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Predicting a cyclic Poisson process

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Abstract We construct and investigate a $(1-\alpha)$-upper prediction bound for a future observation of a cyclic Poisson process using past data. A normal based confidence interval for our upper prediction bound is established. A comparison of the new prediction bound with a simpler nonparametric prediction bound is also given.

Keywords Poisson process · Cyclic intensity function · Prediction upper bound · Confidence interval · Consistency · Asymptotic normality

1 Introduction

Let X be a Poisson process on the real line $\mathbf{R}$ with (unknown) locally integrable intensity function λ. We assume that λ is periodic with period $\tau > 0$ and is positive a.e. w.r.t. Lebesgue measure. We do not assume any parametric form of λ.

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[-n, 0]$.

Our goal in this paper is to propose and investigate a $(1-\alpha)$-upper prediction bound for the time Z of the first event of the Poisson process X after the present time 0, using only a single realization $X(\omega)$ of the cyclic Poisson process X observed in the past,

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i.e. in an interval $[-n, 0]$. A possible application of our $(1-\alpha)$-th upper prediction bounds (cf. (9)) is to forecast daily patient arrivals into an accident and emergency department of a hospital, using past data of previous arrivals. We refer to Au-Yeung, Harder, McCoy, and Knottenbelt (2009) for some applied work in this area. Though the probability model for patient arrivals they propose—a structural time series model with a weekly periodicity—is somewhat different from our semiparametric cyclic Poisson model, their basic statistical set up appears to be very similar to ours. Past real data sets of previous arrivals over a 5 year period (n being large, as in our Theorem 1) are used to predict or forecast the daily arrivals in the near future, say 1–7 days ahead (predicting the first future event, as in (9)). A much simpler but related prediction problem for the homogeneous Poisson process was investigated in Vit (1973): given the number of events in $[-n, 0]$ a prediction interval for the number of events in $[0, y]$ is obtained.

It is well-known that, for any real number $z > 0$, the distribution function of Z is given by :

$$F_Z(z) = \mathbf{P}(Z \leq z) = 1 - \mathbf{P}(Z > z) = 1 - e^{-\Lambda(z)}, \tag{1}$$

with $\Lambda(z) = \int_0^z \lambda(s)\mathrm{d}s$. Let $z_r = z - \tau[\frac{z}{\tau}]$ where for any real number x, $[x]$ denotes the largest integer that less than or equal to x. Then, for any $z > 0$ we have $z = \tau[\frac{z}{\tau}] + z_r$ with $0 < z_r < \tau$. Let $\theta = \tau^{-1}\int_0^\tau \lambda(s)\mathrm{d}s$ be the global intensity of X. Then, for any $z > 0$, we can write

$$\Lambda(z) = \theta\tau\left[\frac{z}{\tau}\right] + \Lambda(z_r). \tag{2}$$

Since $\lambda(s) > 0$ a.e. we also have $\theta > 0$. This latter condition is equivalent to the requirement that, with $\mathbf{P}$-probability one, the realization $X(\omega)$ consists of infinite many points, which is obviously a necessary assumption for obtaining our consistency results.

In view of (1) and (2) our probability model for Z is a semiparametric one, the nonparametric component is given by the function $\Lambda(z_r) = \int_0^{z_r} \lambda(s)\mathrm{d}s$, $0 < z_r < \tau$, whereas the parametric component is described by θ (with known period τ).

Let $\hat{F}_{Z,n}(z)$ denote the empirical counterpart of $F_Z(z)$, using the available past data set at hand, i.e. $X(\omega) \cap [-n, 0]$, the Poisson process X observed in $[-n, 0]$, which is given by

$$\hat{F}_{Z,n}(z) = 1 - e^{-\hat{\Lambda}_n(z)} \tag{3}$$

with

$$\hat{\Lambda}_n(z) = \tau\left[\frac{z}{\tau}\right]\hat{\theta}_n + \hat{\Lambda}_n(z_r) \tag{4}$$

where

$$\hat{\theta}_n = \frac{X([-\tau n_\tau, 0])}{\tau n_\tau}, \tag{5}$$

$$\hat{\Lambda}_n(z_r) = \frac{1}{n_\tau} \sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau]), \tag{6}$$

and $n_\tau = [\frac{n}{\tau}]$.

A $(1-\alpha)$-prediction interval for a future observation of X, i.e. the time of the first event after time 0, is given by $(0, \xi_{Z,1-\alpha})$, where $\xi_{Z,1-\alpha}$ is defined by

$$\xi_{Z,1-\alpha} = \inf\{z : F_Z(z) \geq 1 - \alpha\}, \tag{7}$$

i.e. $\xi_{Z,1-\alpha} = F_Z^{-1}(1-\alpha)$, where F_Z^{-1} denotes the inverse of F_Z. In other words, $\xi_{Z,1-\alpha}$ is nothing but the solution of

$$\mathbf{P}\left(Z \leq \xi_{Z,1-\alpha}\right) = 1 - \alpha. \tag{8}$$

Since the distribution of Z is unknown, we replace equation (7) by its empirical counterpart, i.e. we define $\hat{\xi}_{Z,n,1-\alpha}$ by

$$\hat{\xi}_{Z,n,1-\alpha} = \hat{F}_{Z,n}^{-1}(1-\alpha). \tag{9}$$

As a simple consequence of (9) we have that

$$\hat{F}_{Z,n}(\hat{\xi}_{Z,n,1-\alpha}) = 1 - \alpha + \mathcal{O}_p(n^{-1}),$$

as $n \to \infty$, which in turn easily reduces to the equation

$$\tau \left[\frac{\hat{\xi}_{Z,n,1-\alpha}}{\tau}\right] \hat{\theta}_n + \frac{1}{n_\tau} \sum_{k=1}^{n_\tau} X([-k\tau, \hat{\xi}_{Z,n,1-\alpha,r} - k\tau]) = \ln\left(\frac{1}{\alpha}\right) + \mathcal{O}_p\left(\frac{1}{n}\right), \tag{10}$$

as $n \to \infty$, where $\hat{\xi}_{Z,n,1-\alpha,r} = \hat{\xi}_{Z,n,1-\alpha} - \tau\left[\frac{\hat{\xi}_{Z,n,1-\alpha}}{\tau}\right]$. In other words, $\hat{\xi}_{Z,n,1-\alpha}$ given by (9) is nothing but the (smallest) solution of (10). Note that the non negative $\mathcal{O}_p(n^{-1})$ error term appearing in (10) is due to the fact that $\hat{F}_{Z,n}$ is discrete, a step function with jumps of size $\mathcal{O}_p(n^{-1})$ occuring at points $z = s_i + k\tau$ for positive integers k and events s_i which belong to our past data set $X(\omega) \cap [-n, 0]$.

The density of Z exists and is given by (cf. (1))

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(F_Z(z)\right) = \lambda(z)e^{-\Lambda(z)}.$$

Clearly f_Z is unknown, but we can estimate f_Z at a given point z by

$$\hat{f}_{Z,n}(z) = \hat{\lambda}_{n,K}(z)e^{-\hat{\Lambda}_n(z)}, \tag{11}$$

where, for any $z > 0$, $\hat{\lambda}_{n,K}(z)$ is given by

$$\hat{\lambda}_{n,K}(z) = \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_{-n}^{0} K\left(\frac{x-(z+k\tau)}{h_n}\right) X(\mathrm{d}x), \tag{12}$$

which is the kernel-type estimator of the intensity function λ of X introduced in Helmers, Mangku, and Zitikis (2003) and investigated also in Helmers, Mangku, Zitikis (2005). Here, h_n is a sequence of positive real numbers such that $h_n \downarrow 0$, as $n \to \infty$, and K denotes a kernel function $K : \mathbf{R} \to [0, \infty)$ satisfying the following properties: (K.1) K is a probability density function, (K.2) K is bounded, and (K.3) K has support in $[-1, 1]$. The estimator $\hat{\lambda}_{n,K}(z)$ (cf. (12)) will be used in (16).

The rest of the paper is organized as follows. In Sect. 2, we present our main results. Some asymptotics, which are needed for proving the main results, are given in Sect. 3. Proofs of the main results are presented in Sects. 4 and 5.

2 Main results

The main result of this paper is the following theorem:

Theorem 1 *Suppose that λ is periodic and locally integrable. Let $\hat{\xi}_{Z,n,1-\alpha}$ given by (9), i.e. the smallest solution of (10).*

(i) *(Consistency) We have*

$$\mathbf{P}\left(Z \leq \hat{\xi}_{Z,n,1-\alpha}\right) \ \to \ 1-\alpha, \tag{13}$$

as $n \to \infty$.

(ii) *(Asymptotic Normality) We have*

$$\frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}} \left(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}\right) \ \xrightarrow{d} \ N(0,1) \tag{14}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, where for any $z > 0$

$$q(z) = \left[\frac{z}{\tau}\right]^2 \tau\theta + \left(1 + 2\left[\frac{z}{\tau}\right]\right) \Lambda(z_r) \tag{15}$$

with $z_r = z - \tau[\frac{z}{\tau}]$.

(iii) *(Studentization) Let $\hat{\lambda}_{n,K}$ be the kernel-type estimator of λ given by (12), then we have*

$$\frac{\sqrt{n_\tau}\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})}{\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}} \left(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}\right) \ \xrightarrow{d} \ N(0,1) \tag{16}$$

as $n \to \infty$, provided $h_n \downarrow 0$, $nh_n \to \infty$, $\lambda(\xi_{Z,1-\alpha}) > 0$ and $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, where for any $z > 0$

$$\hat{q}_n(z) = \left[\frac{z}{\tau}\right]^2 \tau\hat{\theta}_n + \left(1 + 2\left[\frac{z}{\tau}\right]\right) \hat{\Lambda}_n(z_r). \tag{17}$$

Note that, a point z is called a Lebesgue point of λ if $\lim_{h\downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(z+x) - \lambda(z)|dx = 0$. This assumption is a rather mild one since the set of all Lebesgue point of λ is dense in $\mathbf{R}$, whenever λ is assumed to be locally integrable. The Lebesgue point assumption also occurs in Helmers et al. (2003, 2005).

It is easy to check (cf. (72) and (73)) that $q(\xi_{Z,1-\alpha})$ appearing in (14) reduces to $\Lambda(\xi_{Z,1-\alpha,r})$, with $\xi_{Z,1-\alpha,r} = \xi_{Z,1-\alpha} - \tau[\frac{\xi_{Z,1-\alpha}}{\tau}]$, whenever $\xi_{Z,1-\alpha} < \tau$ which happens if and only if $\theta\tau > \ln(1/\alpha)$ (cf. (74)). In other words

$$q(\xi_{Z,1-\alpha}) = \Lambda(\xi_{Z,1-\alpha,r}) = \Lambda(\xi_{Z,1-\alpha}) \quad <=> \quad \theta\tau > \ln(1/\alpha). \tag{18}$$

We note in passing that $q(\xi_{Z,1-\alpha}) = \Lambda(\xi_{Z,1-\alpha,r})$ also holds true in the case that θ is assumed to be known. To check this is an easy matter in view of (4); i.e. $\hat{\Lambda}_n(z)$ now reduces to $\tau[\frac{z}{\tau}]\theta + \hat{\Lambda}_n(z_r)$.

An important statistical application of (16) is that it enables one to construct a confidence interval for the $(1-\alpha)$−upper prediction bound $\xi_{Z,1-\alpha}$ as follows:

Corollary 1 *For any significance level p, $0 < p < 1$, a normal based confidence interval for $\xi_{Z,1-\alpha}$ with approximate coverage probability $1 - p$ is given by*

$$I_n = \left(\hat{\xi}_{Z,n,1-\alpha} - \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}}{\sqrt{[\frac{n}{\tau}]}\,\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})},\ \hat{\xi}_{Z,n,1-\alpha} + \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}}{\sqrt{[\frac{n}{\tau}]}\,\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})} \right), \tag{19}$$

where Φ denote the distribution function of a standard normal r.v. and

$$\mathbf{P}\left(\xi_{Z,1-\alpha} \in I_n\right) = 1 - p + o(1), \tag{20}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, $\lambda(\xi_{Z,1-\alpha}) > 0$ and the period τ is known.

The upper prediction bound $\hat{\xi}_{Z,n,1-\alpha}$ can be viewed as an estimator of $\xi_{Z,1-\alpha}$ based on the semiparametric model (1). In contrast, a simple nonparametric estimator of $\xi_{Z,1-\alpha}$ is given by the sample quantile $\hat{\xi}^{NP}_{Z,N,1-\alpha}$, which can be defined as follows. First of all, let the $Z_i's$, $i = 1, 2, \ldots, N$, denote the observed times to the first 'event' in $X(\omega) \cap [-n, 0]$, starting at time $-(n_\tau - i + 1)\tau$, $i = 1, 2, \ldots, n_\tau$, whenever well-defined. For instance, when $X([-n, 0]) = 0$, i.e. the data set at hand is empty, the $Z_i's$ do not exist; i.e. $N = 0$. If there is no 'event' of $X(\omega)$ in the interval $[-(n_\tau - i + 1)\tau, -(n_\tau - i)\tau)$ but there is an 'event' in the next interval $[-(n_\tau - i)\tau, -(n_\tau - i - 1)\tau)$, then we know that $\tau < Z_i < 2\tau$. To obtain Z_{i+1}

we observe the time to the next 'event' of $X(\omega)$ starting from time $-(n_\tau - i - 1)\tau$. More generally, if $N = m$, $m = 0, 1, 2, \ldots, n_\tau$, then precisely m waiting times, say $Z_1, Z_2, \ldots, Z_m$, are observed. Of course, the $Z_i' s$ are i.i.d. with common df F_Z (cf. (1)), due to periodicity of λ.

The sample quantile $\hat{\xi}^{NP}_{Z,N,1-\alpha}$ is defined as

$$\hat{\xi}^{NP}_{Z,N,1-\alpha} = \hat{F}_N^{-1}(1-\alpha) \tag{21}$$

where for any $0 < s < 1$, $\hat{F}_N^{-1}(s) = \inf\{x : \hat{F}_N(x) \geq s\}$, and $\hat{F}_N$ denote the empirical distribution function (df) with random sample size N of $Z_1, Z_2, \ldots, Z_N$, with

$$N = \sum_{i=1}^{n_\tau} \mathbf{I}(X([-(n_\tau - i + 1)\tau, -(n_\tau - i)\tau)) \geq 1) \tag{22}$$

where N has Binomial distribution with parameters n_τ and $1-e^{-\theta\tau}$. Note that for each i, $i = 1, 2, \ldots, n_\tau$, we have $\mathbf{P}(X([-(n_\tau - i + 1)\tau, -(n_\tau - i)\tau)) \geq 1) = 1 - e^{-\theta\tau}$, whereas the summands in (22) are i.i.d.

Using a well-known result for sample quantiles based on a sample with nonrandom sample size (see, e.g., Reiss 1989, p.109) and the fact that $\sqrt{N/(n_\tau(1-e^{-\theta\tau}))} \xrightarrow{p} 1$, as $n \to \infty$, we have

$$\frac{\sqrt{n_\tau(1-e^{-\theta\tau})}\, f_Z(\xi_{Z,1-\alpha})}{\sqrt{\alpha(1-\alpha)}} \left(\hat{\xi}^{NP}_{Z,N,1-\alpha} - \xi_{Z,1-\alpha}\right) \xrightarrow{d} N(0,1) \tag{23}$$

as $n \to \infty$. So, the asymptotic variance of $\hat{\xi}^{NP}_{Z,N,1-\alpha}$ is equal to

$$\frac{\alpha(1-\alpha)}{n_\tau(1-e^{-\theta\tau}) f_Z^2(\xi_{Z,1-\alpha})}, \tag{24}$$

provided $f_Z(\xi_{Z,1-\alpha}) > 0$.

Our prediction bound $\hat{\xi}_{Z,n,1-\alpha}$ uses the whole past data set $X(\omega) \cap [-n, 0]$ at hand. So, in contrast to $\hat{\xi}^{NP}_{Z,N,1-\alpha}$, which based on a Binomial random sample of size N with mean $n_\tau(1-e^{-\theta\tau})$, our proposed prediction bound $\hat{\xi}_{Z,n,1-\alpha}$ is a function of $X([-n, 0])$ data points—a Poisson random sample size with mean $\int_{-n}^{0} \lambda(s)\mathrm{d}s \approx n_\tau \int_0^\tau \lambda(s)\mathrm{d}s = n_\tau\, \theta\tau$. Since for any $\theta\tau > 0$ we have $\theta\tau > (1 - e^{-\theta\tau})$, we use, on the average, a bigger data set in constructing $\hat{\xi}_{Z,n,1-\alpha}$ compared with $\hat{\xi}^{NP}_{Z,N,1-\alpha}$. Comparing (24) with the asymptotic variance of $\hat{\xi}_{Z,n,1-\alpha}$ (cf. (14)) which is equal to

$$\frac{q(\xi_{Z,1-\alpha})}{n_\tau \lambda^2(\xi_{Z,1-\alpha})} = \frac{q(\xi_{Z,1-\alpha}) e^{-2\Lambda(\xi_{Z,1-\alpha})}}{n_\tau f_Z^2(\xi_{Z,1-\alpha})} = \frac{q(\xi_{Z,1-\alpha})\alpha^2}{n_\tau f_Z^2(\xi_{Z,1-\alpha})}, \tag{25}$$

provided $\lambda(\xi_{Z,1-\alpha}) > 0$, one can check—cf. Theorem 2 below—that the variance in (25) is smaller than the variance in (24), as one would perhaps expect.

Theorem 2 *Suppose that λ is periodic and locally integrable. If*

$$\theta\tau > \frac{\ln(1/\alpha)}{3}, \tag{26}$$

then for any $0 < \alpha < 1$, *we have*

$$\frac{q(\xi_{Z,1-\alpha})\alpha^2}{n_\tau f_Z^2(\xi_{Z,1-\alpha})} < \frac{\alpha(1-\alpha)}{n_\tau(1-e^{-\theta\tau})f_Z^2(\xi_{Z,1-\alpha})}, \tag{27}$$

provided $f_Z(\xi_{Z,1-\alpha}) > 0$.

Comparing the r.h.s. of (27) (cf. (24)) with the l.h.s. of (27) in the special case that (18) holds true, i.e. when $q(\xi_{Z,1-\alpha})$ reduces to $\Lambda(\xi_{Z,1-\alpha,r}) = \Lambda(\xi_{Z,1-\alpha}) = \ln(\alpha^{-1})$, a simple calculation shows that

$$\frac{\text{asymp. var }(\hat{\xi}^{NP}_{Z,n,1-\alpha})}{\text{asymp. var }(\hat{\xi}_{Z,n,1-\alpha})} = \frac{\alpha^{-1}-1}{\ln(\alpha^{-1})(1-e^{-\theta\tau})} \tag{28}$$

holds true, provided $\theta\tau > \ln(1/\alpha)$. Condition $\theta\tau > \ln(1/\alpha)$, when $\alpha = 0.05$ (0.10), is equivalent to assuming that, on the average, there are at least 2.9957 (2.3026) events of the process X in any interval of length τ. In particular this means, for instance, when $\alpha = 0.05$ (0.10), the ratio in (28) is bigger or equal to 6.6762 (4.3430), whenever $\theta\tau > 2.9957$ (2.3026).

To obtain a Studentized version of (23) (cf. Ho and Lee 2005; Reiss 1989) one need to estimate θ and $f_Z(\xi_{Z,1-\alpha})$ by $\hat{\theta}_n$ (cf. (5)) and a density estimate $\hat{f}_{Z,n}(\hat{\xi}^{NP}_{Z,N,1-\alpha})$, where $\hat{f}_{Z,n}$ (cf. (11)) denotes an appropriate density estimate of f. For any significance level p, $0 < p < 1$, a normal based confidence interval for $\xi_{Z,1-\alpha}$ with approximate coverage probability $1-p$ is given by

$$I_n^{NP} = \left(\hat{\xi}^{NP}_{Z,N,1-\alpha} - \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\alpha(1-\alpha)}}{\sqrt{[\frac{n}{\tau}](1-e^{-\tau\hat{\theta}_n})}\hat{f}_{Z,n}(\hat{\xi}^{NP}_{Z,N,1-\alpha})},\right.$$
$$\left.\hat{\xi}^{NP}_{Z,N,1-\alpha} + \frac{\Phi^{-1}(1-\frac{p}{2})\sqrt{\alpha(1-\alpha)}}{\sqrt{[\frac{n}{\tau}](1-e^{-\tau\hat{\theta}_n})}\hat{f}_{Z,n}(\hat{\xi}^{NP}_{Z,N,1-\alpha})}\right)$$

where

$$\mathbf{P}\left(\xi_{Z,1-\alpha} \in I_n^{NP}\right) = 1 - p + o(1), \tag{29}$$

as $n \to \infty$, provided $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, $\lambda(\xi_{Z,1-\alpha}) > 0$ and the period τ is known.

Ho and Lee (2005) recently obtained an iterated smoothed bootstrap-t method for setting confidence interval for quantiles like $\hat{\xi}^{NP}_{Z,N,1-\alpha}$ for a nonrandom sample size n, with coverage error of order $n^{-58/57}$, i.e. the classical normal error $\mathcal{O}(n^{-1/2})$, which one would expect in (29), is replaced by a much smaller coverage error $\mathcal{O}(n^{-58/57})$ using an iterated smoothed bootstrap method to approximate the distribution of a Studentized sample quantile. The question remains whether we can obtain such much smaller coverage errors using bootstrap methods for (29) and (20) as well. The authors hope to pursue this matter elsewhere.

In certain cases of interest the intensity function λ is apriori known to be sufficiently smooth and one may estimate $\Lambda(z)$ by $\int_0^z \hat{\lambda}_{n,K}(s)\mathrm{d}s$ instead of $\hat{\Lambda}_n(z)$, for any $z > 0$. In this set up, it might be of interest to construct a confidence region for the function $\Lambda(z)$, $z > 0$ (cf. (2)) using a kernel type estimator for λ, somewhat similar to the methodology used in Helmers, Wang, and Zitikis (2009).

To conclude this section we also want to refer to Helmers and Zitikis (1999) and Helmers and Mangku (2009) for some related statistical work on Poisson intensity functions.

3 Some asymptotics

In this section we investigate the asymptotic behaviour of $\hat{F}_{Z,n}$ (cf. (3)), our estimator of F_Z.

Proposition 1 *Suppose that λ is periodic and locally integrable.*

(i) *(Consistency) For any $z > 0$ we have*

$$\hat{F}_{Z,n}(z) \xrightarrow{p} F_Z(z), \tag{30}$$

as $n \to \infty$.

(ii) *(Asymptotic normality) For any $z > 0$ we have*

$$\frac{\sqrt{n_\tau} e^{\Lambda(z)}}{\sqrt{q(z)}} \left(\hat{F}_{Z,n}(z) - F_Z(z) \right) \xrightarrow{d} N(0,1) \tag{31}$$

as $n \to \infty$, where $N(0,1)$ denotes a standard normal random variable and $q(z)$ is given by (15).

(iii) *(Studentization) For any $z > 0$ we have*

$$\frac{\sqrt{n_\tau} e^{\hat{\Lambda}_n(z)}}{\sqrt{\hat{q}_n(z)}} \left(\hat{F}_{Z,n}(z) - F_Z(z) \right) \xrightarrow{d} N(0,1) \tag{32}$$

as $n \to \infty$, where $\hat{q}_n(z)$ is given by (17).

The error of the normal approximation in (31) is easily seen to be of the classical order $n^{-1/2}$. A correction term of Edgeworth type, correcting not only for bias and skewness but also for the lattice character of the Poisson distribution, can in principle be established using a general result on Edgeworth expansions for lattice distributions due to Kolassa and McCullagh (1990). We also refer to (44) for a simple explicit bias correction term to $\hat{F}_{Z,n}(z)$ of order $n^{-1/2}$.

Next we prove Proposition 1. To check Proposition 1 we need the following lemmas.

Lemma 1 *Suppose that λ is periodic and locally integrable. Then for any $z > 0$ we have*

$$\mathbf{E}\hat{\Lambda}_n(z) = \Lambda(z), \tag{33}$$

$$Var\left(\hat{\Lambda}_n(z)\right) = \frac{q(z)}{n_\tau}, \tag{34}$$

where $q(z)$ is given by (15), and

$$\frac{\sqrt{n_\tau}}{\sqrt{q(z)}}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) \xrightarrow{d} N(0,1) \tag{35}$$

as $n \to \infty$.

Note also that, since $0 \le \Lambda(z_r) \le \tau\theta$, from (34), we have that, for any $z > 0$, $Var(\hat{\Lambda}_n(z)) = \mathcal{O}(n^{-1})$, as $n \to \infty$.

Proof Define $\Lambda^c(z_r) = \int_{z_r}^{\tau} \lambda(s)ds$. Then $\Lambda(z_r) + \Lambda^c(z_r) = \theta\tau$ so that, for any $z > 0$, we have

$$\Lambda(z) = \left(1 + \left[\frac{z}{\tau}\right]\right)\Lambda(z_r) + \left[\frac{z}{\tau}\right]\Lambda^c(z_r) \tag{36}$$

instead of (2). An estimator of $\Lambda^c(z_r)$ is given by

$$\hat{\Lambda}_n^c(z_r) = \frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X((z_r - k\tau, \tau - k\tau)).$$

Note that $\hat{\Lambda}_n(z_r)$ and $\hat{\Lambda}_n^c(z_r)$ are independent and $\hat{\Lambda}_n(z_r) + \hat{\Lambda}_n^c(z_r) = \tau\hat{\theta}_n$. Hence, we can write $\hat{\Lambda}_n(z)$ in (4) as

$$\hat{\Lambda}_n(z) = \left(1 + \left[\frac{z}{\tau}\right]\right)\hat{\Lambda}_n(z_r) + \left[\frac{z}{\tau}\right]\hat{\Lambda}_n^c(z_r), \tag{37}$$

where

$$\mathbf{E}\hat{\Lambda}_n(z_r) = \frac{1}{n_\tau}\sum_{k=1}^{n_\tau}\mathbf{E}X([-k\tau, z_r - k\tau]) = \frac{1}{n_\tau}\sum_{k=1}^{n_\tau}\int_{-k\tau}^{z_r - k\tau}\lambda(x)dx = \Lambda(z_r), \tag{38}$$

similarly $\mathbf{E}\hat{\Lambda}_n^c(z_r) = \Lambda^c(z_r)$, and relation (33) follows.

Next we prove (34). Since $\hat{\Lambda}_n(z_r)$ and $\hat{\Lambda}_n^c(z_r)$ are independent, by (37), we have

$$Var(\hat{\Lambda}_n(z)) = \left(1 + \left[\frac{z}{\tau}\right]\right)^2 Var(\hat{\Lambda}_n(z_r)) + \left[\frac{z}{\tau}\right]^2 Var(\hat{\Lambda}_n^c(z_r)). \tag{39}$$

For any $0 < z_r < \tau$ and any pair of integers (k, j), with $k \neq j$, we have that $X([-k\tau, z_r - k\tau])$ and $X([-j\tau, z_r - j\tau])$ are independent. By a similar calculation as the one in (38), we obtain

$$Var\left(\hat{\Lambda}_n(z_r)\right) = \frac{1}{n_\tau^2} \sum_{k=1}^{n_\tau} Var(X([-k\tau, z_r - k\tau]) = \frac{\Lambda(z_r)}{n_\tau}. \tag{40}$$

Similarly we also have $Var(\hat{\Lambda}_n^c(z_r)) = \Lambda^c(z_r)/n_\tau$. Substituting these variances into the r.h.s. of (39) we obtain

$$Var(\hat{\Lambda}_n(z)) = \frac{(1 + [\frac{z}{\tau}])^2 \Lambda(z_r) + [\frac{z}{\tau}]^2 \Lambda^c(z_r)}{n_\tau}. \tag{41}$$

Since $\Lambda^c(z_r) = \theta\tau - \Lambda(z_r)$, we have

$$\left(1 + \left[\frac{z}{\tau}\right]\right)^2 \Lambda(z_r) + \left[\frac{z}{\tau}\right]^2 \Lambda^c(z_r) = \left[\frac{z}{\tau}\right]^2 \tau\theta + \left(1 + 2\left[\frac{z}{\tau}\right]\right) \Lambda(z_r) = q(z) \tag{42}$$

(cf. (15)). Substituting (42) into (41), we obtain (34).

Next we check (35). An easy calculation using (36), (37), (40) and the line after (40), shows that

$$\begin{aligned}
&\sqrt{n_\tau}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) \\
&= \sqrt{\Lambda(z_r)}\left(1 + \left[\frac{z}{\tau}\right]\right)\left(\frac{\sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau]) - n_\tau \Lambda(z_r)}{\sqrt{n_\tau \Lambda(z_r)}}\right) \\
&\quad + \sqrt{\Lambda^c(z_r)}\left[\frac{z}{\tau}\right]\left(\frac{\sum_{k=1}^{n_\tau} X((z_r - k\tau, \tau - k\tau)) - n_\tau \Lambda^c(z_r)}{\sqrt{n_\tau \Lambda^c(z_r)}}\right).
\end{aligned} \tag{43}$$

Since $\sum_{k=1}^{n_\tau} X([-k\tau, z_r - k\tau])$ is a Poisson random variable with mean $n_\tau \Lambda(z_r)$, the normal approximation for Poisson random variables directly yields that the first term on the r.h.s. of (43) converges in distribution to $N(0, (1 + [\frac{z}{\tau}])^2 \Lambda(z_r))$, as $n \to \infty$, and similarly the second term on the r.h.s. of (43) converges in distribution to $N(0, [\frac{z}{\tau}]^2 \Lambda^c(z_r))$, as $n \to \infty$. Since these two normal r.v.'s are independent, we obtain

$$\sqrt{n_\tau}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) \xrightarrow{d} N(0, \left(1 + \left[\frac{z}{\tau}\right]\right)^2 \Lambda(z_r) + \left[\frac{z}{\tau}\right]^2 \Lambda^c(z_r))$$

as $n \to \infty$, and consequently also (35). This completes the proof of Lemma 1. $\square$

Lemma 2 *Suppose that λ is periodic and locally integrable. Then for any $z > 0$ we have*

$$\mathbf{E}\left(\hat{F}_{Z,n}(z)\right) = F_Z(z) - \frac{q(z)e^{-\Lambda(z)}}{2n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{44}$$

and

$$Var\left(\hat{F}_{Z,n}(z)\right) = \frac{q(z)e^{-2\Lambda(z)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{45}$$

as $n \to \infty$.

Proof First we check (44). By (3), (37) and noting that $\hat{\Lambda}_n(z_r)$ and $\hat{\Lambda}_n^c(z_r)$ are independent, we obtain

$$\mathbf{E}\left(\hat{F}_{Z,n}(z)\right) = 1 - \mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)}\mathbf{E}e^{-[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)}. \tag{46}$$

Using the moment generating function of a Poisson r.v. we obtain

$$\mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)} = \exp\left(n_\tau\Lambda(z_r)\left(e^{-(1+[\frac{z}{\tau}])/n_\tau} - 1\right)\right). \tag{47}$$

A Taylor expansion yields

$$e^{-(1+[\frac{z}{\tau}])/n_\tau} = 1 - \frac{(1+[\frac{z}{\tau}])}{n_\tau} + \frac{(1+[\frac{z}{\tau}])^2}{2n_\tau^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \tag{48}$$

as $n \to \infty$. Substituting (48) into the r.h.s. of (47), we find after some calculations

$$\begin{aligned}\mathbf{E}e^{-(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)} &= \exp\left(n_\tau\Lambda(z_r)\left(-\frac{(1+[\frac{z}{\tau}])}{n_\tau} + \frac{(1+[\frac{z}{\tau}])^2}{2n_\tau^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right)\right)\\ &= e^{-(1+[\frac{z}{\tau}])\Lambda(z_r)} + \frac{(1+[\frac{z}{\tau}])^2\Lambda(z_r)e^{-(1+[\frac{z}{\tau}])\Lambda(z_r)}}{2n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right),\end{aligned} \tag{49}$$

as $n \to \infty$. Similarly we have

$$\mathbf{E}e^{-[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)} = e^{-[\frac{z}{\tau}]\Lambda^c(z_r)} + \frac{[\frac{z}{\tau}]^2\Lambda^c(z_r)e^{-[\frac{z}{\tau}]\Lambda^c(z_r)}}{2n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{50}$$

as $n \to \infty$. Combining (49) and (50) with (46) and using (42), we obtain (44).

Next we verify (45).

$$Var\left(\hat{F}_{Z,n}(z)\right) = \mathbf{E}\left(e^{-\hat{\Lambda}_n(z)}\right)^2 - \left(\mathbf{E}e^{-\hat{\Lambda}_n(z)}\right)^2.$$

Since

$$\mathbf{E}\left(e^{-\hat{\Lambda}_n(z)}\right)^2 = \mathbf{E}\left(e^{-2\hat{\Lambda}_n(z)}\right) = \mathbf{E}e^{-2(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)}\,\mathbf{E}e^{-2[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)},$$

a similar calculation as the one in (47)–(49), with $-(1+[\frac{z}{\tau}])/n_\tau$ replaced by $-2(1+[\frac{z}{\tau}])/n_\tau$, yields that $\mathbf{E}e^{-2(1+[\frac{z}{\tau}])\hat{\Lambda}_n(z_r)}$ is equal to the r.h.s. of (49) with $(1+[\frac{z}{\tau}])$ replaced by $2(1+[\frac{z}{\tau}])$. Similarly, $\mathbf{E}e^{-2[\frac{z}{\tau}]\hat{\Lambda}_n^c(z_r)}$ is equal to the r.h.s. of (50) with $[\frac{z}{\tau}]$ replaced by $2[\frac{z}{\tau}]$. Combining these results with (42), we obtain

$$\mathbf{E}\left(e^{-\hat{\Lambda}_n(z)}\right)^2 = e^{-2\Lambda(z)} + \frac{2q(z)\,e^{-2\Lambda(z)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

as $n \to \infty$. From (44) we easily obtain

$$\left(\mathbf{E}e^{-\hat{\Lambda}_n(z)}\right)^2 = e^{-2\Lambda(z)} + \frac{q(z)\,e^{-2\Lambda(z)}}{n_\tau} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

as $n \to \infty$. Together these results yield (45). This completes the proof of Lemma 2. □

Proof of Proposition 1 By Lemma 2, i.e. $\mathbf{E}(\hat{F}_{Z,n}(z) - F_Z(z)) = \mathcal{O}(n^{-1})$ and $Var(\hat{F}_{Z,n}(z)) = \mathcal{O}(n^{-1})$, as $n \to \infty$, Chebychev inequality yields part (i) of Proposition 1.

To prove part (ii) of Proposition 1, we argue as follows. First we write the l.h.s. of (31) as follows

$$\frac{\sqrt{n_\tau}e^{\Lambda(z)}}{\sqrt{q(z)}}\left(e^{-\Lambda(z)} - e^{-\hat{\Lambda}_n(z)}\right) = \frac{\sqrt{n_\tau}}{\sqrt{q(z)}}\left(1 - e^{-(\hat{\Lambda}_n(z)-\Lambda(z))}\right). \tag{51}$$

By a Taylor expansion

$$e^{-(\hat{\Lambda}_n(z)-\Lambda(z))} = 1 - \left(\hat{\Lambda}_n(z) - \Lambda(z)\right) + \frac{1}{2!}\left(\hat{\Lambda}_n(z) - \Lambda(z)\right)^2 - \cdots$$

and from (35) we know

$$\left(\hat{\Lambda}_n(z) - \Lambda(z)\right) = \frac{\sqrt{q(z)}}{\sqrt{n_\tau}}N(0,1) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{52}$$

as $n \to \infty$, and consequently

$$\left(1 - e^{-(\hat{\Lambda}_n(z)-\Lambda(z))}\right) = \frac{\sqrt{q(z)}}{\sqrt{n_\tau}}N(0,1) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{53}$$

as $n \to \infty$, directly follows. Substituting (53) into r.h.s. of (51), we obtain

$$\frac{\sqrt{n_\tau} e^{\Lambda(z)}}{\sqrt{q(z)}} \left(\hat{F}_{Z,n}(z) - F_Z(z) \right) = N(0,1) + o_p(1),$$

as $n \to \infty$. This completes the proof of part(ii) of Proposition 1.

To establish part (iii) of Proposition 1, it suffices to check, for any $z > 0$,

$$\sqrt{\frac{q(z)}{\hat{q}_n(z)}}\, e^{(\hat{\Lambda}_n(z) - \Lambda(z))} \xrightarrow{p} 1, \tag{54}$$

as $n \to \infty$. By (35) we have $(\hat{\Lambda}_n(z) - \Lambda(z)) = \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. From (52) we know that $(\hat{\Lambda}_n(z_r) - \Lambda(z_r)) = \mathcal{O}_p(n^{-1/2})$ as $n \to \infty$. A simple calculation also shows that

$$(\hat{\theta}_n - \theta) = \mathcal{O}_p(n^{-1/2}), \tag{55}$$

as $n \to \infty$. Consequently $\hat{q}_n(z) = q(z) + \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$, and (54) is immediate. This completes the proof of Proposition 1. $\square$

4 Proof of Theorem 1 and relation (18)

First we prove part (i) of Theorem 1. To check this, we write the l.h.s. of (13) as

$$\mathbf{P}\left(Z \le \xi_{Z,1-\alpha} + (\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \right) = \mathbf{P}\left(Z - (\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \le \xi_{Z,1-\alpha} \right)$$

Then, by (8), proving (13) is equivalent to showing that

$$\mathbf{P}\left(Z - (\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \le \xi_{Z,1-\alpha} \right) \to \mathbf{P}\left(Z \le \xi_{Z,1-\alpha} \right), \tag{56}$$

as $n \to \infty$. To prove (56), it suffices to check

$$(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) \xrightarrow{p} 0, \tag{57}$$

as $n \to \infty$. By (7) and (9), to verify (57), it suffices to show

$$\left(\inf\left\{ x : F_Z(x) + (\hat{F}_{Z,n}(x) - F_Z(x)) \ge 1 - \alpha \right\} - \inf\left\{ x : F_Z(x) \ge 1 - \alpha \right\} \right) \xrightarrow{p} 0, \tag{58}$$

as $n \to \infty$. By part (i) of Proposition 1 (cf. Sect. 3) and the fact that F_Z is continuous in a neighborhood of $\xi_{Z,1-\alpha}$, we obtain (58). This completes the proof of part (i) of Theorem 1.

Next we prove part (ii) of Theorem 1. To verify this, by (7) and (9), we write the l.h.s. of (14) as

$$\frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\inf\left\{x : \hat{F}_{Z,n}(x) \geq 1-\alpha\right\} - \inf\left\{x : F_Z(x) \geq 1-\alpha\right\}\right). \quad (59)$$

By part (ii) of Proposition 1 we can write

$$\hat{F}_{Z,n}(x) = F_Z(x) + \frac{N(0,1)\sqrt{q(x)}}{\sqrt{n_\tau}e^{\Lambda(x)}} + o_p\left(\frac{1}{\sqrt{n}}\right), \quad (60)$$

as $n \to \infty$. By (58), we know from the proof of part (i) of Theorem 1 that

$$\inf\left\{x : \hat{F}_{Z,n}(x) \geq 1-\alpha\right\} - \xi_{Z,1-\alpha} = o_p(1),$$

as $n \to \infty$. Hence, to prove part (ii) of Theorem 1 we only need to consider x in a shrinking neighborhood of $\xi_{Z,1-\alpha}$. Next we show that, for any sequence $\{x_n\}$, such that $|x_n - \xi_{Z,1-\alpha}| = o_p(1)$, as $n \to \infty$, $N(0,1)\sqrt{q(x_n)}e^{-\Lambda(x_n)}/\sqrt{n_\tau}$ in (60) can be replaced by $N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}e^{-\Lambda(\xi_{Z,1-\alpha})}/\sqrt{n_\tau}$. To verify this we have to show

$$\left(\sqrt{q(x_n)}\, e^{-\Lambda(x_n)} - \sqrt{q(\xi_{Z,1-\alpha})}\, e^{-\Lambda(\xi_{Z,1-\alpha})}\right) = o_p(1), \quad (61)$$

as $n \to \infty$. To prove (61), we write the l.h.s. of (61) as

$$\sqrt{q(x_n)}\left(e^{-\Lambda(x_n)} - e^{-\Lambda(\xi_{Z,1-\alpha})}\right) + e^{-\Lambda(x_n)}\left(\sqrt{q(x_n)} - \sqrt{q(\xi_{Z,1-\alpha})}\right). \quad (62)$$

Since $|x_n - \xi_{Z,1-\alpha}| = o_p(1)$ as $n \to \infty$, a simple argument show that, the quantity in (62) is of order $o_p(1)$ as $n \to \infty$, provided

$$\Lambda(x_n) - \Lambda(\xi_{Z,1-\alpha}) = o_p(1), \quad (63)$$

as $n \to \infty$. To verify (63) we note that the l.h.s. of (63) is equal to

$$\int_{\xi_{Z,1-\alpha}}^{x_n} \lambda(s)\mathrm{d}s = \int_0^{(x_n-\xi_{Z,1-\alpha})} \lambda(s+\xi_{Z,1-\alpha})\mathrm{d}s = \lambda(\xi_{Z,1-\alpha})(x_n - \xi_{Z,1-\alpha})$$
$$+ (x_n - \xi_{Z,1-\alpha})\left(\frac{1}{(x_n - \xi_{Z,1-\alpha})}\int_0^{(x_n-\xi_{Z,1-\alpha})} (\lambda(s+\xi_{Z,1-\alpha}) - \lambda(\xi_{Z,1-\alpha}))\mathrm{d}s\right). \quad (64)$$

Since $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ and $|x_n - \xi_{Z,1-\alpha}| = o_p(1)$, then the r.h.s. of (64) is $o_p(1)$, as $n \to \infty$. Hence we have (61).

Next, substituting (60) with $N(0,1)\sqrt{q(x)}e^{-\Lambda(x)}/\sqrt{n_\tau}$ replaced by $N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}e^{-\Lambda(\xi_{Z,1-\alpha})}/\sqrt{n_\tau}$ into (59), we obtain that the l.h.s. of (14) is equal to

$$
\begin{aligned}
&\frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\inf\left\{x : F_Z(x) + \frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\, e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right) \geq 1-\alpha\right\}\right. \\
&\left. - \inf\{x : F_Z(x) \geq 1-\alpha\}\right) \\
&= \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(F_Z^{-1}\left(1-\alpha + \frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\, e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right) - F_Z^{-1}(1-\alpha)\right) \\
&= \frac{\sqrt{n_\tau}\lambda(\xi_{Z,1-\alpha})}{\sqrt{q(\xi_{Z,1-\alpha})}}\left(\frac{N(0,1)\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{n_\tau}\, e^{\Lambda(\xi_{Z,1-\alpha})}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{1}{f_Z(F_Z^{-1}(1-\alpha))} + o_p(1)\right) \\
&= N(0,1) + o_p(1),
\end{aligned}
$$

as $n \to \infty$, where for any $0 < s < 1$, $F_Z^{-1}(s) = \inf\{x : F_Z(x) \geq s\}$. This completes the proof of part (ii) of Theorem 1.

Next we prove part (iii) of Theorem 1. To check this, by (14), it suffices to show

$$
\frac{\sqrt{q(\xi_{Z,1-\alpha})}}{\sqrt{\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha})}} \xrightarrow{p} 1, \tag{65}
$$

and

$$
\frac{\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha})}{\lambda(\xi_{Z,1-\alpha})} \xrightarrow{p} 1, \tag{66}
$$

as $n \to \infty$.

First we consider (65). By writing $\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha}) = q(\xi_{Z,1-\alpha}) + (\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha}) - q(\xi_{Z,1-\alpha}))$, to prove (65), it suffices to check

$$
\left(\hat{q}_n(\hat{\xi}_{Z,n,1-\alpha}) - q(\xi_{Z,1-\alpha})\right) \xrightarrow{p} 0,
$$

as $n \to \infty$. By (15) and (17), and since $(\hat{\xi}_{Z,n,1-\alpha} - \xi_{Z,1-\alpha}) = \mathcal{O}_p(n^{-1/2})$ (cf. (14)) and $(\hat{\theta}_n - \theta) = \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$ (cf. (55)), a simple argument shows that, to prove (65), it suffices to verify

$$
\left(\hat{\Lambda}_n(\hat{\xi}_{Z,n,1-\alpha,r}) - \Lambda(\xi_{Z,1-\alpha,r})\right) \xrightarrow{p} 0, \tag{67}
$$

as $n \to \infty$. To verify (67), we write the l.h.s. of (67) as

$$
\left(\hat{\Lambda}_n(\hat{\xi}_{Z,n,1-\alpha,r}) - \hat{\Lambda}_n(\xi_{Z,1-\alpha,r})\right) + \left(\hat{\Lambda}_n(\xi_{Z,1-\alpha,r}) - \Lambda(\xi_{Z,1-\alpha,r})\right). \tag{68}
$$

By (52) with z replaced by $\xi_{Z,1-\alpha,r}$, we have the second term of (68) is of order $\mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. Next we show the first term of (68) is of the same order, i.e.

$$\left(\hat{\Lambda}_n(\hat{\xi}_{Z,n,1-\alpha,r}) - \hat{\Lambda}_n(\xi_{Z,1-\alpha,r})\right) = \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right), \tag{69}$$

as $n \to \infty$. To verify (69), note that by (14), we have $\hat{\xi}_{Z,n,1-\alpha} = \xi_{Z,1-\alpha} + \mathcal{O}_p(n^{-1/2})$, which also implies $\hat{\xi}_{Z,n,1-\alpha,r} = \xi_{Z,1-\alpha,r} + \mathcal{O}_p(n^{-1/2})$, as $n \to \infty$. The l.h.s. of (69) can be written as

$$\frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X([-k\tau, \hat{\xi}_{Z,n,1-\alpha,r} - k\tau]) - \frac{1}{n_\tau}\sum_{k=1}^{n_\tau} X([-k\tau, \xi_{Z,1-\alpha,r} - k\tau]) = \mathcal{O}_p\left(\frac{1}{\sqrt{n}}\right),$$

as $n \to \infty$, since clearly $X([\xi_{Z,1-\alpha,r} - k\tau, \xi_{Z,1-\alpha,r} - k\tau + \mathcal{O}_p(n^{-1/2})]) = \mathcal{O}_p(n^{-1/2})$ uniformly in k, because λ is periodic and $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ. Hence we have (69). Therefore, we obtain (67).

Next we prove (66). By writing $\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) = \lambda(\xi_{Z,1-\alpha}) + (\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) - \lambda(\xi_{Z,1-\alpha}))$, to prove (66), it suffices to check

$$\left(\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) - \lambda(\xi_{Z,1-\alpha})\right) \xrightarrow{p} 0, \tag{70}$$

as $n \to \infty$. To verify (70), we write the l.h.s. of (70) as

$$\left(\hat{\lambda}_{n,K}(\hat{\xi}_{Z,n,1-\alpha}) - \hat{\lambda}_{n,K}(\xi_{Z,1-\alpha})\right) + \left(\hat{\lambda}_{n,K}(\xi_{Z,1-\alpha}) - \lambda(\xi_{Z,1-\alpha})\right). \tag{71}$$

Since $\xi_{Z,1-\alpha}$ is a Lebesgue point of λ, Theorem 2.1. of Helmers et al. (2003) for the case τ is known yields that the second term of (71) is $o_p(1)$, as $n \to \infty$. By a similar argument as the one used to prove (69), we also obtain the first term of (71) is $o_p(1)$, as $n \to \infty$. Hence we have (70) and (66) follows. This completes the proof of Theorem 1. □

Proof of (18) To begin with, first we show

$$\xi_{Z,1-\alpha} < \tau \quad \text{if and only if} \quad \theta\tau > \ln\left(\frac{1}{\alpha}\right). \tag{72}$$

To verify (72) we argue as follows. By (8) we have $\mathbf{P}\left(Z > \xi_{Z,1-\alpha}\right) = \alpha$. Note also that

$$\xi_{Z,1-\alpha} < \tau \quad <=> \quad \mathbf{P}(Z > \tau) < \mathbf{P}\left(Z > \xi_{Z,1-\alpha}\right) \quad <=> \quad \mathbf{P}(Z > \tau) < \alpha. \tag{73}$$

Since $\mathbf{P}(Z > \tau) = e^{-\theta\tau}$, the statement in (73) is equivalent to

$$e^{-\theta\tau} < \alpha \quad <=> \quad \theta\tau > \ln\left(\frac{1}{\alpha}\right). \tag{74}$$

Combining (73) and (74) we obtain (72). By the l.h.s. of (72) we have $\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right] = 0$. Substituting $\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right] = 0$ into $q(\xi_{Z,1-\alpha})$ we obtain the l.h.s. of (18). This completes the proof of (18). □

5 Proof of Theorem 2

Since $\Lambda(\xi_{Z,1-\alpha,r}) = \Lambda(\xi_{Z,1-\alpha}) - \left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\theta\tau$, $q(\xi_{Z,1-\alpha})$ can also be written as

$$q(\xi_{Z,1-\alpha}) = \left(1 + 2\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\right)\Lambda(\xi_{Z,1-\alpha}) - \left(1 + \left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\right)\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\tau\theta,$$

instead of (15). Then, proving (27) is equivalent to checking that

$$\begin{aligned}&\left\{\left(1 + 2\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\right)\Lambda(\xi_{Z,1-\alpha}) - \left(1 + \left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\right)\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right]\tau\theta\right\}\\ &\quad < \frac{(1-\alpha)}{\alpha(1-e^{-\theta\tau})}.\end{aligned} \tag{75}$$

To prove (75), we split up condition (26) into three cases, namely, case (i) $\theta\tau > \ln(1/\alpha)$, case (ii) $\ln(1/\alpha)/2 < \theta\tau \leq \ln(1/\alpha)$ and case (iii) $\ln(1/\alpha)/3 < \theta\tau \leq \ln(1/\alpha)/2$.

First we consider case (i). In this case, by (72), we have $\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right] = 0$. Since $(1 - e^{-\theta\tau}) < 1$, proving (75) in this case, it suffices to check

$$\Lambda(\xi_{Z,1-\alpha,r}) < \frac{(1-\alpha)}{\alpha}. \tag{76}$$

By noting that $\Lambda(\xi_{Z,1-\alpha,r}) \leq \Lambda(\xi_{Z,1-\alpha}) = -\ln(\alpha)$, to prove (76), it suffices to check

$$-\ln(\alpha) < \frac{(1-\alpha)}{\alpha} \quad <=> \quad \ln(\alpha) + \frac{1}{\alpha} - 1 > 0.$$

Let $h(\alpha) = \ln(\alpha) + 1/\alpha - 1$. We have to show, for all $0 < \alpha < 1$, $h(\alpha) > 0$. To do this, note that $h(1) = 0$ and $h'(\alpha) = \alpha^{-1}(1 - \alpha^{-1})$. Since $0 < \alpha < 1$, we have $\alpha^{-1} > 0$ and $(1 - \alpha^{-1}) < 0$, which implies $h'(\alpha) < 0$ for all $0 < \alpha < 1$. Hence, $h(\alpha)$ is monotone decreasing to 0 in interval (0, 1), which implies $h(\alpha) > 0$ for all $0 < \alpha < 1$. Therefore we obtain (27).

Next we consider case (ii). By a similar argument as the proof of (72), we have

$$\frac{\ln(1/\alpha)}{2} < \theta\tau \le \ln\left(\frac{1}{\alpha}\right) <=> \theta\tau \le \ln\left(\frac{1}{\alpha}\right) < 2\theta\tau \quad \text{if and only if} \quad \tau \le \xi_{Z,1-\alpha} < 2\tau. \tag{77}$$

By (77) we have $\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right] = 1$. Since $\theta\tau \le \ln(1/\alpha)$, we have $(1-\alpha)/(1-e^{-\theta\tau}) \ge 1$. Then to prove (75) in this case, it suffices to check

$$\{3\Lambda(\xi_{Z,1-\alpha}) - 2\tau\theta\} < \frac{1}{\alpha}. \tag{78}$$

By noting that $\Lambda(\xi_{Z,1-\alpha}) = \ln(1/\alpha)$ and $2\theta\tau > \ln(1/\alpha)$ (cf. (77)), to prove (78), it suffices to verify

$$\{3\ln(1/\alpha) - \ln(1/\alpha)\} < \frac{1}{\alpha} <=> \alpha(\ln(1/\alpha)) < \frac{1}{2}. \tag{79}$$

Since the maximum value of $\alpha(\ln(1/\alpha))$ is e^{-1} (when $\alpha = e^{-1}$) which is less than $1/2$, we have (79).

Next we consider case (iii). Similarly to (77), we now have

$$\frac{\ln(1/\alpha)}{3} < \theta\tau \le \frac{\ln(1/\alpha)}{2} <=> 2\theta\tau \le \ln\left(\frac{1}{\alpha}\right) < 3\theta\tau \quad \text{iff} \quad 2\tau \le \xi_{Z,1-\alpha} < 3\tau. \tag{80}$$

By (80) we have $\left[\frac{\xi_{Z,1-\alpha}}{\tau}\right] = 2$. Next to prove (75) in this case, it suffices to check

$$\{5\Lambda(\xi_{Z,1-\alpha}) - 6\tau\theta\} < \frac{(1-\alpha)}{\alpha(1-e^{-\theta\tau})}. \tag{81}$$

Since $\theta\tau \le \ln(1/\alpha)/2$, we have $(1-e^{-\theta\tau}) \le (1-\alpha^{1/2})$. By noting that $\Lambda(\xi_{Z,1-\alpha}) = \ln(1/\alpha)$ and $\theta\tau > \ln(1/\alpha)/3$ (cf. (80)), to prove (81), it suffices to verify

$$\{5\ln(1/\alpha) - 2\ln(1/\alpha)\} < \frac{(1-\alpha)}{\alpha(1-\alpha^{1/2})} <=> \frac{(1-\alpha)}{3(1-\alpha^{1/2})} + \alpha\ln(\alpha) > 0.$$

Define

$$f_3(\alpha) = \frac{(1-\alpha)}{3(1-\alpha^{1/2})} + \alpha\ln(\alpha) = \frac{1}{3} + \frac{\sqrt{\alpha}}{3} + \alpha\ln(\alpha).$$

It remains to show that $f_3(\alpha) > 0$ for all $0 < \alpha < 1$. To verify this, first note that $f_3'(\alpha) = 1/(6\sqrt{\alpha}) + \ln(\alpha) + 1$ and $f_3''(\alpha) = -1/(12\alpha^{3/2}) + 1/\alpha$. Since the first derivative f_3' is monotone increasing on (0, 1) with $f_3'(0) = -\infty$ and $f_3'(1) = 7/6$, the function f_3' is equal to zero for exactly one value of α, namely 0.266351 ... Because

$f_3''(0.266351) = 3.148215 > 0$, we can conclude that $f_3(0.266351) = 0.152998$ is the minimum value of f_3 on $(0, 1)$. Hence $f_3(\alpha) > 0$ for all $0 < \alpha < 1$. This completes the proof of Theorem 2. □

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