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A least squares estimator for discretely observed Ornstein–Uhlenbeck processes driven by symmetric α-stable motions

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Abstract We study the problem of parameter estimation for Ornstein–Uhlenbeck processes driven by symmetric α-stable motions, based on discrete observations. A least squares estimator is obtained by minimizing a contrast function based on the integral form of the process. Let h be the length of time interval between two consecutive observations. For both the case of fixed h and that of $h \to 0$, consistencies and asymptotic distributions of the estimator are derived. Moreover, for both of the cases of h, the estimator has a higher order of convergence for the Ornstein–Uhlenbeck process driven by non-Gaussian α-stable motions ($0 < \alpha < 2$) than for the process driven by the classical Gaussian case ($\alpha = 2$).

Keywords Stable law · Ornstein–Uhlenbeck · Parametric estimation · Consistency · Asymptotic distribution · Least squares method

1 Introduction

A stationary process $\{X_t, t \geq 0\}$ is defined to be an Ornstein–Uhlenbeck (O–U) process driven by a symmetric α-stable motion if it is the stationary solution of the stochastic differential equation (SDE)

$$\mathrm{d}X_t = -\theta X_t \,\mathrm{d}t + \sigma d Z_t, \tag{1}$$

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where $\theta > 0$ and $\sigma > 0$ are parameters, and $\{Z_t, t \geq 0\}$ is a standard symmetric α-stable Lévy process with starting value $Z_0 = 0$. If the characteristic exponent α of the process Z_t is 2, i.e. Z_t is a Brownian motion, the stationary solution of SDE (1) will have a Gaussian marginal distribution; if $0 < \alpha < 2$, the stationary solution will have an α-stable marginal distribution with α being equal to that of Z_t. The O–U process with α-stable marginal distribution belongs to non-Gaussian O–U processes. The modeling via the use of these processes has received considerable attention in applications such as finance and econometrics (see, e.g., Barndorff-Nielsen and Shephard 2001; Cariboni and Schoutens 2009; Benth et al. 2007).

Due to growing practical interest, many researchers have proposed statistical inference methods for discretely sampled non-Gaussian O–U processes. As is well-known, the use of non-Gaussian marginal distributions makes likelihood analysis of these processes unfeasible for virtually all cases of interest. Non-parametric estimation of the Lévy measure of a hidden Lévy process driving a stationary O–U process is considered in Jongbloed et al. (2005). Compared with the study of non-parametric methods, that of parametric methods is more active. The asymptotic behavior of a so-called cumulant M-estimator of the O–U process induced by the subordinator is analyzed in Jongbloed and Van Der Meulen (2006). Brockwell et al. (2007) have also studied the parametric inference of the O–U process induced by the subordinator, but they have used a discrete approximation to the exact integral representation of the driven process. In Sun and Zhang (2009), an empirical likelihood estimation procedure for parameters of the discretely sampled process of O–U type is presented. By minimizing some distance function between an empirical estimator of the characteristic function and a model-based one, Taufer and Leonenko (2009) have studied the asymptotic properties of the characteristic function estimation method.

In most of the above-mentioned literatures, the driving Lévy process of the O–U process has finite moments. However, when the driving Lévy process has infinite variance, few papers are concerned with parametric inference in this case except for that of Hu and Long (2009). For the case $1 < \alpha < 2$ and $\sigma = 1$, the consistency and the asymptotic distribution of a least squares estimator of parameter θ for the O–U process defined in SDE (1) are studied in Hu and Long (2009). But they do not consider the case $\alpha \in (0, 1]$. The contrast function, which the LSE is obtained by minimizing, is constructed by the discretization of SDE (1). The main focus of this paper is the study of the consistency and the asymptotic distribution of another least squares estimator of parameter θ for the O–U process driven by the α-stable motion, but for all the cases of $0 < \alpha < 2$ and $\sigma > 0$. However, the contrast function is constructed by the integral representation of SDE (1).

Assume that the O–U process is observed at equidistant discrete times $\{t_k = kh, k = 0, 1, 2, \ldots\}$. It is well-known (see Lemma 17.1 of Sato 1999) that the stationary solution of (1) satisfies the difference equations,

$$\begin{aligned} X_{kh} &= \mathrm{e}^{-\theta h} X_{(k-1)h} + \sigma Z_{k,h} \quad \text{with} \quad Z_{k,h} = \int_{(k-1)h}^{kh} \mathrm{e}^{-\theta(kh-s)}\, \mathrm{d}Z(s) \\ &\stackrel{d}{=} \left(\frac{1-\mathrm{e}^{-\theta\alpha h}}{\theta\alpha}\right)^{1/\alpha} S_k, \end{aligned} \tag{2}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and $\{S_k, k = 1, 2, \ldots\}$ is an independent identically distributed (i.i.d.) sequence of random variables with a symmetric α-stable distribution. Our proposed estimator $\hat{\theta}_n$ is the value of θ which minimizes $\sum_{k=1}^{n} |X_{kh} - \mathrm{e}^{-\theta h} X_{(k-1)h}|^2$, i.e.

$$\hat{\theta}_n = -\frac{1}{h} \log\left(\frac{\sum_{k=1}^{n} X_{(k-1)h} X_{kh}}{\sum_{k=1}^{n} X_{(k-1)h}^2} \right). \tag{3}$$

The estimator $\tilde{\theta}_n$ of Hu and Long (2009) is the value of θ which minimizes $\sum_{k=1}^{n} |X_{kh} - (1 - \theta h) X_{(k-1)h}|^2$, i.e.

$$\tilde{\theta}_n = -\frac{\sum_{k=1}^{n} (X_{kh} - X_{(k-1)h}) X_{(k-1)h}}{h \sum_{k=1}^{n} X_{(k-1)h}^2}. \tag{4}$$

As is well-known, $\hat{\theta}_n$ is the exact MLE when $\{Z_t, t \geq 0\}$ is a standard Brownian motion. For the case that $\{Z_t, t \geq 0\}$ is not a Brownian motion, based on the sample $(X_{kh})_{k=0}^{n}$, we shall study the asymptotic properties of $\hat{\theta}_n$ under the following three conditions.

Condition 1 As $n \to \infty$, $h > 0$ is fixed.

Condition 2 As $n \to \infty$, $h \to 0$ and $nh \to \infty$. (For clarity, we omit the dependence of h on n.)

Condition 3 As $n \to \infty$, $h \to 0$ and $nh^{2+\rho} \to \infty$ for some $\rho > 0$ small enough. (For clarity, we omit the dependence of h on n. To keep the condition to be as weak as possible, ρ can be chosen to be a fixed positive small enough number.)

Under either Condition 1 or 2, we shall give the strong and the weak consistencies of $\hat{\theta}_n$, respectively for the case of $1 \leq \alpha < 2$ and $0 < \alpha < 1$. Further, under either Condition 1 or 3, we shall give the asymptotic distributions of $\hat{\theta}_n$. And the asymptotic distributions are independent of the parameter σ.

For the case $1 < \alpha < 2$, the strong consistency and the asymptotic distribution of $\tilde{\theta}_n$ are given in Hu and Long (2009), respectively, under Condition 2 and condition (A1) of Hu and Long (2009). Condition (A1) of Hu and Long (2009) depends on the value of α. On the contrary, Condition 3 of this paper is independent of the value of α. Actually, by the series representation of the logarithm and Theorem 2.1 of Hu and Long (2009), we have for the case $1 < \alpha < 2$,

$$\hat{\theta}_n = -\frac{1}{h} \log(1 - h\tilde{\theta}_n) = \tilde{\theta}_n + o_{a.s.}^{(n)}(h), \tag{5}$$

where $o_{a.s.}^{(n)}(h)/h \to 0$ almost surely as $h \to 0$ and $nh \to \infty$.

Since the consistency of $\tilde{\theta}_n$ does not hold under Condition 1, it is obvious that $\hat{\theta}_n$ is superior to $\tilde{\theta}_n$ for fixed positive h. It seems that the asymptotic behaviors of $\hat{\theta}_n$ and $\tilde{\theta}_n$ are same as $h \to 0$. Actually, the asymptotic distribution of $\tilde{\theta}_n$ depends on the condition $nh^{1+\alpha}/\log n \to 0$. However, the asymptotic behavior of $\hat{\theta}_n$ is independent of this condition.

The remainder of the paper is organized as follows. In Sect. 2, we provide several asymptotic results of the estimator $\hat{\theta}_n$. All proofs are in Sect. 3. In Sect. 4, we present some simulation examples to evidence our theoretical results.

2 Notation and results

2.1 General setting and notation

We assume that $(\mathcal{F}_t)$ is the right-continuous filtration generated by $\{Z_t, t \geq 0\}$ and that $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. Our O–U process $\{X_t, t \geq 0\}$ is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. In the SDE (1), we let $\theta_0 \in (0, \infty)$ denote the true unknown parameter value, which we wish to estimate.

We use the symbol $S_\alpha(\sigma, \beta, \mu)$ to denote an α-stable distribution with the characteristic exponent (index) $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$ and location parameter $\mu \in (-\infty, \infty)$ (cf. Lévy 1934; Hall 1981 for the details on stable distributions). If a random variable X follows the $S_\alpha(\sigma, \beta, \mu)$ distribution, we write $X \sim S_\alpha(\sigma, \beta, \mu)$. The standard symmetric α-stable Lévy process Z_t satisfies $Z_1 \sim S_\alpha(1, 0, 0)$, i.e. the characteristic function of Z_1 is $\phi_{Z_1}(u) = \exp\{-|u|^\alpha\}$. It follows from Theorem 4.1 of Sato and Yamazato (1984) that the process $X(t)$ has a limit distribution $S_\alpha((1/(\theta_0\alpha))^{1/\alpha}\sigma, 0, 0)$. We assume that the starting value X_0 follows this limit distribution.

The statement $h(x) \sim g(x)$ as $x \to a$ means $\lim_{x\to a} h(x)/g(x) = 1$. Convergence in distribution, convergence in probability and convergence almost surely are denoted by $\rightsquigarrow$, $\xrightarrow{\mathbf{P}}$ and $\xrightarrow{\text{a.s.}}$, respectively.

2.2 Consistency and asymptotic behavior of the LSE for the case of fixed h

Essentially, if h is fixed, the difference equations (2) can be treated as an $AR(1)$ process, whose noise is assumed to be α-stable. For a general $AR(1)$ process with α-stable noise, the LSE of the autocorrelation parameter is weak consistent for its true value (see Davis and Resnick 1986). But it is not sure that the strong consistency of the LSE holds. However, for the difference equations (2), the noise has the specified form $\int_{(k-1)h}^{kh} \mathrm{e}^{-\theta(kh-s)}\, \mathrm{d}Z(s)$, which ensures that $\hat{\theta}_n$ is strong consistent for θ_0 in the case $1 \leq \alpha < 2$. The consistency of $\hat{\theta}_n$ for all $\alpha \in (0, 2)$ is stated as follows.

Theorem 1 *Under Condition 1, we have*

(1) *if* $1 \leq \alpha < 2$, *then* $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta_0$;
(2) *if* $0 < \alpha < 1$, *then* $\hat{\theta}_n \xrightarrow{\mathbf{P}} \theta_0$.

For $0 < \alpha < 2$, let $C_\alpha = (\int_0^\infty x^{-\alpha} \sin(x)\, \mathrm{d}x)^{-1} = 2\sin(\pi\alpha/2)\Gamma(\alpha)/\pi$, and let $\sigma_1 = 2^\alpha C_{\alpha/2}^{-2/\alpha}$, $\sigma_2 = C_\alpha^{-1/\alpha}$. The asymptotic behavior of the estimator $\hat{\theta}_n$ is stated next.

Theorem 2 *Suppose* α $(0 < \alpha < 2)$ *be known. Then, under Condition 1, we have*

$$\left(\frac{n}{\log n}\right)^{1/\alpha} (\hat{\theta}_n - \theta_0) \rightsquigarrow \frac{e^{\theta_0 h}(1 - e^{-2\theta_0 h})}{h(1 - e^{-\alpha\theta_0 h})^{1/\alpha}} \frac{\tilde{Y}}{Y_0}, \tag{6}$$

where Y_0 *and* $\tilde{Y}$ *are independent stable variables. Further,* $Y_0 \sim S_{\alpha/2}(\sigma_1, 1, 0)$ *and* $\tilde{Y} \sim S_\alpha(\sigma_2, 0, 0)$.

Remark 1 Theorem 2 states that the rate at which $\hat{\theta}_n$ converges to θ_0 is $(\log n/n)^{1/\alpha}$, which is considerably faster than the rate $n^{-1/2}$ in the classical Brownian motion case. Moreover, the asymptotic distribution of the estimator $\hat{\theta}_n$ is independent of the parameter σ.

2.3 Consistency and asymptotic behavior of the LSE for the case of $h \to 0$

Theorem 3 *Under Condition 2, we have*

(1) *if* $1 \le \alpha < 2$, *then* $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta_0$;
(2) *if* $0 < \alpha < 1$, *then* $\hat{\theta}_n \xrightarrow{\mathbf{P}} \theta_0$.

Theorem 4 *Suppose* α $(0 < \alpha < 2)$ *be known. Then, under Condition 3, we have*

$$\left(\frac{n}{\log n}\right)^{1/\alpha} h^{1/\alpha}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \frac{2\theta_0(\alpha\theta_0)^{-1/\alpha}\tilde{Y}}{Y_0}, \tag{7}$$

where Y_0 *and* $\tilde{Y}$ *are as specified in Theorem 2.*

Remark 2 It is not surprising due to (5) that the asymptotic distributions of Theorem 4 are the same as that of Theorem 3.1 of Hu and Long (2009). Theorem 4 states that the rate at which $\hat{\theta}_n$ converges to θ_0 is $(\log n/(nh))^{1/\alpha}$, which is considerably faster than the rate $(nh)^{-1/2}$ in the classical Gaussian O–U process case. Moreover, the asymptotic distribution of the estimator $\hat{\theta}_n$ is independent of the parameter σ. However, Condition 3 is stronger than Condition 2, which can ensure the asymptotic normality of $\hat{\theta}_n$ in the classical Gaussian O–U process case.

Remark 3 The asymptotic behavior of $\tilde{\theta}_n$ depends on the condition $nh^{1+\alpha}/\log n \to 0$ (see the proof of Lemma 3.4 of Hu and Long 2009). On the contrary, the asymptotic behavior of $\hat{\theta}_n$ is independent of this condition. For convenience of verification, Condition 3 is deliberately chosen to be independent of α. Or else, it can be relaxed. Actually, from the proof of Theorem 4, ρ can be chosen to be any fixed positive number. To keep the condition to be as weak as possible, ρ can be chosen to be a fixed positive small enough number.

3 Proofs

3.1 Proofs for the consistencies

We shall firstly establish several preliminary lemmas.

Lemma 1 *Let* $\delta > 0$ *be any positive number. Then, under Condition 1, if* $0 < \alpha < 1$, *we have*

$$\frac{1}{n^{1+\delta}} \sum_{k=1}^{n} |X_{(k-1)h}|^{\alpha} \xrightarrow{\text{a.s.}} 0. \tag{8}$$

Proof By (2) and the C_r-inequality (see, e.g., p. 97 of Lin and Bai 2010), we have $|X_{kh}|^{\alpha} = |\mathrm{e}^{-\theta h} X_{(k-1)h} + \sigma Z_{k,h}|^{\alpha} \le \mathrm{e}^{-\alpha\theta h}|X_{(k-1)h}|^{\alpha} + \sigma^{\alpha}|Z_{k,h}|^{\alpha}$. Summing $k = 1$ to n and rearranging gives

$$(1 - \mathrm{e}^{-\alpha\theta h})\frac{1}{n^{1+\delta}} \sum_{k=1}^{n} |X_{(k-1)h}|^{\alpha} \le \sigma^{\alpha}\frac{1}{n^{1+\delta}} \sum_{k=1}^{n} |Z_{k,h}|^{\alpha} + \frac{1}{n^{1+\delta}}(|X_0|^{\alpha} - |X_{kh}|^{\alpha}). \tag{9}$$

Obviously, as $n \to \infty$, $(|X_0|^{\alpha} - |X_{kh}|^{\alpha})/n^{1+\delta} \to 0$ almost surely. Since $\{Z_{k,h}, k = 1, 2, \ldots, n\}$ are i.i.d. random variables with the common $S_{\alpha}(((1-\mathrm{e}^{-\theta\alpha h})/\theta\alpha)^{1/\alpha}, 0, 0)$ law, by Theorem 1.12 of Nolan (2010), we have as $n \to \infty$, $\mathbf{P}(|Z_{1,h}|^{\alpha} > n^{1+\delta}) = \mathbf{P}(|Z_{1,h}| > n^{(1+\delta)/\alpha}) \sim ((1 - \mathrm{e}^{-\theta\alpha h})/(\theta\alpha))C_{\alpha}n^{-(1+\delta)}$. Hence, $\sum_{n=1}^{\infty} \mathbf{P}(|Z_{1,h}|^{\alpha} > n^{1+\delta}) < \infty$. Since $\mathbf{E}[|Z_{1,h}|^{\alpha}] = \infty$ and $\sum_{k=1}^{n} |Z_{k,h}|^{\alpha}/n^{1+\delta} \ge 0$, it follows from the result of Feller (1946) (see also p. 66 of Durrett 2004) that $\lim_{n\to\infty} \sum_{k=1}^{n} |Z_{k,h}|^{\alpha}/n^{1+\delta} = 0$ almost surely. Due to (9), this establishes (8). □

The following lemma, which is a special case of Theorem 3.3 of Davis and Resnick (1986), is also given as Lemma 3.6 of Hu and Long (2009).

Lemma 2 *Suppose* $\{S_i\}_{i=0}^{\infty}$ *be i.i.d. with the same stable distribution* $S_{\alpha}(1, 0, 0)$. *Then for* $a_n = (C_{\alpha}n)^{1/\alpha}$ *and* $\tilde{a}_n = C_{\alpha}^{2/\alpha}(n \log n)^{1/\alpha}$, *we have for* $m \in \mathbb{N}$,

$$\left(a_n^{-2} \sum_{i=1}^{n} S_i^2, \tilde{a}_n^{-1} \sum_{i=1}^{n} S_i S_{i+1}, \ldots, \tilde{a}_n^{-1} \sum_{i=1}^{n} S_i S_{m+1}\right) \rightsquigarrow (Y_0, Y_1, \ldots, Y_m),$$

where $Y_0, Y_1, \ldots, Y_m$ *are independent stable random variables,* Y_0 *is positive* $\alpha/2$-*stable with distribution* $S_{\alpha/2}(\sigma_1, 1, 0)$, *and* $Y_1, \ldots, Y_m$ *are i.i.d. symmetric* α-*stable with distribution* $S_{\alpha}(\sigma_2, 0, 0)$.

The precise values of σ_1 and σ_2 are not provided explicitly in Davis and Resnick (1986). Mikosch et al. (1995) have determined their values, but with an incorrect value of σ_1; it is necessary to replace $C_{\alpha/2}^{-2/\alpha}$ by $2^{\alpha}C_{\alpha/2}^{-2/\alpha}$ to obtain the correct value of σ_1 that was used in Mikosch et al. (1995) and Hu and Long (2009).

Since we have expression (2), using Lemma 2, we therefore obtain the following lemma.

Lemma 3 *Suppose* α $(0 < \alpha < 2)$ *be known,* $Z_{k,h}$ $(k = 1, 2, \ldots, n)$ *be defined in* (2) *and* Y_0 *be defined in Theorem* 2. *Then, we have*

(1) *under Condition* 1, $n^{-2/\alpha} \sum_{k=1}^{n} Z_{k,h}^2 \rightsquigarrow ((1 - \mathrm{e}^{-\theta\alpha h})/(\theta\alpha))^{2/\alpha} C_\alpha^{2/\alpha} Y_0$ *and*

(2) *under Condition* 2, $(nh)^{-2/\alpha} \sum_{k=1}^{n} Z_{k,h}^2 \rightsquigarrow C_\alpha^{2/\alpha} Y_0$.

The following lemma is a result of Corollary 3.1 of Rosinski and Woyczynski (1986).

Lemma 4 *Suppose* $0 < \alpha < 2$ *and the process* $\{\phi(t), t \geq 0\}$ *be a real measurable* $(\mathcal{F}_t)$*-adapted process such that for every* $T > 0$, $\int_0^T |\phi(t)|^\alpha \,\mathrm{d}t < \infty$ *almost surely. Let* $\tau(u) = \int_0^u |\phi(t)|^\alpha \,\mathrm{d}t$, *and let* $\phi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ *be increasing. Then, if* $\tau(u) \to \infty$ *almost surely as* $u \to \infty$ *and* $\int_1^\infty \varphi^{-\alpha}(t)\,\mathrm{d}t < \infty$, *then*

$$\limsup_{t\to\infty} \frac{\left|\int_0^t \phi(s)\,\mathrm{d}Z_s\right|}{\varphi(\tau(t))} = 0 \quad a.s.$$

The idea of the proof of Theorem 1 is analogous to that of Theorem 2.1 in Hu and Long (2009), but the details are more involved, so we give a complete proof.

Proof of Theorem 1 Substitution of expression (2) of X_{kh} leads to

$$\mathrm{e}^{-\hat{\theta}_n h} = \frac{\sum_{k=1}^{n} X_{(k-1)h} X_{kh}}{\sum_{k=1}^{n} X_{(k-1)h}^2} = \mathrm{e}^{-\theta_0 h} + \sigma \frac{\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^{n} X_{(k-1)h}^2}. \tag{10}$$

By (10), to prove Theorem 1, it suffices to show that under Condition 1,

$$\frac{\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^{n} X_{(k-1)h}^2} \xrightarrow{\text{a.s.}} 0 \quad \text{or} \quad \xrightarrow{\mathbf{P}} 0 \tag{11}$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$.

From the strict stationarity of X_t and Theorem 4.3 of Masuda (2004), we note that X_t is ergodic. Thus, it follows from the ergodic theorem and Corollary 25.8 of Sato (1999) that for $0 < \alpha < 2$ and any $r \geq \alpha$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |X_{(k-1)h}|^r = \mathbf{E}[X_\infty^r] = \infty \tag{12}$$

almost surely. Let

$$\phi_n(t) = \begin{cases} \sum_{k=1}^{n} X_{(k-1)h} \mathrm{e}^{-\theta_0(kh-t)} 1_{((k-1)h,kh]}(t), & \text{if } 1 < \alpha < 2, \\ \frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} X_{(k-1)h} \mathrm{e}^{-\theta_0(kh-t)} 1_{((k-1)h,kh]}(t), & \text{if } 0 < \alpha \leq 1, \end{cases}$$

and let $\tau(nh) = \int_0^{nh} |\phi_n(t)|^\alpha \, \mathrm{d}t$. Then it is easy to find

$$\tau(nh) = \begin{cases} \left(\frac{1-\mathrm{e}^{-\alpha\theta_0 h}}{\alpha\theta_0}\right) \sum_{k=1}^n |X_{(k-1)h}|^\alpha, & \text{if } 1 < \alpha < 2, \\ \left(\frac{1-\mathrm{e}^{-\alpha\theta_0 h}}{\alpha\theta_0}\right) \frac{1}{n} \sum_{k=1}^n |X_{(k-1)h}|^\alpha, & \text{if } 0 < \alpha \leq 1. \end{cases} \tag{13}$$

Equality (12) implies that for fixed $h > 0$, $\tau(nh) \overset{\text{a.s.}}{\to} \infty$ as $n \to \infty$. Let $\delta \in (0, 1)$ be a fixed real number, and let

$$\varphi(t) = \begin{cases} t, & \text{if } \ 1 < \alpha < 2, \\ t^{1+\delta}, & \text{if } \ \alpha = 1, \\ t^{(2+\alpha)/\alpha}, & \text{if } \ 0 < \alpha < 1. \end{cases}$$

Note that $\int_1^\infty \varphi^{-\alpha}(t)\, \mathrm{d}t < \infty$. It is clear that

$$\frac{\sum_{k=1}^n X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^n X_{(k-1)h}^2} = \frac{\int_0^{nh} \phi_n(s)\, \mathrm{d}Z_s}{\varphi(\tau(nh))} \frac{\varphi(\tau(nh))}{b_n \sum_{k=1}^n X_{(k-1)h}^2}$$
$$\text{with } b_n = \begin{cases} 1, & \text{if } 1 < \alpha < 2, \\ n^{-1/\alpha}, & \text{if } 0 < \alpha \leq 1. \end{cases} \tag{14}$$

By Lemma 4, we have

$$\limsup_{n\to\infty} \frac{\left|\int_0^{nh} \phi_n(s)\, \mathrm{d}Z_s\right|}{\varphi(\tau(nh))} = 0 \quad \text{a.s.} \tag{15}$$

By (14)–(15), to prove (11), it suffices to show that under Condition 1,

$$\frac{\varphi(\tau(nh))}{b_n \sum_{k=1}^n X_{(k-1)h}^2} \overset{\text{a.s.}}{\longrightarrow} 0 \quad \text{or} \quad \overset{\mathbf{P}}{\longrightarrow} 0 \tag{16}$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$. In the reminder of the proof, conclusion (16) will be proved on three cases of the values of α.

First consider the case $1 < \alpha < 2$. By the Hölder inequality, we have

$$\frac{\varphi(\tau(nh))}{b_n \sum_{k=1}^n X_{(k-1)h}^2} = \frac{\tau(nh)}{\sum_{k=1}^n X_{(k-1)h}^2} \leq \left(\frac{1-\mathrm{e}^{-\alpha\theta_0 h}}{\alpha\theta_0}\right)\left(\frac{1}{n}\sum_{k=1}^n X_{(k-1)h}^2\right)^{-\frac{2-\alpha}{\alpha}}, \tag{17}$$

which converges to zero almost surely as $n \to \infty$, since we have obtained (12). This proves (16) for the case $1 < \alpha < 2$.

Next consider the case $\alpha = 1$. By the C_r-inequality and then by the Hölder inequality, we have $\left(\sum_{k=1}^n |X_{(k-1)h}|\right)^{1+\delta} \leq n^\delta \sum_{k=1}^n |X_{(k-1)h}|^{1+\delta} \leq n^{(1+\delta)/2}$

$(\sum_{k=1}^{n}|X_{(k-1)h}|^2)^{(1+\delta)/2}$. Thus, from (13), we have

$$\frac{\varphi(\tau(nh))}{b_n\sum_{k=1}^{n}X_{(k-1)h}^2}=\frac{(\tau(nh))^{1+\delta}}{\frac{1}{n}\sum_{k=1}^{n}X_{(k-1)h}^2}\le\left(\frac{1-\mathrm{e}^{-\theta_0 h}}{\theta_0}\right)^{1+\delta}\left(\frac{1}{n}\sum_{k=1}^{n}X_{(k-1)h}^2\right)^{-\frac{1-\delta}{2}},\tag{18}$$

which converges to zero almost surely as $n\to\infty$, since we have obtained (12). This proves (16) for the case $\alpha=1$.

Then consider the case $0<\alpha<1$. Note that

$$\frac{\varphi(\tau(nh))}{b_n\sum_{k=1}^{n}X_{(k-1)h}^2}=\frac{(\tau(nh))^{(2+\alpha)/\alpha}}{\frac{1}{n^{1/\alpha}}\sum_{k=1}^{n}X_{(k-1)h}^2}=\left(\frac{1-\mathrm{e}^{-\alpha\theta_0 h}}{\alpha\theta_0}\right)^{(2+\alpha)/\alpha}$$
$$\times\left(\frac{1}{n^{1+1/(2+\alpha)}}\sum_{k=1}^{n}|X_{(k-1)h}|^{\alpha}\right)^{(2+\alpha)/\alpha}\left(\frac{1}{n^{2/\alpha}}\sum_{k=1}^{n}X_{(k-1)h}^2\right)^{-1}.\tag{19}$$

Then, it follows from (8) that we have, under Condition 1,

$$\frac{1}{n^{1+1/(2+\alpha)}}\sum_{k=1}^{n}|X_{(k-1)h}|^{\alpha}\xrightarrow{\text{a.s.}}0.\tag{20}$$

Due to (2), we have $\sigma^2 Z_{k,h}^2=(X_{kh}-\mathrm{e}^{-\theta h}X_{(k-1)h})^2\le 2X_{kh}^2+2\mathrm{e}^{-2\theta h}X_{(k-1)h}^2$. Summing $k=1$ to n and rearranging gives $\sum_{k=1}^{n}X_{(k-1)h}^2\ge 2^{-1}(1+\mathrm{e}^{-2\theta h})^{-1}(\sigma^2\sum_{k=1}^{n}Z_{k,h}^2+2(X_0^2-X_{kh}^2))$. Since as $n\to\infty$, $\frac{1}{n^{2/\alpha}}(X_0^2-X_{kh}^2)\to 0$ almost surely. Then, by Lemma 3, we have under Condition 1,

$$\left(\frac{1}{n^{2/\alpha}}\sum_{k=1}^{n}X_{(k-1)h}^2\right)^{-1}\le 2(1+\mathrm{e}^{-2\theta h})\left(\sigma^2\frac{1}{n^{2/\alpha}}\sum_{k=1}^{n}Z_{k,h}^2+2\frac{1}{n^{2/\alpha}}(X_0^2-X_{kh}^2)\right)^{-1}$$
$$\rightsquigarrow 2(1+\mathrm{e}^{-2\theta h})\sigma^{-2}\left(\frac{1-\mathrm{e}^{-\theta\alpha h}}{\theta\alpha}\right)^{-2/\alpha}C_\alpha^{-2/\alpha}\frac{1}{Y_0}.\tag{21}$$

It follows from (19)–(21) that under Condition 1, $\frac{\varphi(\tau(nh))}{b_n\sum_{k=1}^{n}X_{(k-1)h}^2}\rightsquigarrow 0$.

The convergence also holds in probability, since convergence in probability and convergence in distribution are equivalent when the limit is a constant. This proves (16) for the case $0<\alpha<1$. □

Proof of Theorem 3 In view of expressions (3) and (10), to prove Theorem 3, it suffices to show that

$$\left(1+\frac{\sigma\mathrm{e}^{\theta_0 h}\sum_{k=1}^{n}X_{(k-1)h}Z_{k,h}}{\sum_{k=1}^{n}X_{(k-1)h}^2}\right)^{1/h}\xrightarrow{\text{a.s.}}1\quad\text{or}\xrightarrow{\mathbf{P}}1\tag{22}$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$. It is clear that

$$\left(1+\frac{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^{n} X_{(k-1)h}^2}\right)^{1/h}$$

$$=\left(1+\frac{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^{n} X_{(k-1)h}^2}\right)^{\frac{\sum_{k=1}^{n} X_{(k-1)h}^2}{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}} \frac{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{h\sum_{k=1}^{n} X_{(k-1)h}^2}}.$$

Reviewing the proof of Theorem 1, we find, under Condition 2,

$$\frac{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^{n} X_{(k-1)h}^2} \xrightarrow{\text{a.s.}} 0 \quad \text{or} \quad \xrightarrow{\mathbf{P}} 0$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$. It is equivalent to

$$\left(1+\frac{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{\sum_{k=1}^{n} X_{(k-1)h}^2}\right)^{\frac{\sum_{k=1}^{n} X_{(k-1)h}^2}{\sigma \mathrm{e}^{\theta_0 h}\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}} \xrightarrow{\text{a.s.}} \mathrm{e} \quad \text{or} \quad \xrightarrow{\mathbf{P}} \mathrm{e}$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$. Therefore, to prove (22), it suffices to show

$$\frac{\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{h\sum_{k=1}^{n} X_{(k-1)h}^2} \xrightarrow{\text{a.s.}} 0 \quad \text{or} \quad \xrightarrow{\mathbf{P}} 0 \tag{23}$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$.

Let $\phi_n(\cdot)$, $\tau(\cdot)$ and $\varphi(\cdot)$ be as defined in the proof of Theorem 1. According to (14), we can represent the left side of (23) to be

$$\frac{\sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{h\sum_{k=1}^{n} X_{(k-1)h}^2} = \frac{\int_0^{nh} \phi_n(s)\,\mathrm{d}Z_s}{\varphi(\tau(nh))} \frac{\varphi(\tau(nh))}{b_n h\sum_{k=1}^{n} X_{(k-1)h}^2}$$

with b_n defined in (14). Note that under Condition 2, Equalities (12) and (15) still hold. Thus, to show (23), it suffices to show that under Condition 2,

$$\frac{\varphi(\tau(nh))}{b_n h\sum_{k=1}^{n} X_{(k-1)h}^2} \xrightarrow{\text{a.s.}} 0 \quad \text{or} \quad \xrightarrow{\mathbf{P}} 0 \tag{24}$$

according as $1 \leq \alpha < 2$ or $0 < \alpha < 1$.

From (17), we have

$$\frac{\varphi(\tau(nh))}{b_n h\sum_{k=1}^{n} X_{(k-1)h}^2} \leq \left(\frac{1-\mathrm{e}^{-\alpha\theta_0 h}}{\alpha\theta_0 h}\right)\left(\frac{1}{n}\sum_{k=1}^{n} X_{(k-1)h}^2\right)^{-\frac{2-\alpha}{\alpha}},$$

which converges to zero almost surely under Condition 2, since we have (12) and $(1-\mathrm{e}^{-\alpha\theta_0 h})/(\alpha\theta_0 h) \to 1$. This proves (24) for the case $1<\alpha<2$. From (18), we have

$$\frac{\varphi\left(\tau(nh)\right)}{b_n h\sum_{k=1}^n X_{(k-1)h}^2} \le \left(\frac{1-\mathrm{e}^{-\theta_0 h}}{\theta_0 h}\right)^{1+\delta} h^{\delta}\left(\frac{1}{n}\sum_{k=1}^n X_{(k-1)h}^2\right)^{-\frac{1-\delta}{2}}$$

which converges to zero almost surely under Condition 2, since we have (12) and $((1-\mathrm{e}^{-\theta_0 h})/(\theta_0 h))^{1+\delta}h^{\delta} \to 0$. This proves (24) for the case $\alpha=1$. From (19), we have

$$\begin{aligned}\frac{\varphi\left(\tau(nh)\right)}{b_n h\sum_{k=1}^n X_{(k-1)h}^2} &= \left(\frac{1-\mathrm{e}^{-\alpha\theta_0 h}}{\alpha\theta_0 h}\right)^{(2+\alpha)/\alpha} \\ &\times\left(\frac{1}{n^{1+1/(2+\alpha)}}\sum_{k=1}^n |X_{(k-1)h}|^{\alpha}\right)^{(2+\alpha)/\alpha}\left(\frac{1}{(nh)^{2/\alpha}}\sum_{k=1}^n X_{(k-1)h}^2\right)^{-1}.\end{aligned}$$

Similar to prove (21), by Lemma 3, we have under Condition 2,

$$\begin{aligned}\left(\frac{1}{(nh)^{2/\alpha}}\sum_{k=1}^n X_{(k-1)h}^2\right)^{-1} &\le 2(1+\mathrm{e}^{-2\theta h}) \\ &\times\left(\sigma^2\frac{1}{(nh)^{2/\alpha}}\sum_{k=1}^n Z_{k,h}^2 + 2\frac{1}{(nh)^{2/\alpha}}(X_0^2 - X_{kh}^2)\right)^{-1} \rightsquigarrow 4\sigma^{-2}C_\alpha^{-2/\alpha}\frac{1}{Y_0}.\end{aligned}$$

Note that under Condition 2, the result (20) still holds. This proves (24) for the case $0<\alpha<1$. This completes the proof. □

3.2 Proofs for the asymptotic behaviors

Proof of Theorem 2 The representation (2) can be treated as an $AR(1)$ model. Using the result in Example 5.3 of Davis and Resnick (1986), we have under Condition 1,

$$\left(\frac{n}{\log n}\right)^{1/\alpha}\left(\frac{\sum_{k=1}^n X_{(k-1)h}X_{kh}}{\sum_{k=1}^n X_{(k-1)h}^2} - \mathrm{e}^{-\theta_0 h}\right) \rightsquigarrow \frac{1-\mathrm{e}^{-2\theta_0 h}}{(1-\mathrm{e}^{-\alpha\theta_0 h})^{1/\alpha}}\frac{\tilde{Y}}{Y_0}.$$

By the mean value theorem, $(n/\log n)^{1/\alpha}\left(\mathrm{e}^{-\hat{\theta}_n h} - \mathrm{e}^{-\theta_0 h}\right) = -(n/\log n)^{1/\alpha}\mathrm{e}^{-\bar{\theta}_n h}$ $(\hat{\theta}_n-\theta_0)h$, where $\bar{\theta}_n$ satisfies $|\bar{\theta}_n-\theta_0| \le |\hat{\theta}_n-\theta_0|$. By the consistency of $\hat{\theta}_n$ and the symmetry of random variable $\tilde{Y}$, we obtain (6). □

The proof of Theorem 4 follows some similar ideas as the proof of Theorem 3.1 of Hu and Long (2009). We shall only give an outline of the proof. (The complete proof is available at http://blog.sciencenet.cn/home.php?mod=space&uid=116301&do=blog&id=505695.)

Proof skeleton of Theorem 4 By (10), we have

$$\left(\frac{n}{\log n}\right)^{1/\alpha} h^{1/\alpha-1}(\mathrm{e}^{-\hat{\theta}_n h} - \mathrm{e}^{-\theta_0 h}) = \frac{(n\log n)^{-1/\alpha} h^{-1/\alpha}\sigma \sum_{k=1}^{n} X_{(k-1)h} Z_{k,h}}{n^{-2/\alpha} h^{1-2/\alpha} \sum_{k=1}^{n} X_{(k-1)h}^2} := \frac{\Phi_1(n)}{\Phi_2(n)},$$

where $\Phi_1(n)$ and $\Phi_2(n)$ are as defined in (3.2) of Hu and Long (2009). Under Condition 3, the asymptotic behavior of $\Phi_1(n)/\Phi_2(n)$ is same as that in the proof of Theorem 3.1 of Hu and Long (2009). That is, under Condition 3,

$$\left(\frac{n}{\log n}\right)^{1/\alpha} h^{1/\alpha-1}(\mathrm{e}^{-\hat{\theta}_n h} - \mathrm{e}^{-\theta_0 h}) \rightsquigarrow \frac{2\theta_0(\alpha\theta_0)^{-1/\alpha}\tilde{Y}}{Y_0},$$

where $\tilde{Y} \sim S_\alpha(\sigma_2, 0, 0)$, $Y_0 \sim S_{\alpha/2}(\sigma_1, 1, 0)$, and $\tilde{Y}$ is independent of Y_0. By the mean value theorem, $(n/\log n)^{1/\alpha} h^{1/\alpha-1}(\mathrm{e}^{-\hat{\theta}_n h} - \mathrm{e}^{-\theta_0 h}) = -(n/\log n)^{1/\alpha} h^{1/\alpha} \mathrm{e}^{-\bar{\theta}_n h}(\hat{\theta}_n - \theta_0)$, where $\bar{\theta}_n$ satisfies $|\bar{\theta}_n - \theta_0| \leq |\hat{\theta}_n - \theta_0|$. Due to the consistency of $\hat{\theta}_n$ and the symmetry of random variable $\tilde{Y}$, we establish (7). $\square$

4 Simulation

By Eq. (2) and the simulation algorithm for generating an α-stable random variate (see Chambers et al. 1976), we can obtain the exact simulation algorithm for generating a trajectory of the O–U process driven by the α-stable motion.

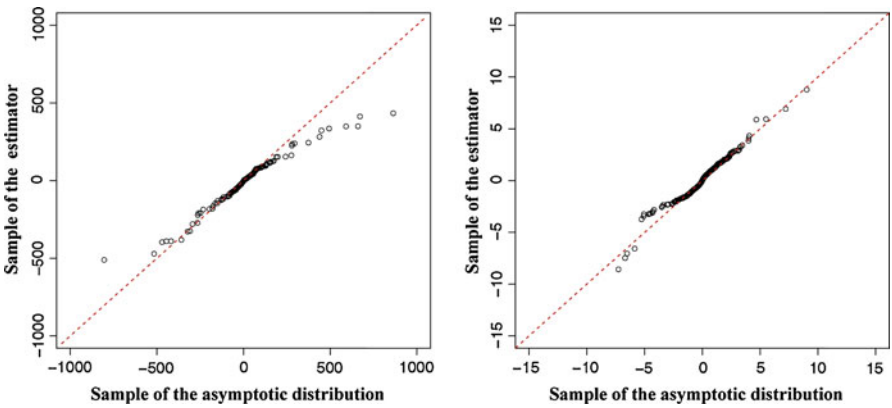

Fig. 1 Two Q–Q plots, each of which is depicted by the empirical distributions obtained from the two data sets. *Left* The case of $\alpha = 0.8$; *right* The case of $\alpha = 1.6$

Table 1 Mean and RMSE of 400 realizations of $\hat{\theta}_n$ (the true parameter $\theta_0 = 1$)

h	T			
	50	100	200	500
$\alpha = 0.8$				
$\sigma = 1$				
1.00	– (–)	1.030 (0.239)	0.993 (0.087)	1.000 (0.063)
0.50	1.007 (0.163)	1.004 (0.131)	1.005 (0.101)	1.005 (0.060)
0.10	1.028 (0.185)	1.005 (0.083)	1.006 (0.081)	1.003 (0.050)
0.01	1.025 (0.171)	1.008 (0.080)	1.006 (0.064)	1.000 (0.031)
$\sigma = 10$				
1.00	– (–)	1.016 (0.220)	1.025 (0.191)	1.001 (0.057)
0.50	1.035 (0.258)	1.023 (0.214)	1.001 (0.076)	1.000 (0.037)
0.10	1.008 (0.150)	1.012 (0.121)	1.005 (0.153)	0.999 (0.029)
0.01	1.032 (0.261)	1.003 (0.091)	1.008 (0.082)	0.999 (0.044)
$\alpha = 1.6$				
$\sigma = 1$				
1.00	– (–)	1.060 (0.311)	1.026 (0.172)	1.003 (0.106)
0.50	1.045 (0.260)	1.032 (0.213)	1.014 (0.122)	0.996 (0.070)
0.10	1.020 (0.187)	1.005 (0.156)	1.012 (0.089)	1.009 (0.061)
0.01	1.046 (0.209)	1.031 (0.149)	1.009 (0.086)	1.002 (0.058)
$\sigma = 10$				
1.00	– (–)	– (–)	1.026 (0.167)	1.018 (0.108)
0.50	1.028 (0.275)	1.023 (0.200)	1.016 (0.126)	1.006 (0.074)
0.10	1.030 (0.220)	1.029 (0.137)	1.018 (0.092)	0.999 (0.058)
0.01	1.024 (0.195)	1.021 (0.126)	1.014 (0.092)	1.004 (0.058)

To evidence the results in Theorems 2 and 4, we provide two Q–Q plots in Fig. 1. In both of the two Q–Q plots, we set $\sigma = 1$ and $\theta_0 = 2$, and depict each of them by the empirical distributions obtained from the two data sets. Each of the two data sets includes 1000 samples. In the left figure, for the case $\alpha = 0.8$, one of the two data sets is sampled from $\frac{e^{\theta_0 h}(1-e^{-2\theta_0 h})}{h(1-e^{-\alpha\theta_0 h})^{1/\alpha}} \frac{\tilde{Y}}{Y_0}$, and the other from $(n/\log n)^{1/\alpha}(\hat{\theta}_n - \theta_0)$ with $n = 20000$ and $h = 1$. In the right figure, for the case $\alpha = 1.6$, one of the two data sets is sampled from $2\theta_0(\alpha\theta_0)^{-1/\alpha}\tilde{Y}/Y_0$, and the other from $(n/\log n)^{1/\alpha}h^{1/\alpha}(\hat{\theta}_n - \theta_0)$ with $n = 4 \times 10^5$ and $h = 0.05$. Each of the realizations of $\hat{\theta}_n$ is obtained by generating a trajectory of the O–U process and calculated by (3). Note that the left figure is a Q–Q plot for two very heavy-tailed distributions. The Q–Q plots in Fig. 1 show that the difference between two empirical distributions is subtle in each of the two cases of $\alpha = 0.8$ and $\alpha = 1.6$.

Table 1 reports the mean values and the square roots of mean square error (RMSEs) of the realizations of $\hat{\theta}_n$ for different sample size n and different length of time interval between two consecutive observations h. In Table 1, we set $T = nh$. As we see from the table, the difference between the mean value of the estimates and the true value

Table 2 Mean and RMSE of 400 realizations of $\hat{\theta}_n$ and $\tilde{\theta}_n$ (the true parameter $\theta_0 = 2$)

h	T			
	50	100	200	500
0.5				
$\hat{\theta}_n$	2.062 (0.498)	2.053 (0.372)	2.009 (0.236)	2.012 (0.157)
$\tilde{\theta}_n$	1.266 (0.753)	1.272 (0.739)	1.262 (0.743)	1.266 (0.736)
0.1				
$\hat{\theta}_n$	2.047 (0.299)	2.031 (0.231)	2.002 (0.150)	2.005 (0.089)
$\tilde{\theta}_n$	1.848 (0.283)	1.836 (0.248)	1.813 (0.223)	1.817 (0.197)
0.05				
$\hat{\theta}_n$	2.066 (0.298)	2.018 (0.198)	2.011 (0.145)	2.008 (0.091)
$\tilde{\theta}_n$	1.961 (0.264)	1.919 (0.195)	1.912 (0.157)	1.910 (0.122)
0.025				
$\hat{\theta}_n$	2.064 (0.278)	2.002 (0.189)	2.000 (0.132)	2.001 (0.085)
$\tilde{\theta}_n$	2.011 (0.256)	1.952 (0.186)	1.951 (0.135)	1.952 (0.094)
0.0125				
$\hat{\theta}_n$	2.038 (0.300)	2.010 (0.193)	2.001 (0.136)	2.004 (0.082)
$\tilde{\theta}_n$	2.011 (0.290)	1.985 (0.188)	1.976 (0.134)	1.979 (0.082)
0.005				
$\hat{\theta}_n$	2.033 (0.291)	2.023 (0.187)	2.006 (0.129)	2.005 (0.083)
$\tilde{\theta}_n$	2.022 (0.287)	2.013 (0.184)	1.996 (0.127)	1.995 (0.082)

θ_0 is very small for all case of n and h. Otherwise, the RMSE becomes lesser with T increasing and h decreasing. Also, we find the value of σ has little effect on the bias and the RMSE of $\hat{\theta}_n$. Missing values in the table is owing to the case that there is no solution in some realizations of $\hat{\theta}_n$, i.e. there exist some negative realizations for $\sum_{k=1}^{n} X_{(k-1)h} X_{kh} / \sum_{k=1}^{n} X_{(k-1)h}^2$.

For fixed $\alpha = 1.8$, Table 2 reports the mean values and the RMSEs of the realizations of $\hat{\theta}_n$ and $\tilde{\theta}_n$ (defined in (4)) for different n and different h. In Table 2, we still set $T = nh$. Table 2 shows that for larger h, the difference between the mean value of the estimates and the true value θ_0 and the RMSE of the estimator $\hat{\theta}_n$ are always less than those of $\tilde{\theta}_n$. However, with T increasing and h decreasing, the difference between $\hat{\theta}_n$ and $\tilde{\theta}_n$ becomes subtle.

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References

Barndorff-Nielsen, O. E., Shephard, N. (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society B-Statistical Methodology, 63*, 167–241.

Benth, F. E., Gorth, M., Kufakunesu, R. (2007). Valuing volatility and variance swaps for a non-Gaussian Ornstein–Uhlenbeck stochastic volatility model. *Applied Mathematical Finance, 14*, 347–363.

Brockwell, P. J., Davis, R. A., Yang, Y. (2007). Estimation for nonnegative Lévy-driven Ornstein–Uhlenbeck processes. *Journal of Applied Probability, 44*, 977–989.

Cariboni, C., Schoutens, W. (2009). Jumps in intensity models: Investigating the performance of Ornstein–Uhlenbeck processes in credit risk modeling. *Metrika, 69*, 173–198.

Chambers, J. M., Mallows, C. L., Stuck, B. W. (1976). A method for simulating stable random variables. *Journal of the American Statistical Association, 71*, 340–344.

Davis, R., Resnick, S. (1986). Limit theory for the sample covariance and correlation functions of moving averages. *Annals of Statistics, 14*, 533–558.

Durrett, R. (2004). *Probability: Theory and examples* (3rd ed). Belmont: Thomson.

Feller, W. (1946). A limit theorem for random variables with infinite moments. *American Journal of Mathematics, 68*, 257–262.

Hall, P. (1981). A comedy of errors: The canonical form for a stable characteristic function. *Bulletin of the London Mathematical Society, 13*, 23–27.

Hu, Y., Long, H. (2009). Least squares estimator for Ornstein–Uhlenbeck processes driven by α-stable motions. *Stochastic Processes and Their Applications, 119*, 2465–2480.

Jongbloed, G., Van Der Meulen, F. H. (2006). Parametric estimation for subordinators and induced OU processes, *Scandinavian Journal of Statistics, 33*, 825–847.

Jongbloed, G., Van Der Meulen, F. H., Van Der Vaart, A. W. (2005). Non-parametric inference for Lévy driven Ornstein-Uhlenbeck processes. *Bernoulli, 11*, 759–791.

Lévy, P. (1934). Sur les intégrales dont les éléments sont des variables aléatoires indépendantes. *Annali della Scuola Normale Superiore di Pisa, 3*, 337–366.

Lin, Z., Bai, Z. (2010). *Probability inequalities*. Beijing: Science Press.

Masuda, H. (2004). On multidimensional Ornstein–Uhlenbeck processes driven by a general Lévy process. *Bernoulli, 10*, 97–120.

Mikosch, T., Gadrich, T., Klüppelberg, C., Adler, R. (1995). Parameter estimation for ARMA models with infinite variance innovations. *Annals of Statistics, 23*, 305–326.

Nolan, J. P. (2010). *Stable distributions—models for heavy tailed data*. Boston: Birkhäuser.

Rosinski, J., Woyczynski, W. A. (1986). On Itô stochastic integration with respect to p-stable motion: Inner clock, integrability of sample paths, double and multiple integrals. *Annals of Probability, 14*, 271–286.

Sato, K. (1999). *Lévy processes and infinitely divisible distributions*. Cambridge: Cambridge University Press.

Sato, K., Yamazato, M. (1984). Operator-self-decomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type. *Stochastic Processes and Their Applications, 17*, 73–100.

Sun, S., Zhang, X. (2009). Empirical likelihood estimation of discretely sampled processes of OU type. *Science China Mathematics, 52*, 908–931.

Taufer, E., Leonenko, N. (2009). Characteristic function estimation of non-Gaussian Ornstein–Uhlenbeck processes. *Journal of Statistical Planning and Inference, 139*, 3050–3063.