o_p(1)\right) \leq O_p\left(n^{-1/2}\right)$ where $\rho_\Gamma^m > 0$ is the largest eigenvalue of Γ and

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{n}\frac{\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}}{\left(1 + \widehat{\lambda}'\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right)} \leq \frac{1}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}} - \Gamma\widehat{\lambda} + \frac{1}{n}\sum_{i=1}^{n}\frac{\left\|\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right\|^3\left\|\widehat{\lambda}\right\|^2}{\left|1 + \widehat{\lambda}'\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right|} \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}} - \Gamma\widehat{\lambda} + o_p\left(n^{-1/2}\right), \end{aligned}$$

which implies $\widehat{\lambda} = \Gamma^{-1}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}/n + o_p\left(n^{-1/2}\right)$. By Taylor expansion about $\widehat{\lambda}'\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}} = o_p(1)$

$$EL_n = n\left(\frac{1}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}}\right)'\left(\sigma_{G_0}^2\Gamma\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\widetilde{W}_i\widehat{\widetilde{\varepsilon}}_{0i\widehat{G}} + o_p(1),$$

so that the conclusion follows by (34) and CMT. ■

Proof of Proposition 6. By the same arguments as those used in the proof of Theorem 4 $|\widehat{\varrho}-\varrho| = o_p(1)$ uniformly in $\mathcal{U}$, hence the conclusion follows directly from the results of Rao and Scott (1981). ■

Proof of Theorem 7. Following Fan and Huang (2005) it suffices to show that $\left|RSS_j - RSS_j^0\right| = o_p(n)$ $(j=0,1)$ where RSS_j^0 is as RSS_j but with G_0 and β_0 assumed known. Note that

$$\left|\frac{RSS_0}{n}-\frac{RSS_0^0}{n}\right| \leq \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{Z}_{i\widehat{G}}-\widetilde{Z}_{iG_0}-\widetilde{W}_i\widehat{\beta}_0+\widetilde{W}_i\beta_0\right)^2+$$
$$\frac{2}{n}\sum_{i=1}^{n}\left|\widetilde{Z}_{i\widehat{G}}-\widetilde{W}_i\widehat{\beta}_0\right|\left|\widetilde{Z}_{i\widehat{G}}-\widetilde{Z}_{iG_0}-\widetilde{W}_i\widehat{\beta}_0+\widetilde{W}_i\beta_0\right| \leq$$
$$\frac{2}{n}\sum_{i=1}^{n}\left(\widetilde{Z}_i\delta_i\frac{\widehat{G}(Z_i)-G_0(Z_i)}{(1-G_0(Z_i))^2}\right)^2+\frac{2}{n}\left(\widehat{\beta}_0-\beta_0\right)'\sum_{i=1}^{n}\widetilde{W}_i\widetilde{W}_i'\left(\widehat{\beta}_0-\beta_0\right)+$$
$$2\left[\sum_{i=1}^{n}\left(\frac{\widehat{\varepsilon}_{i\widehat{G}}}{n}\right)^2\right]^{1/2}\left[\sum_{i=1}^{n}\left(\frac{\widehat{\varepsilon}_{i\widehat{G}}-\varepsilon_{iG_0}}{n}\right)^2\right]^{1/2}+O_p\left(c_n^2\right)=O_p\left(\frac{1}{n}\right)+O_p\left(c_n^2\right),$$

and similarly for RSS_1. Hence

$$PL_n^{\alpha}=\frac{n}{2}\frac{RSS_0^0-RSS_1^0}{RSS_1^0}+o_p(1),$$

and the result follows by the same arguments used by Fan, Zhang and Zhang (2001). ■

Proof of Theorem 8. By the results of Lo and Singh (1986) (see also Akritas (1986)) the bootstrap Kaplan-Meier estimator $\widehat{G}^*$ is $n^{1/2}$ consistent. This together with the bootstrap LLN and CLT of Bickel and Freedman (1981) to show that $n^{1/2}\left(\widehat{\beta}^*-\widehat{\beta}\right)=O_{p^*}(1)$ and $(nh)^{1/2}\left(\widehat{\alpha}^*(u)-\widehat{\alpha}(u)\right)=O_{p^*}(1)$ except if the original sample is on a set with probability tending to 0 as $n\to\infty$. As in the proof of Theorem 5 let $RSS_1^{*0}=RSS_1^*$ with $\widehat{G}^*$ and $\widehat{\beta}^*$ assumed known. Then

$$\left|\frac{RSS_1^*}{n}-\frac{RSS_1^{*0}}{n}\right|\leq 4\sup_{Z_i}\left|\widehat{G}^*(Z_i)-G_0(Z_i)\right|^2\frac{1}{n}\sum_{i=1}^{n}\left(Z_{iG_0}^*\delta_i^*\right)^2+$$
$$4\left\|\widehat{\beta}^*-\widehat{\beta}\right\|^2\left\|\frac{1}{n}\sum_{i=1}^{n}W_iW_i'\right\|\left(1+O_{p^*}\left(c_n^2\right)\right)+\frac{4}{n}\sup_{U_i}\left\|\widehat{\alpha}^*(U_i)-\widehat{\alpha}(U_i)\right\|^2\left\|\frac{1}{n}\sum_{i=1}^{n}X_iX_i'\right\|+$$
$$2\left[\sum_{i=1}^{n}\left(\frac{\widehat{\varepsilon}_{i\widehat{G}}^*}{n}\right)^2\right]^{1/2}\sup_{Z_i}\left|\widehat{G}^*(Z_i)-G_0(Z_i)\right|\left[\sum_{i=1}^{n}\left(\frac{\varepsilon_{iG_0}^*}{n}\right)^2\right]^{1/2}+$$
$$2\left\|\widehat{\beta}^*-\widehat{\beta}\right\|\left[\sum_{i=1}^{n}\left(\frac{\widehat{\varepsilon}_{i\widehat{G}}^*}{n}\right)^2\right]^{1/2}\left[\frac{1}{n}\sum_{i=1}^{n}\left\|W_i\right\|^2\right]^{1/2}\left(1+O_{p^*}(c_n)\right)=o_{p^*}(1),$$

and $\left|RSS_1^{*0}-\sigma_{G_0}^2\right|=o_{p^*}(1)$ except if the original sample is on a set with probability tending to 0 as $n\to\infty$. A similar argument shows that $|RSS_0^*-RSS_0^{*0}|=o_{p^*}(1)$, so that by the same argument as those used by Fan, Zhang and Zhang (2001) we have that $(RSS_1^{*0} - RSS_0^{*0}) = T_n^* + \mu_n + o_{p^*}(1)$ where

$$T_n^* = \frac{h^{1/2}}{2n\sigma_{G_0}^2} \sum_{i \neq j} \widehat{\varepsilon}_{iG_0}^* \widehat{\varepsilon}_{jG_0}^* \left(2K_h(U_j - U_i) - K_h * K_h(U_j - U_i)\right) X_i' \widehat{\Sigma}(U_i)^{-1} X_j,$$

$\mu_n = p|\mathcal{U}| \left(K(0) - \int K^2(v)\, dv\right)/h$, and $\widehat{\varepsilon}_{iG_0}^* = Z_{iG_0}^* - W_i'\beta_0 - X_i'\widehat{\alpha}^*(U_i)$. Since $E^*(T_n) = 0$ and

$$V^*(T_n) = \frac{h}{4n^2} \sum_{i \neq j} \left(\left(2K_h(U_j - U_i) - K_h * K_h(U_j - U_i)\right) X_i' \widehat{\Sigma}(U_i)^{-1} X_j\right)^2 := \sigma_n^2$$

hence by the bootstrap CLT (Bickel and Freedman 1981) $T_n^*/\sigma_n \xrightarrow{d^*} N(0,1)$. As in Fan, Zhang and Zhang (2001), it is possible to show that

$$\sigma_n^2 = \int \left(K(v) - \frac{1}{2} K * K(v)\right)^2 dv p E\left(f(U)^{-1}\right) + O(h).$$

from which the conclusion follows immediately. ■

Proof of Theorem 9. Note that

$$\begin{aligned} n^{1/2}\widehat{v}(u,s,\theta) &= \frac{1}{n^{1/2}} \sum_{i=1}^n \varepsilon_{iG_0} I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) + \\ &\quad \frac{1}{n^{1/2}} \sum_{i=1}^n \left(\widehat{\varepsilon}_{i\widehat{G}} - \varepsilon_{iG_0}\right) I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) := T_{6n} + T_{7n} \end{aligned}$$

and

$$\begin{aligned} T_{7n} &= \frac{1}{n^{1/2}} \sum_{i=1}^n \left(Z_{i\widehat{G}} - Z_{iG_0}\right) I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) - \\ &\quad \frac{1}{n^{1/2}} \sum_{i=1}^n \left(\sum_{j=1}^n S_j(U_i)\right) \left(Z_{i\widehat{G}} - Z_{iG_0}\right) I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) - \\ &\quad \frac{1}{n^{1/2}} \sum_{i=1}^n \left(\sum_{j=1}^n S_j(U_i)(Z_{iG_0} - W_i'\beta_0) - X_i'\Sigma(U_i)^{-1} E\left(X(Z_{G_0} - W'\beta_0)|U_i\right)\right) \times \\ &\quad I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) + \\ &\quad \frac{\left(\widehat{\beta} - \beta_0\right)'}{n^{1/2}} \sum_{i=1}^n \left(W_i - \Omega(U_i)\Sigma(U_i)^{-1} X_i\right) I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) + \\ &\quad \frac{\left(\widehat{\beta} - \beta_0\right)'}{n^{1/2}} \sum_{i=1}^n \left(\Omega(U_i)\Sigma(U_i)^{-1} X_i - \sum_{j=1}^n W_j S_j(U_i)'\right) I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) := \sum_{j=1}^5 T_{7jn}. \end{aligned}$$

Note that the class of functions

$$\{(r,q,t) \to (r - d(q)t) I(r \le u, \theta' t \le s), \quad u, s, \theta \in \Pi\} \tag{35}$$

is Vapnik-Chervonenkis. By Lo and Singh (1986)'s representation of the Kaplan-Meier estimator

$$
\begin{aligned}
T_{71n} &= \frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{\widehat{G}(Z_i) - G_0(Z_i)}{(1 - G_0((Z_i)))} Z_{iG_0} \delta_i I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) + o_p(1) \\
&= \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} \frac{\eta(Z_j, \delta_j; Z_i)}{1 - G_0(Z_i)} \right) Z_{iG_0} \delta_i I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) + o_p(1),
\end{aligned}
$$

and by Lemma 16 and LLN

$$
\begin{aligned}
&\Bigg| T_{72n} - \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left(X_i' \Sigma(U_i)^{-1} \sum_{j=1}^{n} \frac{1}{n} E\left[\frac{\eta(Z_j, \delta_j; Z) X Z_{G_0} \delta}{1 - G_0(Z)} | U_i \right] \right) \times \\
&I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) \Bigg| = O_p(c_n)
\end{aligned}
$$

uniformly in $\mathcal{U}$. For the T_{73n} term let

$$
T_{73n} = \frac{1}{2n^{3/2}} \sum_{i<j} h(U_i, X_i, W_i, Z_i, \delta_i, U_j, X_j, W_j, Z_j, \delta_j; u, s, \theta)
$$

where

$$
\begin{aligned}
&h(U_i, X_i, W_i, Z_i, \delta_i, U_j, X_j, W_j, Z_j, \delta_j; u, s, \theta) = \\
&\Big[X_i' \Sigma(U_i)^{-1} X_i (Z_{iG_0} - W_i' \beta_0) I\left(U_j \le u, \theta' \left[X_j', W_j'\right]' \le s\right) + \\
&X_j' \Sigma(U_j)^{-1} X_j \left(Z_{jG_0} - W_j' \beta_0\right) I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) \Big] K_h(U_i - U_j) - \\
&X_i' \Sigma(U_i)^{-1} E[X(Z_{G_0} - W' \beta_0 | U_i)] I\left(U_i \le u, \theta' [X_i', W_i']' \le s\right) - \\
&X_j' \Sigma(U_j)^{-1} E[X(Z_{G_0} - W' \beta_0 | U_j)] I\left(U_j \le u, \theta' \left[X_j', W_j'\right]' \le s\right).
\end{aligned}
$$

By iterated expectations and some calculations

$$
\begin{aligned}
&E[h(U_i, X_i, W_i, Z_i, \delta_i, U_j, X_j, W_j, Z_j, \delta_j; u, s, \theta)] = X_j' I\left(U_j \le u, \theta' \left[X_j', W_j'\right]' \le s\right) \times \\
&\int \alpha(hv + U_j) K(v) dv + \\
&X_j' \Sigma(U_j)^{-1} X_j \left(Z_{jG_0} - W_j' \beta_0\right) \int F_\theta(s|U) f(U) I(U \le u) K_h(U - U_j) dU - \\
&E\left[X_1' \alpha(U_1) I\left(U_1 \le u, \theta' [X_1', W_1']' \le s\right)\right] - X_j' \alpha(U_j) I\left(U_j \le u, \theta' \left[X_j', W_j'\right]' \le s\right) = \\
&X_j' \Sigma(U_j)^{-1} X_j \left(Z_{jG_0} - W_j' \beta_0\right) F_\theta(s|U_j) f(U_j) I(U_j \le u) - \\
&X_j' \Sigma(U_j)^{-1} X_j \left(Z_{jG_0} - W_j' \beta_0\right) \Big[F_\theta(s|U_j) f(U_j) I(U_j \le u) - \\
&\int F_\theta(s) I(U \le u) K_h(U - U_j) dU \Big] + \\
&X_j' I\left(U_j \le u, \theta' \left[X_j', W_j'\right]' \le s\right) \left[\alpha(U_j) - \int \alpha(hv + U_j) K(v) dv \right] - \\
&E\left[X_1' \alpha(U_1) I\left(U_1 \le u, \theta' [X_1', W_1']' \le s\right)\right] := \sum_{k=1}^{4} T_{73kjn}.
\end{aligned}
$$

By similar arguments as those used by Zhu and Ng (2003)

$$\sup_{u,s,\theta\in\Pi}\left|\frac{1}{n^{1/2}}\sum_{j=1}^{n}T_{732jn}\right| = O_p\left(h^{1/2}\log n\right) \quad j=2,3$$

and hence

$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}E\left[h\left(U_i,X_i,W_i,Z_i,\delta_i,U_j,X_j,W_j,Z_j,\delta_j;u,s,\theta\right)\right] =$$
$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}\left\{X_j'\Sigma\left(U_j\right)^{-1}X_j\left(Z_{jG_0}-W_j'\beta_0\right)F_\theta\left(s|U_j\right)f\left(U_j\right)I\left(U_j\le u\right)-\right.$$
$$\left.E\left[X_1'\alpha\left(U_1\right)I\left(U_1\le u,\theta'\left[X_1',W_1'\right]'\le s\right)\right]\right\}+o_p(1).$$

Thus noting (35) we have by LLN

$$\sup_{u,s,\theta\in\Pi}\left|T_{73n}-\frac{1}{n^{1/2}}\sum_{j=1}^{n}E\left[h\left(U_1,X_1,W_1,Z_1,\delta_1,U_j,X_j,W_j,Z_j,\delta_j;u,s,\theta\right)\right]\right| = o_p(1);$$

furthermore by the results of Theorem 1

$$\begin{aligned}T_{74n} &= n^{1/2}\left(\widehat{\beta}-\beta_0\right)'\frac{1}{n}\sum_{i=1}^{n}\left(W_i-\Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1}X_i\right)I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)=\\ &\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left(W_i-\Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1}X_i\right)'\left(\varepsilon_{iG_0}+\frac{\eta\left(Z_j,\delta_j;Z_i\right)}{\left(1-G_0\left(Z_i\right)\right)^2}Z_i\delta_i\right)\Gamma^{-1}\times\\ &E\left[W-\Omega(U)\Sigma(U)^{-1}XI\left(U\le u,\theta'\left[X',W'\right]'\le s\right)\right]+o_p(1),\end{aligned}$$

and by (35) and A3 we have by LLN that

$$\sup_{u,s,\theta\in\Pi}\left\|\frac{1}{n}\sum_{i=1}^{n}\left(W_i-\Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1}X_i\right)I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)-\right.$$
$$\left.E\left[W-\Omega(U)\Sigma(U)^{-1}XI\left(U\le u,\theta'\left[X',W'\right]'\le s\right)\right]\right\| = o_p(1).$$

Finally by a standard kernel calculation and LLN

$$\left|T_{75n}\right| = O_p(1)\left\|\frac{1}{n}\sum_{i=1}^{n}\left(\Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1}X_i-\sum_{j=1}^{n}W_jS_j\left(U_i\right)'\right)I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)\right\| = O_p(h).$$

Hence

$$
\begin{aligned}
T_{7n} \;=\; & \frac{1}{n^{1/2}}\sum_{i=1}^{n}\left\{\frac{1}{n}\left[\sum_{j=1}^{n}\frac{\eta\left(Z_j,\delta_j;Z_i\right)}{1-G_0\left(Z_i\right)}Z_{iG_0}\delta_i - X_i'\Sigma\left(U_i\right)^{-1}E\left[\frac{\eta\left(Z_j,\delta_j;Z\right)}{1-G_0\left(Z\right)}Z_{G_0}\delta|U_i\right]\right]\right\}\times \\
& I\left(U_i\leq u,X_i\leq x,W_i\leq w\right)+ \qquad (36)\\
& \frac{1}{n^{1/2}}\left(\sum_{i=1}X_i'\Sigma\left(U_i\right)^{-1}X_i\left(Z_{iG_0}-W_i'\beta_0\right)F_\theta\left(s|U_i\right)f\left(U_i\right)I\left(U_i\leq u\right)-\right. \\
& \left.E\left[X'\alpha\left(U\right)I\left(U\leq u,\theta'\left[X_i',W_i'\right]'\leq s\right)\right]\right)+ \\
& \frac{1}{n^{1/2}}\sum_{i=1}^{n}\left(W_i-\Omega\left(U_i\right)\Sigma\left(U_i\right)^{-1}X_i\right)'\left(\varepsilon_{iG_0}+\frac{\eta\left(Z_j,\delta_j;Z_i\right)}{\left(1-G_0\left(Z_i\right)\right)^2}Z_i\delta_i\right)\times \\
& \Gamma^{-1}E\left[W-\Omega\left(U\right)\Sigma\left(U\right)^{-1}XI\left(U\leq u,\theta'\left[X_i',W_i'\right]'\leq s\right)\right]+o_p\left(1\right).
\end{aligned}
$$

The fidi convergence of (36) follows by a standard CLT, whereas the asymptotic equicontinuity follows by a direct application of Theorem 2.5.2 of van der Vaart and Wellner (1996), which implies the weak convergence of (36) to a Gaussian process with the same covariance as that given in (4.5). ∎

Proof of Theorem 11. Note that under the alternative hypothesis (4.9) the same arguments used in the proof of Theorem 1 show that $n^{1/2}\left(\widehat{\beta}-\beta_0\right)$ has the same expansion as that given in (32) plus the additional term

$$\overline{\gamma}=\Gamma^{-1}E\left[\left(W-\Omega\left(U\right)\Sigma^{-1}\left(U\right)X\right)\left(\gamma\left(U,X,W\right)-E\left[\gamma\left(U,X,W\right)|U\right]\right)\right].$$

Thus following the same arguments as those used in the proof of Theorem 7, the projected marked process $n^{1/2}\widehat{v}\left(u,s,\theta\right)$ admits the following expansion

$$n^{1/2}\widehat{v}\left(u,s,\theta\right)=T_{6n}+T_{7n}+T_{\gamma n}+o_p\left(1\right),$$

where

$$
\begin{aligned}
T_{\gamma n} \;=\; & \frac{1}{n}\sum_{i=1}^{n}\left(\gamma\left(U_i,X_i,W_i\right)-E\left[\gamma\left(U,X,W\right)|U_i\right]\right)I\left(U_i\leq u,\theta'\left[X_i',W_i'\right]'\leq s\right)+ \\
& E\left[\left(W-\Omega\left(U\right)\Sigma^{-1}\left(U\right)X\right)'I\left(U\leq u,\theta'\left[X',W'\right]'\leq s\right)\right]\overline{\gamma}.
\end{aligned}
$$

By the LLN $\sup_{u,s,\theta\in\Pi}\left|T_{\gamma n}-\gamma^*\left(u,s,\theta\right)\right|=o_p\left(1\right)$, hence

$$n^{1/2}\widehat{v}\left(u,s,\theta\right)\Rightarrow v_\infty\left(u,s,\theta\right)+\gamma^*\left(u,s,\theta\right),$$

and the conclusion follows by CMT. ∎

Proof of Theorem 12. Let $\sigma_i^*\left(u,s,\theta\right)=\sigma_i^*\left(\widehat{\beta},\widehat{\alpha},\widehat{G};u,s,\theta\right)$ to emphasise the dependence on β, α and G. First we show that

$$\sup_{u,s,\theta\in\Pi}\left|\frac{1}{n^{1/2}}\sum_{i=1}^{n}\sigma_i^*\left(\widehat{\beta},\widehat{\alpha},\widehat{G};u,s,\theta\right)-\frac{1}{n^{1/2}}\sum_{i=1}^{n}\sigma_i^*\left(\beta_0,\alpha_0,G_0;u,s,\theta\right)\right|=o_p\left(1\right).$$

Note that

$$\sup_{u,s,\theta\in\Pi}\left|\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left(\widehat{\varepsilon}_{i\widehat{G}}-\varepsilon_{iG_0}\right)\xi_i I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)\right|\le\sup_{u,s,\theta\in\Pi}\sup_{z}\left|n^{1/2}\frac{\widehat{G}(z)-G_0(z)}{1-G_0(z)}\right|\times$$
$$\sum_{i=1}^{n}\frac{Z_{iG_0}\delta_i\xi_i}{n}I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)+$$
$$\sup_{U\in\mathcal{U}}\left\|\widehat{\alpha}(U)-\alpha_0(U)\right\|\left\|\frac{1}{n^{1/2}}\sum_{i=1}^{n}X_i\xi_i I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)\right\|+$$
$$\left\|\widehat{\beta}-\beta_0\right\|\left\|\frac{1}{n^{1/2}}\sum_{i=1}^{n}W_i\xi_i I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)\right\|=o_p(1)$$

by (35), consistency of $\widehat{\alpha}(U)$, $\widehat{\beta}$ and the fact that $E(\xi)=0$. Similarly it is possible to show the convergence of each of the other terms appearing in $\sigma_i^*\left(\widehat{\beta},\widehat{\alpha},\widehat{G};u,s,\theta\right)$ (see (36)). For example consider the resampled version of the first term in (36), that is

$$T_{71}^*\left(\widehat{G}\right)+T_{72}^*\left(\widehat{G}\right)=\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left\{\frac{\xi_i}{n}\left[\sum_{j=1}^{n}\frac{\eta(Z_j,\delta_j;Z_i)}{1-\widehat{G}(Z_i)}Z_{i\widehat{G}}\delta_i-X_i'\widehat{\Sigma}(U_i)^{-1}E\left(\frac{\eta(Z_j,\delta_j;Z)}{1-\widehat{G}(Z)}Z_{\widehat{G}}\delta|U_i\right)\right]\right\};$$

by (31) and LLN

$$\left|T_{71}^*\left(\widehat{G}\right)-T_{71}^*(G_0)\right|\le\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left|\frac{1}{n}\sum_{j=1}^{n}\left[\frac{\eta(Z_j,\delta_j;Z_i)}{1-\widehat{G}(Z_i)}Z_{i\widehat{G}}\delta_i-\frac{\eta(Z_j,\delta_j;Z_i)}{1-G_0(Z_i)}Z_{iG_0}\delta_i\right]\xi_i\right|\le$$
$$\sup_{0\le z\le\max_i Z_i}\left|\left(\frac{\widehat{G}(z)-G_0(z)}{1-G_0(z)}\right)\right|\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left|\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\eta(Z_j,\delta_j;Z_i)}{1-G_0(Z_i)}Z_{iG_0}\delta_i\xi_i\right)\right|+o_p(1)=O_p(1)\,o_p(1),$$
$$\left|T_{72}^*\left(\widehat{G}\right)-T_{72}^*(G_0)\right|\le\left|\frac{1}{n^{1/2}}\sum_{i=1}^{n}\left\{\frac{1}{n}\sum_{j=1}^{n}X_i'\widehat{\Sigma}(U_i)^{-1}\left[\widehat{E}\left(\frac{\eta(Z_j,\delta_j;Z)}{1-\widehat{G}(Z)}Z_{i\widehat{G}}\delta_i|U_i\right)-\right.\right.\right.$$
$$\left.\left.\left.\widehat{E}\left(\frac{\eta(Z_j,\delta_j;Z)}{1-G_0(Z)}Z_{G_0}\delta|U_i\right)\right]\xi_i\right\}I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)\right|$$
$$\le\sum_{i=1}^{n}\left(\frac{1}{n}X_i'\Sigma(U_i)^{-1}\xi_i\sup_{z}\left|n^{1/2}\left(\frac{\widehat{G}(z)-G_0(z)}{1-G_0(z)}\right)\right|\times\right.$$
$$\left.\sum_{j=1}^{n}\frac{1}{n}\widehat{E}\left[\frac{X_jZ_{jG_0}\delta_j}{1-G_0(Z_j)}|U_i\right]\right)I\left(U_i\le u,\theta'\left[X_i',W_i'\right]'\le s\right)=o_p(1)\left(O_p(1)+O_p(c_n)\right),$$

and thus $\left|T_{71}^*\left(\widehat{G}\right)+T_{72}^*\left(\widehat{G}\right)-T_{71}^*(G_0)-T_{72}^*(G_0)\right|=o_p(1)$. Next by LLN

$$\left|\frac{1}{n}\sum_{i=1}^{n}\sigma_i^*(\beta_0,\alpha_0,G_0;u_1,s_1,\theta_1)\,\sigma_i^*(\beta_0,\alpha_0,G_0;u_2,s_2,\theta_2)-E\left[\sigma(u_1,s_1,\theta_1)\,\sigma(u_2,s_2,\theta_2)\right]\right|=o_p(1),$$

and finally the same arguments used in the proof of Theorem 1 combined with the multiplier CLT (van der Vaart and Wellner 1996, Lemma 2.9.5) imply the fidi convergence of $n^{1/2}\widehat{\sigma}^*(\beta_0, \alpha_0, G_0; u, s, \theta) := \sum_{i=1}^n \sigma_i^*(\beta_0, \alpha_0, G_0; u, s, \theta)/n^{1/2}$ for any fixed u, s and θ, whereas the asymptotic equicontinuity of the process $n^{1/2}\widehat{\sigma}^*(\beta_0, \alpha_0, G_0; u, s, \theta)$ follows as in the proof of Theorem 7. Thus $n^{1/2}\widehat{\sigma}^*(\beta_0, \alpha_0, G_0; u, s, \theta)$ converges weakly to a Gaussian process with the same covariance structure as that given in (4.5), and the result follows by CMT. ■

Proof of Proposition 13. By the results of Lo and Singh (1986), the bootstrap LLN and repeated application of the triangle inequality

$$\left|\frac{RSS_1^*}{n} - \frac{RSS_1}{n}\right| \leq \left|\frac{1}{n}\sum_{i=1}^n \left(\widetilde{Z}_{i\widehat{G}}^* - \widetilde{Z}_{i\widehat{G}} - \widetilde{W}_i'\left(\widehat{\beta}^* - \widehat{\beta}\right)\right)^2\right| +$$

$$\left|\frac{2}{n}\sum_{i=1}^n \left(\widehat{\widetilde{\varepsilon}}_{i\widehat{G}}^*\right)\left(\widetilde{Z}_{i\widehat{G}}^* - \widetilde{Z}_{i\widehat{G}} - \widetilde{W}_i'\left(\widehat{\beta}^* - \widehat{\beta}\right)\right)\right| \leq$$

$$2\sup_{Z_i}\left|\widehat{G}^*(Z_i) - G_0(Z_i)\right|^2 \frac{1}{n}\sum_{i=1}^n (Z_{iG_0}\delta_i)^2 + 2\left\|\left(\widehat{\beta}^* - \widehat{\beta}\right)\right\|^2 \left\|\frac{1}{n}\sum_{i=1}^n \widetilde{W}_i\widetilde{W}_i'\right\| +$$

$$2\sup_{Z_i}\left|\widehat{G}^*(Z_i) - G_0(Z_i)\right| \left|\frac{1}{n}\sum_{i=1}^n \left(\widehat{\widetilde{\varepsilon}}_{i\widehat{G}}\right)\right| + \left\|\left(\widehat{\beta}^* - \widehat{\beta}\right)\right\| \left|\frac{1}{n}\sum_{i=1}^n \left(\widehat{\widetilde{\varepsilon}}_{i\widehat{G}}\right)\right| = o_{p^*}(1)$$

except if the original sample is on a set with probability tending to 0 as $n \to \infty$. The bootstrap CLT and CMT imply

$$n^{1/2}R\left(\widehat{\beta}^* - \widehat{\beta}\right) = R\widehat{\Gamma}^{-1}\frac{1}{n^{1/2}}\sum_{i=1}^n \widetilde{W}_i\widetilde{\varepsilon}_{i\widehat{G}}^* + o_{p^*}(1) \xrightarrow{d^*} N\left(R\Gamma^{-1}\Xi_\beta\Gamma^{-1}R'\right)$$

except if the original sample is on a set with probability tending to 0 as $n \to \infty$, and the conclusion follows immediately. ■

2 Additional figures and tables

Table 1 (continued)

	$h^{cv}/4$			$2h^{cv}$			$4h^{cv}$		
	bias	*std.err.*	*RMSE*	*bias*	*std.err.*	*RMSE*	*bias*	*std.err.*	*RMSE*
				$n = 100$					
$\widehat{\beta}_1^N$	−.307	.165	.348	−.341	.133	.366	−.387	.125	.406
$\widehat{\beta}_1$	−.120	.201	.234	−.143	.180	.229	−.157	.170	.231
$\widehat{\beta}_2^N$	−.105	.176	.204	−.132	.145	.196	−.143	.135	.196
$\widehat{\beta}_2$	−.030	.202	.204	−.045	.186	.191	−.059	.172	.181
				$n = 400$					
$\widehat{\beta}_1^N$	−.302	.070	.310	−.329	.065	.335	−.378	.062	.383
$\widehat{\beta}_1$	−.070	.121	.139	−.076	.095	.121	−.089	.097	.131
$\widehat{\beta}_2^N$	−.100	.103	.143	−.112	.082	.138	−.134	.079	.155
$\widehat{\beta}_2$	−.017	.119	.120	−.023	.108	.110	−.030	.097	.101

Table 2 (continued)

δ	$h^{cv}/4$				$2h^{cv}$				$4h^{cv}$			
	PL_n	EL_n	W_n	PL_n^*	PL_n	EL_n	W_n	PL_n^*	PL_n	EL_n	W_n	PL_n^*
						$n = 100$						
0.0	.065	.062	.057	.048	.081	.067	.063	.055	.088	.077	.069	.058
0.2	.094	.121	.098	.110	.147	.211	.131	.149	.183	.236	.174	.193
0.4	.200	.297	.209	.267	.342	.442	.332	.363	.387	.458	.378	.395
0.8	.578	.645	.586	.599	.691	.770	.685	.695	.712	.796	.723	.721
1.2	.897	.930	.856	.884	.990	1.00	.925	.985	1.00	1.00	1.00	1.00
					$n = 400$							
0.0	.060	.056	.055	.051	.070	.065	.061	.055	.083	.074	.065	.056
0.2	.112	.145	.127	.134	.190	.230	.188	.193	.231	.248	.219	.225
0.4	.287	.322	.329	.352	.467	.500	.452	.470	.472	.534	.458	.489
0.8	.750	.766	.743	.737	.931	1.00	.947	.930	.972	.998	.982	.988
1.2	.935	.946	.855	.987	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 3b. Normal approximation p-values for $PL_n^{\alpha,S}$ for constancy of coefficients

	$H_0^{(1)}$	$H_0^{(2)}$	$H_0^{(3)}$	$H_0^{(4)}$
		$h^{cv/2}$		
PL_n^{α}	.467	2.942	3.001	.321
p-value	.640	.003	.002	.748
		h^{cv}		
PL_n^{α}	.532	2.598	2.697	.391
p-value	.594	.009	.006	.695
		h^{3cv}		
PL_n^{α}	.632	2.259	2.431	.455
p-value	.533	.023	.015	.649

$H_0^{(1)}$:$\alpha_4(U)=\beta_1,\alpha_5(U)=\beta_2$, $H_0^{(2)}$:$\alpha_2(U)=\alpha_2$, $H_0^{(3)}$:$\alpha_3(U)=\alpha_3$
$H_0^{(4)}$:$\alpha_4(U)=\alpha_4$

Table 3b shows that the standardised profile least squares ratio $PL_n^{\alpha,S}$ defined in (3.9) correctly detect the constancy of the linear part of the model while strongly rejecting the constancy of the varying coefficients. Note however that with the large bandwidth the second and third hypothesis cannot be rejected at the 1% significance level.

The dependence of $PL_n^{\alpha,S}$ on the choice of the bandwidth is illustrated in the Q-Q plots of Figures 5 and 6, which show that for a large value of the bandwidth the normal approximation provides a less accurate approximation to the tails of the distribution.

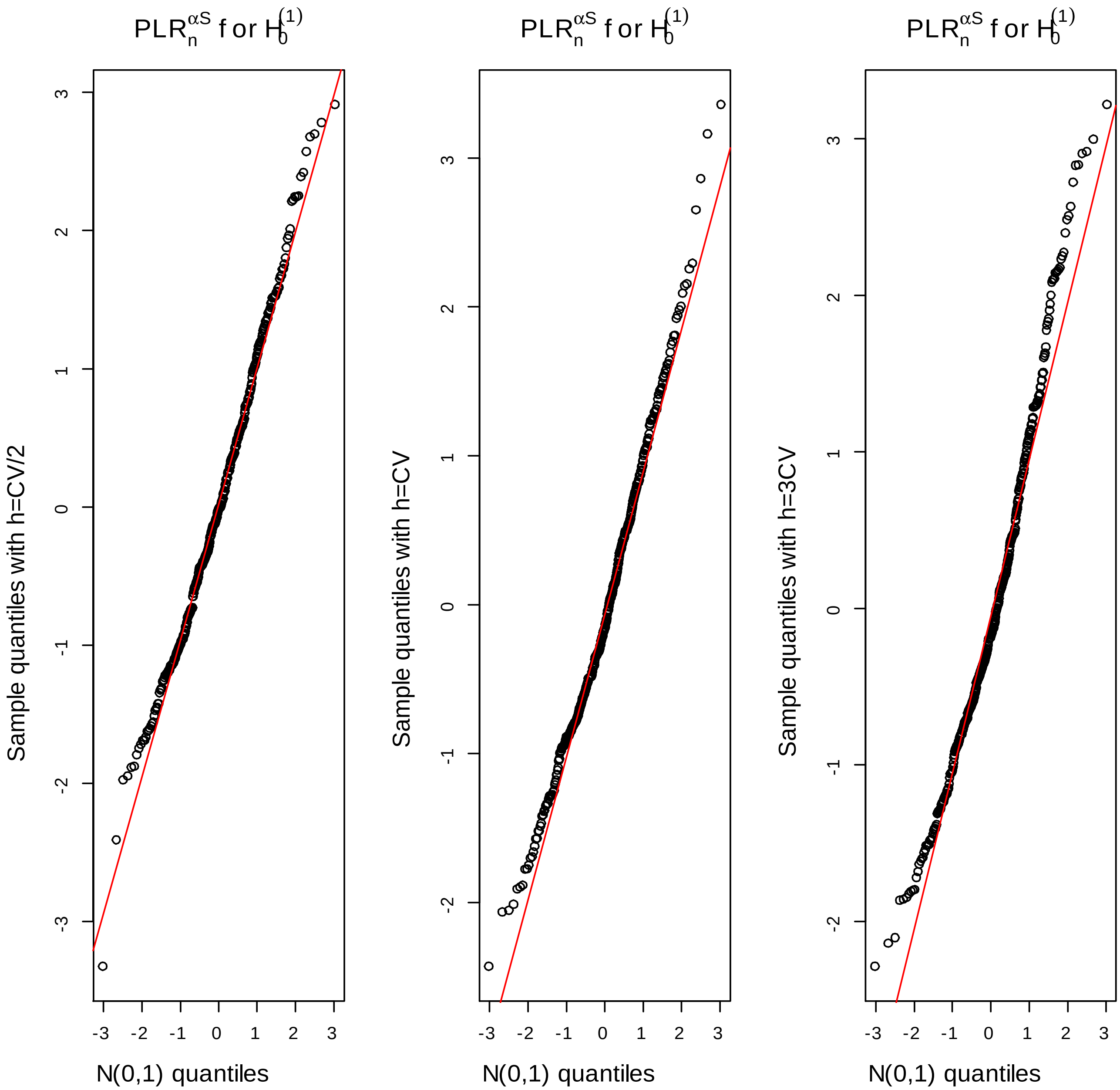

Figure 5: Q-Q plots for standardised $PL_n^{\alpha,S}$ with bandwidths chosen as half (left), equal (centre) and three times (right) the crossvalidated bandwidth.

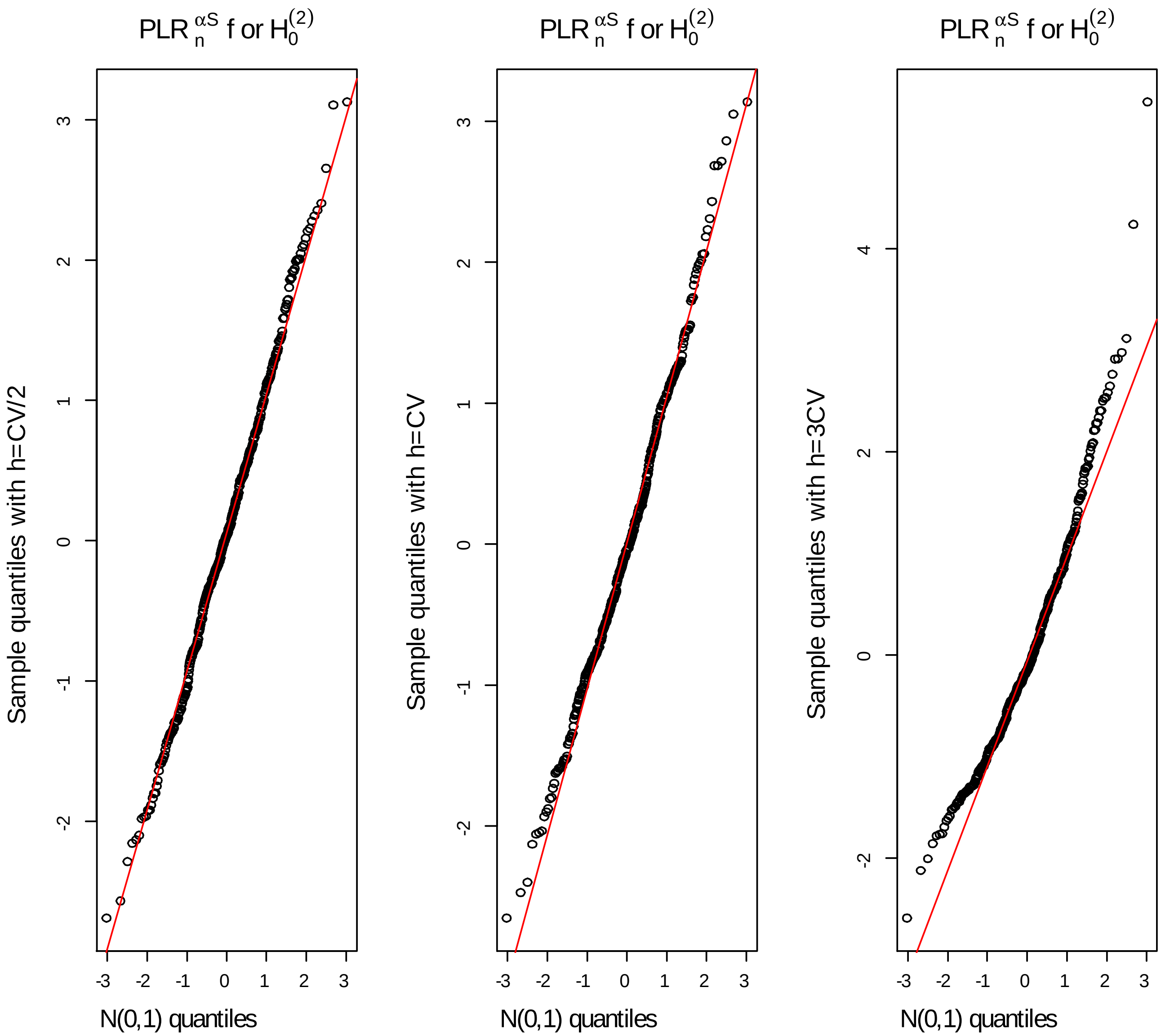

Figure 6: Q-Q plots for standardised $PL_n^{\alpha,S}$ with bandwidths chosen as half (left), equal (centre) and three times (right) the crossvalidated bandwidth.