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Bootstrapping continuous-time autoregressive processes

Peter J. Brockwell · Jens-Peter Kreiss · Tobias Niebuhr

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Abstract We develop a bootstrap procedure for Lévy-driven continuous-time autoregressive (CAR) processes observed at discrete regularly-spaced times. It is well known that a regularly sampled stationary Ornstein–Uhlenbeck process [i.e. a CAR(1) process] has a discrete-time autoregressive representation with i.i.d. noise. Based on this representation a simple bootstrap procedure can be found. Since regularly sampled CAR processes of higher order satisfy ARMA equations with uncorrelated (but in general dependent) noise, a more general bootstrap procedure is needed for such processes. We consider statistics depending on observations of the CAR process at the uniformly-spaced times, together with auxiliary observations on a finer grid, which give approximations to the derivatives of the continuous time process. This enables us to approximate the state-vector of the CAR process which is a vector-valued CAR(1) process, and whose sampled version, on the uniformly-spaced grid, is a multivariate AR(1) process with i.i.d. noise. This leads to a valid residual-based bootstrap which allows replication of CAR(p) processes on the underlying discrete time grid. We show that this approach is consistent for empirical autocovariances and autocorrelations.

Keywords CARMA processes · Lévy process · Bootstrap · Autocovariance

P. J. Brockwell
Department of Statistics, Colorado State University,
Fort Collins, CO 80523-1877, USA
e-mail: peter.brockwell@gmail.com

J.-P. Kreiss · T. Niebuhr (✉)
Institut für Mathematische Stochastik, Technische Universität Braunschweig,
38092 Braunschweig, Germany
e-mail: t.niebuhr@tu-braunschweig.de

J.-P. Kreiss
e-mail: j.kreiss@tu-braunschweig.de

1 Introduction

The modeling of continuous time processes has a long history and has been carried out widely in financial econometrics. Early papers of Doob (1944) and Phillips (1959) deal with representations and properties of Gaussian continuous-time ARMA processes. State-space representations of these processes were exploited by Jones (1980) for dealing with missing values in time series, and by Brockwell (2001) in the study of Lévy-driven continuous time ARMA (CARMA) processes. These allow the modeling of series with a wide variety of marginal distributions including heavy-tailed and asymmetric distributions. Long-memory versions have been developed by Brockwell and Marquardt (2005). One of the important applications of Lévy-driven CARMA processes is in financial econometrics where they have been used as models for spot volatility in stochastic volatility models (Barndorff-Nielsen and Shephard 2001; Brockwell and Lindner 2012). Over the years, the topic of embedding a discrete-time ARMA process in a continuous-time ARMA process has also been studied by a number of authors including Chan and Tong (1987), He and Wang (1989), Huzii (2001), Brockwell (1994), and Brockwell and Lindner (2009), whose results will be important for our work later in this paper. High-frequency sampling of CARMA processes has also been studied by Brockwell et al. (2012) in connection with the extremely high-frequency measurements of turbulent wind speed which are now available. The bootstrap possibilities for this huge class of processes have not previously been investigated.

This article is concerned with bootstrapping statistics of general Lévy-driven CAR processes on general but fixed time grids with spacing $\Delta > 0$. We define the CAR(p) process in Sect. 2 and give an overview of its representations. Afterwards, we will briefly review the results of Cohen and Lindner (2012) who handle equidistant samples of continuous-time moving average processes and give another representation for the limiting variance in their central limit result. This representation will be helpful for proving an analogous bootstrap result later in the paper. Section 3 discusses the Ornstein–Uhlenbeck case and its special characteristics. The bootstrapping of Ornstein–Uhlenbeck processes reduces to a very simple and well-known situation. Unfortunately, this is not the case for higher-order CAR processes. A bootstrap procedure for the general case is then proposed and investigated, concluding with a simulation study in Sect. 4.

2 The CAR model

We define a second-order Lévy-driven CAR(p) process $\{Y(t)\}$ with $p > 0$ and parameters $a_1, \dots, a_p$ to be a stationary solution of the formal differential equation

$$a(D)Y(t) = DL(t), \qquad t \geq 0, \tag{1}$$

where D denotes differentiation with respect to t, L is a second order Lévy process, and the polynomial $a(z)$ is defined by

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p. \tag{2}$$

Since the derivative of the Lévy process $L(t)$ does not exist in the usual sense, we follow the standard approach via the state-space representation of (1) [cf. Brockwell and Lindner 2009 for an overview],

$$Y(t) = \underline{b}^{\mathrm{T}} \underline{X}(t) = X_0(t), \tag{3}$$

$$\mathrm{d}\underline{X}(t) - A\underline{X}(t)\mathrm{d}t = \underline{e}\mathrm{d}L(t), \tag{4}$$

where

$$\underline{X}(t) = \begin{bmatrix} X_0(t) \\ X_1(t) \\ \vdots \\ X_{p-2}(t) \\ X_{p-1}(t) \end{bmatrix}, \quad \underline{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{bmatrix}. \tag{6}$$

Note that (4) is a system of p stochastic differential equations. Except for the last one, all equations are of the same type and give

$$X_j = X_0^{(j)}, \quad j = 0, \dots, p-1. \tag{7}$$

Thus, the components of $\underline{X}$ are the derivatives of the CAR process $Y = X_0$. Every solution of (2.4) satisfies the equation,

$$\underline{X}(t) = \mathrm{e}^{A(t-s)} \underline{X}(s) + \int_s^t \mathrm{e}^{A(t-u)} \underline{e}\, \mathrm{d}L(u) \quad \forall s < t. \tag{8}$$

For the existence of a weakly stationary and causal solution $(Y(t))$ of the equations (2.3) and (2.4) under the assumption $E\, L(1)^2 < \infty$ it is necessary and sufficient that the zeroes $\lambda_1, \lambda_2, \dots, \lambda_p$ of (2) (which coincide with the eigenvalues of the matrix A) all have strictly negative real parts (see Brockwell 2001 or Brockwell 2012, Proposition 1). Under these assumptions the solution is also strictly stationary (cf. Brockwell 2012, Proposition 2). The stationary solution of (2.8) is given by

$$\underline{X}(t) = \int_{-\infty}^t \mathrm{e}^{A(t-u)} \underline{e}\, \mathrm{d}L(u), \tag{9}$$

while the unique weakly and strictly stationary solution of (2.3) and (2.4) reads

$$Y(t) = \underline{b}^{\mathrm{T}}\underline{X}(t) = \int_{-\infty}^{t} \underline{b}^{\mathrm{T}} \mathrm{e}^{A(t-u)}\underline{e}\mathrm{d}L(u) = \int_{-\infty}^{\infty} f(t-u)\mathrm{d}L(u). \tag{10}$$

In (10), the function $f(t) = \underline{b}^{\mathrm{T}}\mathrm{e}^{At}\underline{e}\mathbf{1}_{[0,\infty)}(t)$ is referred to as the kernel of the CAR process $Y(t)$, see e.g. Brockwell et al. (2010) or Cohen and Lindner (2012). This is the reason for making the following assumption.

Assumption 1 (i) The zeroes $\lambda_1, \lambda_2, \ldots, \lambda_p$ of the autoregressive polynomial (2) (which are also the eigenvalues of the matrix A) are all assumed to have strictly negative real parts.
(ii) The driving Lévy process is assumed to have zero mean, variance $\sigma^2 := E\,L(1)^2 < \infty$ and $\eta := \sigma^{-4}\,E\,L(1)^4 < \infty$.

Remark 1 Although $Y = X_0$ is a univariate process, the state representation (3) and (4) characterizes it as the first component of a multivariate state-vector $\underline{X}$, i.e. for $\underline{b}^{\mathrm{T}} = (1, 0, \ldots, 0)$ we have $Y(t) = \underline{b}^{\mathrm{T}}\underline{X}(t)$. This leads us to

$$\varrho_Y(q) = \frac{\gamma_Y(q)}{\gamma_Y(0)} = \frac{\underline{b}^{\mathrm{T}}\underline{\boldsymbol{\Gamma}}(q)\underline{b}}{\underline{b}^{\mathrm{T}}\underline{\boldsymbol{\Gamma}}(0)\underline{b}}, \tag{11}$$

where $\underline{\boldsymbol{\Gamma}}(q)$ denotes the autocovariance matrix of $\underline{X}$ at lag q.

For the estimation and bootstrap procedure, we assume that high-frequency observations are available for the estimation of certain derivatives while our interest lies in the behaviour of the process on a fixed Δ-grid. More detailed comments on the observation structure are given later. For technical reasons in the proof of the ensuing bootstrap procedure, we present another representation of the CAR process $Y(t)$ itself. Using (8) with $t\Delta$ and $(t+1)\Delta$ as bounds of the integral, we obtain

$$\underline{X}((t+1)\Delta) = \mathrm{e}^{A\Delta}\underline{X}(t\Delta) + \int_{t\Delta}^{(t+1)\Delta} \mathrm{e}^{A((t+1)\Delta-u)}\underline{e}\mathrm{d}L(u), \tag{12}$$

that is a vector autoregressive representation (VAR) of order one. Abbreviating the i.i.d. noise sequence by

$$\underline{Z}((t+1-j)\Delta) := \int_{(t-j)\Delta}^{(t+1-j)\Delta} \mathrm{e}^{A((t+1)\Delta-u)}\underline{e}\mathrm{d}L(u)m \quad t \in \mathbb{Z}, \tag{13}$$

and inverting the VAR(1)-representation (12) leads to the following moving average representation of the process $(\underline{X}(t\Delta) : t \in \mathbb{Z})$:

$$\underline{X}((t+1)\Delta) = \sum_{j=0}^{\infty}(\mathrm{e}^{A\Delta})^j\,\underline{Z}((t+1-j)\Delta). \tag{14}$$

Correspondingly, the sampled CAR(p)-process $(Y(t\Delta))$ itself can be written as

$$\begin{aligned} Y((t+1)\Delta) &= \sum_{j=0}^{\infty} \underline{b}^{\mathrm{T}} \mathrm{e}^{A\Delta j} \underline{Z}((t+1-j)\Delta) \\ &= \sum_{j=0}^{\infty} \underline{c}_j^{\mathrm{T}} \underline{Z}((t+1-j)\Delta) \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{p-1} c_{j,i} Z_i((t+1-j)\Delta), \end{aligned} \tag{15}$$

where $\underline{c}_j^{\mathrm{T}} := \underline{b}^{\mathrm{T}} (\mathrm{e}^{A\Delta})^j$, $j \in \mathbb{N}_0$, is a sequence of p-variate coefficients.

Remark 2 It is worth mentioning that the moving average representation (15) varies with Δ, since coefficients $c_{j,i}$ depend on the fixed grid size Δ.

Under the assumption of finite fourth moments and an appropriate Lévy process Cohen and Lindner (2012) investigated continuous-time moving average processes of infinite order if observations are taken on a fixed Δ-grid. Their Theorem 3.3 gives the asymptotic normal distribution for empirical autocovariances and autocorrelations based on observations taken on a fixed Δ-grid. It is worth mentioning that the asymptotic variance–covariance matrix substantially differs from the matrix obtained in discrete time linear process, i.e. discrete time moving average processes of possibly infinite order and, most important, with i.i.d. innovations (cf. Brockwell and Davis 1991, Proposition 7.3.4 and Theorem 7.2.1). If we specialize Theorem 3.3 of Cohen and Lindner (2012) to CAR(p)-processes we obtain:

Proposition 1 *Under Assumption* 1 *we obtain for the empirical autocovariances*

$$\widehat{\gamma}(h) = \frac{1}{n} \sum_{t=0}^{n-h-1} (Y_{t+h} - \overline{Y})(Y_t - \overline{Y}), h = 0, \ldots, n-1, \ \overline{Y} = \frac{1}{n} \sum_{t=0}^{n-1} Y_t, \tag{16}$$

of observations $(Y_t := Y(t\Delta) : t = 0, \ldots, n-1)$ *stemming from a CAR(p) process sampled on a fixed* Δ*-grid*

$$\sqrt{n}(\widehat{\gamma}_Y(0) - \gamma_Y(0), \ldots, \widehat{\gamma}_Y(q) - \gamma_Y(q))^{\mathrm{T}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, V), n \to \infty, \tag{17}$$

where the variance–covariance matrix $V = (v_{q_1,q_2,\Delta})_{q_1,q_2=0,\ldots,q} \in \mathbb{R}^{q+1,q+1}$ *has components*

$$v_{q_1,q_2,\Delta} = \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\sum_{i_3=0}^{p-1}\sum_{i_4=0}^{p-1}\sum_{j=0}^{\infty} c_{j,i_1}c_{j+q_1,i_2}\sum_{r=0}^{\infty} c_{j+r,i_3}c_{j+r+q_2,i_4}$$
$$\cdot\left(\kappa^4_{i_1,i_2,i_3,i_4} - E[Z_{i_1}Z_{i_2}]E[Z_{i_3}Z_{i_4}] - E[Z_{i_1}Z_{i_3}]E[Z_{i_2}Z_{i_4}] - E[Z_{i_1}Z_{i_4}]E[Z_{i_2}Z_{i_3}]\right)$$
$$+\sum_{r=-\infty}^{\infty}\{\gamma_Y(r\Delta)\gamma_Y((q_2+r-q_1)\Delta)+\gamma_Y((r+q_2)\Delta)\gamma_Y((r-q_1)\Delta)\}, \quad (18)$$

and $\kappa^4_{i_1,i_2,i_3,i_4} := E[Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}]$. *We use the abbreviation* $(Z_0,\ldots,Z_{p-1})^{\mathrm{T}} = \underline{Z}(\Delta)$ *and denote by* $c_{j,i}$ *the* i*th component of the vector* $\underline{b}^{\mathrm{T}}\left(e^{A\Delta}\right)^j$.

Remark 3 Of course Proposition 1 together with the delta method immediately leads to asymptotic normality of the empirical autocorrelations, $\widehat{\varrho}(h) = \widehat{\gamma}(h)/\widehat{\gamma}(0)$, $h = 1, 2, \ldots$. Comparing the expression (18) with the asymptotic covariance matrix of the empirical autocovariances of a discrete time linear process (cf. Brockwell and Davis 1991, Proposition 7.3.4), we see that both expressions consist of two summands. Although the second summands coincide, the first summands differ substantially. In Bartlett's formula for discrete time linear processes, the asymptotic covariance matrix depends only on the autocorrelation function of the process. This convenient property fails to hold in the continuous time setting. Thus, in contrast with the expression in Brockwell and Davis (1991), Theorem 7.2.1, we obtain for the limiting covariance matrix

$$\sqrt{n}(\widehat{\varrho}_Y(0)-\varrho_Y(0),\ldots,\widehat{\varrho}_Y(q)-\varrho_Y(q))^{\mathrm{T}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,W),\, n\to\infty, \quad (19)$$

where the limiting covariance matrix $W = (w_{q_1,q_2})_{q_1,q_2=0,\ldots,q} \in \mathbb{R}^{q+1,q+1}$ reads

$$w_{q_1,q_2} = \frac{(\eta-3)\sigma^4}{\gamma_Y(0)^2}\int_0^1\left(\sum_{k=-\infty}^{\infty} f(u+k)f(u+k+q_1) - \varrho_Y(q_1)f(u+k)^2\right)$$
$$\cdot\left(\sum_{l=-\infty}^{\infty} f(u+l)f(u+l+q_2) - \varrho_Y(q_2)f(u+l)^2\right)\mathrm{d}u$$
$$+\sum_{k=1}^{\infty}[\varrho_Y(k+q_1)+\varrho_Y(k-q_1)-2\varrho_Y(q_1)\varrho_Y(k)]$$
$$\cdot[\varrho_Y(k+q_2)+\varrho_Y(k-q_2)-2\varrho_Y(q_2)\varrho_Y(k)]. \quad (20)$$

Here, as above, $f(t) = \underline{b}^{\mathrm{T}}\mathrm{e}^{At}\underline{e}\mathbf{1}_{[0,\infty)}(t)$. For details, we refer to Cohen and Lindner (2012) and, for a corresponding phenomenon for discrete-time linear processes observed at lower frequencies, to Niebuhr and Kreiss (2012). □

3 Bootstrap procedure

First we consider the simplest case, the CAR(1) or stationary Ornstein–Uhlenbeck process. In this case, both the observation and state equation (3) and (4) reduce to a single one-dimensional equation for $Y(t\Delta) = X(t\Delta)$. Namely

$$X((t+1)\Delta) = \mathrm{e}^{-a\Delta} X(t\Delta) + Z((t+1)\Delta), \qquad t = 0, \ldots, n-1. \tag{21}$$

Thus, every equidistantly discretely (fixed Δ-grid) sampled CAR(1) process is a first-order autoregressive process with i.i.d. innovations. This of course is a very well-studied process in time series analysis.

The autoregressive parameter $\mathrm{e}^{-a\Delta}$ can be $\sqrt{n}$-consistently estimated using the Yule-Walker method, which immediately leads to a $\sqrt{n}$-consistent estimator of the continuous time parameter a via

$$\widehat{a} = \frac{-\log\left(\widehat{\gamma}(\Delta)\,/\,\widehat{\gamma}(0)\right)}{\Delta}. \tag{22}$$

Residual-based or wild bootstrap proposals are well understood for such cases and immediately lead to consistent bootstrap procedures for discretely observed Ornstein–Uhlenbeck processes.

Recall that our interest is to setup a bootstrap procedure which is able to consistently approximate distributions of statistics that depend on observations on a fixed Δ-grid of the CAR process, only. Without loss of generality let us assume $\Delta = 1$. Consider as an important example empirical autocovariances $\widehat{\gamma}(h)$, cf. (16), or empirical autocorrelations. As Proposition 1 shows, the asymptotic variance of such quantities depends in a quite complicated way on properties of the underlying continuous time process, which is quite difficult to estimate from discrete time observations. Thus, it appears that there is some room for a bootstrap procedure.

Extending the simple approach described above for CAR(1) processes to deal with CAR(p) processes with $p > 1$ presents serious difficulties. It is well known (see e.g. Brockwell 1994; Huzii 2001) that, from a second-order point of view, every discretely sampled CARMA(p, q) process can be represented as a stationary solution of ARMA(p, q') equations with $q' < p$. Brockwell and Lindner (2009) give the stronger result that the discretely sampled observations of a CARMA(p, q) process satisfy autoregressive equations of order p with driving noise which is $(p-1)$-dependent. Since every $(p-1)$-dependent sequence has a moving average representation of order at most $(p-1)$ driven by white noise which is uncorrelated but not necessarily (except when $p = 1$) independent, our observations will satisfy an ARMA($p, p-1$) equation with innovations which are uncorrelated only. Thus, a residual bootstrap as described in Kreiss and Franke (1992) using a standard ARMA($p, p-1$)-model fit to the observations and a resampling via drawing with replacement from residuals from this fit will lead to consistent results only if statistics are considered whose asymptotic distribution depends only on second order properties, i.e. on the autocovariance structure of the observations. This is because a given ARMA-model has autocovariances which are the same whether the driving noise is independent or simply uncorrelated. A simple example is the sample mean, $\overline{Y}$. Central limit results for $\overline{Y}$ can be established under quite general assumptions, which typically are satisfied for discretely observed CAR(p) processes. Since the asymptotic variance of $\overline{Y}$ depends only on the autocovariance function of Y,every bootstrap proposal which mimics the second-order properties of the underlying observations will work asymptotically. But in all cases in which the asymptotic distribution of a statistic of interest depends on properties that go beyond second-order properties, such a simple ARMA-based residual bootstrap procedure for discretely observed CAR(p)-processes would fail! In Proposition 1 and Remark 3, we have seen that for empirical autocovariances and more interestingly, even for empirical autocorrelations, features of the process beyond second-order properties show up in the asymptotic distribution and this fact therefore directly implies that a standard residual-based ARMA bootstrap does not work at all in such situations.

The block bootstrap (cf. Künsch 1989 and Bühlmann and Künsch 1995), which has been shown to work for rather general strictly stationary processes, is a possibility to overcome this problem. However, we intend to follow in this paper a different approach, which tries to take existing parametric structure as much as possible into account. Moreover block bootstrap techniques have to deal with quite delicate problems around a proper choice of the block length (e.g. Nordman et al. (2007)). Instead of dealing with block bootstrap methods we focus on an i.i.d. based bootstrap proposal influenced by the ideas of Kreiss and Franke (1992) and Paparoditis (1996).

To be able to apply such a residual-based bootstrap, we make use of the vector autoregressive representation obtained from (12), namely

$$\underline{X}((t+1)\Delta) = \mathrm{e}^{A\Delta}\underline{X}(t\Delta) + \underline{Z}((t+1)\Delta). \tag{23}$$

In this vector autoregressive representation, the driving white noise $\underline{Z}(t+1)$ [cf. (13)] indeed is an i.i.d. noise sequence.

Our strategy now is to estimate the first $p-1$ derivatives $X_1(t\Delta), \ldots, X_{p-1}(t\Delta)$ of the CAR process $X_0(\cdot)$, which represent the back $p-1$ components of the vector $\underline{X}(t\Delta)$, and to use them to estimate the autoregressive parameter matrix $\mathrm{e}^{A\Delta}$. Having done this, we immediately are able to define estimated autoregressive residuals on which an asymptotically consistent residual based bootstrap can be set up.

To this end, let us assume that we are able to observe some additional auxiliary high-frequency data, but must point out that no full-time high-frequency data is needed. More precisely, we assume the following observation structure:

$$\begin{array}{llll} Y_{1\Delta-(p-1)h}, & \ldots, & Y_{1\Delta-h}, & Y_{1\Delta}, \\ Y_{2\Delta-(p-1)h}, & \ldots, & Y_{2\Delta-h}, & Y_{2\Delta}, \\ \vdots & \vdots & \ldots & \vdots \\ Y_{n\Delta-(p-1)h}, & \ldots, & Y_{n\Delta-h}, & Y_{n\Delta}. \end{array} \tag{24}$$

In (24), $\Delta > 0$ still is the fixed grid size of our *main* observations $Y_{1\Delta}, \ldots, Y_{n\Delta}$. The auxiliary $p-1$ pre-observations are on a much finer high-frequency grid of mesh size h, for which we will assume later that $h \to 0$ as $n \to \infty$. This results in a local high-frequency aided low-frequency sampling scheme (cf. Fig. 1).

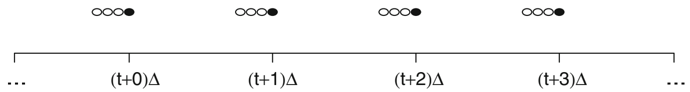

Fig. 1 Local high-frequency aided low frequency sampling scheme

Remark 4 At first glance, the supposed data structure [cf. (24) or Fig. 1] needed for the local high-frequency-aided bootstrap proposal suggested below seems somewhat strange. The following two examples show to what kind of situations our bootstrap proposal is applicable. As a first example from financial econometrics one might be interested in fitting CAR(p) models on the basis of daily return data. Assume that $\Delta = 1$ stands for one-day length. For the application of our bootstrap proposal, it is necessary to be able to additionally observe some more frequent intraday data, e.g. hourly, 30- or 15-min return values. This would lead to values $h = 1/24$, $1/48$ or $1/96$, respectively. Such higher frequency returns are available in many cases (e.g. for currency exchange rates and stock indices such as Dow Jones, S&P 500, FTSE 100, Nikkei or DAX). Alternatively, we may have complete high-frequency intraday return data available, e.g. at equidistant intervals of 15 min. Then our bootstrap proposal allows us to mimic the distribution of autocovariances and autocorrelations for lags $q \cdot \Delta$, $q = 0, 1, \ldots$ on a coarser time grid (e.g. $\Delta = 24$, which corresponds to 6 h). □

For simplicity and easier understanding, we assume all local high-frequency additional observations to be on the same time grid of mesh size h.

Based on the observations (24), the derivatives of the process be estimated consistently by Proposition 5.1 of Brockwell and Schlemm (2011) using iterated difference quotients for the first $p - 1$ derivatives. More precisely, we define

$$\widehat{X_s}(t) := \frac{1}{h^s} \sum_{i=0}^{s} X_0(t - ih)(-1)^i \binom{s}{i} \quad s = 0, \ldots, p-1. \tag{25}$$

If the driving Lévy process L is assumed to have finite second moments then Proposition 5.6 of Brockwell and Schlemm (2011) gives

$$\widehat{X_s}(t) = X_s(t) + \mathcal{O}_{\mathcal{P}}(h) \quad s = 0, \ldots, p-1. \tag{26}$$

The vectors

$$\underline{\widehat{X}}^{\mathrm{T}}(t\Delta) = \left(X_0(t\Delta), \widehat{X}_1(t\Delta), \ldots, \widehat{X}_{p-1}(t\Delta)\right)^{\mathrm{T}} \tag{27}$$

are used to estimate the autoregressive parameter matrix, e.g. by the classical Yule-Walker equations. Thus,

$$\widehat{\mathrm{e}^{A\Delta}} = \underline{\widehat{\Gamma}}(\Delta)\underline{\widehat{\Gamma}}^{-1}(0), \tag{28}$$

where $\underline{\widehat{\Gamma}}(\Delta) = \frac{1}{n}\sum_{t=1}^{n-\Delta}\left(\underline{\widehat{X}}((t+1)\Delta) - \overline{\underline{\widehat{X}}}\right)\left(\underline{\widehat{X}}(t\Delta) - \overline{\underline{\widehat{X}}}\right)^{\mathrm{T}}$.

Even though for our purposes, it would be sufficient to have a consistent estimator of the autoregressive parameter matrix, inspection of the results of Brockwell and Schlemm (2011) leads to the following result

Lemma 1 *Under Assumption* 1 *and if* $h = h(n) \to 0$ *as* $n \to \infty$ *we obtain*

$$\widehat{e^{A\Delta}} = e^{A\Delta} + \mathcal{O}_{\mathcal{P}}(h + n^{-1/2}). \tag{29}$$

Remark 5 We emphasize that the autoregessive parameter matrix $\mathrm{e}^{A\Delta}$ is estimated directly and not via an estimator of the matrix A itself composed with the matrix exponential function $\mathrm{e}^{\cdot\Delta}$. Even if estimation of A was possible, the direct estimation of $\mathrm{e}^{A\Delta}$ via Yule-Walker equations is much simpler in practice. Moreover, it is well known that under very mild conditions the Yule-Walker estimate has eigenvalues with absolute value <1. This fact is a great advantage for the bootstrap procedure to be defined below. Moreover, except in the simple case when $p = 1$, it is not immediately evident that for every fixed Δ the matrix exponential $\mathrm{e}^{\cdot\Delta}$ can be inverted to produce from the estimator $\widehat{\mathrm{e}^{A\Delta}}$ a uniquely defined estimator $\widehat{A}$ of the matrix A such that $\widehat{A}$ satisfies Assumption 1 (i).

The above considerations lead to the following bootstrap algorithm which is used to generate pseudo-observations $Y^*(\Delta), Y^*(2\,\Delta), \ldots, Y^*(n\,\Delta)$ of the continuous-time CAR(p) process $(Y(t) : t \geq 0)$.

Step 1: Let $\widehat{\mathrm{e}^{A\Delta}}$ denote a consistent estimator of $\mathrm{e}^{A\Delta}$. Obtain estimated residuals from

$$\underline{\widehat{Z}}(t\Delta) = \underline{X}(t\Delta) - \widehat{\mathrm{e}^{A\Delta}}\underline{X}((t-1)\Delta),\ t = 1, \ldots n. \tag{30}$$

Step 2: Generate $(\underline{Z}^*(t\,\Delta))$ via drawing with replacement from the centered estimated innovations $\underline{\widehat{Z^c}}(\Delta), \ldots, \underline{\widehat{Z^c}}(n\,\Delta)$, where $\underline{\widehat{Z^c}}(t\,\Delta) = \underline{\widehat{Z}}(t\,\Delta) - 1/n\,\sum_{j=1}^{n} \underline{\widehat{Z}}(j\,\Delta)$.
Step 3: Obtain pseudo-observations $\underline{X}^*(t\Delta), t = 1, \ldots, n$ of the vector autoregressive process from

$$\underline{X}^*(t\Delta) = \widehat{\mathrm{e}^{A\Delta}}\underline{X}^*((t-1)\Delta) + \underline{Z}^*(t\,\Delta). \tag{31}$$

Step 4: Finally obtain pseudo-observations $Y^*(\Delta), Y^*(2\,\Delta), \ldots, Y^*(n\,\Delta)$ according to

$$Y^*(t\,\Delta) = (1, 0, \ldots, 0)\,\underline{X}^*(t\Delta),\ t = 1\ldots, n. \tag{32}$$

Exactly as for the vector autoregressive process (12) we obtain for the bootstrapped vector autoregression $(\underline{X}^*(t\Delta))$ a moving average representation of the form

$$\underline{X}^*((t+1)\Delta) = \sum_{j=0}^{\infty} \widehat{\mathrm{e}^{A\Delta j}}\,\underline{Z}^*((t+1-j)\Delta). \tag{33}$$

This directly leads to a bootstrap analogon of (15)

$$Y^*((t+1)\Delta) = \sum_{j=0}^{\infty}\sum_{i=0}^{p-1} \widehat{c}_{j,i} Z_i^*((t+1-j)\Delta), \tag{34}$$

with $\widehat{\underline{c}}_j^{\mathrm{T}} := \underline{b}^{\mathrm{T}}\widehat{\mathrm{e}^{A\Delta j}}$, $j \in \mathbb{N}_0$.

In Sect. 6, we prove the following result, which states that our bootstrap proposal works asymptotically for statistics depending smoothly on autocovariances or autocorrelations.

Theorem 1 *Let Y be a CAR(p) process and let Assumption 1 be satisfied. Further assume the local high-frequency-aided sampling scheme (24) with h satisfying $h = h(n) \to 0$ as $n \to \infty$ and let $Y(t\,\Delta)^*$ be a bootstrap process generated as described above. Then we have in probability as $n \to \infty$*

(i) *For each $q_1, q_2 \in \mathbb{N}_0$ and $\widehat{\gamma}^*(q_j\Delta) = n^{-1}\sum_{t=1}^{n-h} Y^*(t\Delta)Y^*((t+q_j)\Delta)$, $j = 1, 2$,*

$$\lim_{n\to\infty} n\,\mathrm{Cov}(\widehat{\gamma}^*(q_1\Delta), \widehat{\gamma}^*(q_2\Delta)) \to v_{q_1,q_2,\Delta}, \tag{35}$$

where $v_{q_1,q_2,\Delta}$ is defined as in Proposition 1.

(ii) *Further for each $q \in \mathbb{N}_0$*

$$\sqrt{n}\left(\widehat{\gamma}^*(0) - \gamma_{Y^*}(0), \ldots, \widehat{\gamma}^*(q\Delta) - \gamma_{Y^*}(q\Delta)\right) \stackrel{\mathcal{D}}{\to} \mathcal{N}(0, V), \tag{36}$$

where $\gamma_{Y^*}(\cdot)$ denotes the autocovariance function of the bootstrap process $(Y^*(t\Delta))$ [cf. (32)] and $V = (v_{q_1\Delta,q_2\Delta})_{q_1,q_2=0,\ldots,q}$.

(iii) *Moreover,*

$$\sqrt{n}\left(\widehat{\varrho}^*(0) - \varrho_{Y^*}(0), \ldots, \widehat{\varrho}^*(q\Delta) - \varrho_{Y^*}(q\Delta)\right) \stackrel{\mathcal{D}}{\to} \mathcal{N}(0, W), \tag{37}$$

where $\varrho_{Y^*}(\cdot)$ denotes the autocovariance function of the bootstrap process $(Y^*(t\Delta))$ and W is given in Remark 3. □

Remark 6 Since the bootstrap procedure proposed in Sect. 3 mimics the true underlying vector autoregressive process (including the distribution of the errors), it can be expected that the validity of our bootstrap proposal goes far beyond statistics which are smooth functionals of empirical autocovariances and autocorrelations. Especially for integrated periodograms (cf. Dahlhaus 1985), nonparametric spectral density estimation and the general class of estimators

$$T_n = f\left(\frac{1}{n-m+1}\sum_{t=1}^{n-m+1} g\left(Y(t\Delta), \ldots, Y((t+m-1)\Delta)\right)\right), \tag{38}$$

discussed in Künsch (1989), cf. Example 2.2; for $g : \mathbb{R}^m \to \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$, our proposal will lead to a consistent approximation of the distribution of the corresponding statistics. For the latter class of statistics, Bühlmann (1997) proved validity of the AR-sieve bootstrap under the main assumption of an invertible linear process in discrete time with i.i.d. innovations for the underlying process. One should keep in mind that the proposed bootstrap procedure is aimed at approximate distributions of statistics that can be written as functionals of discretely observed data (fixed Δ-grid) from a CAR(p) process of known order p. The additional high-frequency pre-observations preceding each time point $t\,\Delta$ are only auxiliary values to approximate derivatives of the underlying process at the time points $t\,\Delta$. □

4 Simulation study

In this Section, we present the results of a simulation study for CAR(2) processes. We simulated a CAR(2) process with parameters $a_1 = -1.0525$ and $a_2 = -1.5$. A Wiener process with variance 1 was used as the underlying driving Lévy process. Figure 2 shows a typical realization of such a process. Note that the smooth appearing of the sample path is quite expected because CAR(2) processes are differentiable.

We set $n = 150$ and $\Delta = 1$ and investigated the finite sample distribution of the first-order autocorrelation

$$\sqrt{n}\,(\widehat{\rho}(1) - \rho(1))\,, \tag{39}$$

based on observations as given in (24), and the ability of the proposed bootstrap proposal to approximate this distribution. We simulated (39) 1,500 times to get an appropriate approximation of the finite sample distribution. The histograms in red color in Fig. 3 show this simulated finite sample distribution of (39). The histograms in grey are bootstrap distributions showing average performance. Showing average performance in this context means that we have simulated 1,000 bootstrap distributions and have calculated their distance to the true distribution (histogram in red color). The grey histogram plots now represent bootstrap distributions belonging to the lower quartile, the median and the upper quartile of distances, respectively.

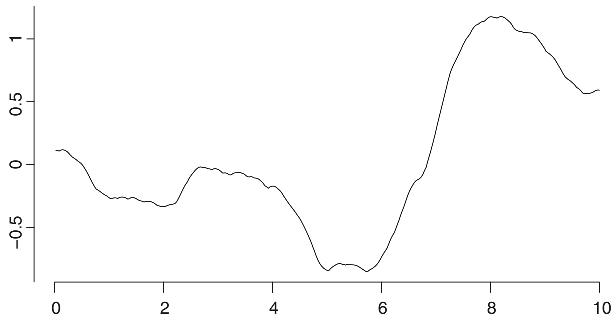

Fig. 2 Typical realization of a CAR(2) process

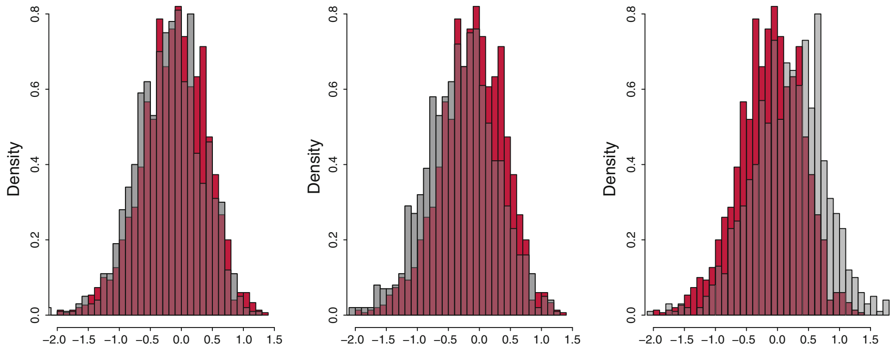

Fig. 3 Average bootstrap performance (lower quartile, median and upper quartile distance). True distribution (*red*) and bootstrap approximations (*light grey*). Overlapping area of both histograms is in *dark grey*

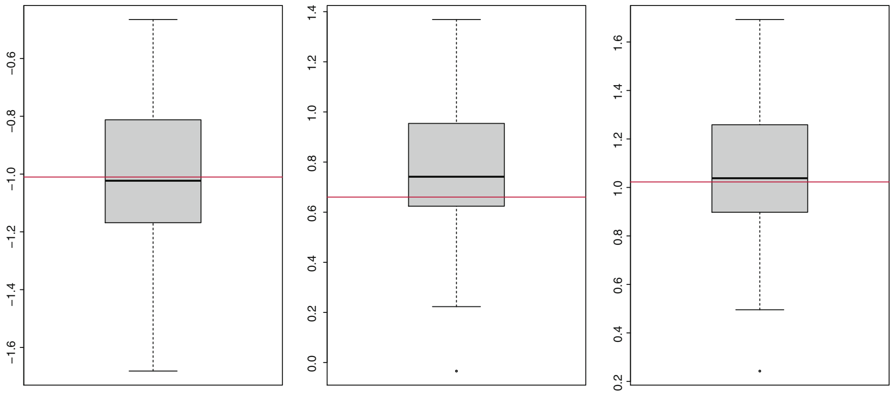

Fig. 4 Boxplots of 5, 95 and 99 % Bootstrap quantiles of $\mathcal{L}(\sqrt{n}(\widehat{\rho}(1) - \rho(1)))$ (*left to right*) with true quantiles (in *red*)

Even if we have not incorporated the limiting normal distribution in the simulation, because the limiting variance (cf. Proposition 1) is quite difficult to compute, it appears that the true distribution shows a significant skewness and thus it can be expected that the limiting normal distribution will possess certain approximation errors. Further, Fig. 4 shows boxplots of generated bootstrap 5, 95 and 99 % quantiles. The added red lines represent the corresponding true quantiles obtained by simulation.

5 Conclusion

We have proposed a bootstrap procedure which is applicable to discrete time (fixed Δ-grid) observations from CAR processes. Starting from the Ornstein–Uhlenbeck process as the simplest CAR process, for which a consistent bootstrap procedure easily can be defined, we have seen that the situation becomes much more complicated for samples from CAR processes of higher order. Using some auxiliary high frequency pre-observations preceding every discrete low frequency time point, we make use of the fact that the process together with its first $p - 1$ derivatives can be written as a vector autoregressive process of order one and, most important for the bootstrap, with i.i.d. innovations. On this basis, a bootstrap procedure has been proposed and the asymptotic validity has been shown for empirical autocovariances and empirical autocorrelations. It has been pointed out, that the normal approximation for the distributions of empirical autocovariances and empirical autocorrelations differs from that for linear time series in discrete time and that even the asymptotic variance of such limiting normal distributions hardly can be estimated from low-frequency data. A small simulation study has shown that the proposed bootstrap proposal works appropriately.

6 Proofs

Proof of Proposition 1 Since all the main arguments for a proof of Proposition 1 have been given in Cohen and Lindner (2012), we restrict ourselves to verify the representation (18) which differs from the representation given in Cohen and Lindner (2012) but is needed for the proof of our main result.

We make heavy use of (15) in the following. Obviously $E[Y(t\Delta)] = 0$. Further, we obtain

$$\gamma_Y(h\Delta) = E[Y(t\Delta)Y((t+h)\Delta)] = \sum_{j=0}^{\infty}\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1} c_{j,i_1}c_{j+h,i_2}E[Z_{i_1}Z_{i_2}]. \quad (40)$$

With the notation $\kappa^4_{i_1,i_2,i_3,i_4} = E[Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}]$, we can compute directly

$$\begin{aligned} &E[Y(t\Delta)Y((t+q_1)\Delta)Y((t+h+q_1)\Delta)Y((t+h+q_1+q_2)\Delta)] \\ &= \sum_{j=0}^{\infty}\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\sum_{i_3=0}^{p-1}\sum_{i_4=0}^{p-1} c_{j,i_1}c_{j+q_1,i_2}c_{j+q_1+h,i_3}c_{j+q_1+h+q_2,i_4} \\ &\quad \cdot \Big(\kappa^4_{i_1,i_2,i_3,i_4} - E[Z_{i_1}Z_{i_2}]E[Z_{i_3}Z_{i_4}] \\ &\quad - E[Z_{i_1}Z_{i_3}]E[Z_{i_2}Z_{i_4}] - E[Z_{i_1}Z_{i_4}]E[Z_{i_2}Z_{i_3}]\Big) \\ &\quad + \gamma_Y(q_1\Delta)\gamma_Y(q_2\Delta) + \gamma_Y((q_1+h)\Delta)\gamma_Y((h+q_2)\Delta) \\ &\quad + \gamma_Y((q_1+h+q_2)\Delta)\gamma_Y(h\Delta). \end{aligned} \quad (41)$$

This last representation corresponds to Eq. (3.5) in Cohen and Lindner (2012). The next step is to compute the asymptotic behaviour of $nCov(\widehat{\gamma}_Y(p\Delta), \widehat{\gamma}_Y(q\Delta))$. Observe that

$$n\,Cov(\widehat{\gamma}_Y(q_1\Delta), \widehat{\gamma}_Y(q_2\Delta)) = \sum_{r=-(n-1)}^{n-1}\left(1-\frac{|r|}{n}\right)T_r + o(1), \quad (42)$$

where

$$T_r = \sum_{j=0}^{\infty}\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\sum_{i_3=0}^{p-1}\sum_{i_4=0}^{p-1} c_{j,i_1}c_{j+q_1,i_2}c_{j+r,i_3}c_{j+r+q_2,i_4}$$
$$\cdot\Big(\kappa^4_{i_1,i_2,i_3,i_4} - E[Z_{i_1}Z_{i_2}]E[Z_{i_3}Z_{i_4}] - E[Z_{i_1}Z_{i_3}]E[Z_{i_2}Z_{i_4}] - E[Z_{i_1}Z_{i_4}]E[Z_{i_2}Z_{i_3}]\Big)$$
$$+\gamma_Y(r\Delta)\gamma_Y((q_2+r-q_1)\Delta) + \gamma_Y((r+q_2)\Delta)\gamma_Y((r-q_1)\Delta). \tag{43}$$

This yields (18). □

Now we come to the proof of our main result.

Proof of Theorem 1 The computation of the asymptotic covariance matrix is exactly as in the proof of Proposition 1. We obtain

$$n\,Cov(\widehat{\gamma}_{Y^*}(q_1\Delta), \widehat{\gamma}_{Y^*}(q_2\Delta))$$
$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\Bigg[\sum_{j=0}^{\infty}\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\sum_{i_3=0}^{p-1}\sum_{i_4=0}^{p-1}\widehat{c}_{j,i_1}\widehat{c}_{j+q_1,i_2}\widehat{c}_{j+s-t,i_3}\widehat{c}_{j+s-t+q_2,i_4}$$
$$\cdot\Big(\kappa^{*4}_{i_1,i_2,i_3,i_4} - E^*[Z^*_{i_1}Z^*_{i_2}]E^*[Z^*_{i_3}Z^*_{i_4}]$$
$$-E^*[Z^*_{i_1}Z^*_{i_3}]E^*[Z^*_{i_2}Z^*_{i_4}] - E^*[Z^*_{i_1}Z^*_{i_4}]E^*[Z^*_{i_2}Z^*_{i_3}]\Big)\Bigg]$$
$$+\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\big[\gamma_{Y^*}((s-t)\Delta)\gamma_{Y^*}((q_2+s-t-q_1)\Delta)$$
$$+\gamma_{Y^*}((s-t+q_2)\Delta)\gamma_{Y^*}((s-t-q_1)\Delta)\big] + o_P(1)$$
$$= \sum_{r=-(n-1)}^{n-1}\left(1-\frac{|r|}{n}\right)T^*_r + o_P(1). \tag{44}$$

Here, $\kappa^{4*}_{i_1,i_2,i_3,i_4} = E^*[Z^*_{i_1}Z^*_{i_2}Z^*_{i_3}Z^*_{i_4}]$ and

$$T^*_r = \sum_{j=0}^{\infty}\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\sum_{i_3=0}^{p-1}\sum_{i_4=0}^{p-1}\widehat{c}_{j,i_1}\widehat{c}_{j+q_1,i_2}\widehat{c}_{j+r,i_3}\widehat{c}_{j+r+q_2,i_4}$$
$$\cdot\Big(\kappa 4^*_{i_1,i_2,i_3,i_4} - E^*[Z^*_{i_1}Z^*_{i_2}]E^*[Z^*_{i_3}Z^*_{i_4}] - E^*[Z^*_{i_1}Z^*_{i_3}]E^*[Z^*_{i_2}Z^*_{i_4}]$$
$$-E^*[Z^*_{i_1}Z^*_{i_4}]E^*[Z^*_{i_2}Z^*_{i_3}]\Big)$$
$$+\gamma_{Y^*}(r\Delta)\gamma_{Y^*}((q_2+r-q_1)\Delta) + \gamma_{Y^*}((r+q_2)\Delta)\gamma_{Y^*}((r-q_1)\Delta). \tag{45}$$

Because of Assumption 1 (i), the parameter matrix $e^{A\Delta}$ only has eigenvalues within the unit circle. The same holds for $\widehat{e^{A\Delta}}$ reasoned by well-known properties of the Yule-Walker method (cf. Remark 5). Thus, the matrix polynomials $I_p - e^{A\Delta}z$ and $I_p - \widehat{e^{A\Delta}}z$

only have roots outside the closed unit circle, and $(I_p - \mathrm{e}^{A\Delta}z)^{-1}$, $(I_p - \widehat{\mathrm{e}^{A\Delta}z})^{-1}$ can be written as power series $\sum_{j=0}^{\infty}(\mathrm{e}^{A\Delta})^j z^j$, $\sum_{j=0}^{\infty}(\widehat{\mathrm{e}^{A\Delta}})^j z^j$, respectively, for all $|z| \leq 1+\delta$ and some $\delta > 0$. Using a multidimensional version of Cauchy's inequality for holomorphic functions, we obtain

$$\sup_{j\in\mathbb{N}_0} (1+\delta)^j \, \|(\widehat{\mathrm{e}^{A\Delta}})^j - (\mathrm{e}^{A\Delta})^j\| = \mathcal{O}_P(h + n^{-1/2}), \tag{46}$$

for some $\delta > 0$ (cf. Jentsch and Kreiss 2010, equation (7.7) for the case $p = 1$), where $\|\cdot\|$ denotes the Euclidean matrix norm. Because $\underline{c}_j$, $\widehat{\underline{c}}_j$ is just the first row of the matrix $(\mathrm{e}^{A\Delta})^j$, $(\widehat{\mathrm{e}^{A\Delta}})^j$, respectively, (46) immediately leads to

$$\sup_{j\in\mathbb{N}_0, i=0,\ldots,p-1} r^j \, |\widehat{c}_{j,i} - c_{j,i}| = \mathcal{O}_P(h + n^{-1/2}). \tag{47}$$

Equation (47) together with consistency of $\kappa^{4*}_{i_1,i_2,i_3,i_4}$ and $E^*[Z_i^* Z_j^*]$ for $\kappa^4_{i_1,i_2,i_3,i_4}$ and $E[Z_i Z_j]$, respectively (both are immediate consequences of the weak law of large numbers), as well as the summability of the coefficients $c_{j,i}$ and $\widehat{c}_{j,i}$ now leads by a direct but tedious computation to the result

$$\sum_{r=-(n-1)}^{n-1} |T_r^* - T_r| = o_P(1), \tag{48}$$

which means by (44) that

$$n \, \mathrm{Cov}(\widehat{\gamma}_{Y^*}(q_1\Delta), \widehat{\gamma}_{Y^*}(q_2\Delta)) \to v_{q_1\Delta, q_2\Delta}. \tag{49}$$

This is part (i) of Theorem 1.

For a proof of part (ii) of Theorem 1, we make use of Brockwell and Davis (1991), Proposition 6.3.9. Recall (34) and define $Y_M^*((t+1)\Delta) = \sum_{j=0}^{M} \widehat{c}_{j,i} Z_i^*((t+1-j)\Delta)$, $M \in \mathbb{N}$. This sequence is M-dependent and a slight extension to triangular arrays (cf. Lemma 2 below) of the CLT for M-dependent sequences stated in Brockwell and Davis (1991), Theorem 6.4.2, leads us to the asymptotic normality

$$\sqrt{n}\left(\widehat{\gamma}_M^*(0) - \gamma_{Y^*,M}(0), \ldots, \widehat{\gamma}_M^*(q\Delta) - \gamma_{Y^*,M}(q\Delta)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_M), \tag{50}$$

where $\widehat{\gamma}_M^*(h)$ and $\gamma_{Y^*,M}(h)$ are defined as $\widehat{\gamma}^*(h)$ and $\gamma_{Y^*}(h)$ with Y^* replaced by Y_M^*. V_M is defined as in (18) with ∞ replaced by M. Since $V_M \to V$ as $M \to \infty$ and for every $\varepsilon > 0$ and every $h \in \mathbb{N}_0$

$$\begin{aligned}\lim_{M\to\infty} \limsup_{n\to\infty} P\Big\{\Big|&\sqrt{n}(\widehat{\gamma}^*(h\Delta) - \gamma_{Y^*}(h\Delta)) \\ &-\sqrt{n}(\widehat{\gamma}_M^*(h\Delta) - \gamma_{Y^*,M}(h\Delta))\Big| > \varepsilon\Big\} = 0\end{aligned} \tag{51}$$

the above-mentioned result (i.e., Brockwell and Davis 1991, Proposition 6.3.9) yields part (ii).

Finally, we obtain part (iii) by the usual delta method from (ii) since autocorrelations are smooth functions of autocovariances.

This concludes the proof of Theorem 1. □

In the proof of Theorem 1, we have made use of the following central limit theorem for triangular arrays of M-dependent sequences. We note that the truncated bootstrap process $Y_M^*(t\Delta)$ is indeed a triangular array of M-dependent random variables since with increasing n the parameters $\widehat{c}_{j,i}$ as well as the distribution of $Z_i^*(j\Delta)$ vary.

Lemma 2 *Suppose that for each $n \in \mathbb{N}$, real-valued, centered and M-dependent ($M \in \mathbb{N}$) random variables $\{U_{t,n} : t = 1, \ldots, n\}$ are given and make the following assumptions.*

(i) *For $h \in \mathbb{N}_0$, we have $E\left(U_{t+h,n}U_{t,n}\right) \to_{n\to\infty} c(h),\ h \in \mathbb{N}_0$, where the function c fulfills $c(0) + 2\sum_{k=1}^{m} c(h) = \tau^2 > 0$.*

(ii) $\frac{1}{n^{1+\delta}} \sum_{t=1}^{n} E\left|U_{t,n}\right|^{2(1+\delta)} \to_{n\to\infty} 0$ *for some $\delta > 0$.*

Then, we have

$$\lim_{n\to\infty} Var\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n} U_{t,n}\right) = \tau^2 \tag{52}$$

and

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} U_{t,n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2). \tag{53}$$

For a proof of this not complicated result, we refer to Kreiss (1997).

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