

Local consistency of Markov chain Monte Carlo methods

Kengo Kamatani

Received: 12 January 2012 / Revised: 8 March 2013 / Published online: 11 April 2013
© The Institute of Statistical Mathematics, Tokyo 2013

Abstract In this paper, we introduce the notion of efficiency (consistency) and examine some asymptotic properties of Markov chain Monte Carlo methods. We apply these results to the data augmentation (DA) procedure for independent and identically distributed observations. More precisely, we show that if both the sample size and the running time of the DA procedure tend to infinity, the empirical distribution of the DA procedure tends to the posterior distribution. This is a local property of the DA procedure, which may be, in some cases, more helpful than the global properties to describe its behavior. The advantages of using the local properties are the simplicity and the generality of the results. The local properties provide useful insight into the problem of how to construct efficient algorithms.

Keywords Monte Carlo · Markov chain · Asymptotic normality

1 Introduction

This paper investigates conditions under which a Markov chain Monte Carlo (MCMC) procedure has a good stability property in the Bayesian context. There is a vast literature related to the sufficient conditions for ergodicity; see reviews such as Tierney (1994) and Roberts and Rosenthal (2004) and textbooks such as Nummelin (1984) and Meyn and Tweedie (1993). The transition kernel of the MCMC procedure is Harris recurrent

Supported in part by Grant-in-Aid for JSPS Fellows (19-3140) and Grant-in-Aid for Young Scientists (B) 22740055.

K. Kamatani (✉)
Graduate School of Engineering Science, Osaka University, Machikaneyama-cho 1-3, Toyonaka,
Osaka 560-0043, Japan
e-mail: kamatani@sigmath.es.osaka-u.ac.jp

under fairly general assumptions. Moreover, it is sometimes geometrically ergodic. In practice, the Foster–Lyapunov type drift condition is commonly used to establish geometric ergodicity and calculation of its rate. This condition is helpful for studying the global property of the MCMC procedure. However, there are some limitations if we want more information for the stability of the MCMC procedure, e.g., this approach has difficulty in comparing two MCMC procedures.

We take another approach to study the stability of the MCMC procedure in the Bayesian context. We will define local consistency as a measure of the performance of the MCMC procedure. The following toy example illustrates our approach.

Assume we have n observation $x_n = \{x^1, \dots, x^n\}$ from a simple model

$$P(X = 1|\theta) = \Phi(\theta), \quad P(X = 0|\theta) = 1 - P(X = 1|\theta)$$

where θ is the parameter and Φ is the cumulative distribution function of the normal distribution. The data augmentation (DA) procedures can be defined by the so-called augmented data model that introduces latent variable y . We consider two DA procedures corresponding to the following augmented data models:

$$y \sim N(0, 1), \quad x = 1_{\{y \leq \theta\}} \quad (1)$$

$$y \sim N(-\theta, 1), \quad x = 1_{\{y \leq 0\}}. \quad (2)$$

Though the models are similar, the performances of the DA procedures are quite different. Figure 1 is a trajectory of the sequences

$$\theta_0, \dots, \theta_{m-1}$$

from the DA procedures with the sample sizes $n = 50$ and $n = 250$. The true value is set to $\theta_0 = 0$.

The simulation result for the sample size $n = 50$ (upper) shows the poor performance of the DA procedure for (1) than that for (2) that may cause the inference bias. Such differences become clear [see $n = 250$ (lower)] and the analysis becomes easier as the sample size grows.

With this observation in mind, we want to define consistency as an asymptotic property as the sample size $n \rightarrow \infty$. For each observation x_n , the MCMC procedure results in a Markov chain $\theta_\infty = \{\theta_0, \theta_1, \dots\}$ that has the invariant probability distribution $p(d\theta|x_n)$. Write \mathbb{P}_n for the probability measure for x_n and θ_∞ . Let

$$I = \int \varphi(\theta) p(d\theta|x_n), \quad I_m = m^{-1} \sum_{i=0}^{m-1} \varphi(\theta_i) \quad (3)$$

for a bounded continuous function φ . The goal of the MCMC procedure was to approximate I by I_m . It would be helpful if

$$|I - I_m| = o_{\mathbb{P}_n}(1) \quad (4)$$

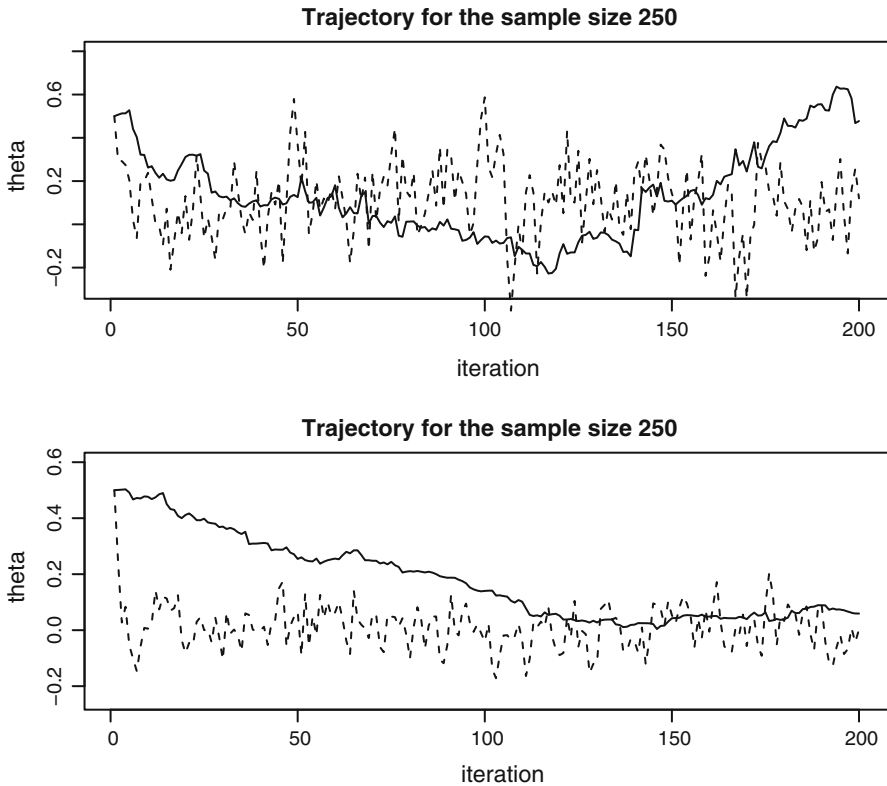


Fig. 1 Trajectory of the DA procedures for sample size $n = 50$ (upper) and $n = 250$ (lower). Solid line is for (1) and dashed lines is for (2)

for any $m_n \rightarrow \infty$. Since the number m corresponds to the iteration counts of the MCMC procedure, smaller is better. For each x_n , usually $\lim_{m \rightarrow \infty} |I - I_m| = 0$ in $\mathbb{P}(\cdot | x_n)$ holds by ergodicity of the Markov chain. However, sometimes the relation (4) does not hold. For example, under a reasonable scaling, the DA procedure that uses the augmented data model (1) does not satisfy (4) for some $m_n \rightarrow \infty$ but it does satisfy if $m_n/n \rightarrow \infty$. If (4) holds for any $m_n \rightarrow \infty$, we call the MCMC procedures consistent. The DA procedure that uses (2) has consistency under the same scaling that will be proved in Theorem 1.

We obtain the following results,

1. The consistency and the local consistency of the MCMC procedures are studied.
2. A reasonable set of sufficient conditions for the local consistency for the DA procedure is addressed for independent identically distributed observations. We only assume (a) the identifiability of parameter, (b) the existence of uniformly consistent test, (c) regularity of prior distribution, and (d) quadratic mean differentiability of the full model.

For the treatment of a large sample setting (with a different motivation), [Belloni and Chernozhukov \(2009\)](#) have recently studied the Metropolis algorithm for increased

parameter dimension d . They obtained the rate of the running time of the Metropolis algorithm for burn-in and after burn-in. To deal with the complex algorithm and to obtain strong results, they assumed strong conditions [C.1, C.2 and (3.5)]. Another paper (Nielsen 2000; Svensson and de Luna 2010) obtained stability properties of the stochastic EM algorithm. Essentially, they studied finite dimensional convergence of $\theta_0, \dots, \theta_k$. However, without tightness arguments, the finite dimensional properties are insufficient to describe the performance of the MCMC procedures. On the other hand, we show the convergence of the law of the process $\{\theta_i; i \in \mathbf{N}_0\}$ with a minimal set of conditions.

It is not our intention to conclude that the DA procedure is always efficient. The conclusion of Theorem 1 is that, under regularity conditions, the DA procedure approximates the posterior distribution in an ordinal running time. On the other hand, it illustrates the causes of the performance bottlenecks of the MCMC procedures. For example, (a) the choice of the initial guess θ_0 is not good, (b) the model has the fragility of the identification, (c) the Fisher information matrix g for the model is too small or that for the hidden information is too large, or (d) the sample size is too small related to its parameter dimension. For example, the DA procedure that uses the model (1) suffers from (c). These studies of regular/non-regular properties are quite important for the elimination of the performance bottlenecks of the MCMC procedures.

The rest of the paper is organized as follows. We prepare in Sect. 2 for needed backgrounds. Consistency will be introduced in Sect. 3. We analyze local consistency of the DA procedure in this section. Concluding remarks are summarized in Sect. 4.

2 Background

2.1 Quadratic mean differentiability

Let (X, \mathcal{X}) be a measurable space. Let $M = \{P_\theta(dx) = p_\theta(x)dx; \theta \in \mathbf{R}^d\}$ be a parametric family on X . The family M is said to be quadratic mean differentiable at θ if

$$\sqrt{p_{\theta+h}(x)} - \sqrt{p_\theta(x)} - h' \tilde{\eta} = o(h^2) \text{ in } L^2(dx) \quad (5)$$

for any $h \rightarrow 0$ and a square integrable function $\tilde{\eta} : X \rightarrow \mathbf{R}^d$ where v' is the transpose of a vector $v \in \mathbf{R}^d$. A matrix $g(\theta) = 4 \int \tilde{\eta} \tilde{\eta}' dx$ is called the Fisher information matrix. In this paper, when $x_n = \{x^1, \dots, x^n\} \sim P_\theta^{\otimes n} = \prod_{i=1}^n P_\theta(dx^i)$, the following random variable is called the normalized score function:

$$\eta = \eta_\theta(x_n) = \sqrt{n}^{-1} \sum_{i=1}^n 2 \frac{\tilde{\eta}(x^i)}{\sqrt{p_\theta(x^i)}}. \quad (6)$$

Suppose now that $M^\dagger = \{P_\theta^\dagger(dxdy) = p_\theta^\dagger(xy)dxdy; \theta \in \mathbf{R}^d\}$ is another model, called the augmented data model on $X \times Y$ that satisfies $P_\theta(dx) = \int_Y P_\theta^\dagger(dxdy)$. According to Proposition 7.4 of Le Cam and Yang (1988), if M^\dagger satisfies quadratic

mean differentiability at θ , then M also does. Let g and g^\dagger be the Fisher information matrices of the models M and M^\dagger and write η and η^\dagger for the normalized score statistics, and write \hat{u} and \hat{u}^\dagger for the maximum likelihood estimators of the models M and M^\dagger under observations x_n and $x_n, y_n = \{y^1, \dots, y^n\}$, respectively.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let \mathcal{G} be a sub σ algebra of \mathcal{F} .

Definition 1 (Stable convergence) A sequence of \mathbf{R}^d -valued \mathcal{F} -measurable random variable Z^n is said to converge \mathcal{G} -stably if there exists a measure μ on $\Omega \times \mathbf{R}^d$ such that

$$\mathbb{E}[f(Z^n)Y] \rightarrow \int \mu(d\omega, dx)Y(\omega) f(x) \tag{7}$$

for any continuous bounded function f and for any bounded \mathcal{G} -measurable random variable Y .

Let $X_i(\omega), Y_i(\omega)$ ($i = 1, \dots, n$) be i.i.d. observation from a probability measure $P_\theta^\dagger(dx dy)$ and set $X(\omega) = \{X_i(\omega); i = 1, \dots\}$ and $\mathcal{G} = \sigma(X)$.

Lemma 1 Let $\eta^* = \eta^\dagger - \eta$ and $g^* = g^\dagger - g$. Then η^* converges \mathcal{G} -stably to $N(0, g^*)$.

Proof By the law of large number, for almost all $\omega \in \Omega$,

$$\mathbb{E}[\eta_i^* | \mathcal{G}] = 0, \mathbb{E} \left[n^{-1} \sum_{i=1}^n \eta_i^* (\eta_i^*)' | \mathcal{G} \right] \rightarrow g^* \tag{8}$$

and

$$\mathbb{E} \left[n^{-1} \sum_{i=1}^n |\eta_i^*|^2 1_{\{n^{-1/2}|\eta_i^*| > \epsilon\}} | \mathcal{G} \right] \rightarrow 0 \tag{9}$$

for $\epsilon > 0$. Write $A_\epsilon \in \mathcal{G}$ for all $\omega \in \Omega$ that satisfies the above three convergences. Then $A = \cap_{i=1}^\infty A_{i-1}$ is still a sure event, and for each of $\omega \in A$, the Lindeberg condition holds for the array $\{\eta_i^*(x_i, Y_i(\omega)); i = 1, \dots, \}$ with probability measure $\mathbb{P}(\cdot | \mathcal{G})_{x=X(\omega)}$. Hence the claim follows. \square

2.2 Key technical lemmas

Let $\{\mathbb{P}_n; n = 1, 2, \dots\}$ be a sequence of probability measures. The following is the key results for the present study. Write θ_∞ for $\{\theta_0, \theta_1, \dots\}$.

Lemma 2 Let θ_∞ be a stationary \mathbb{P}_n -Markov chain with the invariant probability distribution p_n that converges in probability to an ergodic Markov chain. Then for any bounded continuous function φ and for any $m_n \rightarrow \infty$,

$$\int \varphi(\theta) p_n(d\theta) - m_n^{-1} \sum_{i=0}^{m_n-1} \varphi(\theta_i) = o_{\mathbb{P}_n}(1). \tag{10}$$

Proof Let \mathbb{P} be the limit of $\mathbb{P}_n(\theta_\infty \in \cdot)$. Write I^n and $I_{m,n}$ for the first and the second term in the left-hand side of (10), respectively, and write $I_{i,k}$ for $k^{-1} \sum_{j=0}^{k-1} \varphi(\theta_{ik+j})$. Then

$$I_m = \frac{k}{m} \sum_{i=0}^{\lfloor m/k \rfloor - 1} I_{i,k} + \frac{1}{m} \sum_{i=k\lfloor m/k \rfloor}^{m-1} \varphi(\theta_i)$$

where $\lfloor x \rfloor$ is the integer part of $x \in \mathbf{R}$. This relation yields the upper bound of the left-hand side of (10);

$$|I^n - I_m| \leq \frac{k}{m} \sum_{i=0}^{\lfloor m/k \rfloor - 1} |I^n - I_{i,k}| + \frac{1}{m} \sum_{i=k\lfloor m/k \rfloor}^{m-1} |I^n - \varphi(\theta_i)|.$$

By stationarity, each $|I^n - I_{i,k}|$ has the same law under \mathbb{P}_n . Hence for a constant $C > 0$ that satisfies $|\varphi(\theta)| < C$,

$$\mathbb{E}_n[|I^n - I_m|] \leq \frac{k}{m} \left\lfloor \frac{m}{k} \right\rfloor \mathbb{E}_n[|I^n - I_{1,k}|] + 2 \frac{m - k\lfloor m/k \rfloor}{m} C.$$

Since $x - 1 < \lfloor x \rfloor \leq x$, the second term is negligible and the first term has a bound $\mathbb{E}_n[|I^n - I_{1,k}|]$. Write p for the limit of p_n and let $I = \int \varphi(\theta) p(d\theta)$. Then $\mathbb{E}_n[|I^n - I_{1,k}|] \leq \mathbb{E}_n[|I - I_{1,k}|] + |I - I_n|$, but the second term is negligible again by weak convergence of the law of the Markov chain. Thus, the claim follows if $\mathbb{E}_n[|I - I_{1,k}|]$ can be arbitrary small.

Since $\theta_\infty \mapsto I_{1,k}$ is continuous, $\mathbb{E}_n[|I - I_{1,k}|] \rightarrow \mathbb{E}[|I - I_{1,k}|]$, and by the law of large numbers for stationary sequence, the right-hand side tends to 0 as $k \rightarrow \infty$ that proves the claim. \square

We introduce a simple sufficient condition to apply this lemma. Let $\mu(dx)$ be a probability measure and let $K(x, dy)$ be a transition kernel. Let $\mu \otimes K$ be a probability measure defined by

$$(\mu \otimes K)(A \times B) = \int_A K(x, B) \mu(dx).$$

For any probability measures p, q on a measurable space (E, \mathcal{E}) , the total variation distance is

$$\|p - q\| = \sup_{A \in \mathcal{E}} |p(A) - q(A)|. \tag{11}$$

Lemma 3 *Let K and K_n ($n = 1, 2, \dots$) be transition kernels that have the invariant probability distributions p and p_n , respectively. If $\|p_n \otimes K_n - p \otimes K\| \rightarrow 0$, then a Markov chain θ_∞ with transition kernel K_n with the initial distribution p_n converges in law to a Markov chain θ_∞ with transition kernel K with the initial distribution p .*

Proof It suffices to show finite dimensional convergence in law for θ_∞ , and this is completed if we can prove the convergence in total variation distance. Let

$$M_m = p \otimes \overbrace{K \otimes \cdots \otimes K}^m, \quad M_{n,m} = p_n \otimes \overbrace{K_n \otimes \cdots \otimes K_n}^m.$$

The task is now to show $\|M_m - M_{n,m}\| \rightarrow 0$ for any m . For $m = 0, 1$, the convergence is clear and assume that it is true up to $m = k$. For $m = k + 1$, observe that $M_{k+1} - M_{n,k+1}$ equals to

$$(M_k - M_{n,k}) \otimes K + M_{n,k} \otimes (K - K_n).$$

The total variation distance of the former term vanishes by assumption. For that of the latter, since p_n is the invariant probability distribution of K_n , we have a bound

$$\int p_n(d\theta) \|(K - K_n)(\theta, \cdot)\| \leq 4\|p \otimes K - p_n \otimes K_n\| \rightarrow 0$$

by Lemma 12.2.2 of [Le Cam \(1986\)](#). □

2.3 Approximation of the DA procedure

Let p_M be the prior distribution and let $P_n = \int P_\theta^{\otimes n} p_M(d\theta)$. Under some regularity conditions,

$$\hat{u} = \theta + n^{-1/2} g^{-1} \eta(\theta) + o_{P_\theta^{\otimes n}}(1), \quad \|p(d\theta|x_n) - N(\hat{u}, n^{-1} g^{-1})\| = o_{P_\theta^{\otimes n}}(1)$$

where $g = g(\hat{u})$ and $p(\cdot|x_n)$ is the posterior distribution of M . Define $P_\theta^\dagger(dy|x)$ so that $P_\theta^\dagger(dx dy) = P_\theta(dx) P_\theta^\dagger(dy|x)$. Then under some regularity conditions,

$$\hat{u}^\dagger = \theta + n^{-1/2} g^{\dagger-1} \eta^\dagger(\theta) + o_{P_\theta^{\dagger \otimes n}}(1), \quad \|p^\dagger(d\theta|x_n y_n) - N(\hat{u}^\dagger, n^{-1} g^{\dagger-1})\| = o_{P_\theta^{\dagger \otimes n}}(1)$$

where $g^\dagger = g^\dagger(\hat{u}^\dagger)$ (not $g^\dagger(\hat{u}^\dagger)$) and $p^\dagger(\cdot|x_n y_n)$ is the posterior distribution of M^\dagger .

Using models M and M^\dagger , the data augmentation procedure is defined as the iteration of the following:

1. Simulate y_n from $\prod_{i=1}^n P_\theta^\dagger(dy^i|x^i) =: P_\theta^\dagger(dy_n|x_n)$.
2. Simulate θ from $p^\dagger(d\theta|x_n y_n)$,

This procedure results in a Markov chain $\theta_0, \theta_1, \dots$ with the invariant probability distribution $p(d\theta|x_n)$.

It is well known that this procedure is approximated by an auto-regressive process (see [Sahu and Roberts 1999](#); [Meng and van Dyk 1999](#); [Dempster et al. 1977](#)). To explain this approximation, define

$$\eta^*(\theta) = \eta^\dagger(\theta) - \eta(\theta), \quad g^*(\theta) = g^\dagger(\theta) - g(\theta). \tag{12}$$

Then the law of $\eta^* = \eta^*(\theta)$ tends to $N(0, g^*)$ and

$$\hat{u}^\dagger = \theta + n^{-1/2} g^{\dagger-1} (\eta + \eta^*) + o_{P_\theta^{\dagger \otimes n}}(1) \tag{13}$$

$$= \theta + n^{-1/2} g^{\dagger-1} g g^{-1} \eta + n^{-1/2} g^{\dagger-1} \eta^* + o_{P_\theta^{\dagger \otimes n}}(1) \tag{14}$$

$$= \hat{u} + g^{\dagger-1} g^*(\theta - \hat{u}) + n^{-1/2} g^{\dagger-1} \eta^* + o_{P_\theta^{\dagger \otimes n}}(1) \tag{15}$$

where we omit θ in $\eta(\theta)$ and $\eta^*(\theta)$. This calculation yields $n^{1/2}(\hat{u}^\dagger - \hat{u}) = g^{\dagger-1} g^* \tilde{\theta} + g^{\dagger-1} \eta^* + o_{P_\theta^{\dagger \otimes n}}(1)$, where $\tilde{\theta} = \sqrt{n}(\theta - \hat{u})$. With regularity conditions, this approximation results in a Markov chain with a transition kernel defined by

$$K(\tilde{\theta}, \cdot) = N(g^{\dagger-1} g^* \tilde{\theta}, g^{\dagger-1} g^* g^{\dagger-1} + g^{\dagger-1}). \tag{16}$$

Remark 1 (Convergence Rate) In the limit, the matrix $A := g^{\dagger-1} g^* = I - g^{\dagger-1} g$ defines the convergence rate. Let $r \in [0, 1)$ be the spectral radius of A , which is the same as the spectral radius of $g^{\dagger-1/2} g^* g^{\dagger-1/2}$. Then the marginal distribution of $\tilde{\theta}$ converges geometric rate r^2 to the invariant distribution (see Section 16.5.1 of [Meyn and Tweedie 1993](#)). A small value of $g^{\dagger-1} g$ leads to a poor performance.

Remark 2 (Invariant probability distribution) It is easy to check the invariant probability distribution of $K(\tilde{\theta}, \cdot)$ is $p(d\theta) = N(0, g^{-1})$ since

$$g^{-1} = g^{\dagger-1} g^* g^{-1} g^* g^{\dagger-1} + g^{\dagger-1} g^* g^{\dagger-1} + g^{\dagger-1}. \tag{17}$$

Remark 3 (Bayesian paradigm) To show the convergence of the law of $\tilde{\theta}_0, \dots$, Bayesian paradigm will be used efficiently. It is much difficult to show for similar method such as the stochastic EM algorithm. This is probably the reason for the robustness of the DA procedure.

3 Local consistency

3.1 Definitions of the local consistency

Let $(X_n, \mathcal{X}_n, P_n)$ ($n = 1, 2, \dots$) be a sequence of probability spaces. For given $x_n \in X_n$, consider a Markov chain $\theta_\infty = \{\theta_0, \theta_1, \dots\}$ on \mathbf{R}^d with the transition kernel K with the invariant distribution p . Note that they depend on x_n and hence we will write $K(\theta, d\theta^* | x_n)$ and $p(d\theta | x_n)$. Write \mathbb{P}_n for the joint law of x_n and θ_∞ . We call the sequence of the law $\{\mathbb{P}_n(\theta_\infty \in \cdot | x_n); x_n \in X_n\}_{n=1,2,\dots}$ the MCMC procedure.

Definition 2 (Consistency) An MCMC procedure is called consistent if for any $m_n \rightarrow \infty$ and for any bounded continuous function φ ,

$$\int \varphi(\theta) p(d\theta | x_n) - m_n^{-1} \sum_{i=0}^{m_n-1} \varphi(\theta_i) = o_{\mathbb{P}_n}(1). \tag{18}$$

The first term in the left-hand side of (18) corresponds to the amount we are interested in and the second term is the Monte Carlo approximation for this amount. The consistency means that this approximation tends to the targeted value after reasonable number of iteration. However, since the posterior distribution converges to a point mass under a mild condition, the convergence (18) may not have much information. The local consistency claims that the same convergence holds even after the certain scaling. Let $\hat{u} : X_n \rightarrow \mathbf{R}^d$ be a \mathcal{X}_n -measurable map and consider $\theta \mapsto n^{1/2}(\theta - \hat{u})$.

Definition 3 An MCMC procedure is called locally consistent if for any $m_n \rightarrow \infty$ and for any bounded continuous function φ ,

$$\int \varphi(n^{1/2}(\theta - \hat{u}))p(d\theta|x_n) - m_n^{-1} \sum_{i=0}^{m_n-1} \varphi(n^{1/2}(\theta_i - \hat{u})) = o_{\mathbb{P}_n}(1). \tag{19}$$

If each $\mathbb{P}_n(\theta_\infty \in \cdot|x_n)$ ($x_n \in X_n, n = 1, 2, \dots$) is a stationary process, we call the MCMC procedure stationary. In the main theorem in the present paper, stationarity is assumed which is unrealistic in practice. The choice of the initial probability distribution is an important part for designing Monte Carlo method. This choice heavily depends on the structure of the model that prevents from constructing a general framework. However, the following illustrates that a suitable choice of the initial distribution does not change the results. For example, we can choose $N(\tilde{u}_n, n^{-1})$ as the initial distribution $q(\cdot|x_n)$, where \tilde{u}_n is an estimator that satisfies $n^{1/2}(\tilde{u}_n - \hat{u}) = O_{\mathbb{P}_n}(1)$.

For $\epsilon > 0$, when two σ -finite measures μ and ν of (E, \mathcal{E}) satisfies $\mu(A) \leq \nu(A) + \epsilon$ for any $A \in \mathcal{E}$, we write $\mu \leq \nu + \epsilon$.

Lemma 4 Consider a stationary MCMC procedure that has initial distribution $p(d\theta|x_n)$ which is the invariant distribution. Consider another MCMC procedure that replaces the initial distribution $p(d\theta|x_n)$ to $q(d\theta|x_n)$. Then if the former MCMC procedure is consistent and if for any $\epsilon > 0$, there exists $c > 0$ such that

$$\limsup_{n \rightarrow \infty} P_n(\{x_n; q(\cdot|x_n) > cp(\cdot|x_n) + \epsilon\}) < \epsilon, \tag{20}$$

the latter is also consistent.

Proof Let A_ϵ be the event that is measured by P_n in (20). Write \mathbb{Q}_n for \mathbb{P}_n replacing the initial distribution from p to q . For any continuous $[0, 1]$ -valued function $\psi(x_n, \theta_\infty)$, we have

$$\mathbb{Q}_n(\psi) \leq P_n(A_\epsilon) + c\mathbb{P}_n(\psi) + \epsilon.$$

Hence if $\mathbb{P}_n(\psi) \rightarrow 0$, then $\mathbb{Q}_n(\psi) \rightarrow 0$. Take ψ to be the absolute value of the left-hand side of (18). □

3.2 Local consistency of the standard the DA procedure

Let M and M^\dagger be as in Sect. 2.1 and let p_M and P_n be as in Sect. 2.3. A sequence of tests $\psi_n : X_n \rightarrow [0, 1]$ is said to be uniformly consistent for testing $\theta \in \mathbf{R}^d$ against $K^c \subset \mathbf{R}^d$ if

$$P_\theta^{\otimes n}(\psi_n) \rightarrow 0, \sup_{\vartheta \in K^c} P_\vartheta^{\otimes n}(1 - \psi_n) \rightarrow 0. \tag{21}$$

If there exists a uniformly consistent test for each θ with any compact set K that includes θ , then M is said to have a uniformly consistent test.

Assume the following conditions.

- Assumption 1**
1. M^\dagger is quadratic mean differentiable.
 2. The Fisher information matrix g of M is non-singular.
 3. M has a uniformly consistent test.
 4. The prior p_M has a continuous, positive and bounded density.
 5. M is identifiable.

Let

$$K_n(\tilde{\theta}, d\tilde{\theta}^*|x_n) = \int_{Y_n} p^\dagger(\hat{u} + n^{-1/2}d\tilde{\theta}^*|x_n y_n) P_{\hat{u}+n^{-1/2}\tilde{\theta}}^{\dagger \otimes n}(dy_n|x_n) \tag{22}$$

where $\tilde{\theta} = n^{1/2}(\theta - \hat{u})$, $\tilde{\theta}^* = n^{1/2}(\theta^* - \hat{u})$. This is the transition kernel of the DA procedure. The following is the main results for the present paper that says the DA procedure works well under general conditions.

Theorem 1 *Assume $\theta_0 \sim p(d\theta|x_n)$. Then under Assumption 1, the DA procedure has the local consistency.*

Proof By Bernstein–von Mises’s theorem of the model M^\dagger ,

$$\|p^\dagger(d\theta|x_n y_n) - N(\hat{u}^\dagger, n^{-1}g^{\dagger-1})\| = o_{P_\theta^{\dagger \otimes n}}(1) \tag{23}$$

where $g^\dagger = g^\dagger(\hat{u})$. It is possible to replace \hat{u}^\dagger in (23) by the right-hand side of (15) without $o_{P_\theta^{\otimes n}}(1)$ term. By $\theta \mapsto \tilde{\theta} = n^{1/2}(\theta - \hat{u})$, this is mapped to $g^{\dagger-1}g^*\tilde{\theta} + g^{\dagger-1}\eta^*$. We are now in a position to show

$$\left\| \int_{y_n \in Y_n} N(g^{\dagger-1}g^*\tilde{\theta} + g^{\dagger-1}\eta^*, g^{\dagger-1})P_\theta^\dagger(dy_n|x_n) - K(\tilde{\theta}, \cdot) \right\| = o_{P_\theta^{\otimes n}}(1) \tag{24}$$

for each θ . The left-hand side equals to

$$\int_{x \in \mathbf{R}^n} \left| \int \phi(x; g^{\dagger-1}\eta^*, g^{\dagger-1})(P_\theta^\dagger(dy_n|x_n) - \phi(\eta^*; 0, g^*)d\eta^*) \right| dx. \tag{25}$$

For the moment, replace g^\dagger by $g^\dagger(\theta)$ and g^* by $g^*(\theta)$ and fix $x \in \mathbf{R}^d$. Let $\psi_n : X_n \rightarrow [0, 1]$ be the value in the vertical bars in (25) after the replacement and set $\varphi_n \in \{-1, +1\}$ to be the sign of ψ_n . Then for the smooth function $l(z) = \phi(x; g^{\dagger-1}(\theta)z, g^{\dagger-1}(\theta))$, it is sufficient to show

$$P_\theta^{\dagger \otimes n}[\varphi_n l(\eta^*)] - P_\theta^{\otimes n}[\varphi_n \int_{\eta^*} l(\eta^*) \phi(\eta^*; 0, g^*(\theta)) d\eta^*] \rightarrow 0. \tag{26}$$

Since φ_n is $P_\theta^{\dagger \otimes n}$ -tight, by choosing suitable probability space with a probability measure $P_\theta^{\dagger \otimes \infty}$, it is possible to assume $\varphi_n \rightarrow \varphi$ ($n \rightarrow \infty$). Then by replacing φ_n by φ in (26), the convergence follows by the stable convergence of η^* (Lemma 1). Hence we have

$$\|K_n(\tilde{\theta}, d\tilde{\theta}^* | x_n) - K(\tilde{\theta}, d\tilde{\theta}^*)\| = o_{P_\theta^{\otimes n}}(1) \tag{27}$$

where K is defined in (16). By Bernstein–von Mises’s theorem for model M , we have $\|p_n - p\| = o_{P_n}(1)$ where $p_n(d\tilde{\theta} | x_n) = p(\hat{u} + n^{-1/2}d\tilde{\theta} | x_n)$ and $p(d\tilde{\theta}) = N(0, g^{-1})$. Hence Lemma 12.2.2 of Le Cam (1986) shows

$$\|p_n \otimes K_n - p \otimes K\| = o_{P_n}(1) \tag{28}$$

since $P_\theta^{\otimes n}(dx)p_M(d\theta) = p(d\theta | x_n)P_n(dx)$ by integrating the left-hand side of (27) by p_M . Then Lemmas 2 and 3 prove the claim since for each \hat{u} , K defines an ergodic Markov chain and \hat{u} is P_n -tight. □

4 Concluding remarks

4.1 Future work

What we did not discuss in this paper were the following. We believe that the framework we proposed is helpful for these directions.

1. Research for poor performance of the MCMC procedures. The local properties are helpful for identification of the performance bottlenecks of the MCMC procedures. This is studied in two different directions by: Kamatani (2010), the rate of m_n of (4); Kamatani (2013b), the rate of $|\theta_i - \theta_{i-1}|$.
2. Research for constructing new Monte Carlo procedures. Though the present study is for regular Monte Carlo procedures, these results are useful to eliminate the performance bottlenecks of the MCMC procedures. For example, the paper of Kamatani (2013a) studies an efficient MCMC procedure for the cumulative probit model and there are many possibilities for this direction.

4.2 A technical comment

We consider the maximum likelihood estimator \hat{u} . Though it does not always exist, it can be replaced by the central value of the posterior distribution. For a probability measure μ on \mathbf{R} , a central value is a point $\bar{x} \in \mathbf{R}$ satisfying

$$\int_{\mathbf{R}} \arctan(x - \bar{x}) \mu(dx) = 0.$$

Element of \mathbf{R}^p is denoted by $x = (x^1, \dots, x^p)^T$. For a probability measure μ on \mathbf{R}^p , let $\mu^i(A)$ be $\int_{x \in \mathbf{R}} 1_A(x^i) \mu(dx)$ for $A \in \mathcal{B}(\mathbf{R})$. For μ , we call $\bar{x} = (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^p)^T \in \mathbf{R}^p$ central value if each \bar{x}^i is a central value of μ^i . The central value always exists and unique. See Ito (2004).

Acknowledgments This is, essentially, the second part of the author's Ph.D. thesis at Graduate School of Mathematical Sciences, the University of Tokyo. The author wishes to express his thanks to the Ph.D. supervisor, Prof. Nakahiro Yoshida for his several helpful comments and suggestions. The author also thank to the Associate Editor and anonymous referee for constructive comments which helped a lot to improve the paper.

References

- Belloni, A., Chernozhukov, V. (2009). On the computational complexity of MCMC-based estimators in large samples. *Annals of Statistics*, 37(4), 2011–2055.
- Dempster, A. P., Laird, N. M., Rubin, D. B. (1977). Maximum Likelihood from Incomplete Data via the EM Algorithm. *Journal of the Royal Statistical Society. Series B (Methodological)*, 39(1), 1–38.
- Ito, K. (2004). *Stochastic Processes: Lectures Given at Aarhus University*. Berlin: Springer.
- Kamatani, K. (2010). Metropolis-Hastings Algorithm for Mixture Model and its Weak Convergence. In Y. Lechevallier, G. Saporta (Eds.), *Proceedings of COMPSTAT'2010, volume eBook* (pp. 1175–1182).
- Kamatani, K. (2013a). Asymptotic properties of Monte Carlo strategies for cumulative link model (submitted).
- Kamatani, K. (2013b). Order of degeneracy of Markov chain Monte Carlo for categorical data (submitted).
- Le Cam, L. (1986). *Asymptotic Methods in Statistical Decision Theory* (1st ed.). New York: Springer
- Le Cam, L., Yang, G. L. (1988). On the preservation of local asymptotic normality under information loss. *The Annals of Statistics*, 16(2), 483–520.
- Meng, X.-L., van Dyk, D. (1999). Seeking efficient data augmentation schemes via conditional and marginal augmentation. *Biometrika*, 86(2), 301–320.
- Meyn, S. P., Tweedie, R. L. (1993). *Markov Chains and Stochastic Stability*. London: Springer.
- Nielsen, S. F. (2000). The stochastic EM algorithm: Estimation and asymptotic results. *Bernoulli*, 6(3), 457–489.
- Nummelin, E. (1984). General irreducible Markov chains and nonnegative operators. In *Cambridge Tracts in Mathematics* (Vol. 83). Cambridge: Cambridge University Press.
- Roberts, G. O., Rosenthal, J. S. (2004). General state space markov chains and mcmc algorithms. *Probability Surveys*, 1, 20–71.
- Sahu, S. K., Roberts, G. O. (1999). On Convergence of the EM Algorithm and the Gibbs Sampler. *Statistics and Computing*, 9(1), 55–64.
- Svensson, I., de Luna, S. S. (2010). Asymptotic properties of a stochastic em algorithm for mixtures with censored data. *Journal of Statistical Planning and Inference*, 140, 117–127.
- Tierney, L. (1994). Markov Chains for Exploring Posterior Distributions (with discussion). *The Annals of Statistics*, 22(4), 1701–1762.