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Strong large deviations for arbitrary sequences of random variables

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Abstract We establish strong large deviation results for an arbitrary sequence of random variables under some assumptions on the normalized cumulant generating function. In other words, we give asymptotic expansions for the tail probabilities of the same kind as those obtained by Bahadur and Rao (Ann. Math. Stat. 31:1015–1027, 1960) for the sample mean. We consider both the case where the random variables are absolutely continuous and the case where they are lattice-valued. Our proofs make use of arguments of Chaganty and Sethuraman (Ann. Probab. 21:1671–1690, 1993) who also obtained strong large deviation results and local limit theorems. We illustrate our results with the kernel density estimator, the sample variance, the Wilcoxon signed-rank statistic and the Kendall tau statistic.

Keywords Large deviations · Bahadur–Rao theorem · Sample variance · Wilcoxon signed-rank statistic · Kendall tau statistic

1 Introduction

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent and identically distributed real random variables with zero mean and finite variance. Denote by $\overline{X}_n$ the sample mean given by

$$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k.$$

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For $a > 0$, the probability $\mathbb{P}(\overline{X}_n \geq a)$ converges to 0. More precisely, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \geq a) = -I(a), \tag{1}$$

where I is the Fenchel–Legendre dual of the cumulant generating function (c.g.f.) of X_1 (I is usually called the rate function). This result is a consequence of the large deviation principle (LDP) satisfied by $\overline{X}_n$ and gives only the limit for $n^{-1} \log \mathbb{P}(\overline{X}_n \geq a)$ [for the general definition of a large deviation principle, we refer to Dembo and Zeitouni (1998) or Deuschel and Stroock (1989)]. In some cases, one may want to get an asymptotic expansion for $\mathbb{P}(\overline{X}_n \geq a)$. Bahadur and Rao (1960) were among the first to establish such expansions for the sample mean. In particular, under some regularity assumptions on the Taylor expansion of the normalized c.g.f., they proved that for any $p > 0$ and n large enough

$$\mathbb{P}(\overline{X}_n \geq c) = \frac{\exp(-nI(c))}{\sqrt{2\pi n}\sigma_c \tau_c} \left[1 + \sum_{j=1}^{p} \frac{a_j}{n^j} + O\left(\frac{1}{n^{p+1}}\right) \right], \quad c > 0 \tag{2}$$

and

$$\mathbb{P}(\overline{X}_n \leq c) = \frac{\exp(-nI(c))}{\sqrt{2\pi n}\sigma_c \tau_c} \left[1 + \sum_{j=1}^{p} \frac{a_j}{n^j} + O\left(\frac{1}{n^{p+1}}\right) \right], \quad c < 0, \tag{3}$$

where $a_j \in \mathbb{R}$, and $\tau_c > 0$, $\sigma_c > 0$ are parameters depending on c. Such results are referred to as strong large deviation results (Chaganty and Sethuraman 1993) or sharp large deviation principles (see, for instance, Bercu et al. 2000).

In addition to the theorems of Bahadur and Rao (1960) and Chaganty and Sethuraman (1993) (who generalized the Bahadur–Rao Theorem on the sample mean to an arbitrary sequence of random variables), several results pertaining to strong large deviations in asymptotic statistics can be found in the literature. Blackwell and Hodges (1959) treated the lattice case of the Bahadur–Rao result on the sample mean. Generalizing the Bahadur–Rao result, Book (1972) obtained a strong large deviation theorem for weighted sums of i.i.d. random variables. Chaganty and Sethuraman (1996) proved a multidimensional version of their earlier result. Cho and Jeon (1994) established a strong large deviation theorem for the ratio of independent random variables. Bercu et al. (2000) gave a sharp large deviation principle for Gaussian quadratic forms. Florens-Landais et al. (1998) derived strong large deviation results (i.e., (2) and (3) with $p = 0$) for the maximum likelihood estimator in an Ornstein–Uhlenbeck model. Later, Bercu and Rouault (2002) extended the strong result to an arbitrary order. Joutard (2006, 2008) obtained strong large deviation results for nonparametric kernel estimators. Daouia and Joutard (2009) studied strong large deviation properties of the quantile-based frontier estimators.

This paper provides strong large deviation results (i.e., (2) with $p = 0$) for an arbitrary sequence of random variables Z_n. Some assumptions on the normalized c.g.f. are assumed. We consider both the case where Z_n is absolutely continuous or its distribution has an absolutely continuous component, and the case where Z_n is lattice-valued. Our results require, in particular, an asymptotic expansion of the normalized c.g.f. The proofs use techniques from Bahadur and Rao (1960) who derived (2) and (3), and Chaganty and Sethuraman (1993) who also obtained strong large deviation theorems for an arbitrary sequence of random variables T_n. Note, however, that their large deviation expressions cannot generally be computed explicitly in a general frame. That is, one cannot generally derive an explicit asymptotic expression for the tail probability $\mathbb{P}(T_n \geq c)$ that is a function of n. Here, we establish first-order expansions similar to (2) with $p = 0$, where the constant c and the parameter τ_c, used to make an exponential change of measure, do not necessarily depend on n. We illustrate our theorems with several statistical applications. We provide new strong large deviation results for the sample variance, the Wilcoxon signed-rank statistic and the Kendall tau statistic. We also give a new proof of a result obtained in Joutard (2006) for the kernel density estimator. The paper is organized as follows. In Sect. 2, we introduce the framework and assumptions, before giving the main results and discussing the statistical applications. Section 3 deals with the lemmas needed for the proofs of the main results which are deferred to Sect. 4.

Note that these strong large deviation results may be of interest, in particular in some nonparametric tests, to obtain estimates of p-values when the exact values are not available or when their computation is time-consuming.

2 Main results

2.1 Notation and assumptions

Let (Z_n) be a sequence of random variables and let (b_n) be a sequence of real positive numbers such that $\lim_{n\to\infty} b_n = \infty$. Let ϕ_n be the moment generating function (m.g.f.) of $b_n Z_n$,

$$\phi_n(t) = \mathbb{E}\{\exp(t b_n Z_n)\}, \quad t \in \mathbb{R},$$

and let φ_n be the normalized c.g.f. of $b_n Z_n$,

$$\varphi_n(t) = b_n^{-1} \log \mathbb{E}\{\exp(t b_n Z_n)\}.$$

Assume that there exists a differentiable function φ defined on an interval $(-\alpha, \alpha)$, $\alpha > 0$, such that $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ for all $t \in (-\alpha, \alpha)$. Let a be a real such that $|a - \varphi'(0)| > 0$ and assume that there exists $\tau_a \in \{t \in \mathbb{R} : 0 < |t| < \alpha\}$, such that $\varphi'(\tau_a) = a$. The parameter τ_a is used to make an exponential change of measure which allows to sharpen the large deviation result (see the proofs in Sect. 4).

This paper deals with strong large deviation results for the sequence (Z_n), that is, asymptotic expansions for the tail probability $\mathbb{P}(Z_n \geq a)$, where $a > \varphi'(0)$ (the real a does not necessarily depend on n). We distinguish the cases where Z_n is absolutely continuous or its distribution has an absolutely continuous component, and Z_n is lattice-valued. Note that we only give the results for the right tail probabilities, the ones for the left tail probabilities can be obtained in a similar way. The proofs, given in Sect. 4, make use of some techniques that can be found in Bahadur and Rao (1960), who obtained strong large deviation results for the sample mean and in Chaganty and Sethuraman (1993), who generalized the Bahadur and Rao theorem (for the first order). To establish the strong large deviation results, we need to assume several assumptions, in particular on the normalized c.g.f. φ_n and on the m.g.f. ϕ_n, as follows.

(A.1) φ_n is an analytic function in $D_C := \{z \in \mathbb{C} : |z| < \alpha\}$, and there exists $M > 0$ such that $|\varphi_n(z)| < M$ for all $z \in D_C$ and n large enough.

(A.2) There exist $\alpha_0 \in (0, \alpha - \tau_a)$ and a function H such that for each $t \in (\tau_a - \alpha_0, \tau_a + \alpha_0)$ and for n large enough,

$$\varphi_n(t) = \varphi(t) + b_n^{-1} H(t) + o\left(b_n^{-1}\right), \tag{4}$$

where the function φ is three times continuously differentiable in $(\tau_a - \alpha_0, \tau_a + \alpha_0)$, $\varphi''(\tau_a) > 0$, and H is continuously differentiable in $(\tau_a - \alpha_0, \tau_a + \alpha_0)$.

(A.3) There exists $\delta_0 > 0$ such that

$$\sup_{\delta < |t| \le \lambda |\tau_a|} \left| \frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)} \right| = o\left(\frac{1}{\sqrt{b_n}}\right)$$

for any given δ and λ such that $0 < \delta < \delta_0 < \lambda$.

Assumption (A.1) is needed in the proof of Lemma 2, where we make use of Cauchy's inequality to bound the remaining term of a Taylor expansion. Assumption (A.2) guarantees the existence of an asymptotic expansion for the normalized c.g.f. This assumption is necessary to establish the strong large deviation results with rate functions that do not depend on n. It is also used to prove Lemmas 1 and 2. Assumption (A.3) is a version of Condition 3.16 of Chaganty and Sethuraman (1993). It implies a necessary condition which is required to apply Theorem 2.3 in Chaganty and Sethuraman (1993) (see the proof of Theorem 1 in Sect. 4). It plays a similar role to that of the Cramer condition (see, for instance, Hall 1992).

2.2 First-order expansions

In what follows, we give the main results. The first theorem deals with the case of absolutely continuous variables.

Theorem 1 *Assume that* (Z_n) *is a sequence of absolutely continuous random variables or its distribution has an absolutely continuous component. Let a be a real such that* $a > \varphi'(0)$ *and let assumptions* (A.1)–(A.3) *hold. Then, for n large enough,*

$$\mathbb{P}(Z_n \ge a) = \frac{\exp(-b_n I(a) + H(\tau_a))}{\sigma_a \tau_a \sqrt{2\pi b_n}}[1 + o(1)], \tag{5}$$

where $\tau_a > 0$ *is such that* $\varphi'(\tau_a) = a$. *Further,* $I(a) = \tau_a a - \varphi(\tau_a)$ *and* $\sigma_a^2 = \varphi''(\tau_a)$.

Now, let us consider the case where (Z_n) is a sequence of lattice-valued random variables. Recall that a random variable Y is said to be lattice if it takes values in a subset of the lattice set $\{d_0 + ks, k \in \mathbb{Z}\}$. The real $0 \leq d_0 < s$ is called the displacement and the positive real s is the span of Y. Denote by d_n and s_n the displacement and the span of the statistic $b_n Z_n$, respectively. The following assumption is required (see Chaganty and Sethuraman 1993).

(A$'$.3) There exists $\delta_1 > 0$ such that for $0 < \delta < \delta_1$

$$\sup_{\delta<|t|\leq\pi/s_n} \left|\frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)}\right| = o\left(\frac{1}{\sqrt{b_n}}\right).$$

The next theorem assumes that the span s_n goes to zero as $n \to \infty$. As noted in Chaganty and Sethuraman (1993, Remark 3.4), in this case, Assumption (A$'$.3) implies Assumption (A.3). Thus we obtain the same result as the one of Theorem 1.

Theorem 2 *Let assumptions* (A.1)–(A.2) *and* (A$'$.3) *hold. Assume that* (Z_n) *is a sequence of lattice-valued random variables. Furthermore, the span* s_n *of* $b_n Z_n$ *goes to zero as* n *tends to infinity. Then, for* $a > \varphi'(0)$ *and* n *large enough, (5) remains valid.*

One can also establish a first-order expansion similar to expression (3.20) of Chaganty and Sethuraman (1993) in the case where the span s_n of $b_n Z_n$ does not go to zero as n tends to infinity.

2.3 Examples

We present four examples to illustrate the theorems of the preceding section.

Example 1. The kernel density estimator. A large deviations result for the kernel density estimator was obtained by Louani (1998). Later, Joutard (2006) proved a pointwise strong large deviation theorem, in particular by using an Edgeworth expansion. We will show that this result can also be obtained from the application of Theorem 1. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. real random variables with density function f. We recall that the kernel density estimator of f is defined by

$$\hat{f}_n(y) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{y - X_j}{h_n}\right), y \in \mathbb{R},$$

where the kernel $K \geq 0$ is such that $\int_{\mathbb{R}} K(y)\mathrm{d}y = 1$, and $h_n > 0$ is the bandwidth such that $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} nh_n = \infty$. Assume that there exists $c \geq 0$ such that $\lim_{n\to\infty} nh_n^2 = c$. We also assume the following conditions (see Joutard 2006).

1. f is bounded, continuously differentiable with bounded derivative on $\mathbb{R}$.
2. $\kappa_0(t) = \int_{\mathbb{R}} K(y)\exp(tK(y))\mathrm{d}y$ is defined on the open interval $(-\infty, \alpha)$, $\alpha > 0$.
3. $\kappa_1(t) = \int_{\mathbb{R}} |y|K(y)\exp(tK(y))\mathrm{d}y$ is defined on the open interval $(-\infty, \alpha)$.

4. The kernel K has unbounded support.
5. For all $u \in (-\infty, \alpha)$ and all $p \in \mathbb{N}$,

$$\int_{\mathbb{R}} K^{1/p}(y) \exp(uK(y)) \mathrm{d}y < \infty.$$

6. For n_0 sufficiently large,

$$\sup_{n \geq n_0} \sup_{t > 0} \left| \int_{\mathbb{R}} \sin(tK(y)) f(x - h_n y) \mathrm{d}y \right| < \infty. \tag{6}$$

Under these conditions, we have the following result (identical to the one in Joutard 2006; but the proof was different).

Corollary 1 *Let x be a fixed value in $\mathbb{R}$ such that $f(x) > 0$. Let $\hat{f}_n(x)$ and h_n be defined as above and assume that conditions 1-6 hold and there exists $c \geq 0$ such that $\lim_{n\to\infty} nh_n^2 = c$. Then for a real $a > 0$ and n large enough,*

$$\mathbb{P}(\hat{f}_n(x) - f(x) \geq a) = \frac{\exp(-nh_n I_{KD}(a) + H_{KD}(\tau_a))}{\tau_a \sqrt{2\pi n h_n f(x) I_0''(\tau_a)}} [1 + o(1)], \tag{7}$$

where $\tau_a > 0$ is such that $a + f(x) = f(x) I_0'(\tau_a)$, $H_{KD}(t) = -c(f^2(x) I_0^2(t)/2 + f'(x) J_0(t))$ and $I_{KD}(a) = \tau_a(a + f(x)) - f(x) I_0(\tau_a)$. Further, $I_0(t) = \int_{\mathbb{R}} (\exp(tK(y)) - 1) \mathrm{d}y$ and $J_0(t) = \int_{\mathbb{R}} y(\exp(tK(y)) - 1) \mathrm{d}y$.

Proof Let us check that the assumptions of Theorem 1 hold with $Z_n = \hat{f}_n(x) - f(x)$ and $b_n = nh_n$. For Assumption (A.1), we observe that, for $z = t_1 + it_2 \in \mathbb{C}$, the normalized c.g.f. of $b_n Z_n$ is :

$$\begin{aligned}
\varphi_n(z) &= \frac{1}{nh_n} \log \mathbb{E} \{\exp(znh_n Z_n)\} \\
&= \frac{1}{h_n} \log \mathbb{E} \left\{ \exp\left(zK\left(\frac{x - X_1}{h_n} \right) \right) \right\} - zf(x) \\
&= \frac{1}{h_n} \log \left[1 + \mathbb{E} \left\{ \exp\left(t_1 K\left(\frac{x - X_1}{h_n} \right) \right) - 1 \right\} \right. \\
&\quad \left. + \mathbb{E} \left\{ \left(\exp\left(it_2 K\left(\frac{x - X_1}{h_n} \right) \right) - 1 \right) \exp\left(t_1 K\left(\frac{x - X_1}{h_n} \right) \right) \right\} \right] - zf(x).
\end{aligned}$$

Assumption (A.1) will then follow from conditions 1-2. As in Joutard (2006), using conditions 1-3 and the fact that $nh_n^2 = c + o(1)$, we can show that for $t < \alpha$,

$$\varphi_n(t) = \varphi(t) + \frac{1}{nh_n} H(t) + o\left(\frac{1}{nh_n} \right),$$

where $\varphi(t) = f(x) I_0(t) - tf(x)$, $H(t) = -c(f^2(x) I_0^2(t)/2 + f'(x) J_0(t))$, $I_0(t) = \int_{\mathbb{R}} (\exp(tK(y)) - 1) \mathrm{d}y$ and $J_0(t) = \int_{\mathbb{R}} y(\exp(tK(y)) - 1) \mathrm{d}y$. The function I_0 is infinitely differentiable in $(-\infty, \alpha)$ since, by condition 2, the function κ_0 is infinitely differentiable in $(-\infty, \alpha)$ [see Joutard (2006, page 296)]. Likewise, the function J_0 is infinitely differentiable in $(-\infty, \alpha)$ since, by condition 3, the function κ_1 is also infinitely differentiable in $(-\infty, \alpha)$. Given that $\varphi''(\tau_a) = f(x)I_0''(\tau_a) > 0$, Assumption (A.2) with $b_n = nh_n$ is therefore satisfied. Besides, for $a > \varphi'(0) = 0$, there exists $\tau_a \in (0, \alpha)$ such that $\varphi'(\tau_a) = a$, that is, $f(x) + a = f(x)I_0'(\tau_a)$. Finally, let us check Assumption (A.3). Let δ and λ be such that $0 < \delta < \delta_0 < \lambda$. Then, for all $t \in \{t \in \mathbb{R} : \delta < |t| < \lambda\tau_a\}$, there exists $q \in \mathbb{N}$ such that

$$\sqrt{nh_n}\left|\frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)}\right| = \sqrt{nh_n}\left|\frac{\phi(\tau_a + it)}{\phi(\tau_a)}\right|^n \leq \sqrt{nh_n}\sup_{\delta\leq|t|\leq n^q}\left|\frac{\phi(\tau_a + it)}{\phi(\tau_a)}\right|^n,$$

where ϕ is the m.g.f. of $K\left(\frac{x-X_1}{h_n}\right)$. Now using conditions 4-6, we can deduce from Lemma 2.2 of Joutard (2006) that, for any $q \in \mathbb{N}$ and n large enough,

$$\sup_{\delta\leq|t|\leq n^q}\left|\frac{\phi(\tau_a + it)}{\phi(\tau_a)}\right| \leq 1 - C(\delta)h_n,$$

where $C(\delta) > 0$. Consequently,

$$\sqrt{nh_n}\sup_{\delta\leq|t|\leq n^q}\left|\frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)}\right| \leq \sqrt{nh_n}(1 - C(\delta)h_n)^n$$

and the right-hand side goes to zero as n tends to infinity. This ends the verification of Assumption (A.3) and so we can apply Theorem 1 to prove (7). $\square$

Example 2. The sample variance. We consider the sample variance

$$Z_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Assuming that the X_i's have a normal distribution $\mathcal{N}(\mu; \sigma^2)$, $\sigma^2 > 0$, we know that $\sigma^{-2}\sum_{i=1}^{n}(X_i - \overline{X})^2$ follows a chi-square distribution with $n-1$ degrees of freedom. A probability of large deviations for the sample variance was studied by Sievers (1969). Here, we give a strong large deviation result by applying Theorem 1 with $b_n = n$.

Corollary 2 *Let Z_n be defined as above. Then for a real a such that $a > \sigma^2$ and n large enough,*

$$\mathbb{P}(Z_n \geq a) = \frac{\exp(-(n-1)I_{SV}(a))}{2\sqrt{\pi n a}\tau_a}[1 + o(1)], \tag{8}$$

where $I_{SV}(a) = \frac{1}{2}\left(\frac{a}{\sigma^2} - \log\left(\frac{a}{\sigma^2}\right) - 1\right) > 0$ and $\tau_a = \frac{a-\sigma^2}{2a\sigma^2}$.

Proof Since $\sigma^{-2}\sum_{i=1}^n (X_i - \overline{X})^2 \sim \chi^2(n-1)$, we have the following expansion for the normalized c.g.f. of nZ_n. For $t < \frac{n-1}{2n\sigma^2}$,

$$\begin{aligned}
\varphi_n(t) &= \frac{1}{n}\log \mathbb{E}\left\{\exp\left(t\frac{n\sigma^2}{n-1}\sum_{i=1}^n \frac{(X_i-\overline{X})^2}{\sigma^2}\right)\right\} \\
&= -\frac{n-1}{2n}\log\left(1-2\sigma^2 t\left(1+\frac{1}{n-1}\right)\right) \\
&= -\frac{1}{2}\log(1-2\sigma^2 t) - \frac{1}{2}\log\left(1-\frac{2\sigma^2 t}{(1-2\sigma^2 t)(n-1)}\right) + \frac{1}{2n}\log(1-2\sigma^2 t) \\
&\quad +\frac{1}{2n}\log\left(1-\frac{2\sigma^2 t}{(1-2\sigma^2 t)(n-1)}\right) \\
&= -\frac{1}{2}\log(1-2\sigma^2 t) + \frac{1}{n}\left(\frac{1}{2}\log(1-2\sigma^2 t) + \frac{\sigma^2 t}{1-2\sigma^2 t}\right) + o\left(\frac{1}{n}\right).
\end{aligned}$$

The function $z \in \mathbb{C} \mapsto \varphi_n(z)$ is analytic in $D_C = \{z \in \mathbb{C} : |z| < \alpha\}$, $\alpha < \frac{n-1}{2n\sigma^2}$, and one can find $M > 0$ such that for all $z \in D_C$, $|\varphi_n(z)| \le M$ (this implies Assumption (A.1)). Next, the expansion (4) holds for every $t \in (-\infty, \alpha)$ with $b_n = n$,

$$\varphi(t) = -\frac{1}{2}\log(1-2\sigma^2 t) \quad \text{and} \quad H(t) = -\varphi(t) + \frac{\sigma^2 t}{1-2\sigma^2 t}.$$

For a real a such that $\varphi'(0) = \sigma^2 < a < \varphi'(\alpha)$, there exists $\tau_a \in (0,\alpha)$ such that $\varphi'(\tau_a) = a$. Actually, we have $\tau_a = \frac{a-\sigma^2}{2a\sigma^2}$. Assumption (A.2) is fulfilled since the functions φ and H are infinitely differentiable on $(-\infty, 1/2\sigma^2)$, and we have

$$\varphi'(t) = \frac{\sigma^2}{1-2\sigma^2 t}, \qquad \varphi''(t) = \frac{2\sigma^4}{(1-2\sigma^2 t)^2}, \quad \varphi''(\tau_a) = 2a^2 > 0.$$

Now, we check Assumption (A.3). We have

$$\begin{aligned}
\left|\frac{\phi_n(\tau_a+it)}{\phi_n(\tau_a)}\right| &= \left|\frac{\mathbb{E}\left\{\exp\left((\tau_a+it)\frac{n}{n-1}\sum_{i=1}^n (X_i-\overline{X})^2\right)\right\}}{\mathbb{E}\left\{\exp\left(\tau_a\frac{n}{n-1}\sum_{i=1}^n (X_i-\overline{X})^2\right)\right\}}\right| \\
&= \left|\frac{(1-2\sigma^2(\tau_a+it)\frac{n}{n-1})^{-\frac{(n-1)}{2}}}{(1-2\sigma^2\tau_a\frac{n}{n-1})^{-\frac{(n-1)}{2}}}\right| \\
&= \left(\frac{(1-2\sigma^2\tau_a\frac{n}{n-1})^2 + 4\sigma^4 t^2(\frac{n}{n-1})^2}{(1-2\sigma^2\tau_a\frac{n}{n-1})^2}\right)^{-\frac{n-1}{4}} \\
&= \left(1+\frac{4\sigma^4 t^2(\frac{n}{n-1})^2}{(1-2\sigma^2\tau_a\frac{n}{n-1})^2}\right)^{-\frac{n-1}{4}}
\end{aligned}$$

$$= \exp\left[-\frac{n-1}{4}\log\left(1+\frac{4\sigma^4 t^2(1+\frac{1}{n-1})^2}{(a^{-1}\sigma^2-\frac{1}{n-1}(1-a^{-1}\sigma^2))^2}\right)\right],$$

where we have used the fact that $\tau_a = \frac{a-\sigma^2}{2a\sigma^2}$. Assumption (A.3) with $b_n = n$ then follows from the above expression. By applying Theorem 1, we eventually obtain the asymptotic result (8), where we have used the following expressions for $I(a)$ and $H(\tau_a)$

$$I(a) = \tau_a a - \varphi(\tau_a) = \frac{1}{2}\left(\frac{a}{\sigma^2} - \log\left(\frac{a}{\sigma^2}\right) - 1\right),$$
$$H(\tau_a) = -\varphi(\tau_a) + \frac{\sigma^2\tau_a}{1-2\sigma^2\tau_a} = I(a).$$

□

Example 3. The Wilcoxon signed-rank statistic. A large deviations result for the Wilcoxon signed-rank statistic was obtained by Klotz (1965). Later, Chaganty and Sethuraman (1993) provided a strong large deviation result by applying their theorem, but as noted in the introduction, their result is intractable in a general frame (in particular, one cannot derive an explicit asymptotic expression for the tail probability that depends on n). The asymptotic expansion (9) will follow from Theorem 2 with $b_n = n$.

Let $\{X_1, \ldots, X_n\}$ be a sequence of i.i.d. continuous random variables having distribution function F and let R_i be the rank of $|X_i|$, $i = 1, \ldots, n$. In other words, if one arranges $|X_1|, |X_2|, \ldots, |X_n|$ in increasing order of magnitude, R_i denotes the rank of $|X_i|$. Assume that the random variables X_i are symmetric about their median m. The Wilcoxon signed-rank statistic W_n is defined as the sum of the quantities R_i corresponding to the positive $X_i's$, that is,

$$W_n = \sum_{i=1}^{n} \mathbb{I}_{\{X_i>0\}} R_i.$$

The statistic W_n is used to test the null hypothesis $H_0 : m = 0$. Letting $Z_n = \frac{W_n}{n^2}$, under the null hypothesis, we have the following result.

Corollary 3 *Let Z_n be defined as above. Then for a real $a > 1/4$ and n large enough,*

$$\mathbb{P}(Z_n \geq a) = \frac{\exp(-nI_W(a) + H_W(\tau_a))}{\sigma_a\tau_a\sqrt{2\pi n}}[1+o(1)], \tag{9}$$

where $\tau_a > 0$ is such that $\int_0^1 \frac{x}{1+\exp(-\tau_a x)}\mathrm{d}x = a$, $H_W(t) = \frac{1}{2}\log\left(\frac{\exp(t)+1}{2}\right)$ and $\sigma_a^2 = \int_0^1 \frac{x^2\exp(\tau_a x)}{(1+\exp(\tau_a x))^2}\mathrm{d}x$. Further, $I_W(a) = \tau_a a - \varphi_W(\tau_a)$ where

$$\varphi_W(t) = \int_0^1 \log\left(\frac{\mathrm{e}^{tx}+1}{2}\right)\mathrm{d}x.$$

Proof The random variable nZ_n ($b_n = n$) is a lattice random variable with displacement 0 and span $s_n = 1/n$, which tends to zero as $n \to \infty$. The m.g.f. of nZ_n under the null hypothesis H_0 (that is, the median is equal to zero) is given by

$$\phi_n(t) = \prod_{k=1}^{n} [(\exp(tk/n) + 1)/2], \quad t \in \mathbb{R}.$$

Define

$$g_t(x) = \log\left(\frac{\exp(tx) + 1}{2}\right).$$

The normalized c.g.f. of nZ_n is then

$$\begin{aligned} \varphi_n(t) &= \frac{1}{n} \log \phi_n(t) \\ &= \frac{1}{n} \sum_{k=1}^{n} g_t(k/n), \quad t \in \mathbb{R}. \end{aligned} \tag{10}$$

The function $z \in \mathbb{C} \mapsto \varphi_n(z)$ is analytic in $D_C = \{z \in \mathbb{C} : |z| < \pi/2\}$ and for all $z \in D_C$, $|\varphi_n(z)| \le M$, $M > 0$ (Assumption (A.1) is fulfilled). On the other hand, φ_n (as a function of $t \in \mathbb{R}$) is a right Riemann sum and we know that its limit is $\varphi(t) = \int_0^1 \log\left(\frac{e^{tx}+1}{2}\right) dx$ for every $t \in \mathbb{R}$. Now, let us find the function H such that (4) holds.

Consider a function $g : D \mapsto \mathbb{R}$ defined on $D \subset \mathbb{R}$ such that $(0, 1) \subset D$. Denoting the trapezoidal sum, the right Riemann sum and the left Riemann sum by S, S_r and S_l, respectively, we have $S = (S_r + S_l)/2$, where $S_r = \frac{1}{n}\sum_{k=1}^{n} g(k/n)$ and $S_l = \frac{1}{n}\sum_{k=0}^{n-1} g(k/n)$. Furthermore,

$$\left| S - \int_0^1 g(x) dx \right| \le \frac{M_2}{12n^2}, \tag{11}$$

where $M_2 = \sup_{x \in (0,1)} |g''(x)| < \infty$. The right Riemann sum can be written as:

$$S_r = S + \frac{1}{n}\left(\frac{g(1) - g(0)}{2}\right). \tag{12}$$

Now, we apply this to the right Riemann sum φ_n. Noticing that $g_t(0) = 0$, by (11) and (12), (10) becomes

$$\varphi_n(t) = \int_0^1 g_t(x) dx + \frac{g_t(1)}{2n} + O(1/n^2).$$

Then, (4), with $b_n = n$, follows from the above expression with

$$\varphi(t) = \int_0^1 \log\left(\frac{e^{tx}+1}{2}\right) dx \quad \text{and} \quad H(t) = \frac{1}{2}\log\left(\frac{\exp(t)+1}{2}\right). \tag{13}$$

Since $\varphi'(0) = 1/4$, for a real $a \in (1/4, \varphi'(\pi/2))$, there exists $\tau_a \in (0, \pi/2)$ such that $\varphi'(\tau_a) = a$. The functions φ and H are infinitely differentiable on $\mathbb{R}$ and we have

$$\varphi'(t) = \int_0^1 \frac{x}{1+\exp(-tx)} dx, \qquad \varphi''(t) = \int_0^1 \frac{x^2 \exp(tx)}{(1+\exp(tx))^2} dx. \tag{14}$$

Besides, $\varphi''(\tau_a) > 0$ and Assumption (A.2) is then satisfied. According to Chaganty and Sethuraman (1993), Assumption (A$'$.3) holds with $b_n = n$ and $s_n = 1/n$ since there exist $n_0 \in \mathbb{N}$, $\delta_1 > 0$ and $c_0 > 0$ such that

$$\sup_{\delta < |t| \le \pi n} \left| \frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)} \right| \le \exp(-nc_0\delta^2)$$

for $n \ge n_0$ and $0 < \delta < \delta_1$. The asymptotic expansion (9) therefore follows from Theorem 2 with $\varphi, \varphi', \varphi''$ and H given by (13) and (14). $\square$

Example 4. The Kendall tau statistic. Sievers (1969) gave a large deviation result for this nonparametric test of independence. An application of Theorem 2 with $b_n = n$ will yield the strong large deviation result (15). Let $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ be a sequence of i.i.d. continuous random couples having distribution function $F(x, y)$ and let F_X and F_Y be the marginal distributions. The Kendall tau statistic Z_n can be defined by

$$Z_n = 2 \sum_{1 \le i < j \le n} \frac{(\mathbb{I}_{\{X_i \ge X_j\}} - \mathbb{I}_{\{X_i \le X_j\}})(\mathbb{I}_{\{Y_i \ge Y_j\}} - \mathbb{I}_{\{Y_i \le Y_j\}})}{n(n-1)}.$$

It was first used by Kendall (see Kendall and Stuart 1979) to test the null hypothesis $H_0 : F(x, y) = F_X(x)F_Y(y)$ for all x, y. Under the null hypothesis, we have the following corollary.

Corollary 4 *Let Z_n be defined as above. Then for a real $a \in (0, 1)$ and n large enough,*

$$\mathbb{P}(Z_n \ge a) = \frac{\exp(-nI_K(a) + H_K(\tau_a))}{\sigma_a \tau_a \sqrt{2\pi n}}[1 + o(1)], \tag{15}$$

where $\tau_a > 0$ is such that $1 - \frac{1}{\tau_a} + 4\int_0^1 \frac{x}{\exp(4\tau_a x)-1} dx = a$,

$$H_K(t) = 2t - 1 + \frac{3}{2}\log(1 - e^{-4t}) - \frac{1}{2}\log(4t) - \int_0^1 \log(1 - e^{-4tx}) dx,$$

and

$$\sigma_a^2 = \frac{1}{\tau_a^2} - 16\int_0^1 \frac{x^2\exp(4\tau_a x)}{(\exp(4\tau_a x)-1)^2}\mathrm{d}x.$$

Further, $I_K(a) = \tau_a a - \varphi_K(\tau_a)$*, where*

$$\varphi_K(t) = t + 1 - \log(4t) + \int_0^1 \log(1-\mathrm{e}^{-4tx})\mathrm{d}x.$$

Proof The statistic nZ_n ($b_n = n$) is a lattice random variable with displacement n and span $s_n = 4/(n-1)$, which tends to zero as $n \to \infty$. The m.g.f. of nZ_n under the null hypothesis is given by (see Kendall and Stuart 1979)

$$\phi_n(t) = \frac{\mathrm{e}^{nt}}{n!}\prod_{k=1}^{n}\frac{\mathrm{e}^{-4kt/(n-1)}-1}{\mathrm{e}^{-4t/(n-1)}-1}.$$

For $t > 0$, define

$$h_t(x) = \log(1-\exp(-4tx)) - \log(x), \quad x \in \mathbb{R}_+.$$

Then, for $t > 0$, the normalized c.g.f. of nZ_n is

$$\begin{aligned}
\varphi_n(t) &= t - \frac{1}{n}\log(n!) + \frac{1}{n}\sum_{k=1}^{n}\log\left(\frac{\mathrm{e}^{-4kt/(n-1)}-1}{\mathrm{e}^{-4t/(n-1)}-1}\right)\\
&= t - \frac{1}{n}\sum_{k=1}^{n}\log(k/(n-1)) - \log(n-1) + \frac{1}{n}\sum_{k=1}^{n}\{\log(1-\mathrm{e}^{-4kt/(n-1)})\\
&\quad - \log(1-\mathrm{e}^{-4t/(n-1)})\}\\
&= t + \frac{1}{n}\sum_{k=1}^{n}h_t\left(\frac{k}{n-1}\right) - \log(n-1) - \log(1-\mathrm{e}^{-4t/(n-1)}).
\end{aligned} \tag{16}$$

One can find $\alpha > 0$ and $M > 0$ such that the function $z \in \mathbb{C} \mapsto \varphi_n(z)$ is analytic in $D_C = \{z \in \mathbb{C} : |z| < \alpha\}$ and $|\varphi_n(z)| \le M$ for all $z \in D_C$ (Assumption (A.1) is verified). Now, for $t > 0$, we have

$$\log(1-\mathrm{e}^{-4t/(n-1)}) = -\log(n-1) + \log(4t) - \frac{2t}{n-1} + o(1/n). \tag{17}$$

For the Riemann sum, we use the same arguments as in the previous example. Define

$$\tilde{\varphi}_n(t) = \frac{1}{n}\sum_{k=1}^{n}h_t(k/n).$$

Notice that

$$h_t''(x) = \frac{1}{x^2} - \frac{16t^2 e^{-4tx}}{(1 - e^{-4tx})^2},$$

and $\lim_{x \to 0} h_t''(x) = \frac{4}{3}t^2$. Thus, for $t \neq 0$, $M_2(t) = \sup_{x \in (0,1)} |h_t''(x)| < \infty$. This implies that

$$\tilde{\varphi}_n(t) = \int_0^1 h_t(x)\mathrm{d}x + \frac{1}{n}\left(\frac{h_t(1) - h_t(0)}{2}\right) + o\left(\frac{1}{n}\right).$$

It is easy to see that $h_t(1) = \log(1 - e^{-4t})$ and $h_t(0) = \lim_{x \to 0} h_t(x) = \log(4t)$. Then,

$$\tilde{\varphi}_n(t) = \int_0^1 h_t(x)\mathrm{d}x + \frac{1}{2n}\left[\log(1 - e^{-4t}) - \log(4t)\right] + o\left(\frac{1}{n}\right), \tag{18}$$

where $\int_0^1 h_t(x)\mathrm{d}x = \int_0^1 \log(1 - e^{-4tx})\mathrm{d}x + 1$. Finally, we can write (16) as

$$\begin{aligned}\varphi_n(t) = t + \left(1 - \frac{1}{n}\right)\tilde{\varphi}_{n-1}(t) + \frac{h_t\left(\frac{n}{n-1}\right)}{n} \\ - \log(1 - e^{-4t/(n-1)}) - \log(n-1).\end{aligned} \tag{19}$$

The expansion (4), with $b_n = n$, therefore follows from (19), (17) and (18) with

$$\varphi(t) = t + 1 - \log(4t) + \int_0^1 \log(1 - e^{-4tx})\mathrm{d}x \tag{20}$$

and

$$H(t) = 2t - 1 + \frac{3}{2}\log(1 - e^{-4t}) - \frac{1}{2}\log(4t) - \int_0^1 \log(1 - e^{-4tx})\mathrm{d}x. \tag{21}$$

We have $\varphi'(0) = 0$. Then, for a real $a \in (0, \varphi'(\alpha))$, $\varphi'(\alpha) < 1$, there exists $\tau_a \in (0, \alpha)$ such that $\varphi'(\tau_a) = a$. The functions φ and H are infinitely differentiable on $\mathbb{R}$, and $\varphi''(\tau_a) > 0$. Hence, Assumption (A.2) holds and we have

$$\begin{aligned}\varphi'(t) &= 1 - \frac{1}{t} + 4\int_0^1 \frac{x}{\exp(4tx) - 1}\mathrm{d}x, \\ \varphi''(t) &= \frac{1}{t^2} - 16\int_0^1 \frac{x^2\exp(4tx)}{(\exp(4tx) - 1)^2}\mathrm{d}x.\end{aligned} \tag{22}$$

Now, in order to check Assumption (A$'$.3), we use the same approach as in Chaganty and Sethuraman (1985). We only give a sketch of the proof. We define

$$G_n(t) = \mathrm{Real}\,\{\varphi_n(\tau_a) - \varphi_n(\tau_a + it)\} = -\frac{1}{n}\log\left(\left|\frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)}\right|\right).$$

One can then show that there exists $\mu \in (2/3, 1)$ such that

$$\inf_{\delta \le |t| \le \delta(n-1)^{1-\mu}} G_n(t) \ge \frac{2\delta^2}{9} n^{3(\mu-1)}[1+o(1)] \tag{23}$$

and

$$\lim_{n\to\infty} \inf_{\delta(n-1)^{1-\mu} \le |t| \le (n-1)\pi/4} G_n(t) = \infty, \tag{24}$$

where $0 < \delta < \delta_1$ and $\delta_1 < \pi/4$. Combining (23) and (24), for n large enough we get

$$G_n(t) \ge \frac{2\delta^2}{9} n^{3(\mu-1)}[1+o(1)]$$

for all $t \in \{t : \delta \le |t| \le (n-1)\pi/4\}$. This implies Assumption (A$'$.3) with $b_n = n$ and $s_n = 4/(n-1)$. Hence, we can apply Theorem 2 with $\varphi, \varphi', \varphi''$ and H given by (20), (21) and (22), and obtain (15). □

3 Lemmas

In this section, we establish two preliminary lemmas needed for the proofs of Theorems 1 and 2. To do so we first introduce some notation. Denote the distribution function of $b_n Z_n$ by K_n. Let a be a real such that $a > \varphi'(0)$ and there exists $\tau_a \in (0, \alpha)$ satisfying $\varphi'(\tau_a) = a$. Using an exponential change of measure, let

$$H_n(u) = \int_{-\infty<y<u} \exp(y\tau_a - b_n\varphi_n(\tau_a))\mathrm{d}K_n(y)$$

be the distribution function of $b_n Z_n^*$. Define the random variable

$$V_n = \frac{\sqrt{b_n}(Z_n^* - a)}{\sigma_a}, \tag{25}$$

where we recall that $\sigma_a^2 = \varphi''(\tau_a) > 0$ (Assumption (A.2)). The following lemma shows the asymptotic normality of V_n.

Lemma 1 *Let Assumption* (A.2) *hold. Then, the statistic V_n converges in distribution to a standard normal random variable.*

Proof Let M_{V_n} be the m.g.f. of V_n. For $t \in (-\alpha_0, \alpha_0)$, we have

$$\begin{aligned} M_{V_n}(t) &= \mathbb{E}\{\exp(tV_n)\} \\ &= \mathrm{e}^{-ta\sqrt{b_n}/\sigma_a}\mathbb{E}\{\exp(tb_n Z_n^*/(\sqrt{b_n}\sigma_a))\} \\ &= \mathrm{e}^{-ta\sqrt{b_n}/\sigma_a}\frac{\phi_n(\tau_a + \frac{t}{\sqrt{b_n}\sigma_a})}{\phi_n(\tau_a)}. \end{aligned}$$

By Assumption (A.2), we get, for n large enough,

$$\begin{aligned}\log M_{V_n}(t) &= -\frac{ta\sqrt{b_n}}{\sigma_a} + b_n\left[\varphi_n\left(\tau_a + \frac{t}{\sqrt{b_n}\sigma_a}\right) - \varphi_n(\tau_a)\right]\\ &= -\frac{ta\sqrt{b_n}}{\sigma_a} + b_n\left[\varphi\left(\tau_a + \frac{t}{\sqrt{b_n}\sigma_a}\right) - \varphi(\tau_a)\right.\\ &\quad \left. + b_n^{-1}\left(H\left(\tau_a + \frac{t}{\sqrt{b_n}\sigma_a}\right) - H(\tau_a)\right) + o(b_n^{-1})\right].\end{aligned}$$

Using Taylor expansions for the functions φ and H, and the fact that $\varphi'(\tau_a) = a$ and $\sigma_a^2 = \varphi''(\tau_a)$, the following result holds for any $t \in (-\alpha_0, \alpha_0)$ and n large enough,

$$\begin{aligned}\log M_{V_n}(t) &= -\frac{ta\sqrt{b_n}}{\sigma_a} + \frac{t\varphi'(\tau_a)\sqrt{b_n}}{\sigma_a} + \frac{t^2\varphi''(\tau_a)}{2\sigma_a^2} + o(1)\\ &= \frac{t^2}{2} + o(1).\end{aligned}$$

As a consequence, $\log M_{V_n}(t) \to t^2/2$ as $n \to \infty$ and the lemma is proved. □

The proof of the next lemma is similar to that of Chaganty and Sethuraman (1993, Lemma 3.1) (in particular, we make use of Chaganty and Sethuraman (1993, Theorem 2.6)).

Lemma 2 *Let f_n be the characteristic function of V_n and assume that assumptions* (A.1)–(A.2) *are satisfied. Then, there exist $\delta > 0$, $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\sup_{n\geq n_0} |f_n(t)| I(|t| \leq \delta\sqrt{b_n}\sigma_a) \leq \exp(-\gamma t^2). \tag{26}$$

Proof Denote

$$g_n(t) = \frac{1}{b_n\sigma_a^2}\log|f_n(\sqrt{b_n}\sigma_a t)|.$$

To prove the lemma, it is sufficient to check that Condition (2.29) of Chaganty and Sethuraman (1993, Theorem 2.6) is satisfied for all $n \geq n_0$ where $n_0 \in \mathbb{N}$. It is straightforward to see that

$$g_n(t) = \frac{1}{\sigma_a^2}\left[\mathrm{Real}(\varphi_n(\tau_a + it) - \varphi_n(\tau_a))\right].$$

We know from Assumption (A.1) that g_n is infinitely differentiable in $(-\alpha_0, \alpha_0)$, $0 < \alpha_0 < \alpha - \tau_a$. Then, using a Taylor expansion, we have for $t \in (-\alpha_0, \alpha_0)$,

$$\begin{aligned} g_n''(t) &= \frac{-\mathrm{Real}(\varphi_n''(\tau_a + it))}{\sigma_a^2} \\ &= \frac{-\mathrm{Real}(\varphi_n''(\tau_a) + r_n(\tau_a + it))}{\sigma_a^2}, \end{aligned}$$

where the remainder term r_n depends on the third derivative of φ_n. Now, let us show that

$$\varphi_n''(\tau_a) = \sigma_a^2 + o(1). \tag{27}$$

Noting that

$$\varphi_n''(\tau_a) = b_n \left(\frac{\mathbb{E}\{Z_n^2 \exp(\tau_a b_n Z_n)\}}{\mathbb{E}\{\exp(\tau_a b_n Z_n)\}} - \left(\frac{\mathbb{E}\{Z_n \exp(\tau_a b_n Z_n)\}}{\mathbb{E}\{\exp(\tau_a b_n Z_n)\}} \right)^2 \right),$$

and in view of the fact that $b_n Z_n^*$ is distributed according to H_n, it can be easily seen that $\varphi_n''(\tau_a) = b_n \mathrm{Var}(Z_n^*)$. From Lemma 1, we have $\mathrm{Var}(V_n) = \frac{b_n}{\sigma_a^2} \mathrm{Var}(Z_n^*) \to 1$ as $n \to \infty$. Then, (27) holds and we have

$$g_n''(t) \leq -1 + \frac{|r_n(\tau_a + it)|}{\sigma_a^2} + o(1).$$

Furthermore, by Assumption (A.1) (φ_n is analytic) and Cauchy's inequality, the remainder term r_n can be bounded. That is, there exists $\delta > 0$ small enough ($\delta < \alpha_0$) such that for all $|t| \leq \delta$,

$$|r_n(\tau_a + it)| \leq \frac{3! M |t|}{\alpha_0^3}.$$

Therefore, there exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ large enough such that for all $|t| \leq \delta$ and all $n \geq n_0$,

$$g_n''(t) \leq -\gamma.$$

Condition (2.29) of Chaganty and Sethuraman (1993, Theorem 2.6) is thus verified. This completes the proof. □

4 Proofs

We give the proofs of the theorems of Sect. 2.

Proof of Theorem 1. The beginning of the proof of (5) follows the same lines as in Bahadur and Rao (1960) and more recently Bercu et al. (2000) or Joutard (2006). Let $a > \varphi'(0)$. We can write the Fenchel-Legendre transform of φ as

$$I(a) := \sup_{t\in\mathbb{R}}\{ta - \varphi(t)\} = \tau_a a - \varphi(\tau_a),$$

where $\tau_a \in (0, \alpha)$ is such that $\varphi'(\tau_a) = a$. Recall that $b_n Z_n^*$ is a random variable with distribution function

$$H_n(u) = \int_{-\infty<y<u} \exp(y\tau_a - b_n\varphi_n(\tau_a))\mathrm{d}K_n(y).$$

Using Assumption (A.2), the right tail probability may now be written as

$$\begin{aligned}\mathbb{P}(Z_n \geq a) &= \mathbb{E}\{\exp(-\tau_a b_n Z_n^* + b_n\varphi_n(\tau_a))\mathbb{I}_{\{Z_n^*\geq a\}}\}\\ &= \exp(b_n\varphi_n(\tau_a) - b_n\tau_a a)\mathbb{E}\{\exp(-\tau_a b_n(Z_n^* - a))\mathbb{I}_{\{Z_n^*\geq a\}}\}\\ &= \exp(b_n\varphi_n(\tau_a) - b_n\tau_a a)\mathbb{E}\{\exp(-\tau_a\sigma_a\sqrt{b_n}V_n)\mathbb{I}_{\{V_n\geq 0\}}\}\\ &= \mathrm{e}^{b_n(\varphi(\tau_a)-\tau_a a)+H(\tau_a)+o(1)}\mathbb{E}\{\exp(-\tau_a\sigma_a\sqrt{b_n}V_n)\mathbb{I}_{\{V_n\geq 0\}}\}\\ &= \mathrm{e}^{-b_n I(a)+H(\tau_a)}\mathbb{E}\{\exp(-\tau_a\sigma_a\sqrt{b_n}V_n)\mathbb{I}_{\{V_n\geq 0\}}\}(1+o(1)),\end{aligned} \tag{28}$$

where

$$V_n = \frac{\sqrt{b_n}(Z_n^* - a)}{\sigma_a} \quad \text{and} \quad \sigma_a = \sqrt{\varphi''(\tau_a)} > 0. \tag{29}$$

It remains to prove that

$$\lim_{n\to\infty} \tau_a\sigma_a\sqrt{b_n}\mathbb{E}\{\exp(-\tau_a\sigma_a\sqrt{b_n}V_n)\mathbb{I}_{\{V_n\geq 0\}}\} = \frac{1}{\sqrt{2\pi}}. \tag{30}$$

To do this, we will apply Chaganty and Sethuraman (1993, Theorem 2.7) to the sequence of random variables V_n. Lemma 1 and Lemma 2 show that V_n converges in distribution to a standard normal variable and that (26) holds, respectively. Besides, it is easy to see that

$$\sup_{\delta\sqrt{b_n}\sigma_a<|t|\leq\lambda\tau_a\sigma_a\sqrt{b_n}} |f_n(t)| = \sup_{\delta<|t|\leq\lambda\tau_a} \left|\frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)}\right|,$$

where f_n is the characteristic function of V_n and ϕ_n the m.g.f. of $b_n Z_n$. Hence, by Assumption (A.3), we have for n large enough,

$$\sup_{\delta\sqrt{b_n}\sigma_a < t \le \lambda\tau_a\sigma_a\sqrt{b_n}} |f_n(t)| = o(b_n^{-1/2}). \tag{31}$$

The convergence in distribution of Z_n, (26) and (31) allow us to verify the conditions of Chaganty and Sethuraman (1993, Theorem 2.3). Denote the density of V_n (or pseudo density if V_n does not possess a proper density function) by q_n. By Chaganty and Sethuraman (1993, Theorem 2.3), there exists a constant $M_0 > 0$ such that

$$\sup_y q_n(y) \le M_0, \tag{32}$$

and if $z_n \to z$, then

$$q_n(z_n) \to (\sqrt{2\pi})^{-1}\mathrm{e}^{-z^2/2}. \tag{33}$$

Chaganty and Sethuraman (1993, Theorem 2.7) follows directly from (32) and (33). Consequently, we have

$$\lim_{n\to\infty} \tau_a\sigma_a\sqrt{b_n}\mathbb{E}\{\exp(-\tau_a\sigma_a\sqrt{b_n}V_n)\mathbb{I}_{\{V_n\ge 0\}}\} = \left[(\sqrt{2\pi})^{-1}\mathrm{e}^{-z^2/2}\right]_{z=0} = \frac{1}{\sqrt{2\pi}},$$

and (30) holds. Combining (30) and (28), we obtain (5). This ends the proof. □

Proof of Theorem 2. Theorem 2 follows from Theorem 1, since Assumption (A′.3) implies Assumption (A.3) (in view of the fact that $s_n \to 0$ as $n \to \infty$). □

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References

Bahadur, R., Rao, R. (1960). On deviations of the sample mean. *Annals of Mathematical Statistics, 31*, 1015–1027.

Bercu, B., Rouault, A. (2002). Sharp large deviations for the Ornstein-Uhlenbeck process. *SIAM Theory of Probability and Its Applications, 46*, 1–19.

Bercu, B., Gamboa, F., Lavielle, M. (2000). Sharp large deviations for Gaussian quadratic forms with applications. *ESAIM:P&S, 3*, 1–29.

Blackwell, D., Hodges, J. L. (1959). The probability in the extreme tail of a convolution. *Annals of Mathematical Statistics, 30*, 1113–1120.

Book, S. (1972). Large deviation probabilities for weighted sums. *Annals of Mathematical Statistics, 43*, 1221–1234

Chaganty, N. R., Sethuraman, J. (1985). Large deviation local limit theorems for arbitrary sequences of random variables. *Annals of Probability, 13*, 97–114.

Chaganty, N. R., Sethuraman, J. (1993). Strong large deviation and local limit theorems. *Annals of Probability, 21*, 1671–1690.

Chaganty, N. R., Sethuraman, J. (1996). Multidimensional strong large deviation theorems. *Journal of Statistical Planning and Inference, 55*(3), 265–280.

Cho, D., Jeon, J. W. (1994). Strong large deviations theorems for the ratio of the independent random variables. *Journal of the Korean Statistical Society, 23*(2), 239–250.

Daouia, A., Joutard, C. (2009). Large deviation properties for empirical quantile-type production functions. *Statistics, 43*(3), 267–277.

Dembo, A., Zeitouni, O. (1998). *Large deviations techniques and applications* (2nd ed.). Springer, New York.

Deuschel, J. D., Stroock, D. W. (1989). *Large deviations*. Boston: Academic Press.

Florens-Landais, D., Gautherat, E., Nisipasu, E. (1998). Asymptotic expansions and large deviations in estimation of an Ornstein-Uhlenbeck model. *Comptes Rendus De l'Académie Des Sciences, 327*, 205–210.

Hall, P. (1992). *The bootstrap and Edgeworth expansion*. New York: Springer.

Joutard, C. (2006). Sharp large deviations in nonparametric estimation. *Journal of Nonparametric Statistics, 18*(3), 293–306.

Joutard, C. (2008). Sharp large deviation principle for the conditional empirical process. *International Journal of Statistics and Management System, 3*, 74–92.

Kendall, M.G., Stuart, A. (1979). *The advanced theory of statistics* (Vol. II). New-York: MacMillan.

Klotz, J. (1965). Alternative efficiencies for signed rank tests. *Annals of Mathematical Statistics, 36*, 1759–1766.

Louani, D. (1998). Large deviations limit theorems for the kernel density estimator. *Scandinavian Journal of Statistics, 25*, 243–253.

Sievers, G.L. (1969). On the probability of large deviations and exact slopes. *Annals of Mathematical Statistics, 40*, 1908–1921.