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Strictly stationary solutions of multivariate ARMA equations with i.i.d. noise

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Abstract We obtain necessary and sufficient conditions for the existence of strictly stationary solutions of multivariate ARMA equations with independent and identically distributed driving noise. For general ARMA(p, q) equations these conditions are expressed in terms of the coefficient polynomials of the defining equations and moments of the driving noise sequence, while for $p = 1$ an additional characterization is obtained in terms of the Jordan canonical decomposition of the autoregressive matrix, the moving average coefficient matrices and the noise sequence. No a priori assumptions are made on either the driving noise sequence or the coefficient matrices.

Keywords VARMA process · Multivariate ARMA · Heavy tails · Infinite variance · Strict stationarity

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1 Introduction

Let $m, d \in \mathbb{N} = \{1, 2 \ldots, \}$, $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(Z_t)_{t\in\mathbb{Z}}$ be a d-variate noise sequence of random vectors defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ be deterministic complex-valued matrices. Then any m-variate stochastic process $(Y_t)_{t\in\mathbb{Z}}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies almost surely

$$Y_t - \Psi_1 Y_{t-1} - \cdots - \Psi_p Y_{t-p} = \Theta_0 Z_t + \cdots + \Theta_q Z_{t-q}, \quad t \in \mathbb{Z}, \tag{1}$$

is called a solution of the ARMA(p, q) equation (1) (autoregressive moving average equation of autoregressive order p and moving average order q). Such a solution is often called a VARMA (vector ARMA) process to distinguish it from the scalar case, but we shall simply use the term ARMA throughout. Denoting the identity matrix in $\mathbb{C}^{m\times m}$ by Id_m, the *coefficient polynomials* $P(z)$ and $Q(z)$ of the ARMA(p, q) equation (1) are defined as:

$$P(z) := \mathrm{Id}_m - \sum_{k=1}^{p} \Psi_k z^k \text{ and } Q(z) := \sum_{k=0}^{q} \Theta_k z^k \quad \text{for} \quad z \in \mathbb{C}. \tag{2}$$

With the aid of the backwards shift operator B, Eq. (1) can be written more compactly in the form

$$P(B)Y_t = Q(B)Z_t, \quad t \in \mathbb{Z}.$$

There is an evidence to show that, although VARMA(p, q) models with $q > 0$ are more difficult to estimate than VARMA$(p, 0)$ (vector autoregressive) models, significant improvement in forecasting performance can be achieved by allowing the moving average order q to be greater than zero. See, for example, Athanasopoulos and Vahid (2008), where such improvement is demonstrated for a variety of macroeconomic time series.

Much attention has been paid to *weak ARMA processes*, i.e., weakly stationary solutions of (1) if $(Z_t)_{t\in\mathbb{Z}}$ is a weak white noise sequence. Recall that a $\mathbb{C}^r$-valued process $(X_t)_{t\in\mathbb{Z}}$ is *weakly stationary* if each X_t has finite second moment, and if $\mathbb{E}X_t$ and $\mathrm{Cov}(X_t, X_{t+h})$ do not depend on $t \in \mathbb{Z}$ for each $h \in \mathbb{Z}$. If additionally every component of X_t is uncorrelated with every component of $X_{t'}$ for $t \neq t'$, then $(X_t)_{t\in\mathbb{Z}}$ is called *weak white noise*. In the case when $m = d = 1$ and Z_t is weak white noise having non-zero variance, it can easily be shown using spectral analysis, see e.g., Brockwell and Davis (1991), Problem 4.28, that a weak ARMA process exists if and only if the rational function $z \mapsto Q(z)/P(z)$ has only removable singularities on the unit circle in $\mathbb{C}$. For higher dimensions, it is well known that a sufficient condition for weak ARMA processes to exist is that the polynomial $z \mapsto \det P(z)$ has no zeroes on the unit circle (this follows as in Theorem 11.3.1 of Brockwell and Davis 1991), by developing $P^{-1}(z) = (\det P(z))^{-1}\mathrm{Adj}(P(z))$, where $\mathrm{Adj}(P(z))$ denotes the adjugate matrix of $P(z)$, into a Laurent series which is convergent in a neighborhood of the unit circle). However, to the best of our knowledge, necessary and sufficient conditions have not been given in the literature so far. We shall obtain such conditions in terms of the matrix rational function $z \mapsto P^{-1}(z)Q(z)$ in Theorem 3, the proof being an easy extension of the corresponding one-dimensional result.

Weak ARMA processes, by definition, are restricted to have finite second moments. However, financial time series often exhibit apparent heavy-tailed behaviour with asymmetric marginal distributions, so that second-order properties are inadequate to account for the data. To deal with such phenomena we focus in this paper on *strict ARMA processes*, by which we mean strictly stationary solutions of (1) when $(Z_t)_{t\in\mathbb{Z}}$ is supposed to be an independent and identically distributed (i.i.d.) sequence of random vectors, not necessarily with finite variance. A sequence $(X_t)_{t\in\mathbb{Z}}$ is *strictly stationary* if all its finite dimensional distributions are shift invariant. Much less is known about strict ARMA processes, and it was shown only recently for $m = d = 1$ in Brockwell and Lindner (2010) that for i.i.d. non-deterministic noise $(Z_t)_{t\in\mathbb{Z}}$, a strictly stationary solution of (1) exists if and only if $Q(z)/P(z)$ has only removable singularities on the unit circle and Z_0 has finite log moment, or if $Q(z)/P(z)$ is a polynomial. For higher dimensions, it can easily be shown (by the same arguments used to establish the existence of weakly stationary solutions) that the conditions $\mathbb{E}\log^+\|Z_0\| < \infty$ and $\det P(z) \neq 0$ for $|z| = 1$ are *sufficient* for a strictly stationary solution to exist. However, necessary and sufficient conditions have not yet been established. In this paper, we obtain such conditions in Theorem 2, generalizing the results of Brockwell and Lindner (2010) to higher dimensions. A related question was considered by Bougerol and Picard (1992) who, using their powerful results on random recurrence equations, showed in their Theorem 4.1 that if $\mathbb{E}\log^+\|Z_0\| < \infty$ and the coefficient polynomials are left-coprime, meaning that the only common left-divisors of $P(z)$ and $Q(z)$ are unimodular (see Sect. 6 for the precise definitions), then a non-anticipative strictly stationary solution of (1) exists if and only if $\det P(z) \neq 0$ for $|z| \leq 1$. Observe that in characterizing the existence of strict (not necessarily non-anticipative) ARMA processes in the present paper, we shall make no a priori assumptions on log moments of the noise sequence or on left-coprimeness of the coefficient polynomials, but rather obtain related conditions as parts of our characterization. As one application of our main results, we shall obtain in Theorem 4 a slight extension of Theorem 4.1 of Bougerol and Picard (1992) by characterizing all non-anticipative strictly stationary solutions of (1) without any moment assumptions but still assuming left-coprimeness of the coefficient polynomials. Klein et al. (2005) consider the model (1) from a somewhat different point of view. Assuming that $d = m$ and that the equations $\det P(z) = 0$ and $\det Q(z) = 0$ have no roots in the closed unit disc, they show that the Fisher information matrix is singular if and only if these two equations have at least one root in common.

The paper is organized as follows. In Sect. 2, we state the main results of the paper. Theorem 1 gives necessary and sufficient conditions for the multivariate ARMA$(1, q)$ equation,

$$Y_t - \Psi_1 Y_{t-1} = \sum_{k=0}^{q} \Theta_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{3}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence, to have a strictly stationary solution. Elementary considerations will show that it suffices to establish these conditions under the assumption that Ψ_1 is in Jordan block form. Theorem 1 characterizes the existence of strictly stationary solutions of (3) in terms of the Jordan canonical decomposition of Ψ_1, the coefficients Θ_k and properties of Z_0. An explicit solution of (3), assuming its existence, is also derived and the question of uniqueness of this solution is addressed.

Strict ARMA(p, q) processes are treated in Theorem 2. Since every m-variate ARMA(p, q) process can be expressed in terms of a corresponding mp-variate ARMA$(1, q)$ process, questions of existence and uniqueness can, in principle, be resolved by Theorem 1. However, since the Jordan canonical form of the corresponding $mp \times mp$-matrix $\underline{\Psi}_1$ in the higher-dimensional ARMA$(1, q)$ representation is, in general, difficult to handle, another more compact characterization is derived in Theorem 2. This characterization is given in terms of properties of the matrix rational function $P^{-1}(z)Q(z)$ and finite log moments of certain linear combinations of the components of Z_0, extending the conditions obtained by Brockwell and Lindner (2010) for $m = d = 1$ in a natural way. Although the statement of Theorem 2 makes no reference to Jordan canonical forms, its proof makes fundamental use of Theorem 1.

Theorem 3 deals with the corresponding questions for weak ARMA(p, q) processes. The proofs of Theorems 1, 3 and 2 are given in Sects. 3, 4 and 5, respectively. The proof of Theorem 2 makes crucial use of both Theorems 1 and 3.

The main results are discussed in Sect. 6 and, as one application, the characterization of non-anticipative strictly stationary solutions is obtained in Theorem 4, generalizing slightly the result of Bougerol and Picard (1992).

Throughout the paper, vectors will be understood as column vectors and e_i will denote the ith unit vector in $\mathbb{C}^m$. The zero matrix in $\mathbb{C}^{m\times r}$ is denoted by $0_{m,r}$ or simply 0, the zero vector in $\mathbb{C}^r$ by 0_r or simply 0. The transpose of a matrix A is denoted by A^{T}, and its complex conjugate transpose matrix by $A^* = \overline{A}^{\mathrm{T}}$. By $\|\cdot\|$ we denote an unspecific, but fixed vector norm on $\mathbb{C}^s$ for $s \in \mathbb{N}$, as well as the corresponding matrix norm $\|A\| = \sup_{x\in\mathbb{C}^s, \|x\|=1} \|Ax\|$. We write $\log^+(x) := \log\max\{1, x\}$ for $x \in \mathbb{R}$, and denote by $\mathbb{P} - \lim$ limits in probability. All equations and inequalities concerning random variables should be interpreted as holding almost surely, and uniqueness will always mean almost sure uniqueness.

2 Main results

Theorems 1 and 2 give necessary and sufficient conditions for the ARMA$(1, q)$ equation (3) and the ARMA(p, q) equation (1), respectively, to have a strictly stationary solution. In Theorem 1, these conditions are expressed in terms of the i.i.d. noise sequence $(Z_t)_{t\in\mathbb{Z}}$, the coefficient matrices $\Theta_0, \ldots, \Theta_q$ and the Jordan canonical decomposition of Ψ_1, while in Theorem 2 they are given in terms of the noise sequence and the coefficient polynomials $P(z)$ and $Q(z)$ as defined in (2).

As background for Theorem 1, suppose that $\Psi_1 \in \mathbb{C}^{m\times m}$ and choose a (necessarily non-singular) matrix $S \in \mathbb{C}^{m\times m}$ such that $S^{-1}\Psi_1 S$ is in Jordan canonical form. Suppose also that $S^{-1}\Psi_1 S$ has $H \in \mathbb{N}$ Jordan blocks, $\Phi_1, \ldots, \Phi_H$, the hth block beginning in row r_h, where $r_1 := 1 < r_2 < \cdots < r_H < m + 1 =: r_{H+1}$. A Jordan block with associated eigenvalue λ will always be understood to be of the form

$$\begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}, \tag{4}$$

i.e., the entries 1 are below the main diagonal.

Observe that (3) has a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if the corresponding equation for $X_t := S^{-1}Y_t$ namely

$$X_t - S^{-1}\Psi_1 S X_{t-1} = \sum_{j=0}^{q} S^{-1}\Theta_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{5}$$

has a strictly stationary solution. This will be the case only if the equation for the hth block,

$$X_t^{(h)} := I_h X_t, \quad t \in \mathbb{Z}, \tag{6}$$

where I_h is the $(r_{h+1} - r_h) \times m$ matrix with (i, j) components,

$$I_h(i, j) = \begin{cases} 1 & \text{if } j = i + r_h - 1, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

has a strictly stationary solution for each $h = 1, \dots, H$. But these equations are simply

$$X_t^{(h)} - \Phi_h X_{t-1}^{(h)} = \sum_{j=0}^{q} I_h S^{-1}\Theta_j Z_{n-j}, \quad t \in \mathbb{Z}, \quad h = 1, \dots, H, \tag{8}$$

where Φ_h is the hth Jordan block of $S^{-1}\Psi_1 S$.

Conversely if (8) has a strictly stationary solution $X'^{(h)}$ for each $h \in \{1, \dots, H\}$, then we shall see from the proof of Theorem 1 that there exist (possibly different if $|\lambda_h| = 1$) strictly stationary solutions $X^{(h)}$ of (8) for each $h \in \{1, \dots, H\}$, such that

$$Y_t := S(X_t^{(1)\mathrm{T}}, \dots, X_t^{(H)\mathrm{T}})^{\mathrm{T}}, \quad t \in \mathbb{Z}, \tag{9}$$

is a strictly stationary solution of (3).

Existence and uniqueness of a strictly stationary solution of (3) are therefore equivalent to the existence and uniqueness of a strictly stationary solution of the equations (8) for each $h \in \{1, \dots, H\}$. The necessary and sufficient condition for each one will depend on the value of the eigenvalue λ_h associated with Φ_h and in particular on whether (a) $|\lambda_h| \in (0, 1)$, (b) $|\lambda_h| > 1$, (c) $|\lambda_h| = 1$ and $\lambda_h \neq 1$, (d) $\lambda_h = 1$

and (e) $\lambda_h = 0$. These cases will be addressed separately in the proof of Theorem 1, which is given in Sect. 3. The aforementioned characterization in terms of the Jordan decomposition of Ψ_1 now reads as follows.

Theorem 1 [Strict ARMA$(1, q)$ processes] *Let $m, d \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1 \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ be complex-valued matrices. Let $S \in \mathbb{C}^{m\times m}$ be an invertible matrix such that $S^{-1}\Psi_1 S$ is in Jordan block form as above, with H Jordan blocks Φ_h, $h \in \{1, \ldots, H\}$, and associated eigenvalues λ_h, $h \in \{1, \ldots, H\}$. Let $r_1, \ldots, r_{H+1}$ be given as above and I_h as defined by (7). Then the ARMA$(1, q)$ equation (3) has a strictly stationary solution Y if and only if the following statements* (i)–(iii) *hold.*

(i) *For every $h \in \{1, \ldots, H\}$ such that $|\lambda_h| \neq 0$ or 1,*

$$\mathbb{E}\log^+ \left\| \left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_0 \right\| < \infty \tag{10}$$

(ii) *For every $h \in \{1, \ldots, H\}$ such that $|\lambda_h| = 1$, but $\lambda_h \neq 1$, there exists a constant $\alpha_h \in \mathbb{C}^{r_{h+1}-r_h}$ such that*

$$\left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_0 = \alpha_h \quad \text{a.s.} \tag{11}$$

(iii) *For every $h \in \{1, \ldots, H\}$ such that $\lambda_h = 1$, there exists a constant $\alpha_h = (\alpha_{h,1}, \ldots, \alpha_{h,r_{h+1}-r_h})^{\mathrm{T}} \in \mathbb{C}^{r_{h+1}-r_h}$ such that $\alpha_{h,1} = 0$ and (11) holds.*

If these conditions are satisfied, then a strictly stationary solution of (3) *is given by* (9) *with*

$$X_t^{(h)} := \begin{cases} \sum_{j=0}^{\infty} \Phi_h^{j-q} \left(\sum_{k=0}^{j\wedge q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_{t-j}, & |\lambda_h| \in (0, 1), \\ -\sum_{j=1-q}^{\infty} \Phi_h^{-j-q} \left(\sum_{k=(1-j)\vee 0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_{t+j}, & |\lambda_h| > 1 \\ \sum_{j=0}^{m+q-1} \left(\sum_{k=0}^{j\wedge q} \Phi_h^{j-k} I_h S^{-1} \Theta_k \right) Z_{t-j}, & \lambda_h = 0, \\ f_h + \sum_{j=0}^{q-1} \left(\sum_{k=0}^{j} \Phi_h^{j-k} I_h S^{-1} \Theta_k \right) Z_{t-j}, & |\lambda_h| = 1, \end{cases} \tag{12}$$

where $f_h \in \mathbb{C}^{r_{h+1}-r_h}$ is a solution of

$$(\mathrm{Id}_h - \Phi_h) f_h = \alpha_h, \tag{13}$$

which exists for $\lambda_h = 1$ by (iii) *and, for $|\lambda| = 1, \lambda \neq 1$, by the invertibility of* $(\mathrm{Id}_h - \Phi_h)$. *The series in* (12) *converge a.s. absolutely.*

If the necessary and sufficient conditions stated above are satisfied, then, provided the underlying probability space is rich enough to support a random variable which is uniformly distributed on $[0, 1)$ *and independent of* $(Z_t)_{t\in\mathbb{Z}}$*, the solution given by* (9) *and* (12) *is the unique strictly stationary solution of* (3) *if and only if* $|\lambda_h| \neq 1$ *for all* $h \in \{1, \dots, H\}$.

Special cases of Theorem 1 are treated in Corollaries 1, 2 and Remark 1.

It is well-known that every ARMA(p, q) process can be embedded into a higher dimensional ARMA$(1, q)$ process in the sense specified by Proposition 1 of Sect. 5. Hence, in principle, the questions of existence and uniqueness of strictly stationary ARMA(p, q) processes can be resolved by Theorem 1. However, it is generally difficult to obtain the Jordan canonical decomposition of the $(mp \times mp)$-dimensional matrix $\underline{\Phi}$ defined in Proposition 1, which is needed to apply Theorem 1. Hence, a more natural approach is to express the conditions in terms of the coefficient polynomials $P(z)$ and $Q(z)$ of the ARMA(p, q) equation (1). Observe that $z \mapsto \det P(z)$ is a polynomial in $z \in \mathbb{C}$, not identical to the zero polynomial. Hence, $P(z)$ is invertible except for finitely many values of z. Denoting the adjugate matrix of $P(z)$ by $\operatorname{Adj}(P(z))$, it follows from Cramér's inversion rule that the inverse $P^{-1}(z)$ of $P(z)$ can be written as

$$P^{-1}(z) = (\det P(z))^{-1} \operatorname{Adj}(P(z))$$

which is a $\mathbb{C}^{m\times m}$-valued rational function, i.e., all its entries are rational functions. For a general matrix-valued rational function $z \mapsto M(z)$ of the form $M(z) = P^{-1}(z)\widetilde{Q}(z)$ with some matrix polynomial $\widetilde{Q}(z)$, the *singularities* of $M(z)$ are the zeroes of $\det P(z)$, and any singularity, z_0 say, is *removable* if all entries of $M(z)$ have removable singularities at z_0. Also, observe that if $M(z)$ has only removable singularities on the unit circle in $\mathbb{C}$, then $M(z)$ can be expanded in a Laurent series $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$, convergent in a neighborhood of the unit circle. The characterization for the existence of strictly stationary ARMA(p, q) processes now reads as follows.

Theorem 2 [Strict ARMA(p, q) processes] *Let* $m, d, p \in \mathbb{N}$, $q \in \mathbb{N}_0$, *and let* $(Z_t)_{t\in\mathbb{Z}}$ *be an i.i.d. sequence of* $\mathbb{C}^d$*-valued random vectors. Let* $\Psi_1, \dots, \Psi_p \in \mathbb{C}^{m\times m}$ *and* $\Theta_0, \dots, \Theta_q \in \mathbb{C}^{m\times d}$ *be complex-valued matrices, and define the coefficient polynomials as in* (2)*. Define the linear subspace of* $\mathbb{C}^d$,

$$K := \{a \in \mathbb{C}^d : \textit{the distribution of } a^* Z_0 \textit{ is degenerate to a Dirac measure}\},$$

denote by $K^\perp$ *its orthogonal complement in* $\mathbb{C}^d$*, and let* s *denotes the vector–space dimension* $\dim K^\perp$ *of* $K^\perp$*. Let* $U \in \mathbb{C}^{d\times d}$ *be a unitary matrix such that* $U K^\perp = \mathbb{C}^s \times \{0_{d-s}\}$ *and* $U K = \{0_s\} \times \mathbb{C}^{d-s}$*, and define the* $\mathbb{C}^{m\times d}$*-valued rational function* $M(z)$ *by*

$$z \mapsto M(z) := P^{-1}(z) Q(z) U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}. \tag{14}$$

Then there is a constant $u \in \mathbb{C}^{d-s}$ *and a* $\mathbb{C}^s$*-valued i.i.d. sequence* $(w_t)_{t\in\mathbb{Z}}$ *such that*

$$UZ_t = \begin{pmatrix} w_t \\ u \end{pmatrix} \quad a.s. \quad \forall\, t \in \mathbb{Z}, \tag{15}$$

and the distribution of $b^* w_0$ *is not degenerate to a Dirac measure for any* $b \in \mathbb{C}^s \setminus \{0\}$*. Moreover, a strictly stationary solution of the ARMA*(p, q) *equation* (1) *exists if and only if the following statements* (i)–(iii) *hold.*

(i) *All singularities on the unit circle of the meromorphic function* $M(z)$ *are removable.*
(ii) *If* $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ *denotes the Laurent expansion of* M *in a neighbourhood of the unit circle, then*

$$\mathbb{E}\log^+ \|M_j U Z_0\| < \infty \qquad \forall\, j \in \{mp + q - p + 1, \ldots, mp + q\} \cup \{-p, \ldots, -1\}. \tag{16}$$

(iii) *There exist* $v \in \mathbb{C}^s$ *and* $g \in \mathbb{C}^m$ *such that* g *is a solution of the linear equation*

$$P(1)g = Q(1)U^*(v^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}. \tag{17}$$

Further, if (i) *above holds, then condition* (ii) *can be replaced by*

(ii′) *If* $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ *denotes the Laurent expansion of* M *in a neighbourhood of the unit circle, then* $\sum_{j=-\infty}^{\infty} M_j U Z_{t-j}$ *converges almost surely absolutely for every* $t \in \mathbb{Z}$*,*

and condition (iii) *can be replaced by*

(*iii′*) *For all* $v \in \mathbb{C}^s$ *there exists a solution* $g = g(v)$ *to the linear equation* (17).

If the conditions (i)–(iii) *given above are satisfied, then a strictly stationary solution* Y *of the ARMA*(p, q) *equation* (1) *is given by*

$$Y_t = g + \sum_{j=-\infty}^{\infty} M_j (UZ_{t-j} - (v^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}), \quad t \in \mathbb{Z}, \tag{18}$$

the series converging almost surely absolutely. Further, provided that the underlying probability space is rich enough to support a random variable which is uniformly distributed on $[0, 1)$ *and independent of* $(Z_t)_{t\in\mathbb{Z}}$*, the solution given by* (18) *is the unique strictly stationary solution of* (1) *if and only if* $\det P(z) \neq 0$ *for all* z *on the unit circle.*

Special cases of Theorem 2 are treated in Remarks 2, 3 and Corollary 3. Observe that for $m = 1$, Theorem 2 reduces to the corresponding result in Brockwell and Lindner (2010). Also, observe that condition (iii) of Theorem 2 is not implied by condition (i), which can be seen, e.g., by allowing a deterministic noise sequence $(Z_t)_{t\in\mathbb{Z}}$, in which case $M(z) \equiv 0$. The proof of Theorem 2 will be given in Sect. 5 and will make use of both Theorems 1 and 3 given below. The latter is the corresponding characterization for the existence of weakly stationary solutions of ARMA(p, q) equations, expressed in terms of the coefficient polynomials $P(z)$ and $Q(z)$. That $\det P(z) \neq 0$ for all z on the unit circle together with $\mathbb{E}(Z_0) = 0$ is sufficient for the existence of weakly stationary solutions is well-known, but that the conditions given below are necessary and sufficient in higher dimensions seems not to have appeared in the literature so far. The proof of Theorem 3, which is similar to the proof in the one-dimensional case, will be given in Sect. 4.

Theorem 3 [Weak ARMA(p, q) processes] *Let $m, d, p \in \mathbb{N}, q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be a weak white noise sequence in $\mathbb{C}^d$ with expectation $\mathbb{E}Z_0$ and covariance matrix Σ. Let $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$, and define the matrix polynomials $P(z)$ and $Q(z)$ by (2). Let $U \in \mathbb{C}^{d\times d}$ be unitary such that $U\Sigma U^* = \begin{pmatrix} D & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$, where D is a real $(s \times s)$-diagonal matrix with the strictly positive eigenvalues of Σ on its diagonal for some $s \in \{0, \ldots, d\}$. (The matrix U exists since Σ is positive semidefinite.) Then the ARMA(p, q) equation (1) admits a weakly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if the $\mathbb{C}^{m\times d}$-valued rational function*

$$z \mapsto M(z) := P^{-1}(z)Q(z)U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$$

has only removable singularities on the unit circle and if there is some $g \in \mathbb{C}^m$ such that

$$P(1)\, g = Q(1)\, \mathbb{E}Z_0. \tag{19}$$

In that case, a weakly stationary solution of (1) is given by

$$Y_t = g + \sum_{j=-\infty}^{\infty} M_j\, U(Z_{t-j} - \mathbb{E}Z_0), \quad t \in \mathbb{Z}, \tag{20}$$

where $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ is the Laurent expansion of $M(z)$ in a neighbourhood of the unit circle, which converges absolutely there.

It is easy to see that if Σ in the theorem above is invertible, then the condition that all singularities of $M(z)$ on the unit circle are removable is equivalent to the condition that all singularities of $P^{-1}(z)Q(z)$ on the unit circle are removable.

3 Proof of Theorem 1

In this section we give the proof of Theorem 1. In Sect. 3.1, we show that the conditions (i)–(iii) are necessary. The sufficiency of the conditions is proved in Sect. 3.2 and the uniqueness assertion is established in Sect. 3.3.

3.1 The necessity of the conditions

Assume that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of Eq. (3). As observed before, Theorem 1 implies that each of the equations (8) admits a strictly stationary solution, where $X_t^{(h)}$ is defined as in (6). Equation (8) is itself an ARMA$(1,q)$ equation with i.i.d. noise, so that for proving (i)–(iii) we may assume that $H=1$, that $S=\mathrm{Id}_m$ and that $\Phi := \Psi_1$ is an $m\times m$ Jordan block corresponding to an eigenvalue λ. Hence, we assume throughout Sect. 3.1 that

$$Y_t - \Phi Y_{t-1} = \sum_{k=0}^{q} \Theta_k Z_{t-k}, \quad t\in\mathbb{Z}, \tag{21}$$

has a strictly stationary solution with $\Phi\in\mathbb{C}^{m\times m}$ of the form (4), and we need to show that this implies (i) if $|\lambda|\neq 0,1$, (ii) if $|\lambda|=1$ but $\lambda\neq 1$, and (iii) if $\lambda=1$. Before we do this in the following subsections, we observe that iteration of the ARMA$(1,q)$ equation (21) gives, for $n\geq q$,

$$\begin{aligned} Y_t &= \sum_{j=0}^{q-1}\Phi^j\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{t-j} + \sum_{j=q}^{n-1}\Phi^j\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\\ &\quad + \sum_{j=0}^{q-1}\Phi^{n+j}\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{t-(n+j)} + \Phi^n Y_{t-n}. \end{aligned} \tag{22}$$

3.1.1 The case $|\lambda|\in(0,1)$

Suppose that $|\lambda|\in(0,1)$ and $\varepsilon\in(0,|\lambda|)$. Then there are constants $C, C'\geq 1$ such that

$$\|\Phi^{-j}\| \leq C\cdot|\lambda|^{-j}\cdot j^m \leq (C')(|\lambda|-\varepsilon)^{-j} \quad \text{for all} \quad j\in\mathbb{N},$$

as a consequence of Theorem 11.1.1 in Golub et al. (1996). Hence, we have for all $j\in\mathbb{N}_0$ and $t\in\mathbb{Z}$

$$\left\|\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\right\| \leq C'(|\lambda|-\varepsilon)^{-j}\left\|\Phi^j\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\right\|. \tag{23}$$

Now, since $\lim_{n\to\infty}\Phi^n=0$ and since $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ are strictly stationary, an application of Slutsky's lemma to Eq. (22) shows that

$$Y_t = \sum_{j=0}^{q-1}\Phi^j\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{t-j} + \mathbb{P}\text{-}\lim_{n\to\infty}\sum_{j=q}^{n-1}\Phi^j\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}. \tag{24}$$

Hence, the limit on the right hand side exists and, as a sum with independent summands, it converges almost surely. Thus, it follows from Eq. (23) and the Borel–Cantelli lemma that

$$\sum_{j=q}^{\infty} \mathbb{P}\left(\left\| \sum_{k=0}^{q} \Phi^{-k} \Theta_k Z_0 \right\| > C'(|\lambda| - \varepsilon)^{-j} \right)$$
$$\leq \sum_{j=q}^{\infty} \mathbb{P}\left(\left\| \Phi^{j} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{-j} \right\| > 1 \right) < \infty,$$

and hence $\mathbb{E}\left(\log^+ \left\| \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k\right) Z_0 \right\|\right) < \infty$. Obviously, this is equivalent to condition (i).

3.1.2 The case $|\lambda| > 1$

Suppose that $|\lambda| > 1$. Multiplying Eq. (22) by Φ^{-n} gives for $n \geq q$

$$\Phi^{-n} Y_t = \sum_{j=0}^{q-1} \Phi^{-(n-j)} \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} + \sum_{j=1}^{n-q} \Phi^{-j} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-n+j}$$
$$+ \sum_{j=0}^{q-1} \Phi^{j} \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} + Y_{t-n}.$$

Defining $\tilde{\Phi} := \Phi^{-1}$, and substituting $u = t - n$ yields

$$Y_u = -\sum_{j=0}^{q-1} \tilde{\Phi}^{-j} \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{u-j} - \sum_{j=1}^{n-q} \tilde{\Phi}^{j} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{u+j}$$
$$- \sum_{j=0}^{q-1} \tilde{\Phi}^{n-j} \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{u+n-j} + \tilde{\Phi}^{n} Y_{u+n}. \tag{25}$$

Letting $n \to \infty$ then gives condition (i) with the same arguments as in the case $|\lambda| \in (0, 1)$.

3.1.3 The case $|\lambda| = 1$ and symmetric noise (Z_t)

Suppose that Z_0 is symmetric and that $|\lambda| = 1$. Defining

$$J_1 := \Phi - \lambda \operatorname{Id}_m \quad \text{and} \quad J_l := J_1^l \quad \text{for} \quad j \in \mathbb{N}_0,$$

we have

$$\Phi^j = \sum_{l=0}^{m-1} \binom{j}{l} \lambda^{j-l} J_l, \quad j \in \mathbb{N}_0,$$

since $J_l = 0$ for $l \geq m$ and $\binom{j}{l} = 0$ for $l > j$. Further, since for $l \in \{0, \ldots, m-1\}$ we have

$$J_l = (e_{l+1}, e_{l+2}, \ldots, e_m, 0_m, \ldots, 0_m) \in \mathbb{C}^{m \times m},$$

with unit vectors $e_{l+1}, \ldots, e_m$ in $\mathbb{C}^m$, it is easy to see that for $i = 1, \ldots, m$ the ith row of the matrix Φ^j is given by

$$e_i^{\mathrm{T}} \Phi^j = \sum_{l=0}^{m-1} \binom{j}{l} \lambda^{j-l} e_i^{\mathrm{T}} J_l = \sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^{\mathrm{T}}, \quad j \in \mathbb{N}_0. \tag{26}$$

It follows from Eqs. (22) and (26) that for $n \geq q$ and $t \in \mathbb{Z}$,

$$\begin{aligned} e_i^{\mathrm{T}} Y_t = & \sum_{j=0}^{q-1} \left(\sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^{\mathrm{T}} \right) \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & + \sum_{j=q}^{n-1} \left(\sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^{\mathrm{T}} \right) \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & + \sum_{j=0}^{q-1} \left(\sum_{l=0}^{i-1} \binom{n+j}{l} \lambda^{n+j-l} e_{i-l}^{\mathrm{T}} \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & + \sum_{l=0}^{i-1} \binom{n}{l} \lambda^{n-l} e_{i-l}^{\mathrm{T}} Y_{t-n}. \end{aligned} \tag{27}$$

We claim that

$$e_i^{\mathrm{T}} \sum_{k=0}^{q} \Phi^{-k} \Theta_k Z_t = 0 \ \text{ a.s.} \quad \forall\, i \in \{1, \ldots, m\} \quad \forall\, t \in \mathbb{Z}, \tag{28}$$

which clearly gives conditions (ii) and (iii), respectively, with $\alpha = \alpha_1 = 0_m$. Equation (28) will be proved by induction on $i = 1, \ldots, m$. We start with $i = 1$. From equation (27) we know that for $n \geq q$

$$e_1^{\mathrm{T}} Y_t - \lambda^n e_1^{\mathrm{T}} Y_{t-n} - \sum_{j=0}^{q-1} \lambda^j e_1^{\mathrm{T}} \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j}$$
$$- \sum_{j=0}^{q-1} \lambda^{n+j} e_1^{\mathrm{T}} \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} = \sum_{j=q}^{n-1} \lambda^j e_1^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j}. \quad (29)$$

Due to the stationarity of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$, there exists a constant $K_1 > 0$ such that

$$\mathbb{P}\left(\left| e_1^{\mathrm{T}} Y_t - \lambda^n e_1^{\mathrm{T}} Y_{t-n} - \sum_{j=0}^{q-1} \lambda^j e_1^{\mathrm{T}} \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \right.\right.$$
$$\left.\left. - \sum_{j=0}^{q-1} \lambda^{n+j} e_1^{\mathrm{T}} \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \right| < K_1 \right) \geq \frac{1}{2} \quad \forall n \geq q.$$

By (29) this implies that

$$\mathbb{P}\left(\left| \sum_{j=q}^{n-1} \lambda^j e_1^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \right| < K_1 \right) \geq \frac{1}{2} \quad \forall n \geq q. \quad (30)$$

Therefore, $|\sum_{j=q}^{n-1} \lambda^j e_1^{\mathrm{T}} (\sum_{k=0}^{q} \Phi^{-k} \Theta_k) Z_{t-j}|$ does not converge in probability to $+\infty$ as $n \to \infty$. Since this is a sum of independent and symmetric terms, this implies that it converges almost surely (Kallenberg 2002, Theorem 4.17), and the Borel-Cantelli lemma then shows that

$$e_1^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_t = 0, \quad t \in \mathbb{Z},$$

which is (28) for $i = 1$. With this condition, equation (29) simplifies for $t = 0$ and $n \geq q$ to

$$e_1^{\mathrm{T}} Y_0 - \lambda^n e_1^{\mathrm{T}} Y_{-n} = \sum_{j=0}^{q-1} \lambda^j e_1^{\mathrm{T}} \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{-j}$$
$$+ \sum_{j=0}^{q-1} \lambda^{n+j} e_1^{\mathrm{T}} \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{-(n+j)}.$$

Now setting $t = -n$ in the above equation, multiplying it with $\lambda^t = \lambda^{-n}$ and recalling that $e_1^{\mathrm{T}} \Phi^j = \lambda^j e_1^{\mathrm{T}}$ by (26) yields for $t \leq -q$

$$
\begin{aligned}
e_1^{\mathrm{T}} Y_t = & -\sum_{j=0}^{q-1} e_1^{\mathrm{T}} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \\
& + \lambda^t e_1^{\mathrm{T}} \left(Y_0 - \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{-j} \right).
\end{aligned}
$$

For the induction step let $i \in \{2, \ldots, m\}$ and assume that

$$
e_r^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_t = 0 \text{ a.s.}, \quad r \in \{1, \ldots, i-1\}, \ t \in \mathbb{Z}, \tag{31}
$$

together with

$$
\begin{aligned}
e_r^{\mathrm{T}} Y_t = & -e_r^{\mathrm{T}} \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \\
& + \begin{cases} 0 & r \in \{1, \ldots, i-2\}, \ t \leq -rq, \\ \lambda^t e_r^{\mathrm{T}} V_r & r = i-1, \ t \leq -rq, \end{cases}
\end{aligned} \tag{32}
$$

where, for $r \in \{1, \ldots, m\}$,

$$
V_r := \lambda^{(r-1)q} \left(Y_{-(r-1)q} - \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{-j-(r-1)q} \right).
$$

We are going to show that this implies

$$
e_i^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_t = 0 \ \text{ a.s.}, \quad t \in \mathbb{Z}, \tag{33}
$$

and

$$
e_i^{\mathrm{T}} Y_t = -e_i^{\mathrm{T}} \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} + \lambda^t e_i^{\mathrm{T}} V_i \quad \text{a.s.}, \quad t \leq -iq, \tag{34}
$$

together with

$$
e_{i-1}^{\mathrm{T}} V_{i-1} = 0. \tag{35}
$$

This will then imply (28). The first step is to establish the following Lemma.

Lemma 1 *Suppose that $i \in \{2, \ldots, m\}$ and that (31) and (32) hold. Then for $t \leq -(i-1)q$ and $n \geq q$,*

$$\begin{aligned} & e_i^{\mathrm{T}} Y_t - \lambda^n e_i^{\mathrm{T}} Y_{t-n} \\ & = \sum_{j=0}^{q-1} e_i^{\mathrm{T}} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} + \sum_{j=q}^{n-1} \lambda^j e_i^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & \quad + \lambda^n \sum_{j=0}^{q-1} e_i^{\mathrm{T}} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} + n \lambda^{t-1} e_{i-1}^{\mathrm{T}} V_{i-1}, \end{aligned} \tag{36}$$

Proof Assuming that $t \leq -(i-1)q$ and $n \geq q$ and using (32) and (26), we see that the last summand of (27) satisfies

$$\begin{aligned} & \sum_{l=0}^{i-1} \binom{n}{l} \lambda^{n-l} e_{i-l}^{\mathrm{T}} Y_{t-n} - \lambda^n e_i^{\mathrm{T}} Y_{t-n} \\ & = \sum_{r=1}^{i-1} \binom{n}{i-r} \lambda^{n-(i-r)} e_r^{\mathrm{T}} Y_{t-n}, \\ & = - \sum_{j=0}^{q-1} \left(\sum_{r=1}^{i-1} \sum_{l=0}^{r-1} \binom{j}{l} \binom{n}{i-r} \lambda^{n-(i-r)} \lambda^{j-l} e_{r-l}^{\mathrm{T}} \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & \quad + n \lambda^{t-1} e_{i-1}^{\mathrm{T}} V_{i-1} \\ & = - \sum_{j=0}^{q-1} \left(\sum_{s=1}^{i-1} \binom{n+j}{s} \lambda^{n+j-s} e_{i-s}^{\mathrm{T}} \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & \quad + \lambda^n \sum_{j=0}^{q-1} \left(\sum_{s=1}^{i-1} \binom{j}{s} \lambda^{j-s} e_{i-s}^{\mathrm{T}} \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} + n \lambda^{t-1} e_{i-1}^{\mathrm{T}} V_{i-1}, \end{aligned}$$

where we substituted $s := i - r + l$ and $p := s - l$ and used Vandermonde's identity $\sum_{p=1}^{s} \binom{j}{s-p} \binom{n}{p} = \binom{n+j}{s} - \binom{j}{s}$ in the last equation. Inserting this back into equation (27) and using (31), we get for $t \leq -(i-1)q$ and $n \geq q$

$$\begin{aligned} & e_i^{\mathrm{T}} Y_t - \lambda^n e_i^{\mathrm{T}} Y_{t-n} \\ & = \sum_{j=0}^{q-1} \left(\sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^{\mathrm{T}} \right) \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & \quad + \sum_{j=q}^{n-1} \lambda^j e_i^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} + \sum_{j=0}^{q-1} \lambda^{n+j} e_i^{\mathrm{T}} \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \end{aligned}$$

$$+\lambda^n \sum_{j=0}^{q-1} \left(\sum_{s=1}^{i-1} \binom{j}{s} \lambda^{j-s} e_{i-s}^{\mathrm{T}} \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)}$$
$$+n\lambda^{t-1} e_{i-1}^{\mathrm{T}} V_{i-1}.$$

Application of (26) then shows (36), completing the proof of the lemma. □

To continue with the induction step, we first show that (35) holds. Dividing (36) by n and letting $n \to \infty$, the strict stationarity of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ implies that for $t \le -(i-1)q$,

$$n^{-1} \sum_{j=q}^{n-1} \lambda^j e_i^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j}$$

converges in probability to $-\lambda^{t-1} e_{i-1}^{\mathrm{T}} V_{i-1}$. On the other hand, this limit in probability must clearly be measurable with respect to the tail-σ-algebra $\cap_{k\in\mathbb{N}} \sigma(\cup_{l\ge k} \sigma(Z_{t-l}))$, which by Kolmogorov's zero-one law is $\mathbb{P}$-trivial. Hence, this probability limit must be constant, and because of the assumed symmetry of Z_0 it must be symmetric, hence is equal to 0, i.e.,

$$e_{i-1}^{\mathrm{T}} V_{i-1} = 0 \quad \text{a.s.},$$

which is (35). Using this, we get from Lemma 1 that

$$\begin{aligned} e_i^{\mathrm{T}} Y_t - \lambda^n e_i^{\mathrm{T}} Y_{t-n} - \sum_{j=0}^{q-1} e_i^{\mathrm{T}} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ -\lambda^n \sum_{j=0}^{q-1} e_i^{\mathrm{T}} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ = \sum_{j=q}^{n-1} \lambda^j e_i^{\mathrm{T}} \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j}, \quad t \le -(i-1)q. \end{aligned} \tag{37}$$

Again due to the stationarity of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ there exists a constant $K_2 > 0$ such that

$$\begin{aligned} \mathbb{P} \Bigg(\Bigg| e_i^{\mathrm{T}} Y_t - \lambda^n e_i^{\mathrm{T}} Y_{t-n} - \sum_{j=0}^{q-1} e_i^{\mathrm{T}} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ -\lambda^n \sum_{j=0}^{q-1} e_i^{\mathrm{T}} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \Bigg| < K_2 \Bigg) \ge \frac{1}{2} \quad \forall\, n \ge q, \end{aligned}$$

so that

$$\mathbb{P}\left(\left|\sum_{j=q}^{n-1}\lambda^j e_i^{\mathrm{T}}\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\right| < K_2\right) \geq \frac{1}{2} \quad \forall\, n \geq q,\ t \leq -(i-1)q.$$

Therefore $\left|\sum_{j=q}^{n-1}\lambda^j e_i^{\mathrm{T}}\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\right|$ does not converge in probability to $+\infty$ as $n \to \infty$. Since this is a sum of independent and symmetric terms, this implies that it converges almost surely (Kallenberg 2002, Theorem 4.17), and the Borel–Cantelli lemma then shows that $e_i^{\mathrm{T}}\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_t = 0$ a.s. for $t \leq -(i-1)q$ and hence for all $t \in \mathbb{Z}$, which is (33). Equation (37) now simplifies for $t = -(i-1)q$ and $n \geq q$ to

$$\begin{aligned} e_i^{\mathrm{T}} Y_{-(i-1)q} - \lambda^n e_i^{\mathrm{T}} Y_{-(i-1)q-n} &= \sum_{j=0}^{q-1} e_i^{\mathrm{T}}\Phi^j\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{-(i-1)q-j} \\ &\quad + \lambda^n\sum_{j=0}^{q-1} e_i^{\mathrm{T}}\Phi^j\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{-(i-1)q-n-j}. \end{aligned}$$

Multiplying this equation by λ^{-n} and denoting $t := -(i-1)q - n$, it follows that for $t \leq -iq$ it holds

$$\begin{aligned} e_i^{\mathrm{T}} Y_t &= -\sum_{j=0}^{q-1} e_i^{\mathrm{T}}\Phi^j\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j} \\ &\quad + \lambda^{t+(i-1)q} e_i^{\mathrm{T}}\left(Y_{-(i-1)q} - \sum_{j=0}^{q-1}\Phi^j\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{-j-(i-1)q}\right) \\ &= -\sum_{j=0}^{q-1} e_i^{\mathrm{T}}\Phi^j\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j} + \lambda^t e_i^{\mathrm{T}} V_i, \end{aligned}$$

which is Eq. (34). This completes the proof of the induction step and hence of (28). It follows that conditions (ii) and (iii), respectively, hold with $\alpha_1 = 0$ if $|\lambda| = 1$ and Z_0 is symmetric.

3.1.4 The case $|\lambda| = 1$ with not necessarily symmetric noise (Z_t)

As in Sect. 3.1.3, assume that $|\lambda| = 1$, but not necessarily that Z_0 is symmetric. Let $(Y_t', Z_t')_{t\in\mathbb{Z}}$ be an independent copy of $(Y_t, Z_t)_{t\in\mathbb{Z}}$ and denote $\widetilde{Y}_t := Y_t - Y_t'$ and $\widetilde{Z}_t := Z_t - Z_t'$. Then $(\widetilde{Y}_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of $\widetilde{Y}_t - \Phi\widetilde{Y}_{t-1} = \sum_{k=0}^{q}\Theta_k\widetilde{Z}_{t-k}$, and $(\widetilde{Z}_t)_{t\in\mathbb{Z}}$ is i.i.d. with $\widetilde{Z}_0$ being symmetric. It hence follows from Sect. 3.1.3 that

$$\left(\sum_{k=0}^{q}\Phi^{q-k}\Theta_k\right)Z_0-\left(\sum_{k=0}^{q}\Phi^{q-k}\Theta_k\right)Z_0'=\left(\sum_{k=0}^{q}\Phi^{q-k}\Theta_k\right)\widetilde{Z}_0=0.$$

Since Z_0 and Z_0' are independent, this implies that there is a constant $\alpha\in\mathbb{C}^m$ such that $\sum_{k=0}^{q}\Phi^{q-k}\Theta_k Z_0=\alpha$ a.s., which is (11), hence condition (ii) if $\lambda\neq 1$. To show condition (iii) in the case $\lambda=1$, recall that the derivation of (30) in Sect. 3.1.3 did not need the symmetry assumption on Z_0. Hence, by (30) there is some constant K_1 such that $\mathbb{P}(|\sum_{j=q}^{n-1}1^j e_1^{\mathrm{T}}\alpha|<K_1)\geq 1/2$ for all $n\geq q$, which clearly implies $e_1^{\mathrm{T}}\alpha=0$ and hence condition (iii).

3.2 The sufficiency of the conditions

Suppose that conditions (i)–(iii) are satisfied, and let $X_t^{(h)}$, $t\in\mathbb{Z}$, $h\in\{1,\ldots,H\}$, be defined by (12). The fact that $X_t^{(h)}$ as defined in (12) converges a.s. for $|\lambda_h|\in(0,1)$ is in complete analogy to the proof in the one-dimensional case treated in Brockwell and Lindner (2010), but we give the short argument here for completeness. Observe that there are constants $a,b>0$ such that $\|\Phi_h^j\|\leq ae^{-bj}$ for $j\in\mathbb{N}_0$. Hence, for $b'\in(0,b)$ we have

$$\begin{aligned}&\sum_{j=q}^{\infty}\mathbb{P}\left(\left\|\Phi_h^{j-q}\sum_{k=0}^{q}\Phi_h^{q-k}I_hS^{-1}\Theta_kZ_{t-j}\right\|>e^{-b'(j-q)}\right)\\&\leq\sum_{j=q}^{\infty}\mathbb{P}\left(\log^+\left(a\left\|\sum_{k=0}^{q}\Phi_h^{q-k}I_hS^{-1}\Theta_kZ_{t-j}\right\|\right)>(b-b')(j-q)\right)<\infty,\end{aligned}$$

the last inequality being due to the fact that $\|\sum_{k=0}^{q}\Phi_h^{q-k}I_hS^{-1}\Theta_kZ_{t-j}\|$ has the same distribution as $\|\sum_{k=0}^{q}\Phi_h^{q-k}I_hS^{-1}\Theta_kZ_0\|$ and the latter has finite log-moment by (10). The Borel–Cantelli lemma then shows that the event $\{\|\Phi_h^{j-q}\sum_{k=0}^{q}\Phi_h^{q-k}I_hS^{-1}\Theta_k Z_{t-j}\|>e^{-b'(j-q)}$ for infinitely many $j\}$ has probability zero, giving the almost sure absolute convergence of the series in (12). The almost sure absolute convergence of (12) if $|\lambda_h|>1$ is established similarly.

It is obvious that $((X_t^{(1)\mathrm{T}},\ldots,X_t^{(H)\mathrm{T}})^{\mathrm{T}})_{t\in\mathbb{Z}}$ as defined in (12), and hence $(Y_t)_{t\in\mathbb{Z}}$ defined by (9), is strictly stationary, so it only remains to show that $(X_t^{(h)})_{t\in\mathbb{Z}}$ satisfies (8) for each $h\in\{1,\ldots,H\}$. For $|\lambda_h|\neq 0,1$, this is an immediate consequence of (12). For $|\lambda_h|=1$, we have by (12) and the definition of f_h that

$$\begin{aligned}X_t^{(h)}-\Phi_hX_{t-1}^{(h)}&=\alpha_h+\sum_{j=0}^{q-1}\sum_{k=0}^{j}\Phi_h^{j-k}I_hS^{-1}\Theta_kZ_{t-j}-\sum_{j=1}^{q}\sum_{k=0}^{j-1}\Phi_h^{j-k}I_hS^{-1}\Theta_kZ_{t-j}\\&=\alpha_h+\sum_{j=0}^{q-1}I_hS^{-1}\Theta_jZ_{t-j}-\sum_{k=0}^{q-1}\Phi_h^{q-k}I_hS^{-1}\Theta_kZ_{t-q}\end{aligned}$$

$$= I_h S^{-1} \sum_{j=0}^{q} \Theta_j Z_{t-j},$$

where the last equality follows from (11). Finally, if $\lambda_h = 0$, then $\Phi_h^j = 0$ for $j \geq m$, implying that $X_t^{(h)}$ defined by (12) satisfies (8) also in this case.

3.3 The uniqueness of the solution

Suppose that $|\lambda_h| \neq 1$ for all $h \in \{1, \ldots, H\}$ and let $(Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary solution of (3). Then $(X_t^{(h)})_{t\in\mathbb{Z}}$ as defined by (6) is a strictly stationary solution of (8) for each $h \in \{1, \ldots, H\}$. It then follows as in Sect. 3.1.1 that by the equation corresponding to (24), $X_t^{(h)}$ is uniquely determined if $|\lambda_h| \in (0, 1)$. Similarly, $X_t^{(h)}$ is uniquely determined if $|\lambda_h| > 1$. The uniqueness of $X_t^{(h)}$ if $\lambda_h = 0$ follows from the equation corresponding to (22) with $n \geq m$, since then $\Phi_h^j = 0$ for $j \geq m$. We conclude that $((X_t^{(1)\mathrm{T}}, \ldots, X_t^{(H)\mathrm{T}})^{\mathrm{T}})_{t\in\mathbb{Z}}$ is unique and hence so is $(Y_t)_{t\in\mathbb{Z}}$.

Now suppose that there is $h \in \{1, \ldots, H\}$ such that $|\lambda_h| = 1$. Let U be a random variable which is uniformly distributed on $[0, 1)$ and independent of $(Z_t)_{t\in\mathbb{Z}}$. Then $(R_t)_{t\in\mathbb{Z}}$, defined by $R_t := \lambda_h^t(0, \ldots 0, e^{2\pi i U})^{\mathrm{T}} \in \mathbb{C}^{r_{h+1}-r_h}$, is strictly stationary and independent of $(Z_t)_{t\in\mathbb{Z}}$ and satisfies $R_t - \Phi_h R_{t-1} = 0$. Hence, if $(Y_t)_{t\in\mathbb{Z}}$ is the strictly stationary solution of (3) specified by (12) and (9), then

$$Y_t + S(0_{r_2-r_1}^{\mathrm{T}}, \ldots, 0_{r_h-r_{h-1}}^{\mathrm{T}}, R_t^{\mathrm{T}}, 0_{r_{h+2}-r_{h+1}}^{\mathrm{T}}, \ldots, 0_{r_{H+1}-r_H}^{\mathrm{T}})^{\mathrm{T}}, \quad t \in \mathbb{Z},$$

is another strictly stationary solution of (3), violating uniqueness.

4 Proof of Theorem 3

In this section, we shall prove Theorem 3. Define

$$R := U^* \begin{pmatrix} D^{1/2} & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} \text{ and } W_t := \begin{pmatrix} D^{-1/2} & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} U(Z_t - \mathbb{E}Z_0), \quad t \in \mathbb{Z},$$

where $D^{1/2}$ is the unique diagonal matrix with strictly positive eigenvalues such that $(D^{1/2})^2 = D$. Then $(W_t)_{t\in\mathbb{Z}}$ is a white noise sequence in $\mathbb{C}^d$ with expectation 0 and covariance matrix $\begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$. It is also clear that all singularities of $M(z)$ on the unit circle are removable if and only if all singularities of $M'(z) := P^{-1}(z)Q(z)R$ on the unit circle are removable, and in that case, the Laurent expansions of both $M(z)$ and $M'(z)$ converge absolutely in a neighbourhood of the unit circle.

To see the sufficiency of the condition, suppose that (19) has a solution g and that $M(z)$ and hence $M'(z)$ have only removable singularities on the unit circle. Define $Y = (Y_t)_{t\in\mathbb{Z}}$ by (20), i.e.,

$$Y_t = g + \sum_{j=-\infty}^{\infty} M_j \begin{pmatrix} D^{1/2} & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} W_{t-j} = g + M'(B)W_t, \quad t \in \mathbb{Z}.$$

The series converges almost surely absolutely due to the exponential decrease of the entries of M_j as $|j| \to \infty$. Further, Y is clearly weakly stationary, and since the last $(d-s)$ components of $U(Z_t - \mathbb{E}Z_0)$ vanish, having expectation zero and variance zero, it follows that

$$\begin{aligned} RW_t &= U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} U(Z_t - \mathbb{E}Z_0) = U^*U(Z_t - \mathbb{E}Z_0) \\ &= Z_t - \mathbb{E}Z_0, \ t \in \mathbb{Z}. \end{aligned}$$

We conclude that

$$P(B)(Y_t - g) = P(B)M'(B)W_t = P(B)P^{-1}(B)Q(B)RW_t = Q(B)(Z_t - \mathbb{E}Z_0).$$

Since $P(1)g = Q(1)\mathbb{E}Z_0$, this shows that $(Y_t)_{t\in\mathbb{Z}}$ is a weakly stationary solution of (1).

Conversely, suppose that $Y = (Y_t)_{t\in\mathbb{Z}}$ is a weakly stationary solution of (1). Taking expectations in (1) yields $P(1)\,\mathbb{E}Y_0 = Q(1)\,\mathbb{E}Z_0$, so that (19) has a solution. The $\mathbb{C}^{m\times m}$-valued spectral measure μ_Y of Y satisfies

$$P(\mathrm{e}^{-\mathrm{i}\omega})\,\mathrm{d}\mu_Y(\omega)\,P(\mathrm{e}^{-\mathrm{i}\omega})^* = \frac{1}{2\pi} Q(\mathrm{e}^{-\mathrm{i}\omega})\Sigma Q(\mathrm{e}^{-\mathrm{i}\omega})^*\,\mathrm{d}\omega, \quad \omega \in (-\pi, \pi].$$

It follows that, with the finite set $N := \{\omega \in (-\pi, \pi] : P(\mathrm{e}^{-\mathrm{i}\omega}) = 0\}$,

$$\mathrm{d}\mu_Y(\omega) = \frac{1}{2\pi} P^{-1}(e^{-i\omega})Q(\mathrm{e}^{-\mathrm{i}\omega})\Sigma Q(e^{-i\omega})^* P^{-1}(\mathrm{e}^{-\mathrm{i}\omega})^*\,\mathrm{d}\omega \quad \text{on} \quad (-\pi, \pi] \setminus N.$$

Observing that $RR^* = \Sigma$, it follows that the function $\omega \mapsto M'(\mathrm{e}^{-\mathrm{i}\omega})M'(e^{-i\omega})^*$ must be integrable on $(-\pi, \pi] \setminus N$. Now assume that the matrix rational function M' has a non-removable singularity at z_0 with $|z_0| = 1$ in at least one matrix element. This must then be a pole of order $r \geq 1$. Denoting the spectral norm by $\|\cdot\|_2$ it follows that there are $\varepsilon > 0$ and $K > 0$ such that

$$\|M'(z)^*\|_2 \geq K|z - z_0|^{-1} \quad \forall\, z \in \mathbb{C} : |z| = 1, z \neq z_0, |z - z_0| \leq \varepsilon.$$

This may be seen by considering first the row sum norm of $M'(z)^*$ and then using the equivalence of norms. Since the matrix $M'(z)M'(z)^*$ is hermitian, we conclude that

$$\begin{aligned} \|M'(z)M'(z)^*\|_2 &= \sup_{v\in\mathbb{C}^n:|v|=1} |v^*M'(z)M'(z)^*v| \\ &= \sup_{v\in\mathbb{C}^n:|v|=1} |M'(z)^*v|^2 \geq K^2|z - z_0|^{-2} \end{aligned}$$

for all $z \neq z_0$ on the unit circle such that $|z - z_0| \leq \varepsilon$. But this implies that $\omega \mapsto M'(\mathrm{e}^{-\mathrm{i}\omega})M'(e^{-i\omega})^*$ cannot be integrable on $(-\pi, \pi] \setminus N$, giving the desired contradiction. This completes the proof of Theorem 3. □

5 Proof of Theorem 2

In this section, we shall prove Theorem 2. In order to do so we first observe that the existence of an ARMA(p, q) process is equivalent to the existence of a corresponding higher dimensional ARMA((1, q) process, specified in the following well-known proposition. We shall omit the elementary proof.

Proposition 1 *Let $m, d, p \in \mathbb{N}, q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ be complex-valued matrices. Define the matrices $\underline{\Phi} \in \mathbb{C}^{mp\times mp}$ and $\underline{\Theta}_k \in \mathbb{C}^{mp\times d}$, $k \in \{0, \ldots, q\}$, by*

$$\underline{\Phi} := \begin{pmatrix} \Psi_1 & \Psi_2 & \cdots & \Psi_{p-1} & \Psi_p \\ \mathrm{Id}_m & 0_{m,m} & \cdots & 0_{m,m} & 0_{m,m} \\ 0_{m,m} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0_{m,m} & \vdots \\ 0_{m,m} & \cdots & 0_{m,m} & \mathrm{Id}_m & 0_{m,m} \end{pmatrix} \quad \text{and} \quad \underline{\Theta}_k = \begin{pmatrix} \Theta_k \\ 0_{m,d} \\ \vdots \\ 0_{m,d} \end{pmatrix}. \tag{38}$$

Then the ARMA(p, q) equation (1) admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of m-dimensional random vectors Y_t if and only if the ARMA$(1, q)$ equation

$$\underline{Y}_t - \underline{\Phi}\, \underline{Y}_{t-1} = \underline{\Theta}_0 Z_t + \underline{\Theta}_1 Z_{t-1} + \cdots + \underline{\Theta}_q Z_{t-q}, \quad t \in \mathbb{Z}, \tag{39}$$

admits a strictly stationary solution $(\underline{Y}_t)_{t\in\mathbb{Z}}$ of mp-dimensional random vectors $\underline{Y}_t$. More precisely, if $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (1), then

$$(\underline{Y}_t)_{t\in\mathbb{Z}} := ((Y_t^{\mathrm{T}}, Y_{t-1}^{\mathrm{T}}, \ldots, Y_{t-(p-1)}^{\mathrm{T}})^{\mathrm{T}})_{t\in\mathbb{Z}} \tag{40}$$

is a strictly stationary solution of (39), and conversely, if the sequence $(\underline{Y}_t)_{t\in\mathbb{Z}} = ((Y_t^{(1)\mathrm{T}}, \ldots, Y_t^{(p)\mathrm{T}})^{\mathrm{T}})_{t\in\mathbb{Z}}$ with random components $Y_t^{(i)} \in \mathbb{C}^m$ is a strictly stationary solution of (39), then $(Y_t)_{t\in\mathbb{Z}} := (Y_t^{(1)})_{t\in\mathbb{Z}}$ is a strictly stationary solution of (1).

For the proof of Theorem 2 we need some notation: define $\underline{\Phi}$ and $\underline{\Theta}_k$ as in (38). Choose an invertible $\mathbb{C}^{mp\times mp}$ matrix $\underline{S}$ such that $\underline{S}^{-1}\underline{\Phi}\,\underline{S}$ is in Jordan canonical form, with H Jordan blocks $\underline{\Phi}_1, \ldots, \underline{\Phi}_H$, say, the hth Jordan block $\underline{\Phi}_h$ starting in row $\underline{r}_h$, with $\underline{r}_1 := 1 < \underline{r}_2 < \cdots < \underline{r}_H < mp + 1 =: \underline{r}_{H+1}$. Let $\underline{\lambda}_h$ be the eigenvalue associated with $\underline{\Phi}_h$, and, similar to (7), denote by $\underline{I}_h$ the $(\underline{r}_{h+1} - \underline{r}_h) \times mp$-matrix with components $\underline{I}_h(i, j) = 1$ if $j = i + \underline{r}_h - 1$ and $\underline{I}_h(i, j) = 0$ otherwise. For $h \in \{1, \ldots, H\}$ and $j \in \mathbb{Z}$ let

$$N_{j,h} := \begin{cases} \mathbf{1}_{j\geq 0}\underline{\Phi}_h^{j-q}\sum_{k=0}^{j\wedge q}\underline{\Phi}_h^{q-k}\underline{I}_h\underline{S}^{-1}\underline{\Theta}_k, & |\underline{\lambda}_h| \in (0,1), \\ -\mathbf{1}_{j\leq q-1}\underline{\Phi}_h^{j-q}\sum_{k=(1+j)\vee 0}^{q}\underline{\Phi}_h^{q-k}\underline{I}_h\underline{S}^{-1}\underline{\Theta}_k, & |\underline{\lambda}_h| > 1, \\ \mathbf{1}_{j\in\{0,\ldots,mp+q-1\}}\sum_{k=0}^{j\wedge q}\underline{\Phi}_h^{j-k}\underline{I}_h\underline{S}^{-1}\underline{\Theta}_k, & \underline{\lambda}_h = 0, \\ \mathbf{1}_{j\in\{0,\ldots,q-1\}}\sum_{k=0}^{j}\underline{\Phi}_h^{j-k}\underline{I}_h\underline{S}^{-1}\underline{\Theta}_k, & |\underline{\lambda}_h| = 1, \end{cases}$$

and

$$\underline{N}_j := \underline{S}^{-1}(N_{j,1}^{\mathrm{T}}, \ldots, N_{j,H}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{C}^{mp\times d}. \tag{41}$$

Further, let U and K be defined as in the statement of the theorem, and denote

$$W_t := UZ_t, \quad t \in \mathbb{Z}.$$

Then $(W_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence. Equation (15) is then an easy consequence of the fact that for $a \in \mathbb{C}^d$ the distribution of $a^*W_0 = (U^*a)^*Z_0$ is degenerate to a Dirac measure if and only if $U^*a \in K$, i.e., if $a \in UK = \{0_s\} \times \mathbb{C}^{d-s}$. Taking for a the ith unit vector in $\mathbb{C}^d$ for $i \in \{s+1, \ldots, d\}$, we see that W_t must be of the form $(w_t^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$ for some $u \in \mathbb{C}^{d-s}$, and taking $a = (b^{\mathrm{T}}, 0_{d-s}^{\mathrm{T}})^{\mathrm{T}}$ for $b \in \mathbb{C}^s$ we see that b^*w_0 is not degenerate to a Dirac measure for $b \neq 0_s$. The remaining proof of the necessity of the conditions, the sufficiency of the conditions and the stated uniqueness will be given in the next subsections.

5.1 The necessity of the conditions

Suppose that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (1). Define $\underline{Y}_t$ by (40). Then $(\underline{Y}_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (39) by Proposition 1. Hence, by Theorem 1, there is $\underline{f}' \in \mathbb{C}^{mp}$, such that $(\underline{Y}_t')_{t\in\mathbb{Z}}$, defined by

$$\underline{Y}_t' = \underline{f}' + \sum_{j=-\infty}^{\infty}\underline{N}_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{42}$$

is (possibly another) strictly stationary solution of

$$\underline{Y}_t' - \underline{\Phi}\,\underline{Y}_{t-1}' = \sum_{k=0}^{q}\underline{\Theta}_k Z_{t-k} = \sum_{k=0}^{q}\underline{\widetilde{\Theta}}_k W_{t-k}, \quad t \in \mathbb{Z},$$

where $\underline{\widetilde{\Theta}}_k := \underline{\Theta}_k U^*$. The sum in (42) converges almost surely absolutely. Now define $A_h \in \mathbb{C}^{(\underline{r}_{h+1}-\underline{r}_h)\times s}$ and $C_h \in \mathbb{C}^{(\underline{r}_{h+1}-\underline{r}_h)\times(d-s)}$ for $h \in \{1, \ldots, H\}$ such that $|\underline{\lambda}_h| = 1$ by

$$(A_h, C_h) := \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\widetilde{\Theta}}_k. \tag{43}$$

By conditions (ii) and (iii) of Theorem 1, for every such h with $|\underline{\lambda}_h| = 1$ there exists a vector $\underline{\alpha}_h = (\alpha_{h,1}, \dots, \alpha_{h,\underline{r}_{h+1}-\underline{r}_h})^{\mathrm{T}} \in \mathbb{C}^{\underline{r}_{h+1}-\underline{r}_h}$ such that

$$(A_h, C_h) W_0 = \underline{\alpha}_h \quad \text{a.s.}$$

with $\alpha_{h,1} = 0$ if $\underline{\lambda}_h = 1$. Since $W_0 = (w_0^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$, this implies $A_h w_0 = \underline{\alpha}_h - C_h u$, but since $b^* w_0$ is not degenerate to a Dirac measure for any $b \in \mathbb{C}^s \setminus \{0_s\}$, this gives $A_h = 0$ and hence $C_h u = \underline{\alpha}_h$ for $h \in \{1, \dots, H\}$ such that $|\underline{\lambda}_h| = 1$. Now let $v \in \mathbb{C}^s$ and $(W_t'')_{t\in\mathbb{Z}}$ be an i.i.d. $N\left(\begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}\right)$-distributed sequence, and let $Z_t'' := U^* W_t''$. Then

$$(A_h, C_h) W_0'' = C_h u = \underline{\alpha}_h \quad \text{a.s.} \quad \text{for} \quad h \in \{1, \dots, H\} : |\underline{\lambda}_h| = 1$$

and

$$\mathbb{E} \log^+ \left\| \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\widetilde{\Theta}}_k W_0'' \right\| < \infty \quad \text{for} \quad h \in \{1, \dots, H\} : |\underline{\lambda}_h| \neq 0, 1.$$

It then follows from Theorem 1 that there is a strictly stationary solution $\underline{Y}_t''$ of the ARMA$(1, q)$ equation $\underline{Y}_t'' - \underline{\Phi}\, \underline{Y}_{t-1}'' = \sum_{k=0}^{q} \underline{\widetilde{\Theta}}_k W_{t-k}'' = \sum_{k=0}^{q} \underline{\Theta}_k Z_{t-k}''$, which can be written in the form $\underline{Y}_t'' = \underline{f}'' + \sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}''$ for some $\underline{f}'' \in \mathbb{C}^{mp}$. In particular, $(\underline{Y}_t'')_{t\in\mathbb{Z}}$ is a Gaussian process. Again from Proposition 1 it follows that there is a Gaussian process $(Y_t'')_{t\in\mathbb{Z}}$ which is a strictly stationary solution of

$$Y_t'' - \sum_{k=1}^{p} \Psi_k Y_{t-k}'' = \sum_{k=0}^{q} \widetilde{\Theta}_k W_{t-k}'' = \sum_{k=0}^{q} \Theta_k Z_{t-k}'', \quad t \in \mathbb{Z}.$$

In particular, this solution is also weakly stationary. Hence, it follows from Theorem 3 that $z \mapsto M(z)$ has only removable singularities on the unit circle and that (17) has a solution $g \in \mathbb{C}^m$, since $\mathbb{E} Z_0'' = U^*(v^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$. Hence, we have established that (i) and (iii′), and hence (iii), of Theorem 2 are necessary conditions for a strictly stationary solution to exist.

To see the necessity of conditions (ii) and (ii′), we need the following lemma, which is interesting in itself since it expresses the Laurent coefficients of $M(z)$ in terms of the Jordan canonical decomposition of $\underline{\Phi}$.

Lemma 2 *With the notations of Theorem 2 and those introduced after Proposition 1, suppose that condition* (i) *of Theorem 2 holds, i.e., that $M(z)$ has only removable singularities on the unit circle. Denote by $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ the Laurent expansion of $M(z)$ in a neighborhood of the unit circle. Then*

$$\underline{M}_j := (M_j^{\mathrm{T}}, M_{j-1}^{\mathrm{T}}, \ldots, M_{j-p+1}^{\mathrm{T}})^{\mathrm{T}} = \underline{N}_j U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} \quad \forall\, j \in \mathbb{Z}. \tag{44}$$

In particular,

$$\underline{M}_j U Z_{t-j} = \underline{N}_j Z_{t-j} - \underline{N}_j U^* (0_s^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}} \quad \forall\, j, t \in \mathbb{Z}. \tag{45}$$

Proof Define $\Lambda := \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$, let $(Z_t')_{t\in\mathbb{Z}}$ be an i.i.d. $N(0_d, U^*\Lambda U)$-distributed noise sequence and define $Y_t' := \sum_{j=-\infty}^{\infty} M_j U Z_{t-j}'$. Then $(Y_t')_{t\in\mathbb{Z}}$ is a weakly and strictly stationary solution of $P(B)Y_t' = Q(B)Z_t'$ by Theorem 3, and the entries of M_j decrease geometrically as $|j| \to \infty$. By Proposition 1, the process $(\underline{Y}_t')_{t\in\mathbb{Z}}$ defined by $\underline{Y}_t' = (Y_t'^{\mathrm{T}}, Y_{t-1}'^{\mathrm{T}}, \ldots, Y_{t-p+1}'^{\mathrm{T}}) = \sum_{j=-\infty}^{\infty} \underline{M}_j U Z_{t-j}'$ is a strictly stationary solution of

$$\underline{Y}_t' - \underline{\Phi}\, \underline{Y}_{t-1}' = \sum_{j=0}^{q} \underline{\Theta}_j Z_{t-j}', \quad t \in \mathbb{Z}. \tag{46}$$

Denoting $\underline{\Theta}_j = 0_{mp,d}$ for $j \in \mathbb{Z} \setminus \{0, \ldots, q\}$, it follows that $\sum_{k=-\infty}^{\infty} (\underline{M}_k - \underline{\Phi}\, \underline{M}_{k-1}) U Z_{t-k}' = \sum_{k=-\infty}^{\infty} \underline{\Theta}_k Z_{t-k}'$, and multiplying this equation from the right by $Z_{t-j}'^{\mathrm{T}}$, taking expectations and observing that $M(z)\Lambda = M(z)$. We conclude that

$$(\underline{M}_j - \underline{\Phi}\, \underline{M}_{j-1}) U = (\underline{M}_j - \underline{\Phi}\, \underline{M}_{j-1}) \Lambda U = \underline{\Theta}_j U^* \Lambda U \quad \forall\, j \in \mathbb{Z}. \tag{47}$$

Next observe that since $(\underline{Y}_t')_{t\in\mathbb{Z}}$ is a strictly stationary solution of (46), it follows from Theorem 1 that $(\underline{Y}_t'')_{t\in\mathbb{Z}}$, defined by $\underline{Y}_t'' = \sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}'$, is also a strictly stationary solution of (46). By precisely the same argument as above it follows that

$$(\underline{N}_j - \underline{\Phi}\, \underline{N}_{j-1}) U^* \Lambda U = \underline{\Theta}_j U^* \Lambda U \quad \forall\, j \in \mathbb{Z}. \tag{48}$$

Now let $L_j := \underline{M}_j - \underline{N}_j U^* \Lambda$, $j \in \mathbb{Z}$. Then $L_j - \underline{\Phi} L_{j-1} = 0_{mp,d}$ from (47) and (48), and the entries of L_j decrease exponentially as $|j| \to \infty$ since so do the entries of $\underline{M}_j$ and $\underline{N}_j$. It follows that for $h \in \{1, \ldots, H\}$ and $j \in \mathbb{Z}$ we have

$$\begin{aligned} &\underline{I}_h \underline{S}^{-1} L_j - \underline{\Phi}_h \underline{I}_h \underline{S}^{-1} L_{j-1} \\ &= \underline{I}_h \left(\underline{S}^{-1} L_j - \begin{pmatrix} \underline{\Phi}_1 & & \\ & \ddots & \\ & & \underline{\Phi}_H \end{pmatrix} \underline{S}^{-1} L_{j-1} \right) = 0_{\underline{r}_{h+1} - \underline{r}_h, d}. \end{aligned} \tag{49}$$

Since $\underline{\Phi}_h$ is invertible for $h \in \{1, \ldots, H\}$ such that $\underline{\lambda}_h \neq 0$, this gives $\underline{I}_h \underline{S}^{-1} L_0 = \underline{\Phi}_h^{-j} \underline{I}_h \underline{S}^{-1} L_j$ for all $j \in \mathbb{Z}$ and $\underline{\lambda}_h \neq 0$. Since for $|\underline{\lambda}_h| \geq 1$, $\|\underline{\Phi}_h^{-j}\| \leq \kappa j^{mp}$ for all $j \in \mathbb{N}_0$ for some constant κ, it follows that $\|\underline{I}_h \underline{S}^{-1} L_0\| \leq \kappa j^{mp} \|\underline{I}_h \underline{S}^{-1} L_j\|$, which converges to 0 as $j \to \infty$ by the geometric decrease of the coefficients of L_j as $j \to \infty$, so that $\underline{I}_h \underline{S}^{-1} L_k = 0$ for $|\underline{\lambda}_h| \geq 1$ and $k = 0$ and hence for all $k \in \mathbb{Z}$. Similarly, letting $j \to -\infty$, it follows that $\underline{I}_h \underline{S}^{-1} L_k = 0$ for $|\underline{\lambda}_h| \in (0, 1)$ and $k = 0$ and hence for all $k \in \mathbb{Z}$. Finally, for $h \in \{1, \dots, H\}$ such that $\underline{\lambda}_h = 0$ observe that $\underline{I}_h \underline{S}^{-1} L_k = \underline{\Phi}_h^{mp} \underline{I}_h \underline{S}^{-1} L_{k-mp}$ for $k \in \mathbb{Z}$ by (49), and since $\underline{\Phi}_h^{mp} = 0$, this shows that $\underline{I}_h \underline{S}^{-1} L_k = 0$ for $k \in \mathbb{Z}$. Summing up, we have $\underline{S}^{-1} L_k = 0$ and hence $\underline{M}_k = \underline{N}_k U^* \Lambda$ for $k \in \mathbb{Z}$, which is (44). Equation (45) then follows from (15), since

$$\underline{M}_j U Z_{t-j} = \underline{M}_j \begin{pmatrix} w_{t-j} \\ u \end{pmatrix} = \underline{N}_j U^* \begin{pmatrix} w_{t-j} \\ 0_{d-s} \end{pmatrix} = \underline{N}_j U^* \left(U Z_{t-j} - \begin{pmatrix} 0 \\ u \end{pmatrix} \right).$$

□

Returning to the proof of the necessity of conditions (ii) and (ii′) for a strictly stationary solution to exist, observe that $\sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}$ converges almost surely absolutely by (42), and since the entries of $\underline{N}_j$ decrease geometrically as $|j| \to \infty$, this together with (45) implies that $\sum_{j=-\infty}^{\infty} \underline{M}_j U Z_{t-j}$ converges almost surely absolutely, which shows that (ii′) must hold. To see (ii), observe that for $j \geq mp + q$ we have

$$N_{j,h} = \begin{cases} \underline{\Phi}_h^{j-q} \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| \in (0, 1), \\ 0, & |\underline{\lambda}_h| \notin (0, 1), \end{cases}$$

while

$$N_{-1,h} = \begin{cases} \underline{\Phi}_h^{-1-q} \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| > 1, \\ 0, & |\underline{\lambda}_h| \leq 1. \end{cases}$$

Since a strictly stationary solution of (39) exists, it follows from Theorem 1 that $\mathbb{E} \log^+ \|\underline{N}_j Z_0\| < \infty$ for $j \geq mp + q$ and $\mathbb{E} \log^+ \|\underline{N}_{-1} Z_0\| < \infty$. Together with (45) this shows that condition (ii) of Theorem 2 is necessary.

5.2 The sufficiency of the conditions and uniqueness of the solution

In this subsection we shall show that (i), (ii), (iii) as well as (i), (ii′), (iii) of Theorem 2 are sufficient conditions for a strictly stationary solution of (1) to exist, and prove the uniqueness assertion.

(a) Assume that conditions (i), (ii) and (iii) hold for some $v \in \mathbb{C}^s$ and $g \in \mathbb{C}^m$. Then $\mathbb{E} \log^+ \|\underline{N}_{-1} Z_0\| < \infty$ and $\mathbb{E} \log^+ \|\underline{N}_{mp+q} Z_0\| < \infty$ by (ii) and (45). In particular, since $\underline{S}$ is invertible, $\mathbb{E} \log^+ \|N_{-1,h} Z_0\| < \infty$ for $|\underline{\lambda}_h| > 1$ and $\mathbb{E} \log^+ \|N_{mp+q,h} Z_0\| < \infty$ for $|\underline{\lambda}_h| \in (0, 1)$. The invertibility of $\underline{\Phi}_h$ for $\underline{\lambda}_h \neq 0$

then shows that

$$\mathbb{E}\log^+ \left\| \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k Z_0 \right\| < \infty \quad \forall\, h \in \{1,\ldots,H\} : |\underline{\lambda}_h| \in (0,1) \cup (1,\infty). \tag{50}$$

Now let $(W_t''')_{t\in\mathbb{Z}}$ be an i.i.d. $N\left(\begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}\right)$-distributed sequence and define $Z_t''' := U^* W_t'''$. Then $\mathbb{E}Z_t''' = U^*(v^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$. By conditions (i) and (iii) and Theorem 3, $(Y_t''')_{t\in\mathbb{Z}}$, defined by $Y_t''' := P(1)^{-1}Q(1)\mathbb{E}Z_0''' + \sum_{j=-\infty}^{\infty} M_j(W_{t-j}''' - (v^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}})$, is a weakly stationary solution of the equation $Y_t''' - \sum_{k=1}^{p} \Psi_k Y_{t-k}''' = \sum_{k=0}^{q} \Theta_k Z_{t-k}'''$, and obviously, it is also strictly stationary. It now follows in complete analogy to the necessity proof presented in Sect. 5.1 that $A_h = 0$ and $C_h u = (\alpha_{h,1}, \ldots, \alpha_{h,\underline{r}_{h+1}-\underline{r}_h})^{\mathrm{T}}$ for $|\underline{\lambda}_h| = 1$, where (A_h, C_h) is defined as in (43) and $\alpha_{h,1} = 0$ if $\lambda_h = 1$. Hence $\sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \widetilde{\underline{\Theta}}_k W_0 = (\alpha_{h,1}, \ldots, \alpha_{h,\underline{r}_{h+1}-\underline{r}_h})^{\mathrm{T}}$ for $|\underline{\lambda}_h| = 1$. By Theorem 1, this together with (50) implies the existence of a strictly stationary solution of (39), so that a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of (1) exists by Proposition 1.

(b) Now assume that conditions (i), (ii′) and (iii) hold for some $v \in \mathbb{C}^s$ and $g \in \mathbb{C}^m$ and define $Y = (Y_t)_{t\in\mathbb{Z}}$ by (18). Then Y is clearly strictly stationary. Since $UZ_t = (w_t^{\mathrm{T}}, u^{\mathrm{T}})$, we further have, using (iii), that

$$\begin{aligned} P(B)Y_t &= P(1)g - P(1)M(1)\begin{pmatrix} v \\ u \end{pmatrix} + Q(B)U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} \begin{pmatrix} w_t \\ u \end{pmatrix} \\ &= Q(1)U^* \begin{pmatrix} v \\ u \end{pmatrix} - Q(1)U^* \begin{pmatrix} v \\ 0_{d-s} \end{pmatrix} + Q(B)U^* \begin{pmatrix} w_t \\ 0_{d-s} \end{pmatrix} \\ &= Q(B)U^* \begin{pmatrix} w_t \\ u \end{pmatrix} = Q(B)Z_t \end{aligned}$$

for $t \in \mathbb{Z}$, so that $(Y_t)_{t\in\mathbb{Z}}$ is a solution of (1).

(c) Finally, the uniqueness assertion follows from the fact that, by Proposition 1, (1) has a unique strictly stationary solution if and only if (39) has a unique strictly stationary solution. By Theorem 1, the latter is equivalent to the fact that $\underline{\Phi}$ does not have an eigenvalue on the unit circle, which, in turn, is equivalent to $\det P(z) \neq 0$ for z on the unit circle, since $\det P(z) = \det(\mathrm{Id}_{mp} - \underline{\Phi} z)$ (e.g., Gohberg et al. 1982, p. 14). This completes the proof of Theorem 2.

6 Discussion and consequences of the main results

In this section, we shall discuss the main results and consider special cases. Some consequences of the results are also listed. We start with some comments on Theorem 1. If Ψ_1 has only eigenvalues of absolute value in $(0,1) \cup (1,\infty)$, then a much simpler condition for stationarity of (3) can be given:

Corollary 1 *Let the assumptions of Theorem 1 be satisfied and suppose that Ψ_1 has only eigenvalues of absolute value in $(0,1)\cup(1,\infty)$. Then a strictly stationary solution of (3) exists if and only if*

$$\mathbb{E}\log^+\left\|\left(\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k\right)Z_0\right\|<\infty. \tag{51}$$

Proof It follows from Theorem 1 that there exists a strictly stationary solution if and only if (10) holds for every $h\in\{1,\dots,H\}$. But this is equivalent to

$$\mathbb{E}\log^+\left\|\left(\sum_{k=0}^{q}(S^{-1}\Psi_1 S)^{q-k}\mathrm{Id}_m S^{-1}\Theta_k\right)Z_0\right\|<\infty,$$

which, in turn, is equivalent to (51), since S is invertible and hence for a random vector $R\in\mathbb{C}^m$ we have $\mathbb{E}\log^+\|SR\|<\infty$ if and only if $\mathbb{E}\log^+\|R\|<\infty$. □

Remark 1 Suppose that Ψ_1 has only eigenvalues of absolute value in $(0,1)\cup(1,\infty)$. Then $\mathbb{E}\log^+\|Z_0\|$ is a sufficient condition for (3) to have a strictly stationary solution, since it implies (51). But it is not necessary. For example, suppose that $q=1$, $m=d=2$,

$$\Psi_1=\begin{pmatrix}2&0\\0&3\end{pmatrix},\ \Theta_0=\mathrm{Id}_2,\ \text{and}\ \Theta_1=\begin{pmatrix}-1&-1\\1&-4\end{pmatrix},\ \text{so that}\ \sum_{k=0}^{1}\Psi_1^{q-k}\Theta_k=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}.$$

By (51), a strictly stationary solution exists if the i.i.d. noise $(Z_t)_{t\in\mathbb{Z}}$ satisfies $Z_0=(R_0,R_0+R_0')^{\mathrm{T}}$, where R_0' is a random variable with finite log moment and R_0 is a random variable with infinite log moment. This means that $\mathbb{E}\log^+\|Z_0\|$ could be infinite.

An example like the one in Remark 1 cannot occur if $m=d$ and the matrix $\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k$ is invertible. More generally, we have the following result.

Corollary 2 *Let the assumptions of Theorem 1 be satisfied and suppose that Ψ_1 has only eigenvalues of absolute value in $(0,1)\cup(1,\infty)$. Suppose further that $d\le m$ and that $\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k$ has full rank d. Then a strictly stationary solution of (3) exists if and only if $\mathbb{E}\log^+\|Z_0\|<\infty$.*

Proof The sufficiency of the condition has been observed in Remark 1, and for the necessity, observe that with $A:=\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k$ and $U:=AZ_0$ we must have $\mathbb{E}\log^+\|U\|<\infty$ by (51). Since A has rank d, the matrix $A^{\mathrm{T}}A\in\mathbb{C}^{d\times d}$ is invertible and we have $Z_0=(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}U$, i.e., the components of Z_0 are linear combinations of those of U. It follows that $\mathbb{E}\log^+\|Z_0\|<\infty$. □

Next we shall discuss the conditions of Theorem 2 in more detail. The following remark is obvious from Theorem 2. It implies, in particular, the well-known fact that the conditions $\mathbb{E}\log^+\|Z_0\|<\infty$ and $\det P(z)\neq 0$ for all z on the unit circle are sufficient for the existence of a strictly stationary solution.

Remark 2 (a) $\mathbb{E}\log^+\|Z_0\| < \infty$ is a sufficient condition for (ii) of Theorem 2.

(b) $\det P(1) \neq 0$ is a sufficient condition for (iii) of Theorem 2.

(c) $\det P(z) \neq 0$ for all z on the unit circle is a sufficient condition for (i) and (iii) of Theorem 2.

With the notation of Theorem 2, define

$$\widetilde{Q}(z) := Q(z)U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}, \tag{52}$$

so that $M(z) = P^{-1}(z)\widetilde{Q}(z)$. It is natural to ask if conditions (i) and (iii) of Theorem 2 can be replaced by a removability condition on the singularities on the unit circle of $(\det P(z))^{-1}\det(\widetilde{Q}(z))$ if $d = m$. The following corollary shows that this condition is indeed necessary, but it is not sufficient as pointed out in Remark 3.

Corollary 3 *Under the assumptions of Theorem 1, with $\widetilde{Q}(z)$ as defined in (52), a necessary condition for a strictly stationary solution of the ARMA(p,q) equation (1) to exist is that the function $z \mapsto |\det P(z)|^{-2}\det(\widetilde{Q}(z)\widetilde{Q}(z)^*)$ has only removable singularities on the unit circle. If additionally $d = m$, then a necessary condition for a strictly stationary solution to exist is that the matrix rational function $z \mapsto (\det P(z))^{-1}\det(\widetilde{Q}(z))$ has only removable singularities on the unit circle.*

Proof The second assertion is immediate from Theorem 2, and the first assertion follows from the fact that if $M(z)$ as defined in Theorem 2 has only removable singularities on the unit circle, then so does $M(z)M(z)^*$ and hence $\det(M(z)M(z)^*)$. □

Remark 3 In the case $d = m$ and $\mathbb{E}\log^+\|Z_0\| < \infty$, the condition that the matrix rational function $z \mapsto (\det P(z))^{-1}\det\widetilde{Q}(z)$ has only removable singularities on the unit circle is not sufficient for the existence of a strictly stationary solution of (3). For example, suppose that $p = q = 1, m = d = 2, \Psi_1 = \Theta_0 = \mathrm{Id}_2, \Theta_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. standard normal sequence and $U = \mathrm{Id}_2$. Then $\det P(z) = \det\widetilde{Q}(z) = (1-z)^2$, but it does not hold that $\Psi_1\Theta_0 + \Theta_1 = 0$, so that condition (iii) of Theorem 1 is violated and no strictly stationary solution can exist.

Next, we shall discuss condition (i) of Theorem 2 in more detail. Recall (e.g., Kailath 1980) that a $\mathbb{C}^{m\times m}$ matrix polynomial $R(z)$ is a *left-divisor* of $P(z)$, if there is a matrix polynomial $P_1(z)$ such that $P(z) = R(z)P_1(z)$. The matrix polynomials $P(z)$ and $\widetilde{Q}(z)$ are *left-coprime*, if every common left-divisor $R(z)$ of $P(z)$ and $\widetilde{Q}(z)$ is *unimodular*, i.e., the determinant of $R(z)$ is constant in z. In that case, the matrix rational function $P^{-1}(z)\widetilde{Q}(z)$ is also called *irreducible*. With $\widetilde{Q}$ as defined in (52), it is then easy to see that condition (i) of Theorem 2 is equivalent to

(i′) *There exist $\mathbb{C}^{m\times m}$-valued matrix polynomials $P_1(z)$ and $R(z)$ and a $\mathbb{C}^{m\times d}$-valued matrix polynomial $Q_1(z)$ such that $P(z) = R(z)P_1(z)$, $\widetilde{Q}(z) = R(z)Q_1(z)$ for all $z \in \mathbb{C}$ and* $\det P_1(z) \neq 0$ *for all z on the unit circle.*

That (i′) implies (i) is obvious, and that (i) implies (i′) follows by taking $R(z)$ as the greatest common left-divisor (cf. Kailath 1980, p. 377) of $P(z)$ and $\widetilde{Q}(z)$. The remaining right-factors $P_1(z)$ and $Q_1(z)$ are then left-coprime, and since the matrix rational function $M(z) = P^{-1}(z)\widetilde{Q}(z) = P_1^{-1}(z)Q_1(z)$ has no poles on the unit circle, it follows from page 447 in Kailath (1980) that $\det P_1(z) \neq 0$ for all z on the unit circle, which establishes (i′). As an immediate consequence, we have the following result.

Remark 4 In the notation of Theorem 2 and (52), if $P(z)$ and $\widetilde{Q}(z)$ are left-coprime then condition (i) of Theorem 2 is equivalent to $\det P(z) \neq 0$ for all z on the unit circle.

Next we show how a slight extension of Theorem 4.1 of Bougerol and Picard (1992), which characterizes the existence of a strictly stationary *non-anticipative* solution of the ARMA(p, q) equation (1), can be deduced from Theorem 2. By a non-anticipative strictly stationary solution we mean a strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ such that for every $t \in \mathbb{Z}$, Y_t is independent of the sigma algebra generated by $(Z_s)_{s>t}$, and by a *causal* strictly stationary solution we mean a strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ such that for every $t \in \mathbb{Z}$, Y_t is measurable with respect to the sigma algebra generated by $(Z_s)_{s\leq t}$. Clearly, since $(Z_t)_{t\in\mathbb{Z}}$ is assumed to be i.i.d., every causal solution is also non-anticipative. The equivalence of (i) and (iii) in the theorem below was already obtained by Bougerol and Picard (1992) under the additional assumption that $\mathbb{E}\log^+\|Z_0\| < \infty$.

Theorem 4 *If, under the assumptions of Theorem 2, the matrix polynomials $P(z)$ and $\widetilde{Q}(z)$, defined in the theorem and in (52), are left-coprime, then the following are equivalent.*

- (i) *There exists a non-anticipative strictly stationary solution of (1).*
- (ii) *There exists a causal strictly stationary solution of (1).*
- (iii) $\det P(z) \neq 0$ *for all $z \in \mathbb{C}$ such that $|z| \leq 1$ and if $M(z) = \sum_{j=0}^{\infty} M_j z^j$ denotes the Taylor expansion of $M(z) = P^{-1}(z)\widetilde{Q}(z)$, then*

$$\mathbb{E}\log^+\|M_j U Z_0\| < \infty \quad \forall\, j \in \{mp+q-p+1, \ldots, mp+q\}. \tag{53}$$

Proof The implication "(iii) ⇒ (ii)" is immediate from Theorem 2 and Eq. (18), and "(ii) ⇒ (i)" is obvious since $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. Let us show that "(i) ⇒ (iii)": since a strictly stationary solution exists, the function $M(z)$ has only removable singularities on the unit circle by Theorem 2. Since $P(z)$ and $\widetilde{Q}(z)$ are left-coprime, this implies by Remark 4 that $\det P(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$. In particular, by Theorem 2, the strictly stationary solution is unique and given by (18). By assumption, this solution must then be non-anticipative, so that we conclude that the distribution of $M_j U Z_{t-j}$ must be degenerate to a constant for all $j \in \{-1, -2, \ldots\}$. But since $UZ_0 = (w_0^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$ and $M_j = (M_j', 0_{m,d-s})$ with certain matrices $M_j' \in \mathbb{C}^{m,s}$, it follows for $j \leq -1$ that $M_j U Z_0 = M_j' w_0$, so that $M_j' = 0$ since no non-trivial linear combination of the components of w_0 is constant a.s. It follows that $M_j = 0$ for $j \leq -1$, i.e., $M(z)$ has only removable singularities for $|z| \leq 1$. Since $P(z)$ and

$\widetilde{Q}(z)$ are assumed to be left-coprime, it follows from page 447 in Kailath (1980) that $\det P(z) \neq 0$ for all $|z| \leq 1$. Equation (53) is an immediate consequence of Theorem 2. □

It may be possible to extend Theorem 4 to situations in which $P(z)$ and $\widetilde{Q}(z)$ are not left-coprime, but we did not investigate this question. The last result is on the interplay between the existence of strictly and of weakly stationary solutions of (1) when the noise is i.i.d. with finite second moments:

Theorem 5 *Let $m, d, p \in \mathbb{N}, q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors with finite second moment. Let $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$. Then the ARMA$(p, q)$ equation (1) admits a strictly stationary solution if and only if it admits a weakly stationary solution, and in that case, the solution given by (20) is both a strictly stationary and weakly stationary solution of (1).*

Proof It follows from Theorem 3 that if a weakly stationary solution exists, then one such solution is given by (20), which is also clearly strictly stationary. On the other hand, if a strictly stationary solution exists, then by Theorem 2, one such solution is given by (18), which is clearly weakly stationary. □

Finally, we remark that most of the results presented in this paper can be applied also to the case when $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence of $\mathbb{C}^{d\times d'}$ random matrices and $(Y_t)_{t\in\mathbb{Z}}$ is $\mathbb{C}^{m\times d'}$-valued. This can be seen by stacking the columns of Z_t into a $\mathbb{C}^{dd'}$-variate random vector Z_t', those of Y_t into a $\mathbb{C}^{md'}$-variate random vector Y_t', and considering the matrices

$$\Psi_k' := \begin{pmatrix} \Psi_k & & \\ & \ddots & \\ & & \Psi_k \end{pmatrix} \in \mathbb{C}^{md'\times md'} \quad \text{and} \quad \Theta_k' := \begin{pmatrix} \Theta_k & & \\ & \ddots & \\ & & \Theta_k \end{pmatrix} \in \mathbb{C}^{md'\times dd'}.$$

The question of existence of a strictly stationary solution of (1) with matrix-valued Z_t and Y_t is then equivalent to the existence of a strictly stationary solution of $Y_t' - \sum_{k=1}^p \Psi_k' Y_{t-k}' = \sum_{k=0}^q \Theta_k' Z_{t-k}'$.

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