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Unified extension of variance bounds for integrated Pearson family

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Abstract We use some properties of orthogonal polynomials to provide a class of upper/lower variance bounds for a function $g(X)$ of an absolutely continuous random variable X, in terms of the derivatives of g up to some order. The new bounds are better than the existing ones.

Keywords Completeness · Derivatives of higher order · Fourier coefficients · Orthogonal polynomials · Parseval identity · Pearson family of distributions · Rodrigues-type formula · Variance bounds

1 Introduction

Let Z be a standard normal random variable and $g : \mathbb{R} \to \mathbb{R}$ any absolutely continuous (a.c.) function with derivative g'. Chernoff (1981), using Hermite polynomials, proved that [see also the previous papers by Nash (1958); Brascamp and Lieb (1976)]

$$\mathsf{Var}\, g(Z) \leq \mathbb{E}\big(g'(Z)\big)^2,$$

provided that $\mathbb{E}\big(g'(Z)\big)^2 < \infty$, where the equality holds if and only if g is a polynomial of degree at most one—a linear function. This inequality plays an important role in the isoperimetric problem and has been extended and generalized by several authors; see, e.g., Chen (1982); Cacoullos and Papathanasiou (1985); Papathanasiou (1988); Houdré and Kagan (1995); Papadatos and Papathanasiou (2001); Prakasa Rao (2006) and references therein.

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The results of the present paper are related to the following class of random variables [cf. Korwar (1991); Diaconis and Zabell (1991); Johnson (1993); see Afendras et al. (2011); Afendras and Papadatos (2012a,b)].

Definition 1 [*integrated Pearson family*] Let X be a random variable with density f and finite mean $\mu = \mathbb{E}X$. We say that X (or its density) belongs to the integrated Pearson family (or integrated Pearson system) if there exists a quadratic polynomial $q(x) = \delta x^2 + \beta x + \gamma$ (with $\delta, \beta, \gamma \in \mathbb{R}\, |\delta| + |\beta| + |\gamma| > 0$), such that

$$\int_{-\infty}^{x} (\mu - t) f(t)\,\mathrm{d}t = q(x) f(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{1}$$

This fact will be denoted by X or $f \sim \mathrm{IP}(\mu; q)$ or, more explicitly, X or $f \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$.

The definition of this class is sometimes considered as equivalent to the Pearson family; cf. Korwar (1991); Johnson (1993). However, this is not precise. In fact, several properties holding for integrated random variables are not true for all Pearson distributions. For instance, Properties P_3 and P_4 in Ord (1972), pp. 4–5, are not informative for the behavior of moments [see (1.7)], unless the distribution is integrated Pearson. The same is true for eq. (12.45), p. 22, of Johnson et al. (1994). In a review paper by Diaconis and Zabell (1991), the classification of Pearson distributions were related to orthogonal polynomials (see Table 2, p. 296). This implicitly stated family is close to what we call "integrated Pearson family". Its properties have been analyzed in detail in a recent work by Afendras and Papadatos (2012a).

Let $X \sim \mathrm{IP}(\mu; q)$ be a random variable and let us consider a suitable function g. Johnson (1993) established Poincaré-type upper/lower bounds for the variance of $g(X)$ of the form

$$(-1)^n \big(\mathsf{Var}\, g(X) - S_n\big) \geq 0, \quad \text{where} \quad S_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{\mathbb{E} q^k(X) \big(g^{(k)}(X)\big)^2}{k! \prod_{j=0}^{k-1} (1 - j\delta)}. \tag{2}$$

In particular, for the normal see Papathanasiou (1988) and Houdré and Kagan (1995); for the gamma see Papathanasiou (1988).

Afendras et al. (2011), using Bessel's inequality, showed that

$$\mathsf{Var}\, g(X) \geq \sum_{k=1}^{n} \frac{\mathbb{E}^2 q^k(X) g^{(k)}(X)}{k! \mathbb{E} q^k(X) \prod_{j=k-1}^{2k-2} (1 - j\delta)}; \tag{3}$$

for the case $n = 1$ see Cacoullos (1982).

Afendras and Papadatos (2012b) showed that, under appropriate conditions, the following two forms of Chernoff-type upper bounds of the variance of $g(X)$ are valid:

$$S_{n,(\mathrm{str})} = \sum_{i=1}^{n} \frac{\mathbb{E}^2 q^i(X) g^{(i)}(X)}{i! \mathbb{E} q^i(X) \prod_{j=i-1}^{2i-2} (1 - j\delta)} + \frac{\mathbb{E} q^n(X) \big(g^{(n)}(X)\big)^2 - \frac{\mathbb{E}^2 q^n(X) g^{(n)}(X)}{\mathbb{E} q^n(X)}}{(n+1)! \prod_{j=n}^{2n-1} (1 - j\delta)}, \tag{4}$$

$$S_{n,(\mathrm{weak})} = \sum_{i=1}^{n-1} \frac{\mathbb{E}^2 q^i(X) g^{(i)}(X)}{i! \mathbb{E} q^i(X) \prod_{j=i-1}^{2i-2}(1-j\delta)} + \frac{\mathbb{E} q^n(X) \big(g^{(n)}(X)\big)^2}{n! \prod_{j=n-1}^{2n-2}(1-j\delta)}. \tag{5}$$

The equality in (4) holds when g is a polynomial of degree at most $n+1$ and in (5) when g is a polynomial of degree at most n. The bound (5) for beta distributions has been shown by Wei and Zhang (2009), using Jacobi polynomials.

In the present article, we shall apply a technique introduced by Afendras and Papadatos (2012b) to obtain a general class of bounds. Specifically, in Sect. 2 we provide a new class of upper/lower bounds for the variance of $g(X)$. They can be called as “Poincaré-type” of order n and with point balance m. They hold for a subfamily of Pearson distributions. In particular, the bound for $N(\mu, \sigma^2)$ distribution, namely

$$S_{m,n}(g) = \sum_{i=1}^{m} \frac{\binom{m}{i}\sigma^{2i}}{(m+n)_i} \mathbb{E}^2 g^{(i)}(X) + \sum_{i=1}^{n} (-1)^{i-1} \frac{\binom{n}{i}\sigma^{2i}}{(m+n)_i} \mathbb{E}\big(g^{(i)}(X)\big)^2$$

(for $(x)_k$ see Definition 2), satisfies the inequality

$$(-1)^n \big(\mathsf{Var} g(X) - S_{m,n}(g)\big) \geq 0,$$

where the equality holds if and only if g is a polynomial of degree at most $m+n$.

For fixed n, Sect. 3 investigates the bounds $S_{m,n}(g)$ as m increases. It is shown that the bound $S_{m+1,n}(g)$ is better than $S_{m,n}(g)$, i.e.,

$$\big|\mathsf{Var} g(X) - S_{m+1,n}(g)\big| \leq \big|\mathsf{Var} g(X) - S_{m,n}(g)\big|.$$

Also, for any suitable function g, $S_{m,n}(g) \to \mathsf{Var} g(X)$ as $m \to \infty$.

2 Unified extension of variance bounds

This section presents a wide class of variance bounds. First, we prove the following useful lemma.

Lemma 1 *Let $X \sim \mathrm{IP}(\mu; q) \equiv \mathrm{IP}(\mu; \delta, \beta, \gamma)$ and let us consider a positive integer m with $\mathbb{E}X^{2m} < \infty$. Suppose that the function g is defined on the support $J = (\alpha, \omega)$ of X, and assume that $g \in C^{m-1}(J)$ and $g^{(m-1)} := \frac{\mathrm{d}^{m-1} g(x)}{\mathrm{d}x^{m-1}}$ are absolutely continuous with (a.s.) derivative $g^{(m)}$. If $\mathbb{E} q^m(X)|g^{(m)}(X)| < \infty$ then*

$$\mathbb{E} q^i(X)|g^{(i)}(X)| < \infty \quad \text{for all} \quad i = 0, 1, \ldots, m-1.$$

Proof Fix $i \in \{0, 1, \ldots, m-1\}$ and assume that $\mathbb{E} q^{i+1}(X)|g^{(i+1)}(X)| < \infty$. Setting $h := g^{(i)}$, we have that $\mathbb{E} q^{i+1}(X)|h'(X)| < \infty$. Consider the random variable X_i with density $f_i = q^i f / \mathbb{E} q^i(X) \sim \mathrm{IP}(\mu_i; q_i)$, where $\mu_i = (\mu + i\beta)/(1-2i\delta)$, $q_i = q/(1-$

$2i\delta$) and $J(X_i) = J$ (see Appendix A). One can easily see that $\mathbb{E}q_i(X_i)|h'(X_i)| < \infty$. Since $\mathbb{E}X^{2m} < \infty$ we get $\mathbb{E}X_i^{2(m-i)} < \infty$. Hence $\mathbb{E}X_i^2 < \infty$ because $m - i \geq 1$. Using Lemma 2 (for $k = 1$) we have that $\mathbb{E}|P_{1,i}(X_i)h(X_i)| < \infty$, where $P_{1,i}(x) = x - \mu_i$ is the Rodrigues polynomial of degree 1 corresponding to the density f_i. Since $\mu_i \in (\alpha, \omega)$, we can find $\epsilon > 0$ such that $[\mu_i - \epsilon, \mu_i + \epsilon] \subset (\alpha, \omega)$. Thus,

$$
\begin{aligned}
\mathbb{E}|P_{1,i}(X_i)h(X_i)| &= \int_{\alpha}^{\mu_i-\epsilon} (\mu_i - x)|h(x)|f_i(x)\,\mathrm{d}x + \int_{\mu_i-\epsilon}^{\mu_i+\epsilon} |(\mu_i - x)h(x)|f_i(x)\,\mathrm{d}x \\
&\quad + \int_{\mu_i+\epsilon}^{\omega} (x - \mu_i)|h(x)|f_i(x)\,\mathrm{d}x \\
&\geq \epsilon \int_{\alpha}^{\mu_i-\epsilon} |h(x)|f_i(x)\,\mathrm{d}x + \int_{\mu_i-\epsilon}^{\mu_i+\epsilon} |(\mu_i - x)h(x)|f_i(x)\,\mathrm{d}x \\
&\quad + \epsilon \int_{\mu_i+\epsilon}^{\omega} |h(x)|f_i(x)\,\mathrm{d}x.
\end{aligned}
$$

Hence, $\int_{\alpha}^{\mu_i-\epsilon} |h(x)|f_i(x)\,\mathrm{d}x$ and $\int_{\mu_i+\epsilon}^{\omega} |h(x)|f_i(x)\,\mathrm{d}x$ are finite. The function h is continuous in the compact interval $[\mu_i - \epsilon, \mu_i + \epsilon]$ so $\int_{\mu_i-\epsilon}^{\mu_i+\epsilon} |h(x)|f_i(x)\,\mathrm{d}x$ is finite. Therefore, $\mathbb{E}|h(X_i)| = \mathbb{E}|g^{(i)}(X_i)| = \frac{\mathbb{E}q^i(X)|g^{(i)}(X)|}{\mathbb{E}q^i(X)}$ is finite.

We have shown that $\mathbb{E}q^{i+1}(X)g^{(i+1)}(X) < \infty$ implies $\mathbb{E}q^i(X)g^{(i)}(X) < \infty$. Applying this for $i = m-1, m-2, \ldots, 0$, the proof is completed. $\square$

Now, we give the following definitions that will be used in the sequel.

Definition 2 For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ define:

(a) $(x)_k = x(x-1)\cdots(x-k+1)$, with $(x)_0 = 1$.

(b) $[x]_k = x(x+1)\cdots(x+k-1)$, with $[x]_0 = 1$.

Note that $[x]_k = (-1)^k(-x)_k = (x+k-1)_k$.

Definition 3 [cf. Afendras and Papadatos (2012b)] Assume that $X \sim \mathrm{IP}(\mu; q)$ and denote $q(x) = \delta x^2 + \beta x + \gamma$, its quadratic polynomial. Let $J(X) = (\alpha, \omega)$ be the support of X and fix the non-negative integers m, n such that $\mathbb{E}|X|^{2\ell}$ is finite, where $\ell = \max\{m, n\}$. We shall denote by $\mathscr{H}^{m,n}(X)$ the class of Borel functions $g : (\alpha, \omega) \to \mathbb{R}$ satisfying the following properties:

H_1 : $g \in C^{\ell-1}(\alpha, \omega)$ and the function $g^{(\ell-1)}(x) := \frac{\mathrm{d}^{\ell-1}g(x)}{\mathrm{d}x^{\ell-1}}$ is a.c. in (α, ω) with a.s. derivative $g^{(\ell)}$.

H_2 : $\mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2 < \infty$ and $\mathbb{E}q^m(X)|g^{(m)}(X)| < \infty$.

Note that from (21), Lemma 1 and $\mathbb{E}^2 q^i(X)|g^{(i)}(X)| \leq \mathbb{E}q^i(X) \cdot \mathbb{E}q^i(X)\big(g^{(i)}(X)\big)^2$, $i = 1, 2, \ldots, n$, if $m \leq n$ and if $\mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2$ is finite then it is implied that $\mathbb{E}q^m(X)|g^{(m)}(X)|$ is finite. For $m = n = 0$, the property H_1 does not impose any restrictions on g, and

$$
\mathscr{H}^{0,0}(X) \equiv L^2(\mathbb{R}, X) := \big\{g : (\alpha, \omega) \to \mathbb{R} \text{ such that } \mathsf{Var}\, g(X) < \infty\big\}.
$$

Also, it is obvious that $\mathscr{H}^{0,n} = \mathscr{H}^{1,n} = \cdots = \mathscr{H}^{n,n}$.

Furthermore, we shall denote by $\mathscr{H}^{\infty,n}(X)$ and $\mathscr{H}^{\infty}(X) \equiv \mathscr{H}^{m,\infty}(X)$ [m is arbitrary because in this case this index is insignificant] the classes of functions $\mathscr{H}^{\infty,n}(X) := \cap_{m=0}^{\infty}\mathscr{H}^{m,n}(X)$ and $\mathscr{H}^{\infty}(X) := \cap_{n=0}^{\infty}\mathscr{H}^{\infty,n}(X)$; i.e,

$$\begin{aligned}
\mathscr{H}^{\infty,n}(X) =& \Big\{g \in C^{\infty}(J) : \mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2 < \infty \text{ and } \mathbb{E}q^i(X)|g^{(i)}(X)| \\
& < \infty \ \forall i > n\Big\}, \\
\mathscr{H}^{\infty}(X) =& \Big\{g \in C^{\infty}(J) : \mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2 < \infty \ \forall n \in \mathbb{N}\Big\}.
\end{aligned}$$

From Lemma 1 and from (21), we conclude that the (finite or infinite) sequence $\mathscr{H}^{m,n}(X)$ is decreasing in m and in n. In particular, if all moments of X exist then

$$\begin{array}{ccccccccc}
L^2(\mathbb{R}, X) \equiv & \mathscr{H}^{0,0}(X) & & & & & & & \\
& \cup\!| & & & & & & & \\
& \mathscr{H}^{1,0}(X) & \supseteq & \mathscr{H}^{1,1}(X) & & & & & \\
& \cup\!| & & \cup\!| & & & & & \\
& \mathscr{H}^{2,0}(X) & \supseteq & \mathscr{H}^{2,1}(X) & \supseteq & \mathscr{H}^{2,2}(X) & & & \\
& \cup\!| & & \cup\!| & & \cup\!| & & & \\
& \vdots & & \vdots & & \vdots & & & \\
& \cup\!| & & \cup\!| & & \cup\!| & & & \\
& \mathscr{H}^{\infty,0}(X) & \supseteq & \mathscr{H}^{\infty,1}(X) & \supseteq & \mathscr{H}^{\infty,2}(X) & \supseteq & \cdots \supseteq & \mathscr{H}^{\infty}(X).
\end{array}$$

Let $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$ with $\delta \le 0$. Also, consider two (fixed) non-negative integers m, n, with $n \neq 0$, and a function $g \in \mathscr{H}^{m,n}(X)$. According to Parseval's identity we have that

$$\mathsf{Var}\, g(X) = \sum_{k=1}^{\infty} c_k^2, \tag{6}$$

where $c_k = \mathbb{E}g(X)\varphi_k(X)$ are the Fourier coefficients of g with respect to the corresponding (to X) orthonormal polynomial system $\{\varphi_k\}_{k=0}^{\infty}$.

For each $i = 1, 2, \ldots, n$ the function $g^{(i)} \in \mathscr{H}^{m-i,n-i}(X_i)$ and, from Parceval's identity again, $\mathbb{E}\big(g^{(i)}(X_i)\big)^2 = \sum_{k=0}^{\infty}\big(c_k^{(i)}\big)^2$, where $c_k^{(i)} = \mathbb{E}g^{(i)}(X_i)\varphi_{k,i}(X_i)$ are the Fourier coefficients of $g^{(i)}$ with respect to the orthonormal polynomial system $\{\varphi_{k,i}\}_{k=0}^{\infty}$ corresponding to $X_i \sim f_i \propto q^i f$; see Appendix A. Using (22) we have that

$$\mathbb{E}q^i(X)\big(g^{(i)}(X)\big)^2 = \sum_{k=i}^{\infty}\left((k)_i \prod_{j=k-1}^{k+i-2}(1 - j\delta)\right) c_k^2, \quad i = 1, 2, \ldots, n, \tag{7}$$

see [Afendras and Papadatos 2012b, Lemma 3.1, eq. (3.4)], where each coefficient of c_k^2 is positive. Also, from (20),

$$\mathbb{E}q^i(X)g^{(i)}(X) = \left(i!\,\mathbb{E}q^i(X) \prod_{j=i-1}^{2i-2} (1-j\delta) \right)^{1/2} c_i, \quad i = 1, 2, \ldots, m, \tag{8}$$

see [Afendras et al. 2011, Section 3, eq.'s (3.2) and (3.5)].

Let $\boldsymbol{\lambda}_n = (\lambda_{1;n}, \lambda_{2;n}, \ldots, \lambda_{n;n})^{\mathrm{t}} \in \mathbb{R}^n$. According to Tonelli's theorem, we have that $\sum_{i=1}^{n} \sum_{k=i}^{\infty} \left|\lambda_{i;n}(k)_i \prod_{j=k-1}^{k+i-2}(1-j\delta)\right| c_k^2 = \sum_{i=1}^{n} |\lambda_{i;n}| \sum_{k=i}^{\infty} \big[(k)_i \prod_{j=k-1}^{k+i-2} (1-j\delta)\big] c_k^2 = \sum_{i=1}^{n} \left|\lambda_{i;n}\right| \mathbb{E}q^i(X)\big(g^{(i)}\big)^2 < \infty$ and, using Fubini's theorem,

$$\sum_{i=1}^{n} \lambda_{i;n} \mathbb{E}q^i(X)\big(g^{(i)}(X)\big)^2 = \sum_{k=1}^{\infty} \rho_{k;n} c_k^2, \quad \text{where } \rho_{k;n} = \sum_{i=1}^{\min\{k,n\}} \lambda_{i;n}(k)_i \prod_{j=k-1}^{k+i-2} (1-j\delta). \tag{9}$$

We seek a vector $\boldsymbol{\lambda}_{m,n}$ such that $\rho_{m+1,n} = \rho_{m+2,n} = \cdots = \rho_{m+n,n} = 1$. From (7) we obtain the system of equations

$$A_{m,n} \cdot \boldsymbol{\lambda}_{m,n} = \mathbf{1}_n, \tag{10}$$

where the matrix $A_{m,n} \in \mathbb{R}^{n \times n}$ has (r, c)-element which is given by

$$a_{r,c;m,n} = (m+r)_c \prod_{m+r-1}^{m+r+c-2} (1-j\delta)$$

and the vector $\mathbf{1}_n = (1, 1, \ldots, 1)^{\mathrm{t}} \in \mathbb{R}^n$.

The above system has the unique solution, see Appendix B,

$$\lambda_{i;m,n} = (-1)^{i-1}\tbinom{n}{i} \Big/ \left[(m+n)_i \prod_{j=m}^{m+i-1} (1-j\delta) \right], \quad i = 1, 2, \ldots, n. \tag{11}$$

From (9) and (11), using the hypergeometric series (23), we have that $\rho_{k;m,n} = 1 - (m+n-k)_n \prod_{j=m+k}^{m+n+k-1} (1-j\delta) \big/ \big[(m+n)_n \prod_{j=m}^{m+n-1} (1-j\delta)\big]$. Equivalently,

$$\rho_{k;m,n} = \begin{cases} 1 - \frac{(m+n-k)_n \prod_{j=m+k}^{m+n+k-1}(1-j\delta)}{(m+n)_n \prod_{j=m}^{m+n-1}(1-j\delta)}, & 1 \le k \le m, \\ 1, & m < k \le m+n, \\ 1 + (-1)^{n-1} \frac{(k-m-1)_n \prod_{j=m+k}^{m+n+k-1}(1-j\delta)}{(m+n)_n \prod_{j=m}^{m+n-1}(1-j\delta)}, & k > m+n. \end{cases}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^{i-1}\frac{\binom{n}{i}\mathbb{E}q^i(X)\big(g^{(i)}(X)\big)^2}{(m+n)_i\prod_{j=m}^{m+i-1}(1-j\delta)}\\
&\quad=\mathsf{Var}g(X)-\sum_{k=1}^{n}\frac{(m+n-k)_n\prod_{j=m+k}^{m+n+k-1}(1-j\delta)}{(m+n)_n\prod_{j=m}^{m+n-1}(1-j\delta)}c_k^2\\
&\qquad+\sum_{k>m+n}(-1)^{n-1}\frac{(k-m-1)_n\prod_{j=m+k}^{m+n+k-1}(1-j\delta)}{(m+n)_n\prod_{j=m}^{m+n-1}(1-j\delta)}c_k^2.
\end{aligned}\tag{12}$$

The main result of this paper is contained in the following theorem.

Theorem 1 *Let $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta\le 0$. Fix two non-negative integers m,n [with $n\neq 0$] and a function $g\in\mathscr{H}^{m,n}(X)$. Consider the quantity*

$$S_{m,n}(g)=\sum_{i=1}^{m}a_i\mathbb{E}^2q^i(X)g^{(i)}(X)+\sum_{i=1}^{n}(-1)^{i-1}b_i\mathbb{E}q^i(X)\big(g^{(i)}(X)\big)^2,\tag{13}$$

where

$$\begin{aligned}
a_i&:=\frac{\binom{m}{i}\prod_{j=m+i}^{m+n+i-1}(1-j\delta)}{(m+n)_i\mathbb{E}q^i(X)\big(\prod_{j=i-1}^{2i-2}(1-j\delta)\big)\prod_{j=m}^{m+n-1}(1-j\delta)},\\
b_i&:=\frac{\binom{n}{i}}{(m+n)_i\prod_{j=m}^{m+i-1}(1-j\delta)}
\end{aligned}$$

are strictly positive constants (depending only on m,n and X) and the empty sums (when $m=0$) are treated as zero. Then, the following inequality holds:

$$(-1)^n\big(\mathsf{Var}g(X)-S_{m,n}(g)\big)\ge 0,$$

and where $S_{m,n}(g)$ becomes equal to $\mathsf{Var}g(X)$ if and only if g is a polynomial of degree at most $m+n$.

Proof From (13), via (8) and (12), we obtain that $(-1)^n\big(\mathsf{Var}g(X)-S_{m,n}(g)\big)=R_{m,n}(g)$, where the residual

$$R_{m,n}(g)=\sum_{k>m+n}r_{k;m,n}(X)c_k^2:=\sum_{k>m+n}\frac{(k-m-1)_n\prod_{j=m+k}^{m+n+k-1}(1-j\delta)}{(m+n)_n\prod_{j=m}^{m+n-1}(1-j\delta)}c_k^2\tag{14}$$

is non-negative and equals to zero if and only if $c_k=0$ for all $k>m+n$, i.e., the function $g:J(X)\to\mathbb{R}$ is a polynomial of degree at most $m+n$. □

Let $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$ with $\delta \le 0$. Then X is a linear function of a normal, a gamma or a beta random variable, see Afendras and Papadatos (2012a). The bounds $S_{m,n}(g)$ for the three main cases are included in Table 1.

Remark 1 (a) For fixed n and for any function $g \in \mathscr{H}^{M,n}(X)$, where M can be finite or infinite, the variance bounds $\{S_{m,n}(g)\}_{m=0}^{M}$ are of the same type, i.e., upper bound when n is odd and lower bound when n is even.

(b) The bounds $\{S_{m,n}(g)\}_{m=0}^{n}$ require the same conditions on the function g, i.e., $g \in \mathscr{H}^{n,n}(X)$.

Remark 2 (a) The bound $S_{1,1}(g)$ is the bound $S_{1,(\mathrm{str})}$ of (4).

(b) The bounds $S_{0,n}(g)$ are the bounds S_n which are given by (2). Also, for the special case $m = 0$, $n = 1$ observe that $S_{0,1}(g) = S_1 = S_{1,(\mathrm{weak})}$; see (5).

(c) The results shown in Theorem 1 apply to the special case where $n = 0$ (note that the second sum is empty and is treated as zero). In this case, the lower bound $S_{m,0}(g)$ is reduced to the one given by (3).

Remark 3 Regarding the conditions on the function g of Theorem 1, we note that $g \in \mathscr{H}^{\max\{m,n\},n-1}(X) \setminus \mathscr{H}^{\max\{m,n\},n}(X)$ implies that the bound $S_{m,n}(g)$ is trivial, i.e., $+\infty$ when n is odd and $-\infty$ when n is even.

Now, we seek for upper bounds of the non-negative residual $R_{m,n}(g)$.

Proposition 1 *Assume the conditions of Theorem 1 and, further, suppose that $g \in \mathscr{H}^{T,T}(X)$ for some $T \in \{n, \ldots, m+n+1\}$. Then the residual $R_{m,n}(g)$, given by (14), is bounded above by*

$$u_\tau \mathbb{E} q^\tau(X)\big(g^{(\tau)}(X)\big)^2, \quad \tau = n, n+1, \ldots, T, \tag{15}$$

where $u_\tau = u_{m,n,\tau}(X) := \prod_{j=2m+n+1}^{2m+2n}(1 - j\delta) \big/ \big[\binom{m+n}{n}(m+n+1)_\tau \prod_{j=m}^{m+n+\tau-1}(1 - j\delta)\big]$.

Proof Using (7), we write the quantity (15) in the form $\sum_{k=\tau}^{\infty} \pi_{k;\tau} c_k^2$. Next, consider the sequence $\{w_{k;\tau} = \pi_{k;\tau}/r_{k;m,n}(X)\}_{k=m+n+1}^{\infty}$, where $r_{k;m,n}(X)$ are the numbers given by (14), and observe that this sequence is increasing in k, with $w_{m+n+1;\tau} = 1$. □

The upper bounds (when there are at least two) of the residual $R_{m,n}(g)$, given by (15), are not comparable. For example, consider the functions $g_1 = \varphi_\tau$ and $g_2 = \varphi_{m+n+2}$ (both belong to $\mathscr{H}^{\infty}(X)$), where φ_k are the polynomials given by (19), and observe that

$$u_\tau \mathbb{E} q^\tau(X)\big(g_1^{(\tau)}(X)\big) = u_\tau \tau! \prod_{j=k-2}^{2\tau-2}(1 - j\delta) > 0 = u_{\tau+1} \mathbb{E} q^{\tau+1}(X)\big(g_1^{(\tau+1)}(X)\big)$$

and

$$\frac{u_\tau \mathbb{E} q^\tau(X)\big(g_2^{(\tau)}(X)\big)}{u_{\tau+1} \mathbb{E} q^{\tau+1}(X)\big(g_2^{(\tau+1)}(X)\big)} = \frac{(m+n-\tau+1)\big(1 - (m+n+\tau)\delta\big)}{(m+n-\tau+2)\big(1 - (m+n+\tau+1)\delta\big)} < 1,$$

since $\delta \le 0$.

Table 1 Specific form of the bounds $S_{m,n}(g)$ for normal, gamma and beta distributions

Distribution	Parameters / Bounds $S_m, n(g)$	f	$J(X)$	$q(x)$
Normal(μ, σ^2)	$\mu \in \mathbb{R}, \sigma^2 > 0$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$\mathbb{R}$	σ^2
	$\sum_{i=1}^m \frac{\binom{m}{i}\sigma^{2i}}{(m+n)_i} \mathbb{E}^2 g^{(i)}(X) + \sum_{i=1}^n (-1)^{i-1} \frac{\binom{n}{i}\sigma^{2i}}{(m+n)_i} \mathbb{E}\big(g^{(i)}(X)\big)^2$			
Gamma(α, λ)	$\alpha, \lambda > 0$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$(0, \infty)$	x/λ
	$\sum_{i=1}^m \frac{\binom{m}{i}}{(m+n)_i [\alpha]_i} \mathbb{E}^2 X^i g^{(i)}(X) + \sum_{i=1}^n (-1)^{i-1} \frac{\binom{n}{i}}{(m+n)_i \lambda^i} \mathbb{E} X^i \big(g^{(i)}(X)\big)^2$			
Beta(α, β)	$\alpha, \beta > 0$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$(0, 1)$	$x(1-x)\big/(\alpha+\beta)$
	$\sum_{i=1}^m \frac{\binom{m}{i}[\alpha+\beta+m+i]_n [\alpha+\beta]_{2i}}{(m+n)_i [\alpha]_i [\beta]_i [\alpha+\beta+i-1]_i [\alpha+\beta+m]_n} \mathbb{E}^2 X^i (1-X)^i g^{(i)}(X)$ $+ \sum_{i=1}^n (-1)^{i-1} \frac{\binom{n}{i}}{(m+n)_i [\alpha+\beta+m]_i} \mathbb{E} X^i (1-X)^i \big(g^{(i)}(X)\big)^2$			

3 Investigating the bounds $S_{m,n}$ for fixed n

Next, for n fixed, we investigate the bounds $S_{m,n}(g)$ as m increases. We compare the variance bounds $S_{n,n}(g)$ and S_n, given by (13) [for $m=n$] and (2), respectively. Also, we compare the new upper variance bounds $S_{n,1}(g)$ and $S_{n-1,1}(g)$ with the existing Chernoff-type upper variance bounds $S_{n,(\mathrm{str})}$ and $S_{n,(\mathrm{weak})}$, respectively; see (4) and (5).

Theorem 2 *Let* $X \sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ *with* $\delta \le 0$. *Fix the positive integer* n *and consider a function* $g \in \mathscr{H}^{M,n}(X)$, *where* M *can be finite* ($M \ge n$) *or infinite. Then, for each* m_1, m_2 *such that* $0 \le m_1 < m_2 \le M$ *the following inequality holds*

$$\left|\mathsf{Var}\, g(X) - S_{m_1,n}(g)\right| \ge \zeta_{m_1,m_2,n}(\delta)\left|\mathsf{Var}\, g(X) - S_{m_2,n}(g)\right|, \tag{16}$$

where $\zeta_{m_1,m_2,n}(\delta) := \frac{(m_2+n)_n \prod_{j=m_2}^{m_2+n-1}(1-j\delta)}{(m_1+n)_n \prod_{j=m_1}^{m_1+n-1}(1-j\delta)} > 1$. *The equality holds if and only if the function* $g : J(X) \to \mathbb{R}$ *is a polynomial of degree at most* $n+m_1$.

Proof Consider the positive sequence $\left\{\zeta_{k;m_1,m_2,n}(\delta) = r_{k;m_1,n}(X) / r_{k;m_2,n}(X)\right\}_{k>m_2+n}$, where $r_{k;m,n}(X)$ are given by (14). This sequence is decreasing in k. Specifically,

$$\zeta_{k;m_1,m_2,n}(\delta) \searrow \zeta_{m_1,m_2,n}(\delta) \equiv \frac{(m_2+n)_n \prod_{j=m_2}^{m_2+n-1}(1-j\delta)}{(m_1+n)_n \prod_{j=m_1}^{m_1+n-1}(1-j\delta)}, \quad \text{as } k \to \infty.$$

Moreover, we observe that $r_{k;m_1,n}(X) > 0$ and $r_{k;m_2,n}(X) = 0$ for all $k = n+m_1+1, \ldots, n+m_2$. Therefore (16) follows.

We write $\left|\mathsf{Var}\, g(X) - S_{m_1,n}(g)\right| - \zeta_{m_1,m_2,n}(\delta)\left|\mathsf{Var}\, g(X) - S_{m_2,n}(g)\right| = \sum_{k>n+m_1} \theta_k c_k^2$ and we observe that $\theta_k > 0$ for all k. Thus, the equality in (16) holds if and only if g is a polynomial of degree at most $n+m_1$. □

Notice that if $\delta < 0$ then $\prod_{j=m_2}^{m_2+n-1}(1-j\delta) / \prod_{j=m_1}^{m_1+n-1}(1-j\delta) > 1$ for each n and $m_1 < m_2$. Therefore, $\zeta_{m_1,m_2,n}(\delta) \ge \zeta_{m_1,m_2,n}(0) = (m_2+n)_n/(m_1+n)_n$, since $\delta \le 0$.

Remark 4 Assume the conditions of Theorem 2. (a) In view of Remark 1(a), the bounds $\{S_{m,n}(g)\}_{m=0}^{M}$ are of the same type. From (16) it is follows that the bound $S_{m_2,n}(g)$ is better than the bound $S_{m_1,n}(g)$. Now, writing $n = 2r$ (when n is even) or $n = 2r+1$ (when n is odd) we observe that

$$S_{0,2r}(g) \le S_{1,2r}(g) \le \cdots \le \mathsf{Var}\, g(X) \le \cdots \le S_{1,2r+1}(g) \le S_{0,2r+1}(g).$$

(b) For the case $M = \infty$, from (6), (13) and (a) it follows that

$$\underset{\text{[when } n \text{ is even]}}{S_{m,n}(g) \nearrow \mathsf{Var}\, g(X)} \quad \text{or} \quad \underset{\text{[when } n \text{ is odd]}}{S_{m,n}(g) \searrow \mathsf{Var}\, g(X)} \quad \text{as} \quad m \to \infty.$$

Now, we compare the existing variance bounds $S_n\big(\equiv S_{0,n}(g)\big)$ with the best proposed bound shown in this article requiring the same conditions on g, i.e., with the bound $S_{n,n}(g)$; see Remark 1(b).

Corollary 1 *The variance bounds $S_{n,n}(g)$ and S_n, given by (13) (for $m=n$) and (2) respectively, are of the same type and require the same conditions on g. Moreover, the new bound $S_{n,n}(g)$ is better than the existing bound S_n. Specifically,*

$$\big|\mathsf{Var}g(X)-S_n\big| \ge \binom{2n}{n}\frac{\prod_{j=n}^{2n-1}(1-j\delta)}{\prod_{j=0}^{n-1}(1-j\delta)}\big|\mathsf{Var}g(X)-S_{n,n}(g)\big|.$$

The equality holds only in the trivial cases when $\mathsf{Var}g(X)=S_{n,n}(g)=S_n$, i.e., the function $g: J(X)\to\mathbb{R}$ is a polynomial of degree at most n.

Note that, since $\delta\le 0$, $\binom{2n}{n}\prod_{j=n}^{2n-1}(1-j\delta)\big/\prod_{j=0}^{n-1}(1-j\delta)\ge\binom{2n}{n}$.

The quantities $S_{n,(\mathrm{str})}$ and $S_{n,1}(g)$ are upper bounds for $\mathsf{Var}g(X)$. Both bounds are equal to $\mathsf{Var}g(X)$ if and only if the function g is a polynomial of degree at most $n+1$. The quantities $S_{n,(\mathrm{weak})}$ and $S_{n-1,1}(g)$ are upper bounds for $\mathsf{Var}g(X)$. Both bounds are equal to $\mathsf{Var}g(X)$ if and only if the function g is a polynomial of degree at most n. Thus, it is reasonable to compare these bounds.

Theorem 3 *For $n=1,2,\ldots$ and any suitable function g we have that:*

(a) *$S_{n,1}(g)\le S_{n,(\mathrm{str})}$, where the equality holds when $n=1$ or $n>1$ and g is a polynomial of degree at most $n+1$.*
(b) *$S_{n-1,1}(g)\le S_{n,(\mathrm{weak})}$, where the equality holds when $n=1$ or $n>1$ and g is a polynomial of degree at most n.*

Proof (a) From (13) and (4), via (7) and (8), we have that

$$S_{n,(\mathrm{str})}-S_{n,1}(g)=\sum_{k>n+1}\tfrac{k[1-(k-1)\delta]}{(n+1)(1-n\delta)}\left(\frac{\binom{k-1}{n-1}\prod_{j=k}^{n+k-2}(1-j\delta)}{n\prod_{j=n+1}^{2n-1}(1-j\delta)}-1\right)c_k^2, \tag{17}$$

where each coefficient of c_k^2, $k>n+1$, is zero when $n=1$ and is positive when $n>1$. (b) Similarly, from (13) and (5), via (7) and (8), it follows that

$$S_{n,(\mathrm{weak})}-S_{n-1,1}(g)=\textstyle\sum_{k>n}\frac{k[1-(k-1)\delta]}{n[1-(n-1)\delta]}\left(\frac{\binom{k-1}{n-1}\prod_{j=k}^{n+k-2}(1-j\delta)}{\prod_{j=n}^{2n-2}(1-j\delta)}-1\right)c_k^2, \tag{18}$$

where each coefficient of c_k^2, $k>n$, is zero when $n=1$ and is positive when $n>1$. □

Remark 5 For each $n=2,3,\ldots$ it follows that:

(a) The bound $S_{n,1}(g)$ is better than the bound $S_{n,(\mathrm{str})}$; notice that the bound $S_{n,1}(g)$ requires a milder finiteness condition, $g\in\mathscr{H}^{n,1}(X)$, compared to $S_{n,(\mathrm{str})}$ which requires that $g\in\mathscr{H}^{n,n}(X)\subseteq\mathscr{H}^{n,1}(X)$.
(b) The bound $S_{n-1,1}(g)$ is better than the bound $S_{n,(\mathrm{weak})}$; as in (a), the bound $S_{n-1,1}(g)$ requires a weaker finiteness condition, i.e., $g\in\mathscr{H}^{n-1,1}(X)$, rather than $S_{n,(\mathrm{weak})}$, i.e., $g\in\mathscr{H}^{n,n}(X)\subseteq\mathscr{H}^{n-1,1}(X)$.

Final Conclusion The variance bounds given by Theorem 1, for appropriate choices of n and m, either provide existing univariate variance bounds or improvements. Our bounds cover all usual cases, namely:

– Chernoff-type [Nash (1958); Brascamp and Lieb (1976); Chernoff (1981); Cacoullos and Papathanasiou (1985); Papadatos and Papathanasiou (2001); Prakasa Rao (2006); Afendras and Papadatos (2012b)],
– Poincaré-type [Papathanasiou (1988); Cacoullos (1989); Johnson (1993); Houdré and Kagan (1995); Afendras et al. (2011)],
– Bessel-type [Cacoullos (1982); Houdré and Kagan (1995); Afendras et al. (2011)].

Note that no further conditions on the function g are imposed; instead, the new bounds require the same or weaker conditions, see Remarks 1, 2 and 5, Theorem 3, and Corollary 1. Therefore, the new proposed variance bounds outweigh all the existing variance bounds presented in the bibliography.

Appendix A: Some useful properties of the integrated Pearson family

The following properties have been reproduced from Afendras et al. (2011); Afendras and Papadatos (2012a,b) and are stated here for easy reference.

Consider a random variable X with density $f \sim \mathrm{IP}(\mu; q) \equiv \mathrm{IP}(\mu; \delta, \beta, \gamma)$.

Let $a \in (1, +\infty)$. Then $E|X|^a < \infty$ if and only if $\delta < 1/(a-1)$. Notice that X has finite moments of any order if and only if $\delta \le 0$; see Afendras and Papadatos (2012a, Corollary 2.2).

The support of X is the interval $J(X) = (\alpha, \omega)$ and the density $f \in C^{\infty}(\alpha, \omega)$, see Afendras and Papadatos (2012a).

If $\mathbb{E}|X|^{2i+1} < \infty$, $i \in \mathbb{N}^* \equiv \mathbb{N} \setminus \{0\}$ (that is, $\delta < 1/2i$), then the random variable X_i with density function $f_i(x) = q^i(x) f(x)/\mathbb{E}q^i(X)$ follows $\mathrm{IP}(\mu_i; q_i)$ distribution with $\mu_i = (\mu + i\beta)/(1 - 2i\delta)$ and $q_i = q/(1 - 2i\delta)$; see Afendras and Papadatos (2012a, Theorem 5.2). Note that if $\mathbb{E}|X|^{2i} < \infty$ and $\mathbb{E}|X|^{2i+1} = \infty$ then the function f_i is a probability density function; however, $\mathbb{E}|X_i| = \infty$ so X_i does not belong to the integrated Pearson family.

If $\mathbb{E}|X|^{2N} < \infty$, $N \in \mathbb{N}^*$ [that is, $\delta < 1/(2N-1)$], then the quadratic q generates the orthogonal polynomials through the Rodrigues-type formula, Papathanasiou (1995),

$$P_k(x) = \frac{(-1)^k}{f(x)} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\big[q^k(x) f(x)\big], \quad x \in J(X), \quad k = 0, 1, \ldots, N.$$

Afendras et al. (2011) showed an extended Stein-type identity of order n. This identity takes the form $\mathbb{E}P_k(X)g(X) = \mathbb{E}q^k(X)g^{(k)}(X)$, provided that $\mathbb{E}q^k(X)|g^{(k)}(X)| < \infty$. Also, $\mathbb{E}P_k(X)P_m(X) = \delta_{k,m} k!\, \mathbb{E}q^k(X) \times \prod_{j=k-1}^{2k-2}(1 - j\delta)$; thus, the system of polynomials $\{\varphi_k\}_{k=0}^N$ is orthonormal, with respect to density f, where

$$\varphi_k(x) = P_k(x)\left(k!\, \mathbb{E}q^k(X) \prod_{j=k-1}^{2k-2}(1 - j\delta)\right)^{-1/2}. \tag{19}$$

Hence,

$$\mathbb{E}q^k(X)g^{(k)}(X) = \left(k!\,\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta)\right)^{-1/2}\mathbb{E}\varphi_k(X)g(X). \tag{20}$$

Moreover, the system of polynomials $\{\varphi_{k+i}^{(i)}\}_{k=0}^{N-i}$ [where φ_k are the polynomials given by (19) and $\varphi_{k+i}^{(i)}$ is the i-th derivative of φ_{k+i}] is orthogonal with respect to density f_i. Specifically, if $\varphi_{k,i}$ are the orthonormal polynomials corresponding to the density f_i then $\varphi_{k+i}^{(i)}(x) = \nu_k^{(i)}\varphi_{k,i}(x)$, where $\nu_k^{(i)} = \nu_k^{(i)}(X) := \left[(k+i)_i \times \left(\prod_{j=k+i-1}^{k+2i-2}(1-j\delta)\right)/\mathbb{E}q^i(X)\right]^{1/2}$, see Afendras and Papadatos (2012a, Corollary 5.4).

Lemma 2 [Afendras et al. (2011, Theo. 3.1(b), p. 516)] *Let $X \sim \mathrm{IP}(\mu;q)$, with $\mathbb{E}X^{2k} < \infty$, and a suitable function g, with $\mathbb{E}q^k(X)|g^{(k)}(X)| < \infty$, then $\mathbb{E}|P_k(X)g(X)| < \infty$.*

Lemma 3 [Afendras and Papadatos (2012b)] *Let a random variable $X \sim \mathrm{IP}(\mu;q)$ and let us consider the strictly positive integers n and N such that $n \le N$ and $\mathbb{E}|X|^{2N} < \infty$.*

$$If\ g \in \mathscr{H}^{n,n}(X)\ then\ g^{(i)} \in \mathscr{H}^{n-i,n-i}(X_i)\quad for\ each\ i = 0,1,\ldots,n-1. \tag{21}$$

$$\mathbb{E}\varphi_{k,i}(X_i)g^{(i)}(X_i) = \nu_k^{(i)}(X)\mathbb{E}\varphi_{k+i}(X)g(X)\quad for\ each\ \begin{cases} i = 1,2,\ldots,n, \\ k = 0,1,\ldots,N-i. \end{cases} \tag{22}$$

If the parameter δ of q is non-positive, then the moment generating function of X is finite in a neighborhood of zero; thus, the system of polynomials $\{\varphi_k\}_{k=0}^{\infty}$ forms an orthonomal basis of $L^2(\mathbb{R},X)$ and the Parseval identity holds, see Afendras and Papadatos (2012a). Notice that for each $i \in \mathbb{N}$ the parameter $\delta_i = \frac{\delta}{1-2i\delta}$ is also non-positive. Thus, the system of polynomials $\{\varphi_{k,i}\}_{k=0}^{\infty}$ is an orthonomal basis of $L^2(\mathbb{R},X_i)$ and the Parseval identity also holds.

Appendix B: The solution of the system (10)

Consider the determinants $d_{m,n} = \det(A_{m,n})$ and $d_{i;m,n} = \det(A_{i;m,n})$, $i = 1,2,\ldots,n$, where the matrix $A_{i;m,n}$ is formed from $A_{m,n}$ by replacing column i with the vector $\mathbf{1}_n$.

For each $t = 1,2,\ldots,n$ define the matrix $B_{m,n}(t) \in \mathbb{R}^{(n-t+1)\times(n-t+1)}$ [m,n are fixed] which has elements $b_{r,c;m,n}(t) = (m+r-1)_{c-1}\prod_{j=m+r+t-1}^{m+r+c+t-3}(1-j\delta)$, where empty products are treated as one. Observe that $d_{m,n} = (m+n)_n\left(\prod_{j=m}^{m+n-1}(1-j\delta)\right)$ $\det\left(B_{m,n}(1)\right)$and $\det\left(B_{m,n}(t)\right) = (n-t)!\left(\prod_{j=m+1}^{m+n-t}(1-[2j+(t-1)]\delta)\right)\det\left(B_{m,n}(t+1)\right)$, $t = 1,2,\ldots,n-1$.

Thus, it follows that

$$d_{m,n} = (m+n)_n \left[\prod_{j=0}^{n-1} j!\right] \left[\prod_{j=m}^{m+n-1} (1-j\delta)\right] \prod_{t=1}^{n-1} \left(\prod_{j=m+1}^{m+n-t} (1-[2j+(t-1)]\delta)\right) \neq 0.$$

Now, for each $t=1, 2, \ldots, n$ define the matrix $B_{i,m,n}(t) \in \mathbb{R}^{(n-t+1)\times(n-t+1)}$ [i, m, n are fixed integers] which has (r, c)-element $b_{r,c;i,m,n}(t) = (m+r)_{c-1} \prod_{j=m+r+t-2}^{m+r+c+t-4}(1-j\delta)$, when $c = 1, 2, \ldots, i-t$, and $b_{r,c;i,m,n}(t) = (m+r)_c \prod_{j=m+r+t-2}^{m+r+c+t-3}(1-j\delta)$, when $c = i-t+1, i-t+2, \ldots, n-t+1$. Observe that $d_{i;m,n} = (-1)^{i-1} \det\left(B_{i,m,n}(1)\right)$, $\det\left(B_{i,m,n}(t)\right) = \frac{(n-t+1)!}{(i-t+1)}\left(\prod_{j=m+1}^{m+n-t}(1-[2j+(t-1)]\delta)\right) \det\left(B_{i,m,n}(t+1)\right)$, $t = 1, 2, \ldots, i-1$, and $\det\left(B_{i,m,n}(i)\right) = (n-i+1)!\left(\prod_{j=m+1}^{m+n-i}(1-[2j+(i-1)]\delta)\right)(m+n-i)_{n-i}\left(\prod_{j=m+i}^{m+n-1}(1-j\delta)\right) \det\left(B_{m,n}(i+1)\right)$.

Thus, it follows that

$$d_{i;m,n} = (-1)^{i-1} \frac{(m+n-i)_{n-i}}{i!(n-i)!} \left[\prod_{j=0}^{n} j!\right] \left[\prod_{j=m+i}^{m+n-1} (1-j\delta)\right] \prod_{t=1}^{n-1} \left(\prod_{j=m+1}^{m+n-t} (1-[2j+(t-1)]\delta)\right).$$

Therefore, according to Cramér's rule, (11) follows.

Appendix C: A useful hypergeometric series

Lemma 4 *Let $m, n, k \in \mathbb{N}$ and $\delta \leq 0$. Then, the following hypergeometric series holds:*

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} \frac{(k)_i}{(m+n)_i} \frac{\prod_{j=k-1}^{k+i-2}(1-j\delta)}{\prod_{j=m}^{m+i-1}(1-j\delta)} = \frac{(m+n-k)_n \prod_{j=m+k}^{m+n+k-1}(1-j\delta)}{(m+n)_n \prod_{j=m}^{m+n-1}(1-j\delta)}. \tag{23}$$

Proof For the case $\delta = 0$, write (23) as

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} \frac{(k)_i}{(m+n)_i} = \frac{(m+n-k)_n}{(m+n)_n}$$

and observe that this follows from Vandermonde's formula,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(x)_i}{(x+y)_i}=\frac{(y)_n}{(x+y)_n},$$

by replacing x with k and y with $m+n-k$, see Charalambides (2002, p. 125).

For the case $\delta<0$, write (23) as

$$\sum_{i=0}^{n}(-1)^i\frac{(n)_i(k)_i(1/\delta+1-k)_i}{i!(m+n)_i(1/\delta-m)_i}=\frac{(m+n-k)_n(1/\delta-m-k)_n}{(m+n)_n(1/\delta-m)_n}.$$

This follows from Dougall's identity,

$$\sum_{i=0}^{s}\frac{(\alpha)_i(\beta)_i(s)_i}{i![\gamma+1]_i(\alpha+\beta+\gamma+s)_i}=\frac{[\alpha+\gamma+1]_s[\beta+\gamma+1]_s}{[\gamma+1]_s[\alpha+\beta+\gamma+1]_s},$$

using the substitution $\alpha\mapsto k$, $\beta\mapsto(1/\delta+1-k)$, $\gamma\mapsto(-m-n-1)$ and $s\mapsto n$, see [Dougall 1907, eq. (2)]. □

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