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Exact goodness-of-fit tests for censored data

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Abstract The statistic introduced in Fortiana and Grané (J R Stat Soc B 65(1):115–126, 2003) is modified so that it can be used to test the goodness-of-fit of a censored sample, when the distribution function is fully specified. Exact and asymptotic distributions of three modified versions of this statistic are obtained and exact critical values are given for different sample sizes. Empirical power studies show the good performance of these statistics in detecting symmetrical alternatives.

Keywords Goodness-of-fit · Censored samples · Maximum correlation · Exact distribution · *L*-statistics

1 Introduction

The usual way to measure system reliability is to test completed products or components, under conditions that simulate real life, until failure occurs. One may think that the more data available, the more confidence one will have in the reliability level, but this practice is often expensive and time-consuming. Hence, it is of special interest to analyze product life before all test units fail. This situation leads to censored samples that are encountered naturally in reliability studies. The present paper is concerned on goodness-of-fit tests for this type of data with particular censoring schemes.

Let $y_1, \ldots, y_n$ be independent and identically distributed (iid) random variables with cumulative distribution function (cdf) F and consider the ordered sample

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$y_{(1)} < \cdots < y_{(n)}$. In the following, we adopt the notation of Stephens and D'Agostino (1986). When some of the observations are missing the sample is said to be censored. If all the observations less than $y_{(s)}$ ($s > 1$) are missing the sample is left censored and if all the observations greater than $y_{(r)}$ ($r < n$) are missing, it is right censored; in either case the sample is said to be singly censored. If the observations are missing at both ends, the sample is doubly censored. Censoring may occur for random values of s or r (Type I or time censoring) or for fixed values (Type II or failure censoring).

In Fortiana and Grané (2003), we proposed a goodness-of-fit statistic for complete samples to test the null hypothesis $H_0 : F(y) = F_0(y)$, where $F_0(y)$ is a completely specified cdf, or equivalently to test that $x_1, \ldots, x_n$, where $x_i = F_0(y_i)$, are iid random variables uniformly distributed in the [0, 1] interval. The statistic is based on Hoeffding's maximum correlation (see the definition below) between the empirical cdf and the hypothesized. When there is no censoring, we found out that the test based on the proposed statistic can advantageously replace those of Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling for a wide range of alternatives. Recently, Grané and Tchirina (2011) studied the efficiency properties, in the Bahadur sense, of the test based on Q_n and obtain its Bahadur local asymptotic optimality domains.

The purpose of the present paper is to derive the exact distribution of Q_n in the context of censored sampling, with the aim of obtaining a test that performs as well as the uncensored one, and which provides a competitive tool for the reader. However this goal is only possible to achieve under some particular censoring schemes like Type I and Type II, and not under random censoring. Concerning the asymptotic distribution of Q_n, the choice of Type I and Type II censoring schemes is again crucial as it allows the application of L-statistic-type results in the derivation of the asymptotics for Q_n.

Most goodness-of-fit statistics can be regarded as measures of proximity between two distributions: the empirical and the hypothesized. For instance, the Kolmogorov–Smirnov statistic is based on the supremum distance, whereas Cramér–von Mises and Anderson–Darling statistics use a weighted L^2 distance. Another possibility is Kulback–Leibler information measure, that was used by Ebrahimi (2001) for testing uniformity in a general set-up. However, there are few references concerning censored samples. Lim and Park (2007) adapted Kulback–Leibler information measure for Type II censored data, for testing normality or exponentiality, but not for uniformity. Moreover, tests based on Kulback–Leibler information measure (and also on entropy measures) present similar drawbacks both for complete and censored samples. It is not possible to obtain the exact distribution of the test statistic and critical values should be derived by Monte Carlo methods. Sometimes, the asymptotic distribution of the statistic is approximate and so the asymptotic critical values. Hence, when studying the power of the uniformity Q_n-based test, in the context of censored samples, we compare it with the *adapted versions* (adapted, to the censoring schemes considered in this paper) of classical statistics, such as Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling (Barr and Davidson 1973; Pettitt and Stephens 1976; Stephens and D'Agostino 1986).

The remaining of paper proceeds as follows: in Sect. 2, the exact distributions of three modifications of the statistic are deduced and exact critical values are obtained for different sample sizes and different significance levels. In Sect. 3, we give some conditions under which the convergence to the normal distribution can be asserted. In

Sect. 4, we study the power of the exact tests based on these statistics for five parametric families of alternative distributions with support contained in the [0, 1] interval, where we conclude that the tests based on our proposals have a good performance in detecting symmetrical alternatives, whereas the tests based on the Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling statistics are biased for some of these alternatives. In Sect. 5, we give an application in the engineering context.

2 Exact distributions

We start by introducing Hoeffding's maximum correlation and recall how the Q_n statistic is obtained in the case of complete samples.

Let F_1 and F_2 be two cdfs with second order moments. Hoeffding's maximum correlation between F_1 and F_2, henceforth denoted by $\rho^+(F_1, F_2)$, is defined as the maximum of the correlation coefficients of bivariate distributions having F_1 and F_2 as marginals:

$$\rho^+(F_1, F_2) = \frac{1}{\sigma_1\,\sigma_2}\left(\int_0^1 F_1^-(p)\,F_2^-(p)\,\mathrm{d}p - \mu_1\,\mu_2\right), \tag{1}$$

where F_i^- is the left-continuous pseudo-inverses of F_i, μ_i and σ_i^2 are, respectively, the expectation and variance of F_i, $i = 1, 2$ (see, e.g., Cambanis et al. 1976). Since $\rho^+(F_1, F_2)$ equals 1 if and only if $F_1 = F_2$ (almost everywhere) up to a scale and location change, it is a measure of proximity between two distributions and yields a goodness-of-fit test statistic when considering the empirical and hypothesized distributions.

In Fortiana and Grané (2003), we studied the test of uniformity based on

$$Q_n = \frac{s_n}{\sqrt{1/12}}\,\rho^+(F_n, F_U), \tag{2}$$

where F_n is the empirical cdf of n iid real-valued random variables, s_n is the sample standard deviation and F_U is the cdf of a uniform in [0, 1] random variable. The exact distribution of Q_n was obtained and its small and large sample properties were studied (see also Grané and Tchirina 2011, where local Bahadur asymptotic optimality domains for Q_n are obtained).

In the following, we deduce the expressions of the modified Q_n statistic for singly- and doubly censored samples and obtain their exact probability density functions (pdfs) under the null hypothesis of uniformity. For all the statistics, we give tables of exact critical values for different sample sizes and different significance levels.

2.1 Right-censored samples

Let $y_{(1)} < \cdots < y_{(n)}$ be the ordered sample. Suppose that the sample is right censored of Type I; the y_i values are known to be less than a fixed value y^*. The set of available transformed x_i values ($x_i = F_0(y_i)$) is then $x_{(1)} < \cdots < x_{(r)} < t$, where $t = F_0(y^*)$.

If the censoring is of Type II, there are again r values $x_{(i)}$, with $x_{(r)}$ the largest and r fixed.

Proposition 1 *Under the null hypothesis of uniformity:*

(i) *The modified Q_n statistic for Type I right-censored data is*

$$ {}_1Q_{tn} = \sum_{i=1}^{r+1} a_i\, x_{(i)}, \tag{3}$$

where $a_i = 6((2i-1)(r+1)-n^2)/(n^2(r+1))$, for $1 \le i \le r$ and $a_{r+1} = 6r(n^2-r^2-r)/(n^2(r+1))$.

(ii) *The modified Q_n statistic for Type II right-censored data is*

$$ {}_2Q_{rn} = \sum_{i=1}^{r} a_i\, x_{(i)}, \tag{4}$$

where $a_i = 6((2i-1)r-n^2)/(n^2 r)$, for $1 \le i \le r-1$ and $a_r = 6(r-1)(n^2-r(r-1))/(n^2 r)$.

Proof (i) For type I right-censored data, suppose t $(t < 1)$ is the fixed censoring value. This value can be added to the sample set (see Stephens and D'Agostino 1986), and the statistic can be calculated using $x_{(r+1)} = t$. Note that it is possible to have $r = n$ observations less than t, since when the value t is added the new sample has size $n+1$.

From formulas (2) and (1) we have that

$$ Q_n = 12\left(\int_0^1 F_n^-(p)\, F_U^-(p)\, \mathrm{d}p - \frac{1}{2}\,\overline{x}_n\right). \tag{5}$$

Noticing that the pseudo-inverse of the empirical cdf is

$$ F_n^-(p) = \begin{cases} x_{(i)}, & \frac{i-1}{n} < p \le \frac{i}{n}, \quad 1 \le i \le r, \\ x_{(r+1)}, & \frac{r}{n} < p \le 1, \end{cases} $$

for $0 \le p \le 1$, the first summand of (5) is

$$ \begin{aligned} \int_0^1 F_n^-(p)\, F_U^-(p)\, \mathrm{d}p &= \sum_{i=1}^{r} \int_{(i-1)/n}^{i/n} x_{(i)}\, p\, \mathrm{d}p + \sum_{i=r+1}^{n} \int_{(i-1)/n}^{i/n} x_{(r+1)}\, p\, \mathrm{d}p \\ &= \frac{1}{2n^2} \sum_{i=1}^{r} (2i-1)x_{(i)} + \frac{1}{2n^2}(n^2-r^2)x_{(r+1)} \end{aligned} $$

and subtracting the (available) sample mean, the part of (5) between parenthesis is

$$ \frac{1}{2n^2(r+1)} \sum_{i=1}^{r} ((2i-1)(r+1)-n^2)x_{(i)} + \frac{r(n^2-r^2-r)}{2n^2(r+1)} x_{(r+1)}. $$

(ii) For Type II right-censored data the pseudo-inverse of the empirical cdf is

$$F_n^-(p) = \begin{cases} x_{(i)}, & \frac{i-1}{n} < p \le \frac{i}{n}, \quad 1 \le i \le r-1, \\ x_{(r)}, & \frac{r-1}{n} < p \le 1, \end{cases}$$

for $0 \le p \le 1$. Proceeding analogously, the first summand of (5) is

$$\int_0^1 F_n^-(p)\, F_U^-(p)\, \mathrm{d}p = \frac{1}{2n^2} \sum_{i=1}^{r-1} (2i-1) x_{(i)} + \frac{1}{2}\left(1 - \frac{(r-1)^2}{n^2}\right) x_{(r)}$$

and subtracting the (available) sample mean, the part of (5) between parenthesis is

$$\frac{1}{2n^2 r} \sum_{i=1}^{r-1} ((2i-1)r - n^2) x_{(i)} + \frac{(r-1)}{2n^2 r}(n^2 - r(r-1)) x_{(r)}.$$

□

Under the null hypothesis, ${}_1Q_{tn}$ and ${}_2Q_{rn}$ are linear combinations of selected order statistics from the [0, 1] uniform distribution. Therefore their exact probability density functions can be obtained with the following algorithm, proposed by Dwass (1961), Matsunawa (1985) and Ramallingam (1989).

For Type I right-censored data, let

$$b_i = \sum_{l=i}^{r+1} a_l = \frac{6}{n^2}(2i - i^2 - 1) + \frac{6}{r+1}(i-1), \quad i = 1, 2, \ldots, r+1,$$

let k be the number of distinct non-zero b_i's, and $(\nu_1, \ldots, \nu_k)$ be the corresponding multiplicities of $(b_1, \ldots, b_k)$. Defining on $\mathbb{C}$ the functions:

$$G(s) = \left[\prod_{j=1}^{k} \left(s + \frac{1}{b_j}\right)^{\nu_j}\right]^{-1}, \quad G_l(s) = \left(s + \frac{1}{b_l}\right)^{\nu_l} G(s), \quad l = 1, 2, \ldots, k,$$

the exact pdf of ${}_1Q_{tn}$ statistic, under H_0, is given by

$$f_{{}_1Q_{tn}}(s) = \sum_{l=1}^{k} \sum_{m=1}^{\nu_l} \mathrm{sign}(b_l)\, C^{\sharp}_{l,m}\, \chi\left(\frac{s}{b_l}\right) \times \chi\left(1 - \frac{s}{b_l}\right) s^{m-1} \left(1 - \frac{s}{b_l}\right)^{n-m} \Big/ B(m, n-m+1)$$

where $\chi(x)$ is the indicator of the interval $[x > 0]$, $B(a, b)$ is the Beta function,

$$C^{\sharp}_{l,m} = \left(\prod_{j=1}^{k} (b_j)^{-\nu_j}\right) C_{l,m}, \qquad C_{l,m} = \frac{G_l^{(\nu_l - m)}(-1/b_l)}{(\nu_l - m)!},$$

and $G_l^{(j)}$ denotes the j-th derivative of G_l.

Table 1 Lower- and upper-tail critical values of ${}_1Q_{tn}$ and ${}_2Q_{rn}$ for p proportions of data in the sample

p	5% significance level						2.5% significance level					
	$n = 10$		$n = 20$		$n = 30$		$n = 10$		$n = 20$		$n = 30$	
Critical values for ${}_1Q_{tn}$												
0.3	0.2221	1.4577	0.3563	1.3467	0.4288	1.2751	0.1692	1.6314	0.3009	1.4778	0.3760	1.3828
0.4	0.3566	1.6455	0.5260	1.5568	0.6133	1.4925	0.2874	1.8059	0.4590	1.6805	0.5514	1.5952
0.5	0.4867	1.7344	0.6820	1.6820	0.7788	1.6315	0.4071	1.8725	0.6088	1.7917	0.7128	1.7239
0.6	0.5980	1.7228	0.8095	1.7167	0.9104	1.6853	0.5142	1.8326	0.7356	1.8076	0.8450	1.7631
0.7	0.6751	1.6095	0.8935	1.6554	0.9937	1.6465	0.5939	1.6875	0.8243	1.7240	0.9335	1.7067
0.8	0.6994	1.4005	0.9172	1.4956	1.0125	1.5103	0.6278	1.4500	0.8580	1.5414	0.9619	1.5516
0.9	0.6337	1.1368	0.8547	1.2518	0.9443	1.2876	0.5789	1.1735	0.8101	1.2817	0.9067	1.3886
Critical values for ${}_2Q_{rn}$												
0.3	0.0999	1.1678	0.2714	1.2105	0.3659	1.1861	0.0684	1.3418	0.2240	1.3423	0.3172	1.2940
0.4	0.2221	1.4577	0.4418	1.4621	0.5531	1.4284	0.1692	1.6314	0.3800	1.5905	0.4938	1.5334
0.5	0.3566	1.6455	0.6066	1.6303	0.7267	1.5943	0.2874	1.8059	0.5359	1.7478	0.6615	1.6906
0.6	0.4867	1.7344	0.7503	1.7110	0.8712	1.6773	0.4071	1.8725	0.6760	1.8119	0.8051	1.7604
0.7	0.5980	1.7228	0.8579	1.6983	0.9722	1.6701	0.5142	1.8326	0.7857	1.7784	0.9098	1.7364
0.8	0.6751	1.6095	0.9141	1.5876	1.0145	1.5666	0.5939	1.6875	0.8493	1.6446	0.9602	1.6141
0.9	0.6994	1.4005	0.8993	1.3818	0.9789	1.3889	0.6278	1.4500	0.8469	1.4180	0.9367	1.4600

Analogously, for Type II right-censored data, the pdf of ${}_2Q_{rn}$ is found by applying the previous algorithm, taking into account that in this case

$$b_i = \sum_{l=i}^{r} a_l = \frac{6}{n^2}(2i - i^2 - 1) + \frac{6}{r}(i - 1), \quad i = 1, 2, \ldots, r.$$

Remark 1 For left-censored data, note that from the r largest observations one can compute the values $x^*_{(i)} = 1 - x_{(n+1-i)}$, for $i = 1, \ldots, r$, so that the sample becomes right censored. In Type I censoring, the left-censoring fixed value converts to $t^* = 1 - t$, to be used as the right-censoring fixed point.

`Mathematica` programs implementing this algorithm are available from the author. As an illustration of the application, critical values for 5 and 2.5% significance levels are computed to test the null hypothesis of uniformity. They are reproduced in Table 1.

In order to show the effect of right censoring on the distribution of Q_n, in Fig. 1 we depict the density functions of Q_n (complete samples), ${}_1Q_{tn}$ (Type I right-censored data) and ${}_2Q_{rn}$ (Type II right-censored data), for different $p = 0.9, 0.8, 0.7$ proportions of data in samples of size $n = 20$.

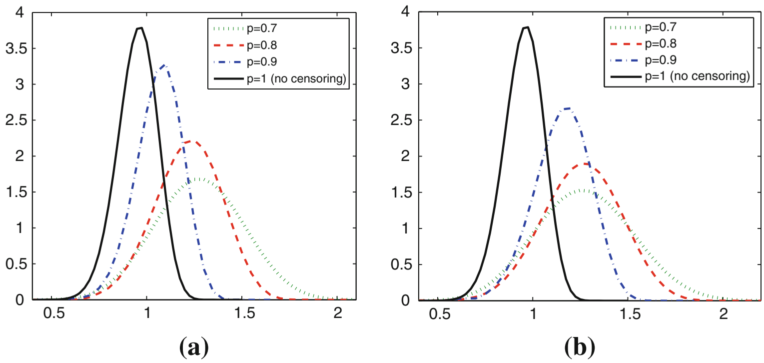


Fig. 1 Density functions for **a** ${}_1Q_{tn}$ (Type I right-censored data) and **b** ${}_2Q_{rn}$ (Type II right-censored data), for different $p = 0.9, 0.8, 0.7$ proportions of data in samples of size $n = 20$

2.2 Doubly censored samples

Let $x_{(s)} < \cdots < x_{(r)}$, with $1 < s < r < n$, be the available x_i values from a Type II doubly censored sample.

Proposition 2 *Under the null hypothesis of uniformity, the modified Q_n statistic for Type II doubly censored data is*

$$ {}_2Q_{sr,n} = \sum_{i=s}^{r} a_i \, x_{(i)}, \tag{6}$$

where $a_s = 6(s - n^2/(r - s + 1))/n^2$, $a_i = 6((2i - 1) - n^2/(r - s + 1))/n^2$, for $s + 1 \le i \le r - 1$ and $a_r = 6(n^2 - (r - 1)^2 - n^2/(r - s + 1))/n^2$.

Proof Since in this case the pseudo-inverse of the empirical cdf is

$$F_n^-(p) = \begin{cases} x_{(s)}, \ 0 < p \le \frac{s}{n}, \\ x_{(i)}, \ \frac{i-1}{n} < p \le \frac{i}{n}, \quad s+1 \le i \le r-1, \\ x_{(r)}, \ \frac{r-1}{n} < p \le 1, \end{cases}$$

for $0 \le p \le 1$, the first summand of (5) is equal to

$$\begin{aligned} \int_0^1 F_n^-(p)\, F_U^-(p)\,\mathrm{d}p &= \sum_{i=1}^{s} \int_{(i-1)/n}^{i/n} x_{(s)}\, p\,\mathrm{d}p \\ &\quad + \sum_{i=s+1}^{r-1} \int_{(i-1)/n}^{i/n} x_{(i)}\, p\,\mathrm{d}p + \sum_{i=r}^{n} \int_{(i-1)/n}^{i/n} x_{(r)}\, p\,\mathrm{d}p \\ &= \frac{1}{2n^2} s\, x_{(s)} + \frac{1}{2n^2} \sum_{i=s+1}^{r-1} (2i-1) x_{(i)} \\ &\quad + \frac{1}{2n^2}\left(n^2 - (r-1)^2\right) x_{(r)}. \end{aligned}$$

Subtracting the (available) sample mean, the part of (5) between parenthesis is

$$\frac{1}{2n^2}\left[\left(s-\frac{n^2}{r-s+1}\right)x_{(s)}+\sum_{i=s+1}^{r-1}\left((2i-1)-\frac{n^2}{r-s+1}\right)x_{(i)}\right.$$
$$\left.+\left(n^2-(r-1)^2-\frac{n^2}{r-s+1}\right)x_{(r)}\right].$$

□

For Type II doubly censored data, the pdf of ${}_2Q_{sr,n}$ is found by applying the algorithm described in Sect. 2.1, taking into account that now

$$b_i=\sum_{l=i}^{r}a_l=\begin{cases}\frac{6}{n^2}s(1-s), & i=s,\\ \frac{6}{n^2}(2i-i^2-1)+\frac{6}{r-s+1}(i-s), & i=s+1\ldots,r.\end{cases}$$

Analogously, it is possible to obtain tables of critical values for different values of r and s and different sample sizes. A `Mathematica` program implementing this algorithm is available from the author. As an illustration of its application, critical values for 5 and 2.5% significance levels are computed to test the null hypothesis of uniformity, for symmetric double censoring, where $p=r/n$, $q=s/n$ and $p=1-q$. They are reproduced in Table 2.

Concerning to the `Mathematica` programs implementing the pdfs of ${}_1Q_{tn}$, ${}_2Q_{rn}$ and ${}_2Q_{sr,n}$, it should be mentioned that the original formula (2.3) of Ramallingam (1989) for the pdf consists on a sum of terms, containing indicator functions of overlapping intervals. There we obtain an alternative expression taking disjoint intervals

Table 2 Lower- and upper-tail critical values of ${}_2Q_{sr,n}$ for different values of $p=r/n$ and $q=s/n$, where $p=1-q$

p	5% sig. level		2.5% sig. level	
$n=10$				
0.80	0.4458	1.3152	0.3798	1.3978
0.90	0.6994	1.4005	0.6278	1.4500
$n=20$				
0.80	0.5297	1.1458	0.4790	1.2065
0.85	0.6518	1.2051	0.6006	1.2535
0.90	0.7627	1.2378	0.7136	1.2750
0.95	0.8547	1.2518	0.8101	1.2817
$n=30$				
0.80	0.5707	1.0752	0.5278	1.1253
0.833	0.6498	1.1195	0.6072	1.1631
0.867	0.7239	1.1519	0.6824	1.1891
0.90	0.7912	1.1736	0.7517	1.2048

which saves computational resources in calculating the critical values. If k is the number of distinct non-zero b_i's and we consider these coefficients ordered in ascending order, $b_{(1)} < \cdots < b_{(k)}$, then the pdf is defined by parts over the partition $0 < b_{(1)} < \cdots < b_{(k)}$, if all the b_i's are positive or either over the partition $b_{(1)} < \cdots < b_{(k)}$ if there are negative b_i's. In all cases, the support of these pdfs is determined by the interval $[\min\{0, b_{(1)}\}, b_{(k)}]$.

2.3 Expectation and variance

In this section, we obtain expressions for the exact expectation and variance of ${}_2Q_{rn}$ and ${}_2Q_{sr,n}$ under the null hypothesis of uniformity. In matrix notation, these statistics can be written as

$$ {}_2Q_{rn} = \mathbf{a}'\,\mathbf{x}_r, \tag{7} $$

where $\mathbf{x}_r = (x_{(1)}, \ldots, x_{(r)})'$, $\mathbf{a} = (a_1, \ldots, a_r)'$, with $a_i = 6((2i-1)r - n^2)/(n^2 r)$, for $1 \le i \le r-1$ and $a_r = 6(r-1)(n^2 - r(r-1))/(n^2 r)$,

$$ {}_2Q_{sr,n} = \mathbf{a}'\,\mathbf{x}_{sr}, \tag{8} $$

where $\mathbf{x}_{sr} = (x_{(s)}, \ldots, x_{(r)})'$, $\mathbf{a} = (a_s, \ldots, a_r)'$, with $a_s = 6(s - n^2/(r-s+1))/n^2$, $a_i = 6((2i-1) - n^2/(r-s+1))/n^2$, for $s+1 \le i \le r-1$ and $a_r = 6(n^2 - (r-1)^2 - n^2/(r-s+1))/n^2$.

It is straightforward to prove that when $s = 1$ and $r = n$ the previous statistics coincide with the Q_n statistic for complete samples (see Proposition 3 of Fortiana and Grané 2003).

Proposition 3 *Under the null hypothesis of uniformity, the exact expectation and variance of* ${}_2Q_{rn}$ *are given by*

$$ E({}_2Q_{rn}|H_0) = \frac{(3n^2 + r - 2r^2)(r-1)}{n^2(n+1)}, \quad var({}_2Q_{rn}|H_0) = \mathbf{a}'\mathbf{C}_r\mathbf{a}, $$

where $\mathbf{a}$ *is the vector of coefficients of* (7) *and matrix* $\mathbf{C}_r = (c_{ij})_{1\le i,j\le r}$ *is defined by the covariances*

$$ c_{ij} = cov((x_{(i)}, x_{(j)})|H_0) = \frac{1}{(n+2)(n+1)^2}\{(n+1)\min(i,j) - ij\}, \quad 1 \le i, j \le r. $$

Proposition 4 *Under the null hypothesis of uniformity, the exact expectation and variance of* ${}_2Q_{sr,n}$ *are given by*

$$ \begin{aligned} E({}_2Q_{sr,n}|H_0) &= \frac{1}{n^2(n+1)}\Big((1 + 3(n-r) - 3n^2 - 4(n-r)^2)(n-r) \\ &\quad + (3n^2 - 1)r + 3r^2 - 2r^3\Big), \\ var({}_2Q_{sr,n}|H_0) &= \mathbf{a}'\mathbf{C}_{sr}\mathbf{a}, \end{aligned} $$

where $\mathbf{a}$ *is the vector of coefficients of* (8) *and matrix* $\mathbf{C}_{sr} = (c_{ij})_{s\le i,j\le r}$*, with covariances* c_{ij} *defined as in Proposition* 3*, but for* $s \le i, j \le r$*.*

Proof (of Propositions 3 and 4). Formulae for the expectation and variance of $\mathbf{x}_r$ under the null can be found in David (1981). Expressions for the expectations and variances of ${}_2Q_{rn}$ and ${}_2Q_{sr,n}$ are obtained from (7) and (8), respectively, after some easy but tedious computations. For example, to get the expectation of ${}_2Q_{rn}$ substitute $E(\mathbf{x}_r|H_0) = \left(\frac{1}{n+1}, \ldots, \frac{r}{n+1}\right)'$ for $\mathbf{x}_r$ in formula (7). □

3 Asymptotic distributions

In this section, we give conditions under which the asymptotic normality of ${}_1Q_{tn}$, ${}_2Q_{rn}$ and ${}_2Q_{sr,n}$ can be established. With the same notation of Sect. 2 and under the null hypothesis of uniformity:

Proposition 5 *If* $r = o(n)$*, the statistic* ${}_2Q_{rn}$ *is asymptotically normally distributed, in the sense of*

$$\sup_{-\infty<t<\infty} |P({}_2Q_{rn} < t) - \Phi_{\mu_n,\sigma_n}(t)| \to 0, \quad (n \to \infty),$$

where Φ_{μ_n,σ_n} *is the cdf of a normal random variable with expectation and variance*

$$\mu_n = \frac{(3n^2 + r - 2r^2)(r-1)}{n^2(n+1)},$$
$$\sigma_n^2 = \frac{6(r+1)}{5n^4(n+1)^2 r}\left(5n^4(2r-1) - 15n^2(r-1)r^2 + r^2(6r^3 - 9r^2 + r + 1)\right).$$

Proof The proof is based on Theorem 4.4 of Matsunawa (1985). Since the assumptions of that theorem hold, to assert the convergence of ${}_2Q_{rn}$ to the normal distribution we must prove that

$$\frac{\max_{1\le i\le r} |b_i|}{(n+1)\,\sigma_n} \to 0, \quad (n \to \infty), \tag{9}$$

where coefficients b_is were defined as $b_i = \frac{6}{n^2}(2i - i^2 - 1) + \frac{6}{r}(i-1)$ for $i = 1, 2, \ldots, r$.

From formulas (4.6) and (4.7) of Matsunawa (1985), we obtain

$$\mu_n = \frac{1}{n+1}\sum_{i=1}^{r} b_i = \frac{(3n^2 + r - 2r^2)(r-1)}{n^2(n+1)},$$
$$\sigma_n^2 = \frac{1}{(n+1)^2} \times \sum_{i=1}^{r} b_i^2 = \frac{6(r+1)(5n^4(2r-1) - 15n^2(r-1)r^2 + r^2(6r^3 - 9r^2 + r + 1))}{5n^4(n+1)^2 r}.$$

If $r = o(n)$, condition (9) is fulfilled, since

$$\max_{1\le i\le r} |b_i| \le \max_{1\le i\le r} \left(\frac{6}{n^2}(i-1)^2 + \frac{6}{r}(i-1)\right) = 6(r-1)\left(\frac{r-1}{n^2} + \frac{1}{r}\right).$$

□

Corollary 1 *Under the assumptions of Proposition 5, an analogous result can be established for the statistic* ${}_1Q_{tn}$ *with conditional mean and variance*

$$\begin{aligned}\mu_n &= (3n^2r - 3r^2 - 2r^3 - r)/(n^2(n+1)),\\ \sigma_n^2 &= 6r(5n^2(2r+1) - 15n^2r(r+1)^2\\ &\quad +(r+1)^2(6r^3+9r^2+r-1))/(5n^4(n^2+1)(r+1)).\end{aligned}$$

Proof The proof is analogous to that of Proposition 5, but taking into account that for Type I right-censored data coefficients b_is are $b_i = \frac{6}{n^2}(2i - i^2 - 1) + \frac{6}{r+1}(i-1)$ for $i = 1, 2, \ldots, r+1$. □

Proposition 6 *If* $r = o(n)$*, the statistic* ${}_2Q_{sr,n}$*, where* $s = n - r$*, is asymptotically normally distributed, in the sense of*

$$\sup_{-\infty<t<\infty} |P({}_2Q_{sr,n} < t) - \Phi_{\mu_n,\sigma_n}(t)| \to 0, \quad (n\to\infty),$$

where Φ_{μ_n,σ_n} *is the cdf of a normal random variable with expectation and variance*

$$\begin{aligned}\mu_n &= \frac{1}{n^2(n+1)}\Big((1+3(n-r) - 3n^2 - 4(n-r)^2)(n-r) + (3n^2-1)r + 3r^2 - 2r^3\Big),\\ \sigma_n^2 &= \frac{6}{5n^4(n+1)^2(n-2r+1)}\Big(19n^6 - 4n^5(21+37r) + 10n^4(9+44r+47r^2)\\ &\quad -10n^3(3+34r+90r^2+76r^3) + n^2(1+80r+470r^2+940r^3+640r^4)\\ &\quad +2r(1+2r+5r^2+40r^3+69r^4+18r^5)\\ &\quad -n(1+4r+60r^2+310r^3+540r^4+258r^5)\Big).\end{aligned}$$

Proof To prove the convergence of ${}_2Q_{sr,n}$ to the normal distribution is equivalent to proving that (see Theorem 4.4. of Matsunawa 1985)

$$\frac{\max_{s\le i\le r} |b_i|}{(n+1)\,\sigma_n} \to 0, \quad (n\to\infty). \tag{10}$$

In the case of doubly censored samples, coefficients b_is were defined as

$$b_i = \begin{cases} \frac{6}{n^2}s(1-s), & i = s,\\ \frac{6}{n^2}(2i - i^2 - 1) + \frac{6}{r-s+1}(i-s), & i = s+1,\ldots,r.\end{cases}$$

Introducing $s = n - r$, we have that

$$\max_{s \le i \le r} |b_i| = \max_{s+1 \le i \le r} |b_i| = \frac{6}{n^2(2r - n + 1)} \max_{s+1 \le i \le r} \Big| - (i-1)^2(2r - n + 1) + n^2(i - n + r)\Big|.$$

The function inside the absolute value is a polynomial of second order in i with a negative leading term. If i takes values from 1 to $n-1$, this function is always positive and the maximum is attained at $i^* = [1 + n^2/(2(2r - n + 1))]$, where $[\cdot]$ denotes the nearest integer. However, since i takes values from $s + 1$ to r, it is possible that the maximum is attained at a certain $i < i^*$. In fact, we have that

$$\text{if } r < i^* \Rightarrow \max_{s \le i \le r} |b_i| = |b_r| < |b_{i^*}|,$$
$$\text{if } r \ge i^* \Rightarrow \max_{s \le i \le r} |b_i| = |b_{i^*}|.$$

In both cases it holds that

$$\max_{s \le i \le r} |b_i| \le |b_{i^*}| = \frac{3\left(4 + 5n^2 + 12r + 8r^2 - 4n(2 + 3r)\right)}{2(n - 2r + 1)^2}.$$

Expressions for the expectation and variance are obtained after applying formulas (4.6) and (4.7) of Matsunawa (1985):

$$\mu_n = \frac{1}{n+1}\left(s\, b_s + \sum_{i=s+1}^{r} b_i\right), \quad \sigma_n^2 = \frac{1}{(n+1)^2}\left(s\, b_s^2 + \sum_{i=s+1}^{r} b_i^2\right).$$

Finally condition (10) is fulfilled since $\sigma_n^2 = O(1/n)$. □

Remark 2 Note that in Propositions 5 and 6, the expectation of the limit distribution is the exact expectation of the statistic. Asymptotic critical values for ${}_2Q_{rn}$ and ${}_2Q_{sr,n}$ can be computed from the limit distribution using the corresponding asymptotic variance given in Propositions 5 and 6, or either the corresponding exact variance given in Propositions 3 and 4 (see Theorem 4.3 of Matsunawa 1985). In Table 3, we show the relative error (percentage of its absolute value) with respect to the 5% exact critical values of ${}_2Q_{rn}$ in either situation, for several p proportions of data in a sample of size $n = 30$.

4 Power study and comparisons

In this section, we study the power of the tests based on ${}_2Q_{rn}$ and ${}_2Q_{sr,n}$ for a set of five parametric families of alternative distributions with support contained in the [0, 1] interval. They have been chosen so that either the mean or the variance differs from

Table 3 Relative error (percentage of its absolute value) of the 5% asymptotic critical values of ${}_2Q_{rn}$ computed from the limit distribution using (a) the exact variance of the statistic and (b) the asymptotic variance of the statistic, with respect to the exact critical values, for p proportions of data in a sample of size $n = 30$

p	(a) Exact (%)		(b) Asymp. (%)	
	Lower	Upper	Lower	Upper
0.3	13.69	3.48	29.73	1.47
0.4	6.73	2.30	22.78	3.91
0.5	3.39	1.46	19.93	6.08
0.6	1.47	0.78	18.76	8.20
0.7	0.25	0.20	18.54	10.45
0.8	0.56	0.32	19.02	13.00
0.9	0.95	0.86	20.19	14.04

those of the null distribution, which in each case is obtained for a particular value of the parameter.

A1. Lehmann alternatives. Asymmetric distributions with cdf $F_\theta(x) = x^\theta$, for $0 \leq x \leq 1$ and $\theta > 0$.
A2. Centered distributions having a U-shaped pdf, for $\theta \in (0, 1)$, or wedge-shaped pdf, for $\theta > 1$, whose cdf is given by

$$F_\theta(x) = \begin{cases} \frac{1}{2}(2x)^\theta, & 0 \leq x \leq 1/2, \\ 1 - \frac{1}{2}(2(1-x))^\theta, & 1/2 \leq x \leq 1. \end{cases}$$

A3. Compressed uniform alternatives in the $[\theta, 1-\theta]$ interval, for $0 \leq \theta < 1/2$.
A4. Centered distributions with parabolic pdf $f_\theta(x) = 1 + \theta(6x(1-x) - 1)$, for $0 \leq x \leq 1$ and $-2 \leq \theta \leq 1$.
A5. Centered distributions with pdf given by

$$f_\theta(x) = \frac{\sqrt{6\theta}}{\sqrt{\pi}\, e^{\theta/2}\, \mathrm{erf}\left(\sqrt{3\theta/2}\right)} \exp\{\theta(6x(1-x) - 1)\},$$

for $0 \leq x \leq 1$ and $\theta > 0$, where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$ is the error function.

The power of the tests increases with the proportion of data in the sample. To illustrate this finding in Fig. 2, we depicted the power functions of the 5% significance level test based on ${}_2Q_{rn}$, for $p = 0.4, 0.6, 0.8$ proportions of data in the sample. For each value of the parameter, the power was estimated from $N = 1000$ simulated samples of size $n = 10$ from alternatives A1–A3 as the relative frequency of values of the statistic in the critical region. For each family, we have taken 30 different values of the corresponding parameter. We used the exact critical values listed in Table 1.

We have compared the power of the test based on ${}_2Q_{rn}$ with those based on classical statistics such as the Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling. Modified versions of these statistics for censored samples, as well as their critical values, can be found in Barr and Davidson (1973), Pettitt and Stephens

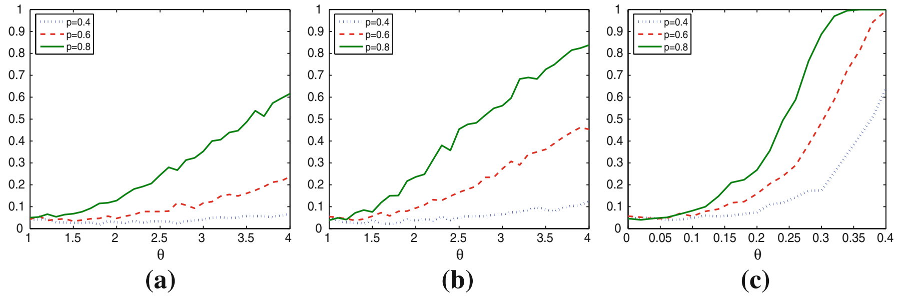


Fig. 2 Power functions of the 5% significance level test based on ${}_2Q_{rn}$, for different p proportions of data in samples of size $n = 10$, for **a** A1 alternative, **b** A2 alternative and **c** A3 alternative

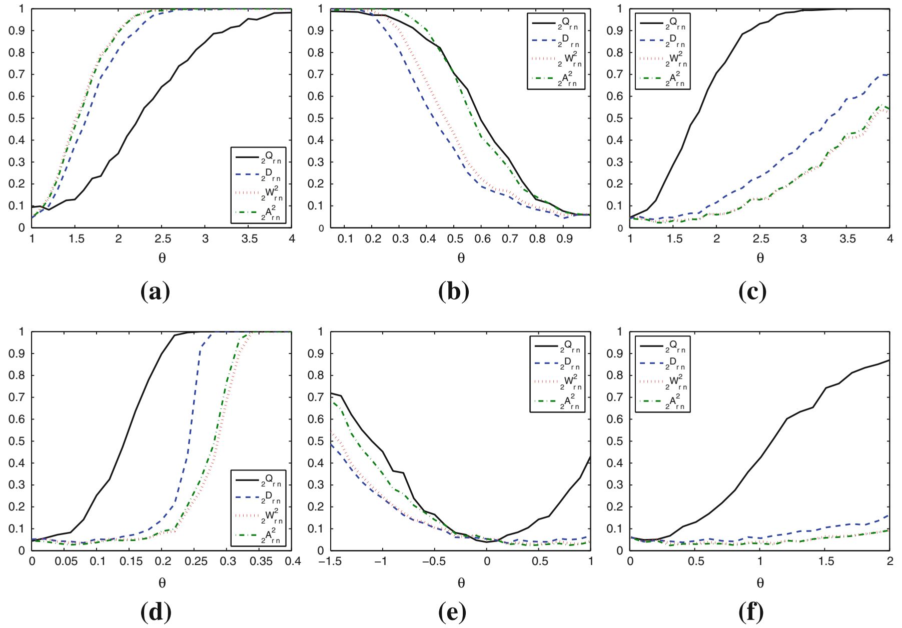


Fig. 3 Power functions of the 5% significance level tests based on ${}_2Q_{rn}$, ${}_2D_{rn}$, ${}_2W^2_{rn}$ and ${}_2A^2_{rn}$ for **a** $p = 0.8$ proportion of data in a sample of size $n = 25$, for **a** A1 alternative, **b**, **c** A2 alternative, **d** A3 alternative, **e** A4 alternative and **f** A5 alternative

(1976) and also in Stephens and D'Agostino (1986). Figure 3 contains the power functions obtained from $N = 1000$ Type II right-censored samples of size $n = 25$ and a proportion of data of $p = 0.8$ for A1–A5 alternatives. For each family, we have taken 30 different values of the corresponding parameter. We have denoted by ${}_2D_{rn}$, ${}_2W^2_{rn}$ and ${}_2A^2_{rn}$ the modified versions of the Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling statistics, respectively. Sample size and data proportion values of $n = 25$ and $p = 0.8$ were chosen so that the critical values reproduced in Stephens and D'Agostino (1986) were appropriate for comparison. For the test based on ${}_2Q_{rn}$, we

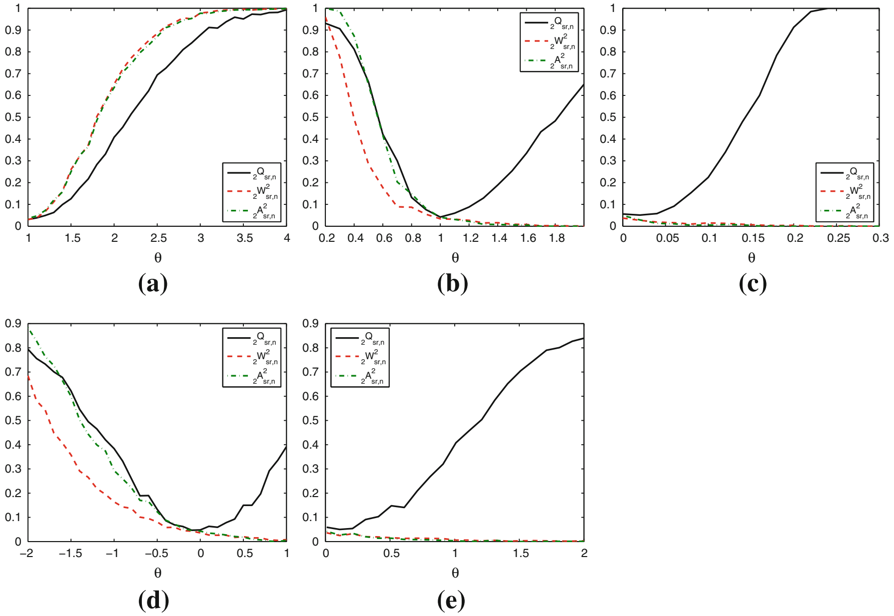


Fig. 4 Power functions of the 5% significance level tests based on ${}_2Q_{sr,n}$, ${}_2W^2_{sr,n}$ and ${}_2A^2_{sr,n}$ for $n = 20$ and $p = 0.9$, for **a** A1 alternative, **b** A2 alternative, **c** A3 alternative, **d** A4 alternative and **e** A5 alternative

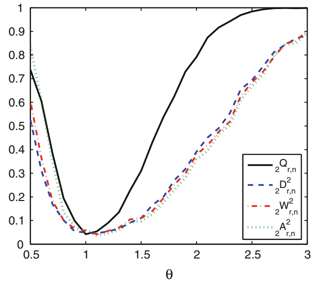
computed the exact critical regions. From Fig. 3, we can observe the good performance of the ${}_2Q_{rn}$ statistic in detecting symmetrical alternatives.

We also compared the power of the test based on ${}_2Q_{sr,n}$ with those based on the Cramér–von Mises and Anderson–Darling statistics. Critical values for the modified versions of these statistics for doubly censored samples were computed by Pettitt and Stephens (1976). Figure 4 contains the power functions obtained from $N = 1000$ Type II doubly censored samples of size $n = 20$ and $p = r/n = 0.9$ ($q = 1 - p, s = q/n$) for A1–A5 alternatives. Once more, for each family we have taken 30 different values of the corresponding parameter. We have denoted by ${}_2W^2_{rs,n}$ and ${}_2A^2_{rs,n}$ the modified versions of the Cramér–von Mises and Anderson–Darling statistics, respectively. For the test based on ${}_2Q_{sr,n}$, we computed the exact critical regions, whereas for ${}_2W^2_{rs,n}$ and ${}_2A^2_{rs,n}$ we considered the asymptotical ones, since Pettitt and Stephens (1976) concluded that the distributions of these statistics converge quickly to the asymptotical ones. From Fig. 4, we can observe the good performance of the ${}_2Q_{sr,n}$ statistic in detecting symmetrical alternatives. Note also that the modified versions of the Cramér–von Mises and Anderson–Darling statistics are biased for A2–A5.

5 Applications

In the context of reliability analysis, the Weibull distribution is perhaps the most widely used lifetime distribution model. The test based on ${}_2Q_{rn}$ turns out to be useful in detecting the Weibull family of distributions from the standard exponential. We

Fig. 5 Power functions of the 5% significance level goodness-of-fit tests based on ${}_2Q_{rn}$, ${}_2D_{rn}$, ${}_2W^2_{rn}$ and ${}_2A^2_{rn}$ for a $p = 0.8$ proportion of data in a sample of size $n = 25$, to detect the Weibull alternative from the standard exponential



illustrate this behavior in Fig. 5, where we depict the power functions obtained from $N = 1000$ Type II right-censored samples of size $n = 25$ and $p = 0.8$ to test the null hypothesis of standard exponentiality versus the alternative of a Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\theta > 0$. These power curves are evaluated on 30 different values of the parameter. Note the good performance of the test based on ${}_2Q_{rn}$ in front of the classical ones based on Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling.

6 Concluding remarks

We adapt the goodness-of-fit test based on Q_n, introduced in Fortiana and Grané (2003), for censored samples. We give tables of exact critical values for different sample sizes and significance levels making these tests easy to implement. The tests based on the modifications of Q_n are consistent for all the families of alternatives studied, and are more powerful than those based on classical statistics, such as the Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling statistics in detecting symmetrical alternatives.

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